UNIFYING CUBICAL AND MULTIMODAL TYPE THEORY

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ABSTRACT. In this paper we combine the principled approach to programming with
modalities of multimodal type theory (MTT) with the computationally well-behaved
identity types of cubical type theory (CTT). The result—cubical modal type theory
(Cubical MTT)—has the desirable features of both systems. In fact, the whole is more
than the sum of its parts: Cubical MTT validates desirable extensionality principles for
modalities that MTT only supported through ad hoc means.

We investigate the semantics of Cubical MTT and provide an axiomatic approach to
producing models of Cubical MTT based on the internal language of topoi and use it to
construct presheaf models. Finally, we demonstrate the practicality and utility of this
axiomatic approach to models by constructing a model of (cubical) guarded recursion in a
cubical version of the topos of trees. We then use this model to justify an axiomatization of
Löb induction and thereby use Cubical MTT to smoothly reason about guarded recursion.

1. INTRODUCTION

Type theorists have produced a plethora of variants of type theory since the introduction
of Martin-Löf type theory (MLTT), each of which attempts to refine MLTT to enhance its
expressivity or convenience. Unfortunately, even extensions of type theory which appear
orthogonal cannot be carelessly combined and so expert attention is frequently necessary to
trust a type theory with the correct suite of extensions. We are particularly interested in two
families of extensions to MLTT: *cubical type theories* [CCHM18, ABC+21] and *(Fitch-style)
modal type theories* [Clo18, BCM+20, GKNB20].

Both of these lines of research aim to increase the expressivity of type theory, but along
different axes. Cubical type theory gives a higher-dimensional interpretation to the identity

type and thereby obtains a more flexible notion of equality along with a computational interpretation of *univalent foundations* [Uni13]. Meanwhile, modal dependent type theory (MTT) extends MLTT with connectives which need not commute with substitution, allowing for type theory to model phenomena such as guarded recursion, axiomatic cohesion, or parametricity [BMSS12, Shu18, ND18].

While combining two complex type theories like this is not a task to be undertaken frivolously, experience has shown that a principled mixture of these two type theories would be useful. Indeed, modalities naturally appear in synthetic homotopy theory [Shu18] but without an apparatus like MTT, these extensions must be handled in an ad-hoc way, which can easily disrupt the desirable properties of type theory. Moreover, cubical type theory's version of equality validates invaluable principles like function extensionality and a cubical variant of MTT would thus eliminate the need to postulate such principles [BBC+19, KMV21].

Prior to discussing how $MTT_\square$ fuses these systems, we set the stage by introducing both cubical type theory and multimodal type theory MTT separately.

Cubical type theory. Cubical type theory originates from the broader class of *homotopy type theories* whose study was instigated by Voevodsky's observation that the intensional identity type could be realized as a (homotopical) path space [KL21]. This shift in perspective justifies the inclusion of the univalence axiom which postulates an equivalence between equalities between elements of a universe and equivalences of the denoted types. While univalence has many pleasant consequences (function extensionality, effectivity of quotients, etc.), the addition of such an axiom disrupts crucial properties of the type theory. In particular, it is not possible to compute in such a theory. In order to rectify this issue, cubical type theory was introduced and shown to simultaneously support computation and validate the univalence axiom.

Cubical type theory extends MLTT with an interval object $\mathbb{I}$ along with a function space—a path space—to hypothesize over it. Intuitively, $\mathbb{I}$ abstracts the interval $[0, 1]$ and this connection is enhanced by the addition of operations, *e.g.* $0, 1 : \mathbb{I}$. Accordingly, $\mathbb{I} \to A$ classifies lines in A and by iterating we obtain squares, cubes, or arbitrary n-cubes in A.

While homotopy type theory recasts the identity type from MLTT as a path in a space, cubical type theory begins with paths and forces them to behave like an identity type. We therefore isolate a subtype $\mathsf{Path}_A(a_0, a_1)$ of paths $\mathbb{I} \to A$ whose values at 0 and 1 are a_0 and a_1. By further equipping A with *Kan operations*, $\mathsf{Path}_A(a_0, a_1)$ becomes a new model for the identity type. The Kan operations are subtle and complex but without them $\mathsf{Path}_A(-, -)$ is not even an equivalence relation. The flexibility afforded by these Kan operations, however, allows cubical type theory to support a computational effective interpretation of univalence.

MTT and Fitch-style modal type theories. We now turn from cubical type theory to modal type theory. Like cubical type theory, the motivations for modalities—a type constructor which does not necessarily respect substitution—are semantic; many models of type theory have further structure which does not directly commute with substitution, but would still be useful to internalize. For instance, the global sections comonad of a presheaf model is frequently essential for working internally to the model [CBGB15, LOPS18, Shu18], but it almost never commutes with substitution.

Unfortunately, much of the convenience of MLTT hinges on the fact that all operators do commute with substitution, so introducing a modality tends to disrupt nearly every

property of importance. In order to cope with this contradiction, modal type theories like MTT [GKNB20] have carefully isolated classes of modalities which can be safely incorporated into type theory while preserving properties such as canonicity and normalization [Gra21]. In fact, MTT can be instantiated with an arbitrary collection of (weak) dependent right adjoints [BCM+20]. The metatheory of MTT applies irrespective of the choice of mode theory, and therefore MTT can be seen as a *general* modal type theory, suitable for instantiation with a wide variety of different modalities to specialize the type theory to capture particular models.

In practice, MTT is parameterized by a mode theory, a 2-category used to describe the modalities in play. The objects of the mode theory correspond to individual copies of MLTT linked together by the modalities, the morphisms of the mode theory. The 2-cells of the mode theory induce natural transformations between modalities. For instance, in order to model the global sections comonad we pick a mode theory with one mode m, one modality μ, and a collection of 2-cells shaping this modality into a comonad, *e.g.*, 2-cells $\mu \longrightarrow \mu \circ \mu$ and $\mu \longrightarrow \mathsf{id}_m$ subject to several equations. Upon instantiating MTT with this mode theory we obtain a type theory with a comonad already known to satisfy many important metatheorems.

Towards Cubical MTT. In [GKNB20], each mode of MTT contains a copy of MLTT. Unfortunately, the type theory therefore inherits the well-known limitations of MLTT: the identity type is difficult to work with, function extensionality is not satisfied. One can resolve these issues by adding equality reflection to MTT, but this disrupts the decidability of type-checking. Moreover, several modalities arise in the context of homotopy type theory [Shu18] and adapting MTT to these models requires simply postulating univalence, thereby conferring the same set of difficulties.

We introduce MTT$_\square$, a unification of Cubical type theory and MTT. To a first approximation, MTT$_\square$ replays the theory of MTT, after replacing MLTT with cubical type theory. One thereby obtains a modal type theory with different modes—now copies of cubical type theory—connected by arbitrary dependent right adjoints. Moreover, each mode now satisfies univalence and function extensionality.

Beyond this, a computation rule for Kan operations in modal types is needed for computation (and thus for normalization), but it is not immediately well-typed. Indeed, a key challenge in combining MTT with CTT is exactly to capture sufficient interactions between modal and cubical aspects for this rule to be well-typed, whilst not making greater demands than the intended models can bear.

The switch from using MLTT to using CTT in MTT also improves modal types. For instance, in [Gra21] special care is taken to include *crisp induction* in order to validate the modal counterpart to function extensionality. While this addition preserves normalization and canonicity, modal extensionality is independent of MTT. In MTT$_\square$, by contrast, the corresponding principle is simply provable (Theorem 3.1).

We show that models of MTT$_\square$ can be assembled from certain models of cubical type theory connected by right adjoints. In particular, given coherent functors $f_\mu : \mathcal{C}_n \longrightarrow \mathcal{C}_m$ there is a model of MTT$_\square$ which realizes context categories as $\mathbf{PSh_{cSet}}(\mathcal{C}_m)$ and modalities as right Kan extension $(f_\mu)_*$. This ensures, for example, that despite the complexity of both MTT and cubical type theory, it is easily shown that MTT$_\square$, appropriately instantiated, models cubical guarded recursion [BBC+19, KMV21]. Indeed, we show that the cubical underpinnings of MTT$_\square$ improve the presentation of guarded recursion in MTT [Gra21].

The development of this theory of models also implies the soundness of $\mathsf{MTT}_\square$. We further conjecture, but do not prove, that the normalization results of [Gra21] for MTT and [SA21] for cubical type theory can be appropriately combined into a normalization proof for $\mathsf{MTT}_\square$.

Contributions. We contribute $\mathsf{MTT}_\square$, a synthesis of cubical type theory and MTT, and a foundation for multimodal and higher-dimensional type theories. In section 2 we recapitulate the basics of cubical type theory and MTT and in section 3 we present the definition of $\mathsf{MTT}_\square$ and further prove the aforementioned modal extensionality principle. Finally, in section 4 we introduce the model theory of $\mathsf{MTT}_\square$ and further show that cubical presheaves and certain essential geometric morphisms assemble into models. We then apply this theory to cubical guarded recursion in section 5 and explore the improved presentation of guarded recursion.

2. Cubical and multimodal type theory

We now recall the essential details of cubical type theory [CCHM18] and MTT [GKNB21]. We focus mostly on the portions relevant for $\mathsf{MTT}_\square$ and refer the reader to the existing literature for a more thorough introduction. Readers familiar with both systems may safely proceed to section 3.

2.1. **Cubical type theory.** CTT begins by extending MLTT with a primitive interval $\mathbb{I}$ and algebraic structure to mimic the real interval $[0, 1]$. A term of type A which assumes *dimension variables* $i, j, k : \mathbb{I}$ corresponds to n-cubes (lines, squares, cubes) in A. Concretely, we add a new context formation rule $\Gamma, i : \mathbb{I}$ and a new syntactic class of *dimension terms* $\Gamma \vdash r : \mathbb{I}$:

   (Abstract interval) $r, s : \mathbb{I}$ $::=$ $i \mid 0 \mid 1 \mid 1 - r \mid r \vee s \mid r \wedge s$

We further identify dimension terms by the equations of De Morgan algebras.

Next, we add *path types*: a dependent product over the interval. The rules for this new connective are given in Figure 1 and—just as with dependent products—path types enjoy β and η rules stating *e.g.* $(\lambda i.\, p)(r) = p[r/i]$. In addition to β and η, paths are equipped with further equalities reducing them at endpoints, *e.g.*, given $p : \mathsf{Path}_A(a, b)$ then $p(0) = a : A$.

Out of the box, paths define a relation on types which is reflexive and symmetric and which validates extensionality principles such as function extensionality. They are not, however, transitive and it is this deficiency that leads to the *Kan composition operation* which forms the backbone of CTT. Intuitively, this composition operation lets us complete an open box (an n-cube missing certain faces) to an n-cube. In order to formalize this geometric intuition we add the face lattice $\mathbb{F}$, a class of propositions, which we use to codify the open boxes to be filled. We therefore add another syntactic class $\Gamma \vdash \phi : \mathbb{F}$:

   (Face lattice) $\phi, \psi : \mathbb{F}$ $::=$ $\bot \mid \top \mid (r = 0) \mid (r = 1) \mid \phi \vee \psi \mid \phi \wedge \psi$

Elements of $\mathbb{F}$ are identified by the equations of distributive lattices as well as the additional equation $(r = 0) \wedge (r = 1) = \bot$.

Elements $\phi : \mathbb{F}$ are used to *restrict* a context by assuming them, denoted Γ, ϕ. This allows us to locally force ϕ to hold so that, *e.g.*, $i : \mathbb{I}, (i = 0) \vdash i = 0 : \mathbb{I}$. We can take advantage of an assumption ϕ in our context through *systems*. The rules for systems are

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathsf{Path}_A(a,b)} \qquad \frac{\Gamma, i : \mathbb{I} \vdash p : A}{\Gamma \vdash \lambda i.\, p : \mathsf{Path}_A(p(0), p(1))} \qquad \frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash p : \mathsf{Path}_A(a,b)}{\Gamma \vdash p(r) : A}$$

$$\frac{\Gamma, \phi_i \vdash a_i : A \qquad \Gamma, \phi_0 \wedge \phi_1 \vdash a_0 = a_1 : A}{\Gamma, \phi_0 \vee \phi_1 \vdash \{\, \phi_0\ a_0, \phi_1\ a_1\, \} : A} \qquad \frac{}{\Gamma, \bot \vdash \{\ \} : A}$$

$$\frac{\Gamma \vdash \phi : \mathbb{F} \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \phi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A[0/i]\,[\phi \mapsto u[0/i]]}{\Gamma \vdash \mathsf{comp}^i_A\,[\phi \mapsto u]\,u_0 : A[1/i]\,[\phi \mapsto u[1/i]]}$$

Figure 1: Selected rules from CTT

given in Figure 1; intuitively they state that to construct an element in $\Gamma, \bigvee_i \phi_i \vdash u : A$, it suffices to construct elements $\Gamma, \phi_i \vdash u_i : A$ that agree on the overlap. An element constructed through this amalgamation restricts appropriately *e.g.*, $\Gamma, \phi_i \vdash u = u_i : A$.

We are frequently concerned with the behavior of a term after some assumption ϕ—its *boundary*—and therefore introduce notation for it. We write $\Gamma \vdash a : A\,[\phi \mapsto u]$ as shorthand for (1) a being a term of A and (2) under the assumption ϕ, $a = u : A$. With this machinery, we can now formulate the Kan composition rule, shown in Figure 1. This one principle is sufficient to prove the properties we expect of identity types, including J (path induction).[1]

We review the proof that path equality is transitive to give the reader a sense of the rule. Let A be a type that does not depend on any interval variables, and suppose $a, b, c : A$, $p : \mathsf{Path}_A(a,b)$, and $q : \mathsf{Path}_A(b,c)$. We form three lines in A: The paths p and q as well as the constant a line. Using these we form a system depending on i and j given by $\{\, (i = 0)\, a, (i = 1)\, q(j)\, \}$. The path p forms an extension of this system at $j = 0$, and so we can form the path $\lambda i.\, \mathsf{comp}^j_A\,[(i = 0) \mapsto a, (i = 1) \mapsto q(j)]\, p(i)$, which will reduce to the $j = 1$ part of our designed system, i.e., a at $i = 0$ and $q(1) = c$ at $i = 1$, thus proving transitivity. We think of the input data as an open box with bottom p and sides given by the system; in this analogy the composition forms a lid completing the outer square.

Remark 2.1. Given $x, y : A$, we write $x \equiv y$ when there exists an element of $\mathsf{Path}_A(x,y)$.

2.2. **Multimodal type theory.** To a first approximation, MTT is a collection of copies of MLTT for each $m \in \mathcal{M}$, connected by dependent adjunctions [BCM$^+$20]. MTT is parameterized by a mode theory $\mathcal{M}$ [LS16], a strict 2-category. Each object $m, n, o : \mathcal{M}$ is assigned to a distinct *mode*: a copy of MLTT complete with its own judgments ($\Gamma\ \mathsf{cx}\,@\,m$, $\Gamma \vdash M : A\,@\,m$, ...). Many of the rules of MTT (dependent sums, inductive types, etc.) are *mode-local* and taken as-is from MLTT; the interesting features of MTT arise from allowing modes to interact with each other.

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[1]Unlike in MLTT, however, path induction reduces on reflexivity only up to a path.

$$\frac{\Gamma, \{\mu\} \vdash A \,@\, n}{\Gamma \vdash \langle \mu \mid A \rangle \,@\, m} \qquad\qquad \frac{\Gamma, \{\mu\} \vdash a : A \,@\, n}{\Gamma \vdash \mathsf{mod}_\mu(a) : \langle \mu \mid A \rangle \,@\, m}$$

$$\frac{\Gamma, \{\mu\} \vdash A \,@\, n}{\Gamma, x : (\mu \mid A) \;\mathsf{cx} \,@\, m \qquad \Gamma, x : (\mu \mid A), \{\mu\} \vdash x : A \,@\, n}$$

$$\frac{\mu : n \to m \qquad \nu : o \to n \qquad \Gamma \;\mathsf{cx} \,@\, m \qquad \Gamma.\{\mu\}, \{\nu\} \vdash A \,@\, o \qquad \Gamma, \{\mu\} \vdash a : \langle \nu \mid A \rangle \,@\, n \\ \Gamma, x : (\mu \mid \langle \nu \mid A \rangle) \vdash B \,@\, m \qquad \Gamma, y : (\mu \circ \nu \mid A) \vdash b : B[\mathsf{mod}_\nu(y)/x] \,@\, m}{\Gamma \vdash \mathsf{let}_\mu \; \mathsf{mod}_\nu(y) \leftarrow a \; \mathsf{in} \; b : B[a/x] \,@\, m}$$

Figure 2: Selected rules from MTT

Modes are connected to each other by *modalities*: a morphism $\mu : n \longrightarrow m$ induces a modality sending types A from mode n to types $\langle \mu \mid A \rangle$ in mode m. The actual presentation of modalities is necessarily complex because of dependence: given a type $\Gamma \vdash A \,@\, n$, there is no obvious choice of context in mode m for $\langle \mu \mid A \rangle$. MTT resolves this tension in Fitch-style [Clo18] and pairs each modality with an adjoint action on contexts. In particular, given a modality $\mu : n \longrightarrow m$, we can obtain a new context $\Gamma, \{\mu\} \;\mathsf{cx} \,@\, n$ from $\Gamma \;\mathsf{cx} \,@\, m$. Further rules turn $-, \{\mu\}$ into a functor between categories of contexts and substitutions at modes n and m; intuitively a left adjoint to the modal type former $\langle \mu \mid - \rangle$. See the introduction and formation rules for $\langle \mu \mid - \rangle$ recorded in Figure 2.

The elimination rule for $\langle \mu \mid - \rangle$ is complex because we cannot 'reverse' the introduction rules without violating the admissibility of substitution. Instead, MTT annotates each variable in the context and replaces $\Gamma, x : A$ with $\Gamma, x : (\mu \mid A)$. One can access a variable annotated with μ if and only if it appears behind precisely $-, \{\mu\}$. The elimination rule uses these annotations to implement a *modal induction principle* and allows one to reduce the process of constructing an element of $B[a/x]$ for some $\Gamma.\{\nu\} \vdash a : \langle \mu \mid A \rangle \,@\, m$ to the case $B[\mathsf{mod}_\mu(y)/x]$ for some fresh $y : (\nu \circ \mu \mid A)$; see the precise rule in Figure 2.

Thus far we have not mentioned the (2-)categorical structure of $\mathcal{M}$, but it is this additional structure which allows us to control the behavior of modalities. In fact, the operation sending a modality μ to $-, \{\mu\}$ is 2-functorial so that, *e.g.*, $\Gamma, \{\mu\}, \{\nu\} = \Gamma, \{\mu \circ \nu\}$. This fact is reflected into types; the assignment $\mu \mapsto \langle - \mid - \rangle$ is essentially pseudo-functorial. Consequently, a 2-cell $\alpha : \mu \longrightarrow \nu$ in $\mathcal{M}$ induces a substitution $\Gamma, \{\nu\} \vdash \{\alpha\}_\Gamma : \Gamma, \{\mu\} \,@\, m$ which in turn produces a collection of functions $\langle \mu \mid - \rangle \to \langle \nu \mid - \rangle$. By modifying the equalities and 2-cells of $\mathcal{M}$, we can force $\langle \mu \mid - \rangle$ to become, *e.g.*, a comonad, a right adjoint, etc.

MTT also extends dependent products to hypothesize over types annotated with modalities other than id, i.e., to abstract over $x : (\mu \mid A)$ [GKNB21]. While these *modal dependent products* are convenient, we refrain from discussing them here for simplicity.

3. MTT□

Cubical multimodal type theory (MTT□) enhances MTT with a better behaved identity type and univalence by combining it with CTT. Like MTT, MTT□ is parameterised by a mode theory, i.e., a 2-category of modes, modalities, and 2-cells. Whereas MTT has a copy of MLTT at each mode, MTT□ has a copy of CTT. A guiding principle in the design of MTT□ is that cubical and modal aspects should be *orthogonal* to each other. In particular,

$$\text{INT/EXC} \quad \frac{\Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma, \{\mu\} \vdash r^\mu : \mathbb{I}_n @ n}$$

$$\text{FACE/EXC} \quad \frac{\Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma, \{\mu\} \vdash \phi^\mu : \mathbb{F}_n @ n}$$

$$\text{SB/EXC-INT} \quad \frac{}{\Gamma, i : \mathbb{I}_m, \{\mu\} \vdash \sigma_\mu : \Gamma, \{\mu\}, i : \mathbb{I}_n @ m}$$

$$\text{SB/EXC-FACE} \quad \frac{\Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma, \phi, \{\mu\} \vdash \tau_\mu : \Gamma, \{\mu\}, \phi^\mu @ m}$$

$$\text{SB/EXC-INT-INV} \quad \frac{}{\Gamma, \{\mu\}, i : \mathbb{I}_n \vdash \sigma^\mu : \Gamma, i : \mathbb{I}_m, \{\mu\} @ n}$$

$$\text{SB/EXC-FACE-INV} \quad \frac{\Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma, \{\mu\}, \phi^\mu \vdash \tau^\mu : \Gamma, \phi, \{\mu\} @ n}$$

$$\text{TERM-EQ/COMP-MOD}$$
$$\frac{\Gamma, i : \mathbb{I}_m, \{\mu\} \vdash A @ n \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m \qquad \Gamma, \phi, i : \mathbb{I}_m, \{\mu\} \vdash u : A @ n \qquad \Gamma, \{\mu\} \vdash u_0 : A[0/i] @ n \qquad \Gamma, \phi, \{\mu\} \vdash u[0/i] = u_0 : A[0/i] @ n}{\Gamma \vdash \mathsf{comp}^i_{\langle \mu | A \rangle} [\phi \mapsto \mathsf{mod}_\mu(u)] \, \mathsf{mod}_\mu(u_0) = \mathsf{mod}_\mu(\mathsf{comp}^i_A [\phi^\mu \mapsto u[\sigma^\mu \circ \tau^\mu]] \, u_0) : \langle \mu | A \rangle[1/i] @ m}$$

Figure 3: Selected rules of $\mathsf{MTT}_\square$, presupposing $\mu : n \longrightarrow m$ and $\Gamma \, \mathsf{cx} @ m$.

the primitives of each system should interact as little as possible with primitives from the other. To realize this, we add certain exchange principles in subsection 3.1 which are then applied in subsection 3.2 to define composition for modal types.

We detail the novel aspects of $\mathsf{MTT}_\square$ and refer to Appendix A for an exhaustive account.

3.1. **Cubical exchange.** The orthogonality principle of $\mathsf{MTT}_\square$ dictates that the interval should be minimally impacted by the action of a modality on the context. Accordingly, we add *exchange operations*. Concretely, given a dimension term $\Gamma \vdash r : \mathbb{I}_m @ m$, we add a new dimension term $\Gamma, \{\mu\} \vdash r^\mu : \mathbb{I}_n @ n$, see INT/EXC in Figure 3. We demand that this operation is a morphism of De Morgan algebras and is lax natural, e.g. $(r \wedge s)^\mu = r^\mu \wedge s^\mu$ and $r^{\mu \circ \nu} = (r^\mu)^\nu$. Using this operation, it is possible to derive the *exchange substitution* $\Gamma, i : \mathbb{I}_m, \{\mu\} \vdash \sigma_\mu : \Gamma, \{\mu\}, i : \mathbb{I}_n @ m$, see SB/EXC-INT. Finally, we add an inverse to this, see SB/EXC-INT-INV, making $\Gamma, i : \mathbb{I}_m, \{\mu\}$ isomorphic to $\Gamma, \{\mu\}, i : \mathbb{I}_n$, once again in accordance with the orthogonality principle.

The case is entirely symmetrical for elements of the face lattice and the corresponding restricted contexts. Concretely, given a face $\Gamma \vdash \phi : \mathbb{F}_m @ m$, we add a new face $\Gamma, \{\mu\} \vdash \phi^\mu : \mathbb{F}_n @ n$, see FACE/EXC. Similarly to before, we require this operation to be a morphism of bounded lattices and be lax natural, but we further require that it matches with the corresponding operation on the interval via the equation $(r = 0)^\mu = (r^\mu = 0)$. Thus, $-^\mu$ commutes with everything but dimension variables, meaning that ϕ^μ is precisely $-^\mu$ applied to every dimension variable in ϕ. Just as before, we can derive a substitution, to which we add an inverse, making $\Gamma, \phi, \{\mu\}$ and $\Gamma, \{\mu\}, \phi^\mu$ isomorphic, see SB/EXC-FACE and SB/EXC-FACE-INV.

As we will see shortly, these rules are sufficient for composition in modal types, but one may still wonder if there would be merit in the addition of inverses to r^μ and ϕ^μ; after all, this would be in line with our orthogonality principle. It turns out, however, that such an addition would lead to a significant restriction of what models are valid, in particular,

it would invalidate our model of guarded recursion in section 5, and we thus refrain from making such an addition.

As mentioned, the exchange operations respect the 2-categorical structure of the mode theory, and since the exchange substitutions are derived from the simpler exchange operations, they inherit this property. We now record one such coherence explicitly for future use. $\mathsf{MTT}_\square$ inherits a *weakening substitution* from CTT: $\uparrow^\phi \colon \Gamma, \phi \longrightarrow \Gamma$. On may show that the two canonical substitutions $\Gamma, \{\mu\}, \phi^\mu \longrightarrow \Gamma, \{\mu\}$ are equal. Explicitly, we equate the direct restriction substitution $\uparrow^{\phi^\mu}$ to $\uparrow^\phi .\{\mu\} \circ \tau^\mu$.

3.2. **Composition in modal types.** Now we can tackle the problem of composition in $\mathsf{MTT}_\square$. Composition is, as the other cubical rules, added to the system locally and satisfies the same computation rules familiar from CTT for standard type formers. Modal types will support a computation principle similar to that of inductive types, allowing us to commute $\mathsf{mod}_\mu(-)$ with comp. Thus the status of composition in modal types is similar to that of natural numbers, where composition is a formal operation that reduces on canonical forms, as opposed to *e.g.* dependent sums.

The desired 'reduction' is TERM-EQ/COMP-MOD. We take a moment to show that the conclusion of this rule is well-typed and that the result has the expected boundary.

Inspecting the assumptions of this rule, we note that all but one are completely equivalent to a composition problem in $\langle \mu \mid A \rangle$ where the input terms are of form $\mathsf{mod}_\mu(u)$ and $\mathsf{mod}_\mu(u_0)$—with the exception that the assumption $u[0/i] = u_0$ is slightly stronger than necessary—so it is clear that the left-hand side of this equality is well-typed. That the right-hand side is well-typed is more subtle. We would like to show that $\mathsf{mod}_\mu(\mathsf{comp}_A [\phi^\mu \mapsto u[\sigma_\mu \circ \tau_\mu]] u_0)$ is well-typed. Inspecting the rule for composition in Figure 1, we see that we must first verify the following:

(1) $\Gamma, \{\mu\} \vdash \phi^\mu : \mathbb{F}_n @ n$
(2) $\Gamma, \{\mu\}, \phi^\mu, i : \mathbb{I}_m \vdash u[\sigma^\mu \circ \tau^\mu] : A[\sigma^\mu] @ n$
(3) $\Gamma, \{\mu\} \vdash u_0 : A[0/i] @ n$
(4) $\Gamma, \phi, \{\mu\} \vdash u[0/i] = u_0 : A[0/i] @ n$

All of these are immediate results of the premises of TERM-EQ/COMP-MOD. In particular, item 4 is precisely the aforementioned stronger premise.

Assured that both sides of TERM-EQ/COMP-MOD are well-typed, we show that that the right-hand side of this equality satisfies the same boundary condition as the left-hand side, i.e., that the right-hand side is equal to $\mathsf{mod}_\mu(u)[1/i]$ under ϕ.[2]

First, we observe that in context $\Gamma, \{\mu\}, \phi^\mu \, \mathsf{cx} @ n$ we have the following:

$$\mathsf{comp}_A [\phi^\mu \mapsto u[\sigma^\mu \circ \tau^\mu]] \, u_0 = u[\sigma^\mu \circ \tau^\mu][1/i] = u[1/i][\tau^\mu]$$

Next, we recall that weakening by an assumption of the face lattice commutes with face exchange. Given that the former is silent in our notation and the latter is not, this leads to the somewhat odd equation $\Gamma, \phi \vdash \mathsf{mod}_\mu(m) = \mathsf{mod}_\mu(m[\tau_\mu]) : \langle \mu \mid A \rangle @ m$ when $\Gamma, \{\mu\} \vdash m : A @ m$. Combining these two equations, we have the following in context Γ, ϕ:

$$\mathsf{mod}_\mu(\mathsf{comp}_A [\phi^\mu \mapsto u[\sigma^\mu \circ \tau^\mu]] \, u_0) = \mathsf{mod}_\mu((\mathsf{comp}_A [\phi^\mu \mapsto u[\sigma^\mu \circ \tau^\mu]] \, u_0)[\tau_\mu]) = \mathsf{mod}_\mu(u[1/i])$$

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[2]This requirement is a further sanity check on the rule; without this equality the right-hand side would not solve the same composition problem as the left and the equation would be highly suspect.

3.3. **Extensionality principles in $\mathsf{MTT}_\square$.** Function extensionality in $\mathsf{MTT}_\square$ follows directly from function extensionality in CTT, since the rules used in CTT are all available mode-locally in $\mathsf{MTT}_\square$. We will now prove *modal extensionality*, which cannot be proven in MTT and cannot be stated in CTT:

Theorem 3.1. *Given $\mu : n \to m$ and a pair of terms $\Gamma \vdash a, b : A @ n$, where A is the elements of some term of the universe, there is an equivalence $\mathsf{modext}_{a,b} : \langle \mu \mid \mathsf{Path}_A(a,b) \rangle \simeq \mathsf{Path}_{\langle \mu | A \rangle}(\mathsf{mod}_\mu(a), \mathsf{mod}_\mu(b))$.*

Proof. We define the map $\mathsf{modext}_{a,b}$ and show it to be an equivalence by constructing an inverse. Fix $\Gamma \vdash m : \langle \mu \mid \mathsf{Path}_A(a,b) \rangle @ m$ and $\Gamma \vdash r : \mathbb{I}_m @ m$. We wish to construct $\mathsf{modext}_{a,b}(m)(r)$. By modal induction it suffices to consider the case where $m = \mathsf{mod}_\mu(p)$ for some $\Gamma, \{\mu\} \vdash p : \mathsf{Path}_A(a,b) @ n$. Because p lives in a locked context whereas r does not, we need an exchange operation. We form $\Gamma, \{\mu\} \vdash r^\mu : \mathbb{I}_n @ n$, and define $\mathsf{modext}_{a,b}(m)(r) = \mathsf{mod}_\mu(p(r^\mu))$. Towards verifying that we obtain an inverse using path induction, note that for $\Gamma, \{\mu\} \vdash c : A @ n$ we have that

$$\mathsf{modext}_{c,c}(\mathsf{mod}_\mu(\mathsf{refl}(c))) = \mathsf{refl}(\mathsf{mod}_\mu(c)).$$

Next we define a map $\mathsf{modext}_{a,b}^{-1}$ in the inverse direction.[3] By based path induction along with careful modal induction, it suffices to define only $\mathsf{modext}_{a,a}^{-1}(\mathsf{refl}(\mathsf{mod}_\mu(a)))$. In this case we define $\mathsf{modext}_{a,a}^{-1}(\mathsf{refl}(\mathsf{mod}_\mu(a))) = \mathsf{mod}_\mu(\mathsf{refl}(a))$. Furthermore, by calculation:

$$\mathsf{modext}_{c,c}^{-1}(\mathsf{refl}(\mathsf{mod}_\mu(c))) \equiv \mathsf{mod}_\mu(\mathsf{refl}(c)).$$

Note that we obtain only a path rather than a judgmental equality because path induction computes only up to a higher path in cubical type theory.

Lastly, we prove that these maps form an equivalence. Let $\Gamma \vdash m : \langle \mu \mid \mathsf{Path}_A(a,b) \rangle @ m$. We are to find a path between $\mathsf{modext}_{a,b}^{-1}(\mathsf{modext}_{a,b}(m))$ and m. It suffices to do so when $m = \mathsf{mod}_\mu(\mathsf{refl}(c))$ for some $\Gamma, \{\mu\} \vdash c : A @ n$ where we compute:

$$\mathsf{modext}_{c,c}^{-1}(\mathsf{modext}_{c,c}(m)) = \mathsf{modext}_{c,c}^{-1}(\mathsf{refl}(\mathsf{mod}_\mu(c))) \equiv m.$$

For the reverse direction, let $\Gamma \vdash p : \mathsf{Path}_{\langle \mu | A \rangle}(\mathsf{mod}_\mu(a), \mathsf{mod}_\mu(b)) @ m$. We need a path between p and $\mathsf{modext}_{a,b}(\mathsf{modext}_{a,b}^{-1}(p))$. We again reduce to the case where $p = \mathsf{refl}(\mathsf{mod}_\mu(c))$ and compute from there: $\mathsf{modext}_{c,c}(\mathsf{modext}_{c,c}^{-1}(p)) \equiv \mathsf{modext}_{c,c}(\mathsf{mod}_\mu(\mathsf{refl}(c))) = p$. $\square$

4. Semantics of $\mathsf{MTT}_\square$

Section 3 toured through $\mathsf{MTT}_\square$ informally, but in fact, $\mathsf{MTT}_\square$ can be presented as a particular *generalized algebraic theory* [Car78]. This automatically gives rise to a category of models—a variant of the standard *categories with families* [Dyb96]—with several desirable properties such as the initiality of syntax. However, $\mathsf{MTT}_\square$ is complex and the induced definition of model is nearly intractable to manipulate, let alone construct.

We fracture the definition of model into more manageable pieces, making heavy use of the natural models of MTT [Awo18, GKNB21]. In order to construct these models, we introduce *cubical MTT cosmoi.* This is a more compact structure supplementing MTT cosmoi [Gra21] with the ingredients necessary to internally construct a model of CTT [OP18, LOPS18]. In

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[3]We will only need path induction and modal induction rather than path abstraction to define $\mathsf{modext}_{a,b}^{-1}$, meaning that it can also be defined in MTT.

practice, cosmoi are easier to obtain and suffice for the most important models *e.g.* those in cubical presheaves.

4.1. **Models of MTT$_\square$.** We now present the definition of a model of MTT$_\square$ with mode theory $\mathcal{M}$. To begin with, we require a strict 2-functor $[\![-]\!] : \mathcal{M}^{\text{coop}} \longrightarrow \mathbf{Cat}$, known as the *modal context structure*. Intuitively, this 2-functor assigns each mode to a category of contexts. From this viewpoint, the functor $[\![\mu]\!] : [\![m]\!] \longrightarrow [\![n]\!]$ sends a morphism $\mu : n \longrightarrow m$ to the adjoint action $-.\{\mu\}$ contexts and the 2-cells $[\![\alpha]\!]$ interpret the natural transformations $\{\alpha\}$. We now specify the remaining structure on top of this functor.

Mode-local structure. Each mode $[\![m]\!]$ should contain a complete model of CTT, and we specify this in the language of natural models [Awo18] which provides a concise description of the connectives of type theory.

As a model of CTT, $[\![m]\!]$ has an interval object $\mathbf{I}_m : [\![m]\!]$. Just as in CTT, we require that $\mathbf{I}_m$ is a De Morgan algebra and that all products $- \times \mathbf{I}_m$ exist.

Next, we require a pair of presheaves $\widetilde{\mathcal{T}}_m, \mathcal{T}_m : \mathbf{PSh}([\![m]\!])$ representing respectively the collection of terms and types in a given context. Moreover, there is a projection map $\tau_m : \widetilde{\mathcal{T}}_m \longrightarrow \mathcal{T}_m$ which sends a term to its type. This universe is closed under dependent sums, products, etc. Each mode also contains an interpretation of the face lattice $\mathbb{F}_m : \mathbf{PSh}([\![m]\!])$ and face restriction which is used to specify the composition operations. While complex, this piece of the model is unchanged from CTT so we omit further details.

Modal types. Next we turn to the modal aspect of a model: modal context extension and modal types. Both of these structures are specified exactly as in MTT [GKNB21], with the small caveat that we require an additional equality for composition operation on modal types.

Cubical exchange. Finally, we must address the interaction of the functors $[\![\mu]\!]$ and the intervals and face lattices. Mirroring the syntax, we require natural transformations $\mathbb{I}_\mu : \mathbf{y}(\mathbf{I}_m) \longrightarrow [\![\mu]\!]^* \mathbf{y}(\mathbf{I}_n)$ that are pointwise morphisms of De Morgan algebras and that assemble with $\mathbf{y}(\mathbf{I}_m)$ into a lax natural transformation. From this, we can define a morphism, which we require to be have an inverse:

$$([\![\mu]\!]\pi_1, \mathbb{I}_{\mu, \Gamma \times \mathbf{I}_m}(\pi_2)) : [\![\mu]\!](\Gamma \times \mathbf{I}_m) \longrightarrow [\![\mu]\!](\Gamma) \times \mathbf{I}_n$$

The above is replayed for face lattices: We require natural transformations $\mathbb{F}_\mu : \mathbb{F}_m \longrightarrow [\![\mu]\!]^* \mathbb{F}_n$ that are pointwise morphisms of bounded lattices and that assemble with $\mathbb{F}_m$ into a lax natural transformation. From this can be defined a canonical morphism $[\![\mu]\!](\Gamma.[\phi]_m) \longrightarrow [\![\mu]\!].[\mathbb{F}_{\mu, \Gamma}(\phi)]_n$, which we require has an inverse.

4.2. **Cubical MTT cosmoi.** Even after the repackaging of models detailed in subsection 4.1, a model of MTT$_\square$ is still a complex object. There are two orthogonal aspects to this complexity: (1) constructing the models of cubical type theory in each mode and (2) constructing the network of modalities and their actions on contexts. Fortunately, there already exists a technique to simplify (1); rather than construct a model of cubical type theory directly, [OP18] and [LOPS18] have shown that any topos satisfying a handful of axioms supports a model of cubical type theory. Moreover, (2) is partially addressed in [Gra21] by the notion of an MTT cosmos which abstracts several of the difficulties of

constructing a model of MTT. We now unify these two ideas to define *cubical MTT cosmoi* and prove that they induce a model of MTT$_\square$.

MTT cosmoi. We will first recall the definition of MTT cosmoi and prove that they induce models of MTT.

Definition 4.1. A cosmos is a pseudofunctor $F : \mathcal{M} \longrightarrow \mathbf{Cat}$ that takes objects to locally cartesian closed categories and morphisms to right adjoints. We denote the left adjoint to $F(\mu)$ by $F_!(\mu)$.

A cosmos abstracts from the basic situation we encountered in subsection 4.1: a 2-functor F picking out categories of contexts and the actions of modalities between them. In this case, we were primarily concerned not with the category $F(m)$, but with *presheaves over* $F(m)$. After all, it is the category of presheaves which hosts types and terms and where we formulate structures like context extension. Careful inspection reveals that we only require the locally Cartesian closed structure of $\mathbf{PSh}(F(m))$ when formulating the rest of the structure of a model, so it is natural to require only that each mode of a cosmos is locally Cartesian closed. Indeed, on top of this skeleton we can transport more of the structure of a model to cosmoi:

Definition 4.2. An extensional MTT cosmos is a cosmos F such that each mode is equipped with a morphism $\tau_m : \widetilde{\mathcal{T}}_m \longrightarrow \mathcal{T}_m$ representing inducing a universe closed under dependent products, sums, booleans, and extensional identity types. We further require that each map $F(\mu) : F(n) \longrightarrow F(m)$ induce a *dependent right adjoint* [BCM+20].[4]

We have leveraged same intuition as natural models to regard $\mathcal{T}_m$ (respectively $\widetilde{\mathcal{T}}_m$) as the collection of types (resp. terms), but without any representability requirements (they cannot be stated in LCCCs). Requiring closure of these universes under the connectives of MTT then ensures that an MTT cosmos induces a model of MTT in the sense of [GKNB21]. Prior to proving this, however, we require the following standard category-theoretic fact:

Lemma 4.3. *Let $\mathcal{C}$ be a 2-category and $F : \mathcal{C} \longrightarrow \mathbf{Cat}$ be a pseudofunctor such that each $F(f)$ is a right adjoint $F_!(f) \dashv F(f)$ then the left adjoints extend to a pseudofunctor $F_! : \mathcal{C}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$.*

Theorem 4.4. *An extensional MTT cosmos induces a model of extensional MTT with modal context structure is pseudonaturally equivalent to the pseudofunctor of left adjoints induced by Lemma 4.3. If the pseudofunctor of left adjoints is a strict 2-functor, it is equal to the modal context structure.*

Proof. Fix an extensional MTT cosmos $F : \mathcal{M} \longrightarrow \mathbf{Cat}$. By Lemma 4.3, the left adjoints $F_!(\mu)$ assemble into a pseudofunctor $F_! : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$. We may strictify this functor to get a strict 2-functor $\widehat{F_!} : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$ and a pseudonatural equivalence of categories $\alpha : F_! \longrightarrow \widehat{F_!}$. We claim that $\widehat{F_!}$ models extensional MTT.

For each mode m, we define the universe of types and terms as the Yoneda embedding of the universe already present in $\widehat{F_!}(m)$: $\widehat{\tau_m} = \mathbf{y}(\alpha_m(\tau_m))$. Because $\widehat{F_!}(m)$ is finitely complete, this is a representable natural transformation. Moreover, since both α_m and $\mathbf{y}$ preserve LCCC structure, this universe is closed under the types in the types cosmos: dependent

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[4]These are essentially the same as modal types in MTT, further equipped with a syntactically ill-behaved but semantically convenient elimination rule.

right adjoints for each μ, dependent sums, products, booleans, and extensional identity types etc. Thus by [GKNB21, Theorem 7.1] $\widehat{F}_!$ models extensional MTT. □

Cubical cosmoi. An MTT cosmos F interprets each mode as an LCCC $F(m)$ because locally Cartesian closed structure is sufficient to specify the connectives of MTT. Unfortunately, it is not sufficient to apply the techniques of [OP18] and [LOPS18] and internally construct a model of CTT. We therefore isolate the notion of *a LOPS topos*,[5] containing precisely the required structure. We further define cubical cosmoi as particularly well-behaved networks of LOPS topoi.

Definition 4.5. A LOPS topos is an elementary topos $\mathcal{E}$ with a hierarchy of universes, an object of cofibrations $\mathbf{F}_{\mathcal{E}} \hookrightarrow \Omega$ and a tiny interval object $\mathbf{I}_{\mathcal{E}}$ subject to the Orton-Pitts axioms.[6]

Theorem 4.6 [LOPS18]**.** *There exists a model of CTT in every LOPS topos.*

Consider a cosmos $F : \mathcal{M} \longrightarrow \mathbf{Cat}$ such that each $F(m)$ is a LOPS topos. Theorem 4.6 then implies that each mode is a model of cubical type theory, but on its own this is insufficient to conclude that F assembles into a model of MTT$_\square$; we must ensure that each $F(\mu)$ properly preserves interval objects and face lattices. In order to isolate what further properties we must impose on F, we briefly revisit how one interprets constructs in cubical type theory such as systems and face restrictions in a LOPS topos.

Extending a context $X : \mathcal{E}$ by an interval variable is given product: $X \times \mathbf{I}_{\mathcal{E}}$. The structure of this context extension and of dimension terms more generally follow directly from the universal property of products along with the De Morgan algebra structure on $\mathbf{I}_{\mathcal{E}}$; a dimension term in context X is realized by a morphism $X \longrightarrow \mathbf{I}_{\mathcal{E}}$. Similarly, an element of the face lattice context X is interpreted by a morphism. $X \longrightarrow \mathbf{F}_{\mathcal{E}}$. Restricting a context by such a face is given by pullback:[7]

$$
\begin{array}{ccc}
\{X \mid \phi\} & \longrightarrow & \mathbf{1} \\
\downarrow & & \downarrow{\scriptstyle \top} \\
X & \xrightarrow{\quad \phi \quad} & \mathbf{F}_{\mathcal{E}}
\end{array}
$$

Returning to our original question, we can now isolate some of the additional structure required by a cosmos valued in LOPS topoi to induce a model of MTT$_\square$. In particular, a right adjoint between LOPS topoi will correctly model a dependent right adjoint which appropriately respects cubical structure when its left adjoint satisfies the following conditions:

Definition 4.7. A morphism of LOPS topoi is a geometric morphism $F_! \dashv F : \mathcal{E} \longrightarrow \mathcal{E}'$ along with an isomorphism of De Morgan algebras $\alpha_F : F_!(\mathbf{I}_{\mathcal{E}'}) \cong \mathbf{I}_{\mathcal{E}}$ and of bounded distributive lattices $\beta_F : F_!(\mathbf{F}_{\mathcal{E}'}) \cong \mathbf{F}_{\mathcal{E}}$. The maps α_F and β_F are required to be compatible in the sense that $\beta_F \circ F_!(- = 0) = (- = 0) \circ \alpha_F$.

---

[5]Named after the authors of [LOPS18]

[6]In fact, we make use of a slight strengthening of axioms presented by [OP18] in order to ensure that $\mathbf{I}_{\mathcal{E}}$ is an internal De Morgan algebra rather than a connection algebra.

[7]We have used the familiar set-comprehension notation for restriction by a face. Because $\mathbf{F}_{\mathcal{E}}$ is a subobject of Ω, this coincides with the standard interpretation of this notation in a topos.

Remark 4.8. Whilst $F(\mathbf{I}_{\mathcal{E}})$ is a De Morgan algebra, we make no assumption that it be isomorphic to $\mathbf{I}_{\mathcal{E}'}$. Doing so would correspond to adding an inverse to INT/EXC, but as mentioned in subsection 3.1, this is not valid in the model of guarded recursion in section 5; explicity, the right adjoint *later* does not preserve the interval. The case is the same for the object of cofibrations.

We now have built up the machinery necessary to define the desired fusion of LOPS topoi and MTT cosmoi:

Definition 4.9. A cubical MTT cosmos $F : \mathcal{M} \longrightarrow \mathbf{Cat}$ is an extensional MTT cosmos satisfying the following additional restrictions:

- $F(m)$ is a LOPS topos for each mode m,
- $F(\mu)$ is a morphism of LOPS topoi for each modality μ,
- The interval and face lattice isomorphisms are pseudonatural.

Theorem 4.10. *Any cubical MTT cosmos F induces a model of $\mathsf{MTT}_{\square}$ with modal context structure pseudonaturally equivalent to the pseudofunctor of left adjoints induced by Lemma 4.3. If the pseudofunctor of left adjoints is a strict 2-functor, it is equal to the modal context structure.*

Proof. A cubical MTT cosmos is in particular an extensional MTT cosmos, meaning that all the rules from extensional MTT can be modelled with the strictified 2-functor $\widehat{F_!}$ by Theorem 4.4. Equivalence preserves being a LOPS topos, and thus $\widehat{F_!}(m) \simeq F_!(m) = F(m)$ is a LOPS topos, implying we can model all the mode-local rules added from CTT (including composition structures for all non-modal types) with Theorem 4.6. It thus remains to construct the exchange principles and composition structures on modal types.

We claim that $\widehat{F_!}$ (or rather, the pseudofunctor of right adjoints $\widehat{F}$ induced by the dual of Lemma 4.3) also has the cubical components of being a cubical MTT cosmoi. We have already argued that $\widehat{F}(m)$ is a LOPS topos since it is equivalent to $F(m)$, the fact the naturality squares of these equivalences commute up to natural isomorphism is enough to show that $\widehat{F}(\mu)$ is a morphism of LOPS topoi, and the pseudonatural coherence of the isomorphisms is preserved since the equivalences cohere pseudonaturally.

As a consequence of this, we have at each mode $m : \mathcal{M}$ a De Morgan algebra and a bounded distributive lattice $\mathbf{I}_{\widehat{F}(m)}, \mathbf{F}_{\widehat{F}(m)} : \widehat{F}(m)$ and for each modality $\mu : n \longrightarrow m$ coherent structure-preserving isomorphisms $\alpha_{\widehat{F}(\mu)} : \widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)}) \cong \mathbf{I}_{\widehat{F}(n)}$ and $\beta_{\widehat{F}(\mu)} : \widehat{F_!}(\mu)(\mathbf{F}_{\widehat{F}(m)}) \cong \mathbf{F}_{\widehat{F}(n)}$.

To define the interval exchange operation, take a dimension term $r : \Gamma \longrightarrow \mathbf{I}_{\widehat{F}(m)}$, and define $\mathbb{I}_{\mu,\Gamma}(r)$ as the composite:

$$\widehat{F_!}(\mu)(\Gamma) \xrightarrow{\widehat{F_!}(\mu)(r)} \widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)}) \xrightarrow{\alpha_{\widehat{F}(\mu)}} \mathbf{I}_{\widehat{F}(n)}$$

The naturality of $\mathbb{I}_\mu$ follows from the functoriality of $\widehat{F_!}(\mu)$, and they cohere lax naturally since the isomorphisms cohere. To see that it defines morphisms of De Morgan algebras, consider the concretely the case of $\wedge$. Preservation is then the commutativity of the following

diagram:

$$\widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)}) \times \widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)}) \xrightarrow{\alpha_{\widehat{F}(\mu)} \times \alpha_{\widehat{F}(\mu)}} \mathbf{I}_{\widehat{F}(n)} \times \mathbf{I}_{\widehat{F}(n)}$$

$$(\widehat{F_!}(\mu)(r), \widehat{F_!}(\mu)(s)) \qquad (\widehat{F_!}(\mu)(\pi_1), \widehat{F_!}(\mu)(\pi_2)) \qquad \wedge$$

$$\widehat{F_!}(\mu)(\Gamma) \xrightarrow{\widehat{F_!}(\mu)((r,s))} \widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)} \times \mathbf{I}_{\widehat{F}(m)}) \xrightarrow{\widehat{F_!}(\mu)(\wedge)} \widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)}) \xrightarrow{\alpha_{\widehat{F}(\mu)}} \mathbf{I}_{\widehat{F}(n)}$$

The right rectangle commutes since the $\alpha_{\widehat{F}(\mu)}$ preserves $\wedge$, and the left triangle commutes by the uniqueness of morphisms to products. Preservation of the other connectives follow similarly.

The final thing to verify for intervals is that the uniquely determined dashed arrow in the following diagram has an inverse:

$$\widehat{F_!}(\mu)(\Gamma) \xleftarrow{\pi_1} \widehat{F_!}(\mu)(\Gamma) \times \mathbf{I}_{\widehat{F}(n)}$$

$$\widehat{F_!}(\mu)(\pi_1) \qquad\qquad\qquad \pi_2$$

$$\widehat{F_!}(\mu)(\Gamma \times \mathbf{I}_{\widehat{F}(m)}) \xrightarrow{\widehat{F_!}(\mu)(\pi_2)} \widehat{F_!}(\mu)(\mathbf{I}_{\widehat{F}(m)}) \xrightarrow{\alpha_{\widehat{F}(\mu)}} \mathbf{I}_{\widehat{F}(n)}$$

This follows from $\alpha_{\widehat{F}(\mu)}$ being invertible and $\widehat{F_!}(\mu)$ preserving finite limits.

Replaying these arguments for the face lattices completes the construction of the exchange principles.

Lastly, we will construct the compositions structures on modal types. For this, we note that the model of extensional MTT obtained from Theorem 4.4 supports an inverse operation to $\mathsf{mod}_\mu(-)$ such that every element of a modal type is of the form $\mathsf{mod}_\mu(a)$. Therefore, the equation TERM-EQ/COMP-MOD can be taken as-is to fully define a composition structure. $\square$

4.3. **Cubical presheaves.** The intended model of cubical type theory is a variant on the standard presheaf mode with types interpreted as a variant of Kan cubical sets [CCHM18]— particular presheaves on the cube category $\square$ realized as the Lawvere theory of De Morgan algebras. One immediate benefit of the internal construction of a model of CTT is to generalize this result from cubical sets to *presheaves valued in cubical sets* [OP18]. Meanwhile, networks of presheaf categories connected by the essential geometric morphisms induced by functors between base categories are known to induce models of MTT [GKNB21, Section 8]. In fact, a consequence of Theorem 4.10 is that these two results can be essentially combined, thereby giving rise to the most important models of MTT$_\square$.

Proposition 4.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, and write F^*, $F_!$, and F_* for precomposition respectively left and right Kan extensions of $F \times \mathsf{id}_\square$.*
(1) *The presheaf categories $\mathbf{PSh}(\mathcal{C} \times \square)$ and $\mathbf{PSh}(\mathcal{D} \times \square)$ are LOPS topoi.*
(2) *The adjunction $F^* \dashv F_*$ induces a morphism of LOPS topoi.*
(3) *If $F_!$ is lex the adjunction $F_! \dashv F^*$ induces a morphism of LOPS topoi.*

Before proving the above proposition we recall some standard lemmas.

Lemma 4.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. Left Kan extension of functors sending $X : \mathcal{C} \longrightarrow \mathbf{Set}$ to $F_!(X) : \mathcal{D} \longrightarrow \mathbf{Set}$ is lex iff $(F \downarrow d)$ is filtered for each $d : \mathcal{D}$. We note that this implies in particular that $(F \downarrow d)$ is connected.*

Proof. Fix a finite diagram of functors $X : \mathcal{I} \longrightarrow [\mathcal{C}, \mathbf{Set}]$. We can identify $F_!(\lim X)(d)$ with the colimit of the diagram $(F \downarrow d) \longrightarrow \mathcal{C} \longrightarrow \mathbf{Set}$ sending $f : F(c) \longrightarrow d$ to $(\lim X)(c)$. By assumption, this is a filtered colimit and thus it commutes with finite limits. Since limits are computed pointwise in $[\mathcal{C}, \mathbf{Set}]$, we may identify $(\lim X)(c) = \lim_i X_i(c)$ and compute:

$$
\begin{aligned}
F_!(\lim X)(d) &= \mathsf{colim}_{(c,f):(F\downarrow d)} \lim_{i:\mathcal{I}} X_i(c) \\
&= \lim_{i:\mathcal{I}} \mathsf{colim}_{(c,f):(F\downarrow d)} X_i(c) \\
&= \lim_{i:\mathcal{I}} F_!(X_i)(d) \qquad\qquad\qquad \square
\end{aligned}
$$

Lemma 4.13. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $F' : \mathcal{C}' \longrightarrow \mathcal{D}'$. Then $(F \times F' \downarrow (d, d'))$ is equivalent to $(F \downarrow d) \times (F' \downarrow d')$ for each $d : \mathcal{D}$ and $d' : \mathcal{D}'$.*

Lemma 4.14. *Consider a diagram $F : A \times B \longrightarrow \mathcal{C}$ where B has a terminal object b_1. Then the colimit of $F(a, b)$ over $A \times B$ is isomorphic to the colimit of $F(a, b_1)$ over A, naturally in A.*

We can now prove proposition 4.11:

Proof. Note first that $\mathbf{PSh}(\mathcal{C} \times \square) = [(\mathcal{C} \times \square)^{\mathsf{op}}, \mathbf{Set}] \cong [\mathcal{C}^{\mathsf{op}}, \mathbf{cSet}] = \mathbf{PSh}_{\mathbf{cSet}}(\mathcal{C})$. Letting $\mathbb{I}, \mathbb{F} : \mathbf{cSet}$ be the interval respectively face lattice from [CCHM18, Section 8.1], we define $\mathbf{I}_{\mathcal{C}}(c, I) = \mathbb{I}(I)$ and $\mathbf{F}_{\mathcal{C}}(c, I) = \mathbb{I}(I)$ for $c : \mathcal{C}$ and $I : \square$.

For (1) these topoi satisfy the Orton-Pitts axioms as noted in [CRS21]. To see that the intervals defined above are tiny we proceed as follows: Using the Yoneda lemma along with the fact that $\mathbb{I}$ is naturally isomorphic to $[-, \{i\}]_{\square}$ shows that $\mathbf{y}(c, I) \times \mathbf{I} \cong \mathbf{y}(c, I + \{i\})$, and we thus calculate:

$$
\begin{aligned}
X^{\mathbf{I}}(c, I) &\cong [\mathbf{y}(c, I), X^{\mathbf{I}}] \\
&\cong [\mathbf{y}(c, I) \times \mathbf{I}, X] \\
&\cong [\mathbf{y}(c, I + \{i\}), X] \\
&\cong X(c, I + \{i\}) \\
&\cong (\mathsf{id}_{\mathcal{C}} \times (- + \{i\}))^*(X)(c, I)
\end{aligned}
$$

The above is natural in X, and thus exponentiation by $\mathbf{I}$ is (naturally isomorphic to) the precomposition functor $(\mathsf{id}_{\mathcal{C}} \times (- + \{i\}))^*$. As this functor has a right adjoint, we have shown that $\mathbf{I}$ is tiny.

We write $\pi_{\mathcal{C}}, \pi_{\mathcal{D}}$ the projections $\mathcal{C} \times \square \longrightarrow \square$ and $\mathcal{D} \times \square \longrightarrow \square$ respectively. For the second (respectively third) requirement we show the following:

- F^* (resp. $F_!$) preserves finite limits.
- $\iota : F^* \circ \pi_{\mathcal{D}}^* \cong \pi_{\mathcal{C}}^*$ $(F_! \circ \pi_{\mathcal{C}}^* \cong \pi_{\mathcal{D}}^*)$
- This isomorphism is an isomorphism of De Morgan algebras at $\mathbb{I}$ and of distributive lattices at $\mathbb{F}$.

The remaining conditions of a morphism of LOPS topoi follow automatically from the naturality α (resp. β).

For the first item, we note that F^* preserves all limits since it is a right adjoint, and that $F_!$ preserves finite limits by assumption.

Next, the desired isomorphism $F^*(\pi_{\mathcal{D}}^*(X)) \cong \pi_{\mathcal{C}}^*(X)$ can be taken to be the identity, justified by the following computation:

$$F^*(\pi_{\mathcal{D}}^*(X))(c, I) = \pi_{\mathcal{D}}^*(X)(F(c), I) = X(I) = \pi_{\mathcal{C}}^*(X)(c, I)$$

It is clear that this isomorphism preserves the De Morgan algebra structure when $X = \mathbb{I}$ and the distributive lattice structure when $X = \mathbb{F}$.

It remains to consider these conditions for $F_!$. We construct $F_!(\pi_{\mathcal{C}}^*(X)) \cong \pi_{\mathcal{D}}^*(X)$ as the composite of a string of natural isomorphisms:

$$
\begin{aligned}
F_!(\pi_{\mathcal{C}}^*(X))(d, I) &\cong \mathsf{colim}_{((c,I'),\_):(F \times \mathsf{id} \downarrow (d,I))} \pi_{\mathcal{C}}^*(X)(c, I') \\
&= \mathsf{colim}_{((c,I'),\_):(F \times \mathsf{id} \downarrow (d,I))} X(I') \\
&\cong \mathsf{colim}_{((c,\_),(I',\_)):(F \downarrow d) \times (\mathsf{id} \downarrow I)} X(I') && \text{lemma 4.13} \\
&\cong \mathsf{colim}_{(c,\_):(F \downarrow d)} X(I) && \text{lemma 4.14} \\
&\cong X(I) \\
&= \pi_{\mathcal{D}}^*(X)(d, I)
\end{aligned}
$$

In the above calculation, the fourth isomorphism is obtained by observing that $\mathsf{colim}_{(c,\_):(F \downarrow d)} X(I)$ is a constant diagram over $(F \downarrow d)$; because $F_!$ is lex lemma 4.12 ensures that $(\mathsf{id}_\square \times F \downarrow (d, I))$ is connected, and hence so is $(F \downarrow d)$. The isomorphism then follows from the observation that colimits of constant, connected diagrams are isomorphic to the value of the diagram.

We must argue that this is an isomorphism of De Morgan algebras when $X = \mathbb{I}$ and of distributive lattices when $X = \mathbb{F}$. Chasing an element through this string of isomorphisms, we send an element of the colimit $\mathsf{in}_{((c,I'),(f,g))}(x)$ to $X(g)(x)$. One can verify that this preserves the relevant structure when X is appropriately specialized. We illustrate the simple case of interval endpoints: The 0 endpoint of $F_!(\mathbf{I}_{\mathcal{C}})$ at (d, I) is given by $\mathsf{in}_{(f_0, \mathsf{id})}(0 : \mathbb{I}(I))$ where f_0 is an arbitrary object of the (necessarily non-empty) category $(F \downarrow d)$. It is clear that this pair is mapped to $0 : \mathbf{I}_{\mathcal{D}}(d, I)$ via the morphisms above. $\qquad\square$

We can package all of the above results into the following:

Theorem 4.15. *Let $f : \mathcal{M} \longrightarrow \mathbf{Cat}$ be a strict 2-functor, write $F^*(\mu)$, $F_!(\mu)$, and $F_*(\mu)$ for the precomposition, left Kan extension, and right Kan extension respectively of $f(\mu) \times \mathsf{id}_\square$, and write F^*, $F_!$, and F_* for the induced pseudofunctors.*

- *The network of morphisms of LOPS topoi given by the adjunctions $F^*(\mu) \dashv F_*(\mu)$ induces a model of $\mathsf{MTT}_\square$ over $\mathcal{M}$ with modal context structure equal to F^*.*
- *The network of morphisms of LOPS topoi given by the adjunctions $F_!(\mu) \dashv F^*(\mu)$ induces a model of $\mathsf{MTT}_\square$ over $\mathcal{M}^{\mathsf{coop}}$ with modal context structure pseudonaturally equivalent to $F_!$ if each $F_!(\mu)$ is lex.*

Proof. By Theorem 4.10, it is sufficient to show that F_* (respectively F^*) is a cubical MTT cosmos. By Proposition 4.11, each $\mathbf{PSh}(f(m) \times \square)$ is a LOPS topos, and each adjunction $F^*(\mu) \dashv F_*(\mu)$ (respectively $F_!(\mu) \dashv F^*(\mu)$) is a morphism of LOPS topoi, and we thus need only verify that the interval and face lattice isomorphisms cohere pseudonaturally.

Consider first the case of the adjunctions $F^*(\mu) \dashv F_*(\mu)$. In this case, each interval (respectiely face lattice) isomorphism is the identity, and thus since the left adjoints form a strict 2-functor, the interval objects (respectively face lattice objects) form a strict 2-natural transformation, which in particular is also a pseudonatural transformation.

Consider next the case of the adjunctions $F_!(\mu) \dashv F^*(\mu)$. To prove pseudonaturality, we need to prove coherence with identity, composition, and 2-cells. The proofs are similar, and we illustrate them with the identity case. Since $F_!(\mu)$ is a pseudofunctor, there is a natural isomorphism $F_!(1_m) \cong \mathsf{id}_{\mathbf{PSh}(f(m)\times\square)}$. We must prove that at the interval (respectively face lattice) object, this is the same isomorphism as the one constructed in Proposition 4.11.

The isomorphism in the proof of Proposition 4.11, $F_!(\mu)(\pi^*_{\mathcal{C}_m}(X))(d, I) \cong \pi^*_{f(m)}(X)(d, I)$, may be specified as follows: It is uniquely defined by its value upon precomposition with the components of the colimit $\mathsf{colim}_{((c,I'),\_):(f(\mu)\times\mathsf{id}\downarrow(d,I))} \pi^*_{f(m)}(X)(c, I')$, and this value is for each $c : f(m)$, $I' : \square$, and $(g, \iota) : (d, I) \longrightarrow (f(\mu)(c), I')$ equal to

$$X(\iota) : \pi^*_{f(m)}(X)(c, I') = X(I') \longrightarrow X(I) = \pi^*_{f(n)}(X)(d, I).$$

The isomorphism $F_!(1_m) \cong \mathsf{id}$ is constructed from the fact that both $F_!(1_m)$ and id are left adjoints to $F^*(1_m) = \mathsf{id}_{\mathbf{PSh}(f(m)\times\square)}$. Concretely, it is the counit:

$$\epsilon_{1_m} : F_!(1_m) = F_!(1_m) \circ F^*(1_m) \longrightarrow \mathsf{id}_{\mathbf{PSh}(f(m)\times\square)}$$

Precomposing with the components of the colimit $\mathsf{colim}_{((c,I'),\_):(f(\mu)\times\mathsf{id}\downarrow(d,I))} \pi^*_{f(m)}(X)(c, I')$, we get $X(\iota)$ as before, which shows that the two isomorphisms are the same, and thus the identity condition for pseudonaturality is satisfied. $\qquad\square$

5. Proving and programming with guarded recursion

We now turn from theory to practice[8] and consider *guarded* $\mathsf{MTT}_\square$. We briefly recall guarded recursion. The core idea of guarded recursion [Nak00] is to use a modality $\blacktriangleright$ (pronounced 'later') to isolate recursively produced data to prevent its use until work is done, thereby ensuring productivity. This modality is equipped with operations making it into an applicative functor [MP08] which satisfies *Löb induction*, a powerful guarded fixed-point principle:

$$\mathsf{next} : A \longrightarrow \blacktriangleright A \qquad (\circledast) : \blacktriangleright(A \to B) \longrightarrow ((\blacktriangleright A) \to (\blacktriangleright B)) \qquad \mathsf{lob} : (\blacktriangleright A \to A) \longrightarrow A$$

In particular, lob allows us to define an element of A recursively but because the recursively computed data is available only as $\blacktriangleright A$, the usual problems with fixed-points are avoided. We consider a variant of guarded recursion which further includes an idempotent comonad $\square$ along with an equivalence $\square \blacktriangleright A \simeq A$. This last property ensures that guarded type theory can construct *coinductive* types through Löb induction [CBGB15].

To encode guarded recursion in $\mathsf{MTT}_\square$, we instantiate the theory with a particular mode theory and extend it with a pair of axioms. As a result, we obtain a highly workable guarded type theory supporting the relevant modalities and operations. Similar work was done for *extensional* MTT in [GKNB21, Section 9]; here we show that the improved notion of equality in $\mathsf{MTT}_\square$ results in an improved experience.

Concretely, we work in the mode theory $\mathcal{M}_g$, a poset-enriched category which is concisely defined by Figure 4. Using the substitutions induced by 2-cells, we define:

$$\blacktriangleright A = \langle \ell \mid A[\{1 \le \ell\}]\rangle \qquad\qquad \mathsf{next}(x) = \mathsf{mod}_\ell(A[\{1 \le \ell\}])$$

While $\circledast$ is likewise definable, lob cannot be defined $\mathsf{MTT}_\square$ and must be axiomatized (Figure 5). In order to justify its inclusion, we provide a model of $\mathsf{MTT}_\square$ over $\mathcal{M}_g$ with Löb induction.

---

[8]Or at least, slightly more practice-adjacent theory!

$$\delta \circ \gamma \leq 1 \quad 1 = \gamma \circ \delta$$
$$1 \leq \ell \quad \gamma = \gamma \circ \ell$$
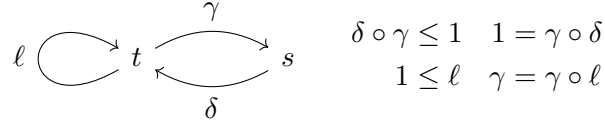
Figure 4: $\mathcal{M}_g$: a mode theory for guarded recursion. Reproduced from fig. 11 of [GKNB21]

$$\frac{\Gamma \; \mathsf{cx} @ t \qquad \Gamma \vdash A @ t}{\Gamma \vdash \mathsf{lob} : (\blacktriangleright A \to A) \to A @ t} \qquad \frac{\Gamma \; \mathsf{cx} @ t \qquad \Gamma \vdash A @ t \qquad \Gamma \vdash M : \blacktriangleright A \to A @ t}{\Gamma \vdash \mathsf{lob}(M) = M(\mathsf{next}(\mathsf{lob}(M))) : A @ t}$$

Figure 5: The rules of Löb induction.

5.1. **Soundness of Löb induction in MTT$_\square$.** Letting ω be the poset category for the first infinite ordinal and 1 the terminal category, we define the strict 2-functor $f : \mathcal{M}_g \longrightarrow \mathbf{Cat}$ by

$$f(t) = \omega \qquad f(s) = 1 \qquad f(\delta)(*) = 0 \qquad f(\ell)(n) = n + 1 \qquad f(\gamma)(n) = *$$

From this, we define the pseudofunctor $F : \mathcal{M}_g \longrightarrow \mathbf{Cat}$ by $F(m) = \mathbf{PSh}(f(m) \times \square)$ and $F(\mu) = (f(\mu) \times \mathsf{id}_\square)_*$, which by Theorem 4.15 induces a model of $\mathsf{MTT}_\square$ $\widehat{F}$. This model is almost the same as the model defined in [GKNB21, Section 9.2], but $\widehat{F}$ uses $\mathbf{cSet}$-valued presheaves. Since the cubical and modal aspects of $\mathsf{MTT}_\square$ are orthogonal, considerations in the $\mathbf{Set}$-based model that do not involve identity types carry over to $\widehat{F}$. In particular, because Löb induction holds in the $\mathbf{Set}$-based model, it also holds in $\widehat{F}$.[9]

5.2. **Programming with guarded MTT$_\square$.** To see that $\mathsf{MTT}_\square$ can not merely replicate but also improve on work done in MTT, we now show that Löb induction not only gives a fixpoint but a unique one. In [GKNB20, Theorem 9.5] this is proven for extensional MTT (by introducing equality reflection), but because of modal extensionality, we can now prove it with nothing but $\mathsf{MTT}_\square$ and Löb induction. Similarly, the results from [GKNB21, Section 9.4] about guarded and coinductive streams in guarded MTT may also be proven in guarded $\mathsf{MTT}_\square$ without equality reflection.

Theorem 5.1. $\mathsf{lob}(M)$ *is the unique guarded fixpoint of* $M : \blacktriangleright \mathsf{El}(A) \longrightarrow \mathsf{El}(A)$, *i.e.*

$$(A : \mathsf{U})(x : \mathsf{El}(A)) \to \mathsf{Path}_{\mathsf{El}(A)}(M(\mathsf{next}(x)), x) \to \mathsf{Path}_{\mathsf{El}(A)}(\mathsf{lob}(M), x)$$

Proof. Supposing $A : \mathsf{U}$, we intend to use Löb induction to find a term of

$$(x : \mathsf{El}(A)) \to \mathsf{Path}_{\mathsf{El}(A)}(M(\mathsf{next}(x)), x) \to \mathsf{Path}_{\mathsf{El}(A)}(\mathsf{lob}(M), x)$$

To this end, given terms $f : \blacktriangleright((x : \mathsf{El}(A)) \to \mathsf{Path}_{\mathsf{El}(A)}(M(\mathsf{next}(x)), x) \to \mathsf{Path}_{\mathsf{El}(A)}(\mathsf{lob}(M), x))$, $x : \mathsf{El}(A)$, and $p : \mathsf{Path}_{\mathsf{El}(A)}(M(\mathsf{next}(x)), x)$, we must define a term of $\mathsf{Path}_{\mathsf{El}(A)}(\mathsf{lob}(M), x)$. We can construct the term

$$f \circledast \mathsf{next}(x) \circledast \mathsf{next}(p) : \blacktriangleright \mathsf{Path}_{\mathsf{El}(A)}(\mathsf{lob}(M), x).$$

By Theorem 3.1, this gives a term of $\mathsf{Path}_{\blacktriangleright \mathsf{El}(A)}(\mathsf{next}(\mathsf{lob}(M)), \mathsf{next}(x))$. Using that function application preserves paths and that $\mathsf{lob}(M)$ is a guarded fix point we then obtain the paths

$$\mathsf{lob}(M) \equiv M(\mathsf{next}(\mathsf{lob}M)) \equiv M(\mathsf{next}(x)) \equiv x. \qquad \square$$

---

[9]This can also be verified by hand as is done in [BBC$^+$19].

6. Related work

MTT$_\square$ builds upon two distinct strands of work: cubical and modal type theories. Even though both of these lines of research are ongoing, several proposals have already been made which combine elements of both.

Modal homotopy type theory. Several version of homotopy type theory extended with modalities have been proposed [Shu18, RFL21]. These type theories aim to increase the expressivity of HoTT and allow it to better capture some aspects of homotopy theory. Unlike MTT$_\square$, however, these theories tend to be specialized to various modal situations. They build in the structure of one specific modality and are hand-crafted to have manageable syntax for that situation. In contrast, MTT$_\square$ follows MTT and works for a *class* of modalities, and provides usable syntax in each case. Moreover, prior type theories in this tradition expand "book HoTT" [Uni13] and therefore do not enjoy the good computational properties we conjecture for MTT$_\square$.

Modal cubical type theory. In order to extend cubical type theory with an *internal* notion of parametricity, Cavallo [Cav21, Part IV] has proposed a variant of (cartesian) cubical type theory extended with connectives and a handful of modalities to capture parametricity. Like MTT$_\square$, this *cohesive parametric cubical type theory* combines cubical type theory with Fitch-style modalities. While morally the system is a specialization of MTT$_\square$ to a cohesive collection of modalities, Cavallo takes advantage of several specifics of the intended model to add various equations to the theory.

Separately, another Fitch-style type theory, *clocked type theory*, has been extended to a cubical basis [KMV21]. This theory is used to present guarded recursion, similarly to section 5. Unlike guarded MTT$_\square$, guarded clocked type theory includes several specialized axioms, a more sophisticated collection of guarded modalities, and an account of the interaction of HITs with parts of the modal machinery.

The extra equations and properties of the modalities in both systems prevent MTT$_\square$ from directly recovering either parametric cubical type theory or clocked cubical type theory. The core aspects of both, however, are similar to MTT$_\square$ and we believe that MTT$_\square$ gives a means of systematically generalizing these calculi to other modal situations.

7. Conclusions

We contribute MTT$_\square$, a general modal type theory based on cubical type theory and MTT. The system can be instantiated to a number of modal situations while still maintaining computationally effective interpretations of univalence and function extensionality.

While in this work we have introduced the theory and characterized a class of models for it, in the future we hope to investigate further metatheoretic properties of the system. In particular, both MTT and cubical type theory enjoy normalization [Gra21, SA21], and we conjecture that these proofs can be combined and generalized to apply to MTT$_\square$. The introduction of cubical cosmoi takes the first step in this direction: cosmoi are a crucial ingredient of the proof of normalization for MTT. In a separate direction, we hope to investigate the behavior of more of the mode-local structure of cubical type theory such as *higher inductive types* and other novelties of cubical type theories.

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Appendix A. Rules of MTT□

We here present the official syntax and rules of $\mathsf{MTT}_\square$. For the sake of brevity, we omit a number the rules, especially those lifted from MTT or CTT; in particular, we omit the following:

- The rules for dependent sums, booleans, universes.
- The equations stating that the interval is a De Morgan algebra.
- The equations stating that $(-)^\mu$ for interval terms is a morphism of De Morgan algebras.
- The equations stating that that face lattice is a bounded distributive lattice,
- The equations stating that $(-)^\mu$ for faces is a morphism of bounded lattices,
- Miscellaneous equations commuting substitutions past term formers or governing the composition of substitutions.

At the end there is a section on derived definitions some of which we will use throughout to ease notation.

Context formation.

$$
\frac{}{\mathbf{1}\ \mathsf{cx} @ m}\ \text{CX/EMP}
\qquad
\frac{\mu : n \longrightarrow m \qquad \Gamma\ \mathsf{cx} @ m}{\Gamma.\{\mu\}\ \mathsf{cx} @ n}\ \text{CX/LOCK}
\qquad
\frac{\mu : n \longrightarrow m \qquad \Gamma\ \mathsf{cx} @ m \qquad \Gamma.\{\mu\} \vdash A @ n}{\Gamma.(\mu \mid A)\ \mathsf{cx} @ m}\ \text{CX/EXT-TYPE}
$$

$$
\frac{\Gamma\ \mathsf{cx} @ m}{\Gamma.\mathbb{I}_m\ \mathsf{cx} @ m}\ \text{CX/EXT-INT}
\qquad
\frac{\Gamma\ \mathsf{cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.[\phi]\ \mathsf{cx} @ m}\ \text{CX/FACE-RES}
$$

Context equality.

$$\frac{\mu : n \longrightarrow m \qquad \nu : o \longrightarrow n \qquad \Gamma \ \mathsf{cx} @ m}{\Gamma.\{\mu \circ \nu\} = \Gamma.\{\mu\}.\{\nu\} \ \mathsf{cx} @ o} \ {\small \text{CX-EQ/COMP-LOCK}}$$

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma.\{1\} = \Gamma \ \mathsf{cx} @ m} \ {\small \text{CX-EQ/ID-LOCK}}$$

Substitution formation.

$$\frac{\Gamma, \Delta, \Xi \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Delta \vdash \xi : \Xi @ m}{\Gamma \vdash \xi \circ \delta : \Xi @ m} \ {\small \text{SB/COMP}}$$

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma \vdash \mathsf{id} : \Gamma @ m} \ {\small \text{SB/ID}} \qquad \frac{\Gamma \ \mathsf{cx} @ m}{\Gamma \vdash \ ! : \mathbf{1} @ m} \ {\small \text{SB/EMP}}$$

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m \qquad \Gamma.\{\mu\} \vdash A @ n}{\Gamma.(\mu \mid A) \vdash \ \uparrow : \Gamma @ m} \ {\small \text{SB/WEAK-TYPE}}$$

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma.\mathbb{I}_m \vdash \uparrow^i : \Gamma @ m} \ {\small \text{SB/WEAK-INT}}$$

$$\frac{\Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.[\phi] \vdash \uparrow^\phi : \Gamma @ m} \ {\small \text{SB/WEAK-RES}}$$

$$\frac{\mu : n \longrightarrow m \qquad \Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m}{\Gamma.\{\mu\} \vdash \delta.\{\mu\} : \Delta.\{\mu\} @ n} \ {\small \text{SB/LOCK}}$$

$$\frac{\mu, \nu : n \longrightarrow m \qquad \alpha : \nu \longrightarrow \mu \qquad \Gamma \ \mathsf{cx} @ m}{\Gamma.\{\mu\} \vdash \{\alpha\}_\Gamma : \Gamma.\{\nu\} @ n} \ {\small \text{SB/KEY}}$$

{\small SB/EXT-TYPE}
$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \overset{\mu : n \longrightarrow m}{\Delta.\{\mu\} \vdash A @ n} \qquad \Gamma.\{\mu\} \vdash a : A[\delta.\{\mu\}] @ n}{\Gamma \vdash \delta.a : \Delta.(\mu \mid A) @ m}$$

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash \delta.r : \Delta.\mathbb{I}_m @ m} \ {\small \text{SB/EXT-INT}}$$

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Delta \vdash \phi : \mathbb{F}_m @ m \qquad \Gamma \vdash \phi[\delta] = \top : \mathbb{F}_m @ m}{\Gamma \vdash \delta.[\phi] : \Delta.[\phi] @ m} \ {\small \text{SB/FACE-RES}}$$

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m}{\Gamma.\{\mu\}.\mathbb{I}_n \vdash \sigma^\mu : \Gamma.\mathbb{I}_m.\{\mu\} @ n} \ {\small \text{SB/EXC-INT-INV}}$$

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.\{\mu\}.[\phi^\mu] \vdash \tau^\mu : \Gamma.[\phi].\{\mu\} @ n} \ {\small \text{SB/EXC-FACE-INV}}$$

Substitution equality.

SB-EQ/COMP-LOCK
$$\frac{\mu : n \longrightarrow m \qquad \nu : o \longrightarrow n \qquad \Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta \, @ \, m}{\Gamma.\{\mu \circ \nu\} \vdash \delta.\{\mu \circ \nu\} = \delta.\{\mu\}.\{\nu\} : \Delta.\{\mu \circ \nu\} \, @ \, o}$$

SB-EQ/ID-LOCK
$$\frac{\Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta \, @ \, m}{\Gamma \vdash \delta.\{1\} = \delta : \Delta \, @ \, m}$$

SB-EQ/LOCK-COMP
$$\frac{\mu : n \longrightarrow m \qquad \Gamma, \Delta, \Xi \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta \, @ \, m \qquad \Delta \vdash \xi : \Xi \, @ \, m}{\Gamma.\{\mu\} \vdash (\xi \circ \delta).\{\mu\} = \xi.\{\mu\} \circ \delta.\{\mu\} : \Xi.\{\mu\} \, @ \, n}$$

SB-EQ/LOCK-ID
$$\frac{\mu : n \longrightarrow m \qquad \Gamma \; \mathsf{cx} \, @ \, m}{\Gamma.\{\mu\} \vdash \mathsf{id}.\{\mu\} = \mathsf{id} : \Gamma.\{\mu\} \, @ \, n}$$

SB-EQ/ID-KEY
$$\frac{\mu : n \longrightarrow m \qquad \Gamma \; \mathsf{cx} \, @ \, m}{\Gamma.\{\mu\} \vdash \{1_\mu\}_\Gamma = \mathsf{id} : \Gamma.\{\mu\} \, @ \, n}$$

SB-EQ/NAT-KEY
$$\frac{\mu, \nu : n \longrightarrow m \qquad \alpha : \nu \longrightarrow \mu \qquad \Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta \, @ \, m}{\Gamma.\{\mu\} \vdash \{\alpha\}_\Delta \circ \delta.\{\mu\} = \delta.\{\nu\} \circ \{\alpha\}_\Gamma : \Delta.\{\nu\} \, @ \, n}$$

SB-EQ/COMP-KEY
$$\frac{\mu, \nu, \rho : n \longrightarrow m \qquad \alpha : \nu \longrightarrow \mu \qquad \beta : \rho \longrightarrow \nu \qquad \Gamma \; \mathsf{cx} \, @ \, m}{\Gamma.\{\mu\} \vdash \{\alpha \circ \beta\}_\Gamma = \{\alpha\}_\Gamma \circ \{\beta\}_\Gamma : \Gamma.\{\rho\} \, @ \, n}$$

SB-EQ/WHISK-KEY
$$\frac{\mu_0, \mu_1 : n \longrightarrow m \qquad \nu_0, \nu_1 : o \longrightarrow n \qquad \alpha : \mu_1 \longrightarrow \mu_0 \qquad \beta : \nu_1 \longrightarrow \nu_0 \qquad \Gamma \; \mathsf{cx} \, @ \, m}{\Gamma.\{\mu_0 \circ \nu_0\} \vdash \{\alpha \star \beta\}_\Gamma = \{\alpha\}_\Gamma.\{\nu_1\} \circ \{\beta\}_{\Gamma.\{\mu_0\}} : \Gamma.\{\mu_1 \circ \nu_1\} \, @ \, o}$$

SB-EQ/EXT-TYPE-BETA
$$\frac{\Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta \, @ \, m \qquad \Delta.\{\mu\} \vdash A \, @ \, n \qquad \Gamma.\{\mu\} \vdash a : A[\delta.\{\mu\}] \, @ \, n}{\Gamma \vdash \uparrow \circ \delta.a = \delta : \Delta \, @ \, m}$$

SB-EQ/EXT-TYPE-ETA
$$\frac{\Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Delta.\{\mu\} \vdash A \, @ \, n \qquad \Gamma \vdash \delta : \Delta.(\mu \mid A) \, @ \, m}{\Gamma \vdash \delta = (\uparrow \circ \delta).\mathbf{v}_0[\delta] : \Delta.(\mu \mid A) \, @ \, m}$$

SB-EQ/EXT-INT-BETA
$$\frac{\Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta \, @ \, m \qquad \Gamma \vdash r : \mathbb{I}_m \, @ \, m}{\Gamma \vdash \uparrow^i \circ \delta.r = \delta : \Delta \, @ \, m}$$

SB-EQ/EXT-INT-ETA
$$\frac{\Gamma, \Delta \; \mathsf{cx} \, @ \, m \qquad \Gamma \vdash \delta : \Delta.\mathbb{I}_m \, @ \, m}{\Gamma \vdash \delta = (\uparrow^i \circ \delta).\mathbf{v}_0^i[\delta] : \Delta.\mathbb{I}_m \, @ \, m}$$

SB-EQ/EXC-INT-LEFT-INV
$$\frac{\mu : n \longrightarrow m \qquad \Gamma \; \mathsf{cx} \, @ \, m}{\Gamma.\mathbb{I}_m.\{\mu\} \vdash \sigma^\mu \circ \sigma_\mu = \mathsf{id} : \Gamma.\mathbb{I}_m.\{\mu\} \, @ \, n}$$

SB-EQ/EXC-INT-RIGHT-INV
$$\frac{\mu : n \longrightarrow m \qquad \Gamma \; \mathsf{cx} \, @ \, m}{\Gamma.\{\mu\}.\mathbb{I}_m \vdash \sigma_\mu \circ \sigma^\mu = \mathsf{id} : \Gamma.\{\mu\}.\mathbb{I}_m \, @ \, n}$$

SB-EQ/EXC-FACE-LEFT-INV

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.[\phi].\{\mu\} \vdash \tau^\mu \circ \tau_\mu = \mathsf{id} : \Gamma.[\phi].\{\mu\} @ n}$$

SB-EQ/EXC-FACE-RIGHT-INV

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.\{\mu\}.[\phi^\mu] \vdash \tau_\mu \circ \tau^\mu = \mathsf{id} : \Gamma.\{\mu\}.[\phi^\mu] @ n}$$

SB-EQ/FACE-RES-UNIQ

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Delta \vdash \phi : \mathbb{F}_m @ m \qquad \Gamma \vdash \delta : \Gamma.[\phi] @ m}{\Gamma \vdash \delta = (\uparrow^\phi \circ \delta).[\phi] : \Gamma.[\phi] @ m}$$

SB-EQ/FACE-RES-BIN

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta, \xi : \Delta @ m \qquad \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m}{\Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \qquad \Gamma.[\phi] \vdash \delta \circ \uparrow^\phi = \xi \circ \uparrow^\phi : \Delta @ m \qquad \Gamma.[\psi] \vdash \delta \circ \uparrow^\psi = \xi \circ \uparrow^\psi : \Delta @ m}{\Gamma \vdash \delta = \xi : \Delta @ m}$$

SB-EQ/FACE-RES-NULL

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta, \xi : \Delta @ m \qquad \Gamma \vdash \bot = \top : \mathbb{F}_m @ m}{\Gamma \vdash \delta = \xi : \Delta @ m}$$

Interval formation.

INT/JOIN

$$\frac{\Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash r, s : \mathbb{I}_m @ m}{\Gamma \vdash r \vee s : \mathbb{I}_m @ m}$$

INT/MEET

$$\frac{\Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash r, s : \mathbb{I}_m @ m}{\Gamma \vdash r \wedge s : \mathbb{I}_m @ m}$$

INT/BOT

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma \vdash 0 : \mathbb{I}_m @ m}$$

INT/TOP

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma \vdash 1 : \mathbb{I}_m @ m}$$

INT/INV

$$\frac{\Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash (1 - r) : \mathbb{I}_m @ m}$$

INT/EXC

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma.\{\mu\} \vdash r^\mu : \mathbb{I}_n @ n}$$

INT/VAR

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma.\mathbb{I}_m \vdash \mathbf{v}_0^i : \mathbb{I}_m @ m}$$

INT/SB

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Delta \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash r[\delta] : \mathbb{I}_m @ m}$$

Interval equality.

INT-EQ/EXT-INT-BETA

$$\frac{\Gamma, \Delta \; \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash \mathbf{v}_0^i[\delta.r] = r : \mathbb{I}_m @ m}$$

INT-EQ/RES-EQ

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma.[(r=0)] \vdash r[\uparrow^{(r=0)}] = 0 : \mathbb{I}_m @ m}$$

INT-EQ/EXC-COMP

$$\frac{\mu : n \longrightarrow m \qquad \nu : o \longrightarrow n \qquad \Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma.\{\mu\}.\{\nu\} \vdash r^{\mu \circ \nu} = (r^\mu)^\nu : \mathbb{I}_o @ o}$$

INT-EQ/EXC-ID

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash r^1 = r : \mathbb{I}_m @ m}$$

INT-EQ/EXC-KEY

$$\frac{\mu, \nu : n \longrightarrow m \qquad \alpha : \nu \longrightarrow \mu \qquad \Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma.\{\mu\} \vdash r^\nu[\{\alpha\}_\Gamma] = r^\mu : \mathbb{I}_n @ n}$$

INTE-EQ/EXC-SUB

$$\frac{\nu : n \longrightarrow m \qquad \Gamma, \Delta \; \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma.\{\mu\} \vdash r^\mu[\delta.\{\mu\}] = r[\delta]^\mu : \mathbb{I}_n @ n}$$

INT-EQ/FACE-RES-BIN

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r, s : \mathbb{I}_m @ m \qquad \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m \qquad \Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \qquad \Gamma.[\phi] \vdash r[\uparrow^\phi] = s[\uparrow^\phi] : \mathbb{I}_m @ m \qquad \Gamma.[\psi] \vdash r[\uparrow^\psi] = s[\uparrow^\psi] : \mathbb{I}_m @ m}{\Gamma \vdash r = s : \mathbb{I}_m @ m}$$

INT-EQ/FACE-RES-NULL

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r, s : \mathbb{I}_m @ m \qquad \Gamma \vdash \bot = \top : \mathbb{F}_m @ m}{\Gamma \vdash r = s : \mathbb{I}_m @ m}$$

Face formation.

FACE/EQ

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash (r = 0) : \mathbb{F}_m @ m}$$

FACE/JOIN

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m}{\Gamma \vdash \phi \vee \psi : \mathbb{F}_m @ m}$$

FACE/MEET

$$\frac{\Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m}{\Gamma \vdash \phi \wedge \psi : \mathbb{F}_m @ m}$$

FACE/BOT

$$\frac{\Gamma \; \mathsf{cx} @ m}{\Gamma \vdash \bot : \mathbb{F}_m @ m}$$

FACE/TOP

$$\frac{\Gamma \; \mathsf{cx} @ m}{\Gamma \vdash \top : \mathbb{F}_m @ m}$$

FACE/EXC

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \; \mathsf{cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.\{\mu\} \vdash \phi^\mu : \mathbb{F}_n @ n}$$

FACE/SB

$$\frac{\Gamma, \Delta \; \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Delta \vdash \phi : \mathbb{F}_m @ m}{\Gamma \vdash \phi[\delta] : \mathbb{F}_m @ m}$$

Face equality.

FACE-EQ/NON-CONTR
$$\frac{\Gamma \text{ cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash (r = 0) \wedge ((1 - r) = 0) = \bot : \mathbb{F}_m @ m}$$

FACE-EQ/EXC-COMP
$$\frac{\mu : n \longrightarrow m \qquad \nu : o \longrightarrow n \qquad \Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.\{\mu\}.\{\nu\} \vdash \phi^{\mu \circ \nu} = (\phi^\mu)^\nu : \mathbb{F}_o @ o}$$

FACE-EQ/EXC-ID
$$\frac{\Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma \vdash \phi^1 = \phi : \mathbb{F}_m @ m}$$

FACE-EQ/EXC-KEY
$$\frac{\mu, \nu : n \longrightarrow m \qquad \alpha : \nu \longrightarrow \mu \qquad \Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.\{\mu\} \vdash \phi^\nu[\{\alpha\}_\Gamma] = \phi^\mu : \mathbb{I}_n @ n}$$

FACE-EQ/EXC-EQ
$$\frac{\mu : n \longrightarrow m \qquad \Gamma \text{ cx} @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash (r = 0)^\mu = (r^\mu = 0) : \mathbb{F}_n @ n}$$

FACE-EQ/EXC-SUB
$$\frac{\mu : n \longrightarrow m \qquad \Gamma, \Delta \text{ cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.\{\mu\} \vdash \phi^\mu[\delta.\{\mu\}] = \phi[\delta]^\mu : \mathbb{F}_n @ n}$$

FACE-EQ/RES-EQ-TOP
$$\frac{\Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m}{\Gamma.[\phi] \vdash \phi[\uparrow^\phi] = \top : \mathbb{F}_m @ m}$$

FACE-EQ/EQ-ZERO
$$\frac{\Gamma \text{ cx} @ m}{\Gamma \vdash (0 = 0) = \top : \mathbb{F}_m @ m}$$

FACE-EQ/FACE-RES-BIN
$$\frac{\Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi, \psi, \chi_0, \chi_1 : \mathbb{F}_m @ m \qquad \Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \qquad \Gamma.[\phi] \vdash \chi_0[\uparrow^\phi] = \chi_1[\uparrow^\phi] : \mathbb{F}_m @ m \qquad \Gamma.[\psi] \vdash \chi_0[\uparrow^\psi] = \chi_1[\uparrow^\psi] : \mathbb{F}_m @ m}{\Gamma \vdash \chi_0 = \chi_1 : \mathbb{F}_m @ m}$$

FACE-EQ/FACE-RES-NULL
$$\frac{\Gamma \text{ cx} @ m \qquad \Gamma \vdash \chi_0, \chi_1 : \mathbb{F}_m @ m \qquad \Gamma \vdash \bot = \top : \mathbb{F}_m @ m}{\Gamma \vdash \chi_0 = \chi_1 : \mathbb{F}_m @ m}$$

Type formation.

TYPE/PI
$$\frac{\mu : n \longrightarrow m \qquad \Gamma \text{ cx} @ m \qquad \Gamma.\{\mu\} \vdash A @ n \qquad \Gamma.(\mu \mid A) \vdash B @ m}{\Gamma \vdash (\mu \mid A) \to B @ m}$$

TYPE/PATH
$$\frac{\Gamma \text{ cx} @ m \qquad \Gamma.\mathbb{I}_m \vdash A @ m \qquad \Gamma \vdash a : A[\text{id}.0] @ m \qquad \Gamma \vdash b : A[\text{id}.1] @ m}{\Gamma \vdash \text{Path}_A(a, b) @ m}$$

TYPE/MOD

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \ \mathsf{cx} @ m \qquad \Gamma.\{\mu\} \vdash A @ n}{\Gamma \vdash \langle \mu \mid A \rangle @ m}$$

TYPE/SYS

$$\frac{\begin{array}{ccc} \Gamma \ \mathsf{cx} @ m & \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m & \Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \\ \Gamma.[\phi] \vdash A @ m \qquad \Gamma.[\psi] \vdash B @ m \qquad \Gamma.[\phi \wedge \psi] \vdash A[\uparrow^{\phi \wedge \psi} .[\phi]] = B[\uparrow^{\phi \wedge \psi} .[\psi]] @ m \end{array}}{\Gamma \vdash \{ \ \phi \ A, \psi \ B \ \} @ m}$$

TYPE/SB

$$\frac{\Gamma, \Delta \ \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Delta \vdash A @ m}{\Gamma \vdash A[\delta] @ m}$$

Type equality.

TYPE-EQ/SYS-TOP

$$\frac{\Gamma \ \mathsf{cx} @ m}{\Gamma \vdash \phi : \mathbb{F}_m @ m \qquad \Gamma.[\top] \vdash A @ m \qquad \Gamma.[\phi] \vdash B @ m \qquad \Gamma.[\phi] \vdash A[\uparrow^{\phi} .[\top]] = B @ m}{\Gamma \vdash \{ \ \top \ A, \phi \ B \ \} = A[\mathsf{id}.[\top]] @ m}$$

TYPE-EQ/FACE-RES-BIN

$$\frac{\begin{array}{ccc} \Gamma \ \mathsf{cx} @ m & \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m & \Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \\ \Gamma \vdash A, B @ m \qquad \Gamma.[\phi] \vdash A[\uparrow^{\phi}] = B[\uparrow^{\phi}] @ m \qquad \Gamma.[\psi] \vdash A[\uparrow^{\psi}] = B[\uparrow^{\psi}] @ m \end{array}}{\Gamma \vdash A = B @ m}$$

TYPE-EQ/FACE-RES-NULL

$$\frac{\Gamma \ \mathsf{cx} @ m \qquad \Gamma \vdash \bot = \top : \mathbb{F}_m @ m \qquad \Gamma \vdash A, B @ m}{\Gamma \vdash A = B @ m}$$

Term formation.

TERM/PI-LAM

$$\frac{\begin{array}{cccc} & & \mu : n \longrightarrow m & \\ \Gamma \ \mathsf{cx} @ m & \Gamma.\{\mu\} \vdash A @ n & \Gamma.(\mu \mid A) \vdash B @ m & \Gamma.(\mu \mid A) \vdash b : B @ m \end{array}}{\Gamma \vdash \lambda(b) : (\mu \mid A) \rightarrow B @ m}$$

TERM/PI-APP

$$\frac{\begin{array}{cccc} & \mu : n \longrightarrow m & \Gamma \ \mathsf{cx} @ m & \\ \Gamma.\{\mu\} \vdash A @ n & \Gamma.(\mu \mid A) \vdash B @ m & \Gamma \vdash f : (\mu \mid A) \rightarrow B @ m & \Gamma.\{\mu\} \vdash a : A @ n \end{array}}{\Gamma \vdash f(a) : B[\mathsf{id}.a] @ m}$$

TERM/PATH-ABS

$$\frac{\Gamma \ \mathsf{cx} @ m \qquad \Gamma.\mathbb{I}_m \vdash A @ m \qquad \Gamma.\mathbb{I}_m \vdash a : A @ m}{\Gamma \vdash \lambda(a) : \mathsf{Path}_A(a[\mathsf{id}.0], a[\mathsf{id}.1]) @ m}$$

TERM/PATH-APP

$$\frac{\Gamma \vdash a : A[\mathsf{id}.0] @ m \qquad \Gamma \vdash b : A[\mathsf{id}.1] @ m \qquad \Gamma \vdash p : \mathsf{Path}_A(a,b) @ m \qquad \Gamma \vdash r : \mathbb{I}_m @ m}{\Gamma \vdash p(r) : A[\mathsf{id}.r] @ m}$$

with premises $\Gamma \, \mathsf{cx} @ m \qquad \Gamma.\mathbb{I}_m \vdash A @ m$

TERM/MOD-MOD

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \, \mathsf{cx} @ m \qquad \Gamma.\{\mu\} \vdash A @ n \qquad \Gamma.\{\mu\} \vdash a : A @ n}{\Gamma \vdash \mathsf{mod}_\mu(a) : \langle \mu \mid A \rangle @ m}$$

TERM/MOD-LET

$$\frac{\begin{array}{c} \mu : n \longrightarrow m \qquad \nu : o \longrightarrow n \qquad \Gamma \, \mathsf{cx} @ m \qquad \Gamma.\{\mu\}.\{\nu\} \vdash A @ o \qquad \Gamma.\{\mu\} \vdash a : \langle \nu \mid A \rangle @ n \\ \Gamma.(\mu \mid \langle \nu \mid A \rangle) \vdash B @ m \qquad \Gamma.(\mu \circ \nu \mid A) \vdash b : B[\uparrow.\mathsf{mod}_\nu(\mathbf{v}_0)] @ m \end{array}}{\Gamma \vdash \mathsf{let}_\mu \, \mathsf{mod}_\nu(\_) \leftarrow a \, \mathsf{in} \, b : B[\mathsf{id}.a] @ m}$$

TERM/SYS-BIN

$$\frac{\begin{array}{c} \Gamma \, \mathsf{cx} @ m \\ \Gamma \vdash A @ m \qquad \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m \qquad \Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \qquad \Gamma.[\phi] \vdash a : A[\uparrow^\phi] @ m \\ \Gamma.[\psi] \vdash b : A[\uparrow^\psi] @ m \qquad \Gamma.[\phi \wedge \psi] \vdash a[\uparrow^{\phi \wedge \psi} .[\phi]] = b[\uparrow^{\phi \wedge \psi} .[\psi]] : A[\uparrow^{\phi \wedge \psi}] @ m \end{array}}{\Gamma \vdash \{ \, \phi \, a, \psi \, b \, \} : A @ m}$$

TERM/SYS-NULL

$$\frac{\Gamma \, \mathsf{cx} @ m \qquad \Gamma \vdash A @ m \qquad \Gamma \vdash \bot = \top : \mathbb{F}_m @ m}{\Gamma \vdash \{ \; \} : A @ m}$$

TERM/COMP

$$\frac{\begin{array}{c} \Gamma \, \mathsf{cx} @ m \qquad \Gamma.\mathbb{I}_m \vdash A @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m \qquad \Gamma.[\phi].\mathbb{I}_m \vdash u : A[\uparrow^\phi .\mathbb{I}_m] @ m \\ \Gamma \vdash u_0 : A[\mathsf{id}.0] @ m \qquad \Gamma.[\phi] \vdash u[\mathsf{id}.0] = u_0[\uparrow^\phi] : A[\uparrow^\phi.0] @ m \end{array}}{\Gamma \vdash \mathsf{comp} \, [\phi \mapsto u] \, u_0 : A[\mathsf{id}.1] @ m}$$

TERM/VAR

$$\frac{\mu : n \longrightarrow m \qquad \Gamma \, \mathsf{cx} @ m \qquad \Gamma.\{\mu\} \vdash A @ n}{\Gamma.(\mu \mid A).\{\mu\} \vdash \mathbf{v}_0 : A[\uparrow.\{\mu\}] @ n}$$

TERM/SB

$$\frac{\Gamma, \Delta \, \mathsf{cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m \qquad \Delta \vdash a : A @ m}{\Gamma \vdash a[\delta] : A[\delta] @ m}$$

Term equality.

TERM-EQ/PI-BETA

$$\dfrac{\mu : n \longrightarrow m \qquad \Gamma\,\mathsf{cx}\,@\,m}{\Gamma.\{\mu\} \vdash A\,@\,n \qquad \Gamma.(\mu \mid A) \vdash B\,@\,m \qquad \Gamma.\{\mu\} \vdash a : A\,@\,n \qquad \Gamma.(\mu \mid A) \vdash b : B\,@\,m}$$
$$\overline{\Gamma \vdash \lambda(b)(a) = b[\mathsf{id}.a] : B[\mathsf{id}.a]\,@\,m}$$

TERM-EQ/PI-ETA

$$\dfrac{\mu : n \longrightarrow m}{\Gamma\,\mathsf{cx}\,@\,m \qquad \Gamma.\{\mu\} \vdash A\,@\,n \qquad \Gamma.(\mu \mid A) \vdash B\,@\,m \qquad \Gamma \vdash f : (\mu \mid A) \to B\,@\,m}$$
$$\overline{\Gamma \vdash f = \lambda(f[\uparrow](\mathbf{v}_0)) : (\mu \mid A) \to B\,@\,m}$$

TERM-EQ/PATH-BETA

$$\dfrac{\Gamma\,\mathsf{cx}\,@\,m \qquad \Gamma.\mathbb{I}_m \vdash A\,@\,m \qquad \Gamma.\mathbb{I}_m \vdash a : A\,@\,m \qquad \Gamma \vdash r : \mathbb{I}_m\,@\,m}{\Gamma \vdash \lambda(a)(r) = a[\mathsf{id}.r] : A[\mathsf{id}.r]\,@\,m}$$

TERM-EQ/PATH-ETA

$$\dfrac{\Gamma\,\mathsf{cx}\,@\,m \qquad \Gamma.\mathbb{I}_m \vdash A\,@\,m}{\Gamma \vdash a_0 : A[\mathsf{id}.0]\,@\,m \qquad \Gamma \vdash a_1 : A[\mathsf{id}.1]\,@\,m \qquad \Gamma \vdash p : \mathsf{Path}_A(a_0, a_1)\,@\,m}$$
$$\overline{\Gamma \vdash p = \lambda(p[\uparrow^i](\mathbf{v}_0^i)) : \mathsf{Path}_A(a_0, a_1)\,@\,m}$$

TERM-EQ/MOD-BETA

$$\dfrac{\mu : n \longrightarrow m \qquad \nu : o \longrightarrow n \qquad \Gamma\,\mathsf{cx}\,@\,m \qquad \Gamma.\{\mu\}.\{\nu\} \vdash A\,@\,o \qquad \Gamma.(\mu \mid \langle \nu \mid A \rangle) \vdash B\,@\,m}{\Gamma.\{\mu\}.\{\nu\} \vdash a : A\,@\,o \qquad \Gamma.(\mu \circ \nu \mid A) \vdash b : B[\uparrow.\mathsf{mod}_\nu(\mathbf{v}_0)]\,@\,m}$$
$$\overline{\Gamma \vdash \mathsf{let}_\mu\,\mathsf{mod}_\nu(\_) \leftarrow \mathsf{mod}_\nu(a)\,\mathsf{in}\,b = b[\mathsf{id}.a] : B[\mathsf{id}.\mathsf{mod}_\nu(a)]\,@\,m}$$

TERM-EQ/EXT-TYPE-BETA

$$\dfrac{\Gamma, \Delta\,\mathsf{cx}\,@\,m \qquad \Gamma \vdash \delta : \Delta\,@\,m \qquad \Delta.\{\mu\} \vdash A\,@\,n \qquad \Gamma.\{\mu\} \vdash a : A[\delta.\{\mu\}]\,@\,n}{\Gamma.\{\mu\} \vdash \mathbf{v}_0[\delta.a.\{\mu\}] = a : A[\delta.\{\mu\}]\,@\,m}$$

TERM-EQ/SYS-TOP

$$\dfrac{\Gamma\,\mathsf{cx}\,@\,m \qquad \Gamma \vdash A\,@\,m \qquad \Gamma \vdash \phi : \mathbb{F}_m\,@\,m}{\Gamma.[\top] \vdash a : A[\uparrow^\top]\,@\,m \qquad \Gamma.[\phi] \vdash b : A[\uparrow^\phi]\,@\,m \qquad \Gamma.[\phi] \vdash a[\uparrow^\phi.[\top]] = b : A\,@\,m}$$
$$\overline{\Gamma \vdash \{\,\top\,a, \phi\,b\,\} = a[\mathsf{id}.[\top]] : A\,@\,m}$$

TERM-EQ/COMP-FACE

$$\dfrac{\Gamma\,\mathsf{cx}\,@\,m \qquad \Gamma.\mathbb{I}_m \vdash A\,@\,m \qquad \Gamma \vdash \phi : \mathbb{F}_m\,@\,m \qquad \Gamma.[\phi].\mathbb{I}_m \vdash u : A[\uparrow^\phi.\mathbb{I}_m]\,@\,m}{\Gamma \vdash u_0 : A[\mathsf{id}.0]\,@\,m \qquad \Gamma.[\phi] \vdash u[\mathsf{id}.0] = u_0[\uparrow^\phi] : A[\uparrow^\phi.0]\,@\,m \qquad \Gamma \vdash \phi = \top : \mathbb{F}_m\,@\,m}$$
$$\overline{\Gamma \vdash \mathsf{comp}\,[\phi \mapsto u]\,u_0 = u[\mathsf{id}.[\phi].1] : A[\mathsf{id}.1]\,@\,m}$$

TERM-EQ/COMP-MOD

$$\dfrac{\Gamma\,\mathsf{cx}\,@\,m \qquad \mu : n \longrightarrow m}{\Gamma.\mathbb{I}_m.\{\mu\} \vdash A\,@\,n \qquad \Gamma \vdash \phi : \mathbb{F}_m\,@\,m \qquad \Gamma.[\phi].\mathbb{I}_m.\{\mu\} \vdash u : A[\uparrow^\phi.\mathbb{I}_m.\{\mu\}]\,@\,n}$$
$$\dfrac{\Gamma.\{\mu\} \vdash u_0 : A[\mathsf{id}.0.\{\mu\}]\,@\,n \qquad \Gamma.[\phi].\{\mu\} \vdash u[\mathsf{id}.0.\{\mu\}] = u_0[\uparrow^\phi.\{\mu\}] : A[\uparrow^\phi.0.\{\mu\}]\,@\,n}{\Gamma \vdash \mathsf{mod}_\mu(\mathsf{comp}\,[\phi^\mu \mapsto u[\sigma_\mu \circ \tau_\mu]]\,u_0) = \mathsf{comp}\,[\phi \mapsto \mathsf{mod}_\mu(u)]\,\mathsf{mod}_\mu(u_0) : \langle \mu \mid A \rangle[\mathsf{id}.1]\,@\,m}$$

TERM-EQ/COMP-PI

$$\Gamma \text{ cx} @ m \qquad \mu : n \longrightarrow m$$

$$\Gamma.\mathbb{I}_m \vdash (\mu \mid A) \to B @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m \qquad \Gamma.[\phi].\mathbb{I}_m \vdash f : ((\mu \mid A) \to B)[\uparrow^\phi .\mathbb{I}_m] @ m$$

$$\Gamma \vdash f_0 : ((\mu \mid A) \to B)[\text{id}.0] @ m \qquad \Gamma.[\phi] \vdash f[\text{id}.0] = f_0[\uparrow^\phi] : ((\mu \mid A) \to B)[\uparrow^\phi .0] @ m$$

$$\Gamma.\{\mu\} \vdash a_1 : A[\text{id}.1.\{\mu\}] @ m \qquad w := (\text{hfill}^i [] \, a_1[\sigma_\mu])[\sigma^\mu] : A[i/1-i] \qquad v := w(1-i) : A$$

$$\overline{\Gamma \vdash (\text{comp} \, [\phi \mapsto f] \, f_0)(a_1) = \text{comp} \, [\phi \mapsto f(v[\uparrow^\phi .\uparrow^i .\{\mu\}])] \, f_0(v[\text{id}.0.\{\mu\}]) : B[\text{id}.1] @ m}$$

TERM-EQ/FACE-RES-BIN

$$\Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi, \psi : \mathbb{F}_m @ m \qquad \Gamma \vdash \phi \vee \psi = \top : \mathbb{F}_m @ m \qquad \Gamma \vdash A @ m$$

$$\Gamma \vdash a, b : A @ m \qquad \Gamma.[\phi] \vdash a[\uparrow^\phi] = b[\uparrow^\phi] : A[\uparrow^\phi] @ m \qquad \Gamma.[\psi] \vdash a[\uparrow^\psi] = b[\uparrow^\psi] : A[\uparrow^\psi] @ m$$

$$\overline{\Gamma \vdash a = b : A @ m}$$

TERM-EQ/FACE-RES-NULL

$$\Gamma \text{ cx} @ m \qquad \Gamma \vdash \bot = \top : \mathbb{F}_m @ m \qquad \Gamma \vdash A @ m \qquad \Gamma \vdash a, b : A @ m$$

$$\overline{\Gamma \vdash a = b : A @ m}$$

Derived Definitions.

SB/PLUS-INT

$$\Gamma, \Delta \text{ cx} @ m \qquad \Gamma \vdash \delta : \Delta @ m$$

$$\overline{\Gamma.\mathbb{I}_m \vdash \delta.\mathbb{I}_m := (\delta \circ \uparrow^i).\mathbf{v}_0^i : \Delta.\mathbb{I}_m @ m}$$

SB/EXC-INT

$$\mu : n \longrightarrow m \qquad \Gamma \text{ cx} @ m$$

$$\overline{\Gamma.\mathbb{I}_m.\{\mu\} \vdash \sigma_\mu := \uparrow^i .\{\mu\}.\mathbf{v}_0^\mu : \Gamma.\{\mu\}.\mathbb{I}_n @ m}$$

SB/EXC-FACE

$$\mu : n \longrightarrow m \qquad \Gamma \text{ cx} @ m \qquad \Gamma \vdash \phi : \mathbb{F}_m @ m$$

$$\overline{\Gamma.[\phi].\{\mu\} \vdash \tau_\mu := \uparrow^\phi .\{\mu\}.[\phi^\mu] : \Gamma.\{\mu\}.[\phi^\mu] @ m}$$

APPENDIX B. MODELS OF MTT$_\square$

Definition B.1. A modal context structure on a mode theory $\mathcal{M}$ is a strict 2-functor $[\![-]\!] : \mathcal{M}^{\text{coop}} \longrightarrow \mathbf{Cat}$ such that for each mode $m : \mathcal{M}$, $[\![m]\!]$ has a terminal object.

Definition B.2. A modal natural model on a modal context structure consists of
- for each mode $m : \mathcal{M}$, a presheaf $\mathcal{T}_m : \mathbf{PSh}([\![m]\!])$,
- for each mode $m : \mathcal{M}$, a presheaf $\widetilde{\mathcal{T}}_m : \mathbf{PSh}([\![m]\!])$,
- for each mode $m : \mathcal{M}$, a natural transformation $\tau_m : \widetilde{\mathcal{T}}_m \longrightarrow \mathcal{T}_m$,

such that
- for any modes $m, n : \mathcal{M}$ and each modality $\mu : n \longrightarrow m$, it holds that $[\![\mu]\!]^* \tau_n : [\![\mu]\!]^* \widetilde{\mathcal{T}}_n \longrightarrow [\![\mu]\!]^* \mathcal{T}_n$ is a representable natural transformation.

The type formers are the same as those in [GKNB21, Section 5.2] and [Awo18] except for identity types which we do not have and path types which will come later.

Definition B.3. A modal interval structure on a modal context structure consists of
- for each mode $m : \mathcal{M}$, a De Morgan algebra $\mathbf{I}_m : [\![m]\!]$,
- for any modes $m, n : \mathcal{M}$ and each modality $\mu : n \longrightarrow m$, a natural transformation of De Morgan algebras $\mathbb{I}_\mu : \mathbf{y}(\mathbf{I}_m) \longrightarrow [\![\mu]\!]^* \mathbf{y}(\mathbf{I}_n)$,

such that

- the preshaves $\mathbf{y}(\mathbf{I}_m)$ and morphisms $\mathbb{I}_\mu$ assemble into a lax natural transformation $[\![-]\!] \longrightarrow \mathbf{Set}^{\mathsf{op}} : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$, where $\mathbf{Set}^{\mathsf{op}} : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$ is the functor constantly equal to $\mathbf{Set}^{\mathsf{op}}$,
- for each mode $m : \mathcal{M}$ and context $\Gamma : [\![m]\!]$, the product $\Gamma \times \mathbf{I}_m$ exists,
- for any modes $m, n : \mathcal{M}$, each modality $\mu : n \longrightarrow m$, and each context $\Gamma : [\![m]\!]$, the uniquely determined dashed arrow in the following diagram has an inverse:

$$
\begin{array}{ccc}
\Gamma & & [\![\mu]\!]\Gamma \xleftarrow{\ \pi_1\ } [\![\mu]\!]\Gamma \times \mathbf{I}_n \\
\pi_1 \uparrow \quad \xmapsto{\ [\![\mu]\!]\ } \quad [\![\mu]\!]\pi_1 \uparrow \qquad \qquad \Big\downarrow \pi_2 \\
\Gamma \times \mathbf{I}_m & & [\![\mu]\!](\Gamma \times \mathbf{I}_m) \xrightarrow[\mathbb{I}_{\mu,\Gamma\times\mathbf{I}_m}(\pi_2)]{} \mathbf{I}_n \\
& & \qquad \qquad \Big\updownarrow \mathbb{I}_{\mu,\Gamma\times\mathbf{I}_m} \\
& & \Gamma \times \mathbf{I}_m \xrightarrow[\pi_2]{} \mathbf{I}_m
\end{array}
$$

Definition B.4. A modal face structure on a modal interval structure consists of

- a lax natural transformation $\mathbb{F} : [\![-]\!] \longrightarrow \mathbf{Set}^{\mathsf{op}} : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$, where $\mathbf{Set}^{\mathsf{op}} : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$ is the functor constantly equal to $\mathbf{Set}^{\mathsf{op}}$,
- for each mode $m : \mathcal{M}$, a natural transformation $\mathsf{Eq}_m^0 : \mathbf{y}(\mathbf{I}_m) \longrightarrow \mathbb{F}_m{}^{\mathsf{op}}$,

such that

- $\mathbb{F}$ factors through $\mathbf{BDisLat}^{\mathsf{op}}$, the functor constantly equal to the opposite of the category of bounded distributive lattices,
- for each mode $m : \mathcal{M}$, each context $\Gamma : [\![m]\!]$, and each interval term $r : \Gamma \longrightarrow \mathbf{I}_m$, it holds that $\mathbb{F}_{\mu,\Gamma}(\mathsf{Eq}_{m,\Gamma}^0(r)) = \mathsf{Eq}_{n,[\![\mu]\!](\Gamma)}^0(\mathbb{I}_{\mu,\Gamma}(r))$,
- for each mode $m : \mathcal{M}$ and each context $\Gamma : [\![m]\!]$, it holds that $\mathsf{Eq}_{m,\Gamma}^0(0) = \top$, where 0 is from $\mathbf{I}_m$ being a De Morgan algebra, and $\top$ is from $\mathbb{F}_m(\Gamma)$ being a bounded lattice,
- for each mode $m : \mathcal{M}$, each context $\Gamma : [\![m]\!]$, and each interval term $r : \Gamma \longrightarrow \mathbf{I}_m$, it holds that $\mathsf{Eq}_{m,\Gamma}^0(r) \wedge \mathsf{Eq}_{m,\Gamma}^0((1-r)) = \bot$, where $(1-r)$ is from $\mathbf{I}_m$ being a De Morgan algebra, and $\wedge$ and $\bot$ are from $\mathbb{F}_m(\Gamma)$ being a bounded lattice.

Definition B.5. A modal restriction structure on a modal face structure consists of

- for each mode $m : \mathcal{M}$, each context $\Gamma : [\![m]\!]$, and each face $\phi : \mathbf{y}(\Gamma) \longrightarrow \mathbb{F}_m$, a choice of pullback of the form:

$$
\begin{array}{ccc}
\mathbf{y}(\Gamma.[\phi]_m) & \longrightarrow & 1 \\
\mathbf{y}(\uparrow_{m,\Gamma}^\phi) \Big\downarrow & & \Big\downarrow \top \\
\mathbf{y}(\Gamma) & \xrightarrow{\ \phi\ } & \mathbb{F}_m
\end{array}
$$

such that

- for all modes $m, n : \mathcal{M}$, each modality $\mu : n \longrightarrow m$, each context $\Gamma : [\![m]\!]$, and each face $\phi : \mathbf{y}(\Gamma) \longrightarrow \mathbb{F}_m$, the uniquely determined dashed arrow in the following diagram has an

inverse:

$$
\begin{array}{c}
\mathbf{y}(\llbracket\mu\rrbracket(\Gamma.[\phi]_m)) \\
\end{array}
$$

Here, the commutativity of the outer square follows from the following calculation:

$$
\begin{aligned}
\mathbb{F}_{\mu,\Gamma}(\phi) \circ \mathbf{y}(\llbracket\mu\rrbracket \uparrow^{\phi}_{m,\Gamma}) &= \mathbb{F}_n(\llbracket\mu\rrbracket \uparrow^{\phi}_{m,\Gamma})(\mathbb{F}_{\mu,\Gamma}(\phi)) \\
&= ((\mathbb{F}_n \circ \llbracket\mu\rrbracket)(\uparrow^{\phi}_{m,\Gamma}) \circ \mathbb{F}_{\mu,\Gamma})(\phi) \\
&= (\mathbb{F}_{\mu,\Gamma.[\phi]_m} \circ \mathbb{F}_m(\uparrow^{\phi}_{m,\Gamma}))(\phi) \\
&= \mathbb{F}_{\mu,\Gamma.[\phi]_m}(\mathbf{y}(\uparrow^{\phi}_{m,\Gamma}) \circ \phi) \\
&= \mathbb{F}_{\mu,\Gamma.[\phi]_m}(\top) \\
&= \top.
\end{aligned}
$$

Remark B.6. For each mode $m : \mathcal{M}$, each context $\Gamma : \llbracket m \rrbracket$, and any faces $\phi, \psi : \mathbf{y}(\Gamma) \longrightarrow \mathbb{F}_m$ with $\phi \leq \psi$, consider the following diagram:

$$
\begin{array}{c}
\mathbf{y}(\Gamma.[\phi]_m) \\
\end{array}
$$

We can calculate

$$
\begin{aligned}
\psi \circ \mathbf{y}(\uparrow^{\phi}_{m,\Gamma}) &= \mathbb{F}_m(\uparrow^{\phi}_{m,\Gamma})(\psi) \\
&\geq \mathbb{F}_m(\uparrow^{\phi}_{m,\Gamma})(\phi) \\
&= \phi \circ \mathbf{y}(\uparrow^{\phi}_{m,\Gamma}) \\
&= \top,
\end{aligned}
$$

and thus $\psi \circ \mathbf{y}(\uparrow^{\phi}_{m,\Gamma}) = \top$, implying the outer square commutates, and we thus get a canonical morphism $\Gamma.[\phi]_m \longrightarrow \Gamma.[\psi]_m$.

Definition B.7. A modal face sturcture and a modal natural model (both on the same modal context sturcture) has systems if

- for each mode $m : \mathcal{M}$, each context $\Gamma : [\![m]\!]$, any faces $\phi, \psi : \mathbb{F}_m(\Gamma)$, each presheaf $X : \mathbf{PSh}([\![m]\!])$, and each commuting diagram

$$
\begin{array}{ccc}
\mathbf{y}(\Gamma.[\phi \wedge \psi]_m) & \longrightarrow & \mathbf{y}(\Gamma.[\psi]_m) \\
\downarrow & & \downarrow \\
\mathbf{y}(\Gamma.[\phi]_m) & \longrightarrow & \mathbf{y}(\Gamma.[\phi \vee \psi]_m) \\
& & \searrow \\
& & X
\end{array}
$$

where the arrows in the inner square are the canonical morphisms following from $\phi \wedge \psi \le \phi$, $\phi \wedge \psi \le \psi$, $\phi \le \phi \vee \psi$, and $\psi \le \phi \vee \psi$, if X is representable or $\mathbb{F}_m$ there exists at most one morphism $\mathbf{y}(\Gamma.[\phi \vee \psi]_m) \longrightarrow X$ such that the diagram commutes, and if X is $\mathcal{T}_m$ or $\widetilde{\mathcal{T}}_m$ there exists exactly one such morphism,
- for each mode $m : \mathcal{M}$, each context $\Gamma : [\![m]\!]$, and each presheaf $X : \mathbf{PSh}([\![m]\!])$, if X is representable, $\mathbb{F}_m$, or $\mathcal{T}_m$ there exists at most one morphism $\mathbf{y}(\Gamma.[\bot]_m) \longrightarrow X$, and if X is $\widetilde{\mathcal{T}}_m$ there exists exactly one such morphism.

Definition B.8. A path structure on modal interval structure and a modal natural model (both on the same modal context structure) is a direct translation of the rules for path types, and we will thus not give the details.

Definition B.9. A composition structure on modal restriction structure is a direct translation of the rules for composition, and we will thus not give the details.