ON THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

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Abstract. In these notes we present a short, mostly self-contained proof of the independence of the continuum hypothesis. The development is topos-theoretic: we shall be presenting a specific topos which can model ZFC for which the continuum hypothesis fails. This demonstrates the connection between the logical methods of forcing, Beth semantics and Grothendieck toposes. This development closely follows that given in MacLane and Moerdijk [16, Chapter 6], a simplification of the original proof given in Cohen [2]. With these notes I hope to illustrate the elegant topos theoretic approach to forcing results that has become increasingly common in logic, type theory, and mathematics at large. The concluding section 5 contains a number of interesting and related developments for the curious reader.

1. Background

These notes are not meant to serve as a complete introduction to topos theory. Therefore, the background section of these notes, rather than being the first 5 chapters of MacLane and Moerdijk [16] or Johnstone [15] will contain more or less an accumulation of definitions and lemmas that we will need. These will be more useful for ensuring that I have things to reference than for the reader to learn. It also comes with the moderate advantage that I get to inflict the my peculiarities upon you dear reader.

We begin by defining the notion of an *elementary topos*.

Definition 1.1. *An elementary topos $\mathcal{E}$ is a category that*

- *has all finite limits*
- *is cartesian closed*
- *has a subobject classifier*

The topos that we are interested in will be ones that satisfy certain characteristics making them into a model of ZFC. Modeling the axiom of choice, the law of the excluded middle, and the existence of infinite sets in particular present challenges. The next set of definitions are the categorical analogs of these traits.

Definition 1.2. *A natural number object (NNO) is an object $\mathsf{N} \in \mathbb{C}$ with arrows $\mathsf{s} : \mathsf{N} \to \mathsf{N}$ and $z : 1 \to \mathsf{N}$ so that for any object $\mathsf{A} \in \mathbb{C}$ with $\mathsf{f} : 1 \to \mathsf{A}$ and $\mathsf{g} : \mathsf{A} \to \mathsf{A}$, there exists a unique h so*

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that

$$\begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "f"'] & N \arrow[r, "s"] \arrow[d, dashed, "h"] & N \arrow[d, dashed, "h"] \\ & A \arrow[r, "g"] & A \end{tikzcd}$$

Example 1.3. *In the category of sets,* **Set***, the set of all natural numbers* $\mathbb{N}$ *forms a natural number object.*

Example 1.4. *In any presheaf* $\widehat{\mathbb{C}}$*, the constant presheaf* $A \mapsto \mathbb{N}$ *forms a natural number object.*

Example 1.5. *NNOs are reflected by geometric morphism. In particular, any reflective subtopos inherits an NNO from the full topos.*

Having categorified the definition of the set of natural numbers, we now turn to the logical aspects of a topos. The internal logic of a topos is, in general, intuitionistic and thus validates neither the law of the excluded middle nor the axiom of choice. We shall need both of these for the topos we're constructing. Therefore, we turn to defining what toposes *do* satisfy these principles.

Definition 1.6. *A topos is said to be boolean if the subobject classifier* Ω *forms an internal boolean algebra.*

Booleanness can be captured in several equivalent in useful ways.

Lemma 1.7. *The following conditions are equivalent*

(1) $\mathcal{E}$ *is boolean*

(2) $\neg\neg = 1 : \Omega \to \Omega$

(3) *The Heyting algebra* $\mathrm{Sub}(A)$ *for an* $A \in \mathcal{E}$ *is a boolean algebra*

(4) $\Omega \cong 1 + 1$ *with* $[\mathrm{true}, \neg \circ \mathrm{true}]$.

Proof.

- $1 \iff 2$
  The condition that $\neg\neg a = a$ for all a is equivalent to being boolean for any boolean algebra. $a \leqslant \neg\neg a$ in any Heyting algebra, therefore all we need to show is that $\neg\neg a \leqslant a$. Since $1 \leqslant a \vee \neg a$, it suffices to show that $\neg\neg a \wedge (a \vee \neg a) \leqslant a$. Since all Heyting algebras are distributive,

$$\begin{aligned} \neg\neg a \wedge (a \vee \neg a) \leqslant a &\Leftrightarrow \\ (\neg\neg a \wedge a) \vee (\neg\neg a \wedge \neg a) \leqslant a &\Leftrightarrow \\ a \vee \bot \leqslant a &\Leftrightarrow \\ a \leqslant a & \end{aligned}$$

For the other direction, it then suffices to show that $1 \leqslant \neg\neg(a \vee \neg a)$

$$\begin{aligned} 1 \leqslant \neg\neg(a \vee \neg a) &\Leftrightarrow \\ \neg(a \vee \neg a) \leqslant 0 &\Leftrightarrow \\ \neg a \wedge \neg\neg a \leqslant 0 &\Leftrightarrow \\ 0 \leqslant 0 &\Leftrightarrow \end{aligned}$$

so we're done.

- $1 \iff 3$
  We know that the Heyting algebra $\mathrm{Sub}(A)$ for any $A \in \mathcal{E}$ corresponds naturally to $\hom(A, \Omega)$. This is a boolean algebra precisely when Ω forms one internally. For the reverse direction, if we have $\neg\neg = 1$ in every subobject preorder, $\mathrm{Sub}(A)$, naturally in A, then by Yoneda this holds internally to Ω.

- $1 \iff 4$
  For this, let us first show that $4 \implies 1$. If $i = [\mathtt{true}, \neg \circ \mathtt{true}] : 1 + 1 \cong \Omega$, then it is clear that

$$\begin{tikzcd} 1+1 \arrow[r, "i"] \arrow[d, "tw"] & \Omega \arrow[d, "\neg"] \\ 1+1 \arrow[r, "i"] & \Omega \end{tikzcd}$$

commutes where $tw : 1 + 1 \to 1 + 1$ is the canonical twisting map. It satisfies the properties that $tw \circ \mathrm{inl} = \mathrm{inr}$ and vice versa. Now this means that $i \circ tw \circ i^{-1} = \neg$ but then

$$\neg\neg = i \circ tw \circ i^{-1} \circ i \circ tw \circ i^{-1} = i \circ tw \circ tw \circ i^{-1} = 1$$

so we indeed have that Ω is an internal boolean algebra. Next, we must show that $1 \implies 4$. We know that $\mathrm{true} \wedge \neg\,\mathrm{true} = 0$ so $\mathrm{true} \vee \neg\,\mathrm{true} = \mathrm{true} + \neg\,\mathrm{true}$ as subobjects of Ω. Let us call $m = \mathrm{true} \vee \neg\,\mathrm{true}$, then pictorially, we have that

$$\begin{tikzcd} 1+1 \arrow[dr, dashed, "m"] & 1 \arrow[l] \arrow[d, "\mathrm{true}"] \\ 1 \arrow[u] \arrow[r, "\neg\,\mathrm{true}"] & \Omega \end{tikzcd}$$

However, we know that since Ω is boolean, $a \vee \neg a = 1$. Therefore, $\mathrm{true} \vee \neg\,\mathrm{true} = 1$. This tells us that $1 + 1 \cong \Omega$ as required.

$\square$

For our purposes of using a topos to model ZFC, booleaness will be essential. A boolean topos will validate the law of excluded middle[1] which is crucial for validating the rules of

[1]Exercise, show that in a boolean topos $[\![\forall x.\ x \vee \neg x]\!]$ holds

ZFC. There then arises the natural question of taking an existing topos and modifying it so that it is boolean. This is easily done using a Lawvere-Tierney topology

Definition 1.8. *A Lawvere-Tierney Topology is a map* $j : \Omega \to \Omega$ *so that*

(1) $j \circ \mathrm{true} = \mathrm{true}$

(2) $j \circ j = j$

(3) $\wedge \circ (j, j) = j \circ \wedge$

While the subject of Lawvere-Tierney topologies on an elementary topos is a rich one, we shall limit ourselves to the essentials for our purposes here. It will be important to know that given a Lawvere-Tierney topology we can internally define a notion of "sheaf" which matches up to that of a sheaf on a Grothendieck topology. In order to do this, first we define

Definition 1.9. *For a topology* j *on* $\mathcal{E}$*, the closure of* $A \rightarrowtail B$ *is the subobject of* B *classified by* $j \circ \mathrm{char}(A)$*,* $\bar{A}$*. A subobject is said to be closed if* $\bar{A} = A$*.*

We can define an alternative subobject classifier which internalizes a Lawvere-Tierney topology as

$$\begin{tikzcd} \Omega_j \arrow[r, dashed] & \Omega \arrow[r, shift left, "1"] \arrow[r, shift right, "j"'] & \Omega \end{tikzcd}$$

This has the nice property that it classifies closed subobjects.

Lemma 1.10. *If* $A \rightarrowtail B$ *is a closed subobject under* j*, then* $\mathrm{char}(A)$ *factors through* Ω_j*.*

Proof. Suppose that A is a closed subobject, then we know that $\mathrm{char}(A) \circ j = \mathrm{char}(A)$ by definition. Therefore,

$$\begin{tikzcd} A \arrow[d, dashed] \arrow[dr, "\mathrm{char}(A)"] & & \\ \Omega_j \arrow[r] & \Omega \arrow[r, shift left, "1"] \arrow[r, shift right, "j"'] & \Omega \end{tikzcd}$$

by the universal property of an equalizer. □

Definition 1.11. *A subobject* $A \rightarrowtail B$ *is said to be dense if* $\bar{A} \cong B$

We are now in a position to define the internalization of sheaves. The inspiration for this definition comes from the fact that in a Grothendieck topology, a sieve is a dense subfunctor of $\mathbf{y}(C)$. We therefore can define sheaves purely in terms of dense subobjects.

Definition 1.12. *An object* $F \in \mathcal{E}$ *is said to be a sheaf if for every dense subobject* $A \rightarrowtail B$*, a morphism* $f : A \to F$ *has a unique extension* $g : B \to F$ *and every* $g : B \to F$ *corresponds to a unique* $f : A \to F$*. The former is captured by the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "m"'] & F \\ B \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Lemma 1.13. Ω_j *is a* j*-sheaf.*

Proof. Suppose we have a dense map $m : A \rightarrowtail B$ and $f : A \to \Omega_j$, we wish to construct $g : B \to \Omega_j$ so that $g \circ m = f$. However, a map into Ω_j composes with $\iota : \Omega_j \rightarrowtail \Omega$ to give a map $\iota \circ f : A \to \Omega$ which must correspond to a subobject $M \rightarrowtail A$. Furthermore, since $\mathrm{char}(\bar{M}) = j \circ j \circ f = \mathrm{char}(M)$. It must be that M is closed. This in turn gives us a closed subobject $M \rightarrowtail B$ by simple composition. Therefore, we have a corresponding map $\mathrm{char}_B(M)$ so that $\mathrm{char}_B(M) = j \circ \mathrm{char}_B(M)$. This implies that $\mathrm{char}_B(M)$ must factor through Ω_j because Ω_j is the equalizer of j and 1.

Finally, we have the diagram

$$\begin{tikzcd}
M \arrow[rr, "!"] \arrow[d, tail] & & 1 \arrow[dd, "\mathrm{true}"] \\
A \arrow[d, tail] \arrow[dr, "f"] & & \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{char}_B(M)"'] & \Omega_j \arrow[r] & \Omega
\end{tikzcd}$$

so it is clear that $\mathrm{char}_B(M)$ is the extension we were looking for. ☐

Sheaves enjoy several nice properties with regards to closure which will be useful for establishing some facts later on.

Lemma 1.14. *For any* $A \rightarrowtail B$ *if* A *is a sheaf then* A *is closed.*

Proof. We know that by definition that the mono $m : A \rightarrowtail \bar{A}$ is dense and since A is a sheaf we must have

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, tail, "m"] & A \\
B \arrow[ur, dashed] &
\end{tikzcd}$$

so in particular m must be an isomorphism. ☐

Finally, while the proof is too far out of scope for these notes, it is important to note that sheaves on a topology j form a full subcategory of a topos, $\mathcal{E}$. This category, denoted $\mathbf{Sh}_j(\mathcal{E})$, is in fact a reflective subcategory with

$$\begin{tikzcd}
\mathbf{Sh}_j(\mathcal{E}) \arrow[rr, bend left, "\iota"] \arrow[rr, phantom, "\top"] & & \mathcal{E} \arrow[ll, bend left, "\mathbf{a}"]
\end{tikzcd}$$

The curious reader is referred to either MacLane and Moerdijk [16] or Johnstone [15]. Additionally Jon Sterling gave a talk on this and ought to distribute his notes. Please go bother him if you really want a proof. Now in our case, we shall be interested in one particular topology, the double negation topology.

Example 1.15. *For any topos,* $\neg\neg : \Omega \to \Omega$ *forms a topology.*

The useful property of the double-negation topology for our purposes is that $\mathbf{Sh}_{\neg\neg}(\mathcal{E})$ will always form a boolean subtopos of $\mathcal{E}$. Later on, we will use this to construct a boolean topos out of a topos modeling the forcing construction we wish to implement.

Lemma 1.16. $\mathbf{Sh}_{\neg\neg}(\mathcal{E})$ *is boolean.*

Proof. By 1.7 it suffices to show that for each sheaf A that $\mathrm{Sub}(A)$ is a boolean lattice. If $B \rightarrowtail A$ is a subobject of A, it must be that B closed as it too must be a sheaf by 1.14. But closure in under $\neg\neg$ means precisely that $\neg\neg B = B$ so $\mathrm{Sub}(A)$ is boolean as required. □

An interesting aside at this point is that the double-negation topology on a presheaf topos has a well known Grothendieck analog: the dense topology. That is, the topology given by

$$J(C) = \{S \mid \forall f : A \to C.\ \exists h.\ f \circ h \in S\}$$

I will not prove this but will use the phrase "dense topology" and "double negation topology" interchangeably in the notes[2].

We next turn to the topos-theoretic analog of the axiom of choice. Here we are presented with two possible ways. The first is a direct formulation of the principle from the category of sets. We wish to generalize *every surjection has a section*. We can generalize this to a topos by replacing surjection with epimorphism to get

Definition 1.17. *A topos satisfies the axiom of choice if every* $e : A \twoheadrightarrow B$ *has a section* $s : B \to A$ *so that* $es = 1$

However, it is often preferable to state a version termed the internal axiom of choice but the differences are out of scope for these notes. I will settle for merely stating it.

Definition 1.18. *A topos satisfies the internal axiom of choice if* $(-)^A$ *preserves epimorphisms.*

Remark 1.19. *Any topos that satisfies the axiom of choice satisfies the internal version. Any topos that satisfies the internal version is boolean. This is due to Diaconescu [6].*

This characterization of the axiom of choice is awkward to work with however in many toposes. A cleaner characterization can be given in terms of the what objects *generate* the topos.

Definition 1.20. *A collection of objects* S *is said to generate a category* $\mathbb{C}$ *if for any parallel arrows* $f \neq g : A \to B$*, there exists an arrow* $h : C \to A$ *for some* $C \in S$ *so that* $fh \neq gh : C \to B$*. In particular, if a topos is generated by* 1 *and is nontrivial then it is* well-pointed*.*

[2] It's because I'm a mean-spirited person

Generation by a collection of objects captures a large number of logical principles that we will need for our topos. Most important among these is that for a boolean Grothendieck topos generated by subobjects of 1, we may automatically derive the validity of the axiom of choice. This is because in a Grothendieck topos the subobject preorders are all *complete* partial orders. We demonstrate this in the next lemma.

Lemma 1.21. *If a boolean topos $\mathcal{E}$, is generated by subobjects of* 1 *and has complete boolean algebras for subobject posets then $\mathcal{E}$ satisfies the axiom of choice.*

Proof. We wish to show that $e : A \twoheadrightarrow B$ has a section $B \to A$. We then wish to use Zorn's lemma to find a maximal subobject $m : M \rightarrowtail B$ so that e has a section $s : M \to A$. That is, so that $es = m$. In order to apply Zorn's lemma we wish to show that the subset of $\mathrm{Sub}(B)$ is closed under chains. First we note that any such section is necessarily mono. This is because if $sf = sg$, then we must have $esf = esg$ so $mf = mg$ so $f = g$ as m is mono. Now let us show the chain condition.

Suppose that $(M_i)_i \subseteq \mathrm{Sub}(B)$ so that for each M_i, we have a section s_i and this forms a chain. Now we note that each M_i is a subobject of $B \times A$ under $\langle m_i, s_i \rangle$. Since we assumed that $\mathcal{E}$ had complete boolean algebras for subobject lattices, let us take $M = \bigvee_i M_i$ taken as subobjects of $B \times A$. Now this tells us that $M \rightarrowtail B \times A$ so we must have $m : M \to B$ and $s : M \to A$. We must show that $e \circ s = m$ now.

Let us note that A is a subobject of $B \times A$ under $\langle e, id \rangle$. Furthermore, by s we have that $M_i \leqslant A$ for all i. Therefore, by the definition of a least upper bound, we must have $M \leqslant A$. Therefore, we have a mono so that

$$\begin{tikzcd} M \arrow[rr, tail, "s'"] \arrow[dr, tail, "\langle m,s \rangle"'] & & A \arrow[dl, tail, "\langle e,id \rangle"] \\ & B \times A & \end{tikzcd}$$

this tells us that $s' = s$ so we have our desired section. Now we want to conclude that $m : M \to B$ is indeed a mono, however, all we know is that $\langle m, s \rangle$ is a mono. Therefore, we want to show that

$$(x, x', y \mid m(x) = y \wedge m(x') = y \vdash x = x')$$

in the internal logic. Since $m' = \langle m, s \rangle$ is a mono, we know that

$$(x, x', z : B \times A \mid m'(x) = z \wedge m(x') = z \vdash x = x')$$

Therefore, if we can show that

$$(a, a', b \times M(b, a), M(b, a') \vdash a = a')$$

then we could rewrite the assumption that m' is mono to

$$(x, x', b : B \mid \pi_1(m'(x)) = b \wedge \pi_1(m(x')) = b \vdash x = x')$$

and since $m = \pi_1 \circ m'$ by definition, this gives us our desired result. We know that since each m_i is mono, that if

$$(x, x', b \mid m_i(x) = b \wedge m_i(x') = b \vdash x = x')$$

Furthermore, this tells us that if

$$(x, x', b, a, a' \mid (\langle m_i, s_1 \rangle)(x) = (b, a) \wedge (\langle m_i, s_1 \rangle)(x') = (b, a') \vdash x = x')$$

so by functionality, since we have $f(x) = (b, a) \wedge f(x) = (b, a')$ we must have $a = a'$ as well. We know that for all i.

$$(a, a', b \times M_i(b, a), M_i(b, a') \vdash a = a')$$

Now, stepping outside the internal logic, we wish to show that

$$\begin{tikzcd}
\pi_{12}^*(M) \wedge \pi_{13}^*(M) \arrow[rr, tail] \arrow[dr] & & B \times A \arrow[dl, "1\times\delta"] \\
& B \times A \times A &
\end{tikzcd}$$

Let us now work with $\mathrm{Sub}(B \times A \times A)$.

$$\begin{aligned}
\pi_{12}^*(M) \wedge \pi_{13}^*(M) &= \pi_{12}^*(\bigvee_i M_i) \wedge \pi_{13}^*(\bigvee_i M_i) \\
&= \bigvee_i \pi_{12}^*(M_i) \wedge \bigvee_i \pi_{13}^*(M_i) \\
&= \bigvee_i \bigvee_j \pi_{12}^*(M_i) \wedge \pi_{13}^*(M_j) \\
&= \bigvee_i \pi_{12}^*(M_i) \wedge \pi_{13}^*(M_i) \\
&= \bigvee_i \delta \\
&= \delta
\end{aligned}$$

Where we have made use of the fact that clearly

$$\bigvee_i \pi_{12}^*(M_i) \wedge \pi_{13}^*(M_i) \leqslant \bigvee_i \bigvee_j \pi_{12}^*(M_i) \wedge \pi_{13}^*(M_j)$$

but since M_i forms a chain, supposing that $i \leqslant j$ without loss of generality, we then have that

$$\pi_{12}^*(M_i) \wedge \pi_{13}^*(M_j) \leqslant \pi_{12}^*(M_j) \wedge \pi_{13}^*(M_j)$$

This means that the reverse inclusion holds. All told, this means that we have indeed constructed an M dominating the chain with a partial section for e as required. Thus, Zorn's lemma gives us a maximal such subobject.

Now suppose we have such a maximal subobject $m : M \rightarrowtail B$ with a section $s : M \to A$. Let suppose that $M \neq A$, for it is then we're done. We note that M must have a nonzero complement as it is not B. Since $\neg M$ is nonzero, it has two different subobjects of its own, 0 and 1. Thus $\neg M$ has two different characteristic maps and since subobjects of 1 generate, for some $U \rightarrowtail 1$ we must have $t : U \to \neg M$. Consider

$$\begin{tikzcd} A' \arrow[rr] \arrow[d, two heads, "e'"] & & A \arrow[d, two heads, "e"] \\ V \arrow[r, tail, "t"] & \neg M \arrow[r, tail] & B \end{tikzcd}$$

Next, we note that $V \neq 0$ because it has an arrow to a nonzero object. From this, it follows that A' must not be 0 as $A' \twoheadrightarrow V$ and $V \overset{f}{\underset{g}{\rightrightarrows}} \Omega$ for some $f \neq g$. If A' was zero then it must be that $fe = ge$ which would give us $f = g$, a contradiction.

Now since A' is nonzero, we can once again get a subterminal object which maps into it, $W \rightarrowtail 1$ with $w : W \to A'$. Now we immediately have $e'w : W \to V$ so it follows that $W \rightarrowtail V$. Now we take the image factorization of $\mathfrak{I}(te'w) = e'' : W \to t(W)$. Now since $W \rightarrowtail 1$, it must be that e'' is also mono and since $\mathcal{E}$ is balanced, we then have that e'' is an iso. Now as $t(W) \rightarrowtail \neg M$, it must be that $t(W) \wedge M = 0$ so $t(W) \vee M = t(W) + M$. Moreover, $we''^{-1} : t(W) \to A'$ so we know that $t(W) \to A$ so we can form a section $M + t(W) \to A$. This contradicts the maximality of M. $\square$

With this we are in a position to start our proof because a topos which satisfies the AoC and has a NNO is powerful enough to provide a model of ZFC that we will use to validate the independence of the continuum hypothesis.

2. An Overview of the Proof

Before we dive into the details of this proof, I think it is helpful to step back and summarize it at a high level. The proof using a method called forcing[3]. The basic premise of the proof is that we want to construct a model of ZFC which forces a set much larger than 2^N, B, to have an injection into 2^N in the topos $\mathbf{Sh}_{\neg\neg}(P)$ for some poset P. It doesn't particularly matter what B is so long as it's strictly larger than 2^N. For simplicity, let us fix

$$B = \mathcal{P}(\mathcal{P}(N))$$

for the remainder of the paper. Since this cannot be done ordinarily we use forcing to gradually introduce fragments of such an injection so that, internally to the topos, we may manipulate this arrow as if it were a complete injection. The "fragments" of the function will constitute the poset P that we're defining sheaves over. This is the essence of topos-theoretic forcing. This topos is developed in section 3.

Now in our model, we then have $\mathbf{a}(\Delta N) \rightarrowtail \mathbf{a}(\Delta B) \rightarrowtail \Omega_{\neg\neg}^N$. Having done this, we can actually force an object to appear strictly between N and $\Omega_{\neg\neg}^N$. This object however isn't $\mathbf{a}(\Delta B)$! Instead, we use $\mathbf{a}(\Delta(2^N))$, the powerset of the NNO from the presheaf topos included into $\mathbf{Sh}_{\neg\neg}(P)$. Since B was specifically chosen to be much larger than 2^N, we get an inclusion

$$\mathbf{a}(N) \rightarrowtail \mathbf{a}(\Delta 2^N) \rightarrowtail \mathbf{a}(B) \rightarrowtail \Omega_{\neg\neg}^N$$

All of these inequalities are developed in section 3. Moreover, those first two inclusions are strict in $\widehat{P}$. We then prove that posets like P satisfy what is called the Souslin property

[3]Forcing was in fact developed to solve this problem

and that in this case $\mathbf{a}(-)$ preserves the strictness of inclusions. This is the most technical portion of the proof and is developed in 4.

Having done this, by transitivity we have then forced the existence of an object $\mathbf{a}(2^{\mathsf{N}})$ which lies between N and $\mathcal{P}(\mathsf{N})$ as required. Now the astute reader might object that we have not actually provided a model of ZFC. This is largely the construction for a model of ZFC is easily found in Fourman [10].

3. The Cohen Topos

3.1. **The Partial Order P.**

For the remainder of this development, we will work with the Cohen topos. This is a subtopos of presheaves on a particular partial order. As mentioned above, we wish to design this partial order in order to help us construct a monomorphism from $\mathsf{B} \to \Omega^{\mathsf{N}}$. Now by transposition, such a morphism can always be represented as $\mathsf{B} \times \mathsf{N} \to \Omega$. This gives us an indication for how to construct P, it can simply be subsets of $\mathsf{B} \times \mathsf{N}$. Now in order for a function $\mathsf{B} \to \Omega^{\mathsf{N}}$ (transposed as $\mathsf{B} \times \mathsf{N} \to \Omega$) to be a monomorphism, it must be that for any p and any $b \neq b' \in \mathsf{B}$ that there is some n so that $p(b, n) \neq p(b', n)$.

Therefore, our partial order P shall be a collection

$$\mathsf{P} = \mathcal{P}_{\mathtt{fin}}\left(\{f \in \mathsf{B} \times \mathsf{N} \times \{0, 1\} \mid f \text{ functional}\}\right)$$

For any $p \in \mathsf{P}$, we shall treat p as a member of $\mathsf{B} \times \mathsf{N} \rightharpoonup \{0, 1\}$ which is defined on only a finite number of inputs. Accordingly, we shall use $p(b, n)\downarrow$ and $p(b, n)\uparrow$ to indicate whether or not p is or isn't defined on a particular input respectively. P is called the collection of forcing conditions and each element is thus a condition. It should be thought of as a "constraint" on the map that we are trying to construct from $\mathsf{B} \to \Omega^{\mathsf{N}}$. If we are working at forcing condition p we are in effect stating that while we do not know the full contents of the map $\mathsf{B} \to \Omega^{\mathsf{N}}$, we know that it is at least a completion of p.

The finiteness of each set is crucial. It is used to imply that each completion *could* be part of a monomorphism. This is because for any $b, b' \in \mathrm{Dom}(p)$ for some $p \in \mathsf{P}$, there exists an n so that $p(b, n)\uparrow$ and $p(b', n)\uparrow$. Therefore, it is always the case that a future condition may add some data to distinguish b and b'. It remains to define an order on P however. Let us say

$$q \leqslant p \triangleq \forall (b, n) \in \mathsf{B} \times \mathsf{N}.\ p(b, n)\downarrow \implies p(b, n) = q(b, n)$$

This is the opposite of the traditional order that partial functions are endowed with and is clearly reflexive, antisymmetric, and transitive. This reversal is typical in forcing developments and may seem slightly confusing. The reason for it though is quite straight forward, each p is thought of representing our knowledge about some map from $\mathsf{B} \to \Omega^{\mathsf{N}}$. The more defined p is, the smaller it is according to our ordering, the fewer maps it corresponds to. The ordering is in effect the traditional subset ordering on the possibilities each condition allows. Hence, a larger condition has more information, it permits fewer possibilities and is therefore smaller.

3.2. **Sheaves on** $\mathbb{P}$.

Let us now turn our attention to the Cohen Topos, or $\mathbf{Sh}_{\neg\neg}(\mathbb{P})$. We want to develop the framework necessary for us to construct our cardinal inequalities that we hinted at in the overview 2. Before we can turn our attention to that though, we would like to establish that our topos is a boolean topos that satisfies the axiom of choice with a natural number object. Since $\mathbf{Sh}_{\neg\neg}(\mathbb{P})$ is a subtopos of $\widehat{\mathbb{P}}$, it inherits its NNO from $\widehat{\mathbb{P}}$. We know from lemma 1.16 that $\mathbf{Sh}_{\neg\neg}(\mathbb{P})$ is boolean. In order to show that it satisfies the axiom of choice, we would like to apply lemma 1.21. In order to do this, we must show that for each $F \in \mathbf{Sh}_{\neg\neg}(\mathbb{P})$ that $\mathrm{Sub}(F)$ is a complete boolean Heyting algebra, but this follows from the fact that $\mathbf{Sh}_{\neg\neg}(\mathbb{P})$ is a Grothendieck topos and lemma 1.7. This means all we must show is

Lemma 3.1. *Subobjects of* 1 *generate* $\mathbf{Sh}_{\neg\neg}(\mathbb{P})$.

Proof. In order to show this, we will show that $\mathbf{a}(\mathbf{y}(p))$ generates $\mathbb{P}$ and that these sheaves are subobjects of one. First, the latter. Since $\mathbf{y}(p) \rightarrowtail 1$ clearly holds in $\widehat{\mathbb{P}}$, it suffices to note that, since $\mathbf{a}$ is left exact $\mathbf{a}(1) = 1$ and it preserves monomorphisms. Therefore, $\mathbf{a}(\mathbf{y}(p)) \rightarrowtail \mathbf{a}(1) = 1$ as required.

Now we must show that $\mathbf{a}(\mathbf{y}(p))$ generates. It suffices to show that $\mathbf{y}(p)$ generates $\widehat{\mathbb{P}}$. To see this, suppose that $f \neq g : F \to G$ in $\mathbf{Sh}_{\neg\neg}(\mathbb{P})$. In this case, we know that there is a p and $t : \mathbf{y}(p) \to F$ so that $ft \neq gt$. Therefore, $\mathbf{a}(ft) \neq \mathbf{a}(gt)$ but $\mathbf{a}(ft) = f\mathbf{a}(t) : \mathbf{a}(\mathbf{y}(p)) \to F$ and likewise for gt. Now to see that $y(p)$ generates it suffices to note that any presheaf is the colimit of representables. Therefore, if $f \neq g : P_1 \to P_2$ then this implies that there are two distinct cones on P_2 by the universality of a colimit. Therefore, one of the legs of the cones is different. Simply factoring that leg through P_1 gives our desired distinguishing map. $\square$

Now that we have established that this topos is of the variety we need, let us investigate what sheaves are actually found in it. First we will demonstrate that the dense topology is *subcanonical*. This means that all yoneda embeddings form sheaves, something that will prove useful later on.

Lemma 3.2. *For any* $p \in \mathbb{P}$, $\mathbf{y}(p)$ *is a sheaf.*

Proof. In order to do this, we will directly use the definition of sieve on Grothendieck topology. Suppose we have S, a covering sieve for the dense topology on q. Furthermore, suppose we have a matching family for S, $(x_f)_{f\in S}$. We now wish to find a unique amalgamation $x \in \mathbf{y}(p)(q)$. Since $\mathbb{P}$ is a poset, it suffices to show that such an amalgamation exists, as $y(p)(q)$ contains either one or zero elements. Now, since $x_d \in y(p)(d)$ for each $d \leqslant q \in S$, we know that $d \leqslant p$.

All we need to show then is that $q \leqslant p$. Suppose not. Then there must be a (b, n) so that $q(b, n) \neq p(b, n)$ or that $q(b, n) \uparrow$. Let us define $q' = q[(b, n) \mapsto \neg p(b, n)]$ so that it is clear that $q'(b, n) \neq p(b, n)$. Then, since S is a cover in the dense topology and $q' \leqslant q$ there must be a $d \leqslant q'$ so that $d \in S$. However, we then have $d \leqslant p$ by assumption so $d(b, n) = p(b, n)$, a contradiction. Therefore, it must be that $q \leqslant p$ holds. $\square$

Having fleshed out a few basic sheaves, let us now begin to construct $\mathbf{a}(\Delta B)$ and the monomorphism from it to $\Omega_{\neg\neg}^{\mathbf{a}(\Delta N)}$. Here we use ΔX to indicate the constant presheaf $A \mapsto X$. First, we note that in order to construct a morphism

$$g : \mathbf{a}(\Delta B) \times \mathbf{a}(\Delta N) \to \Omega_{\neg\neg}$$

In order to construct this, we will look to construct a similar morphism in $\widehat{P}$. Let us define a subobject of $A \rightarrowtail \Delta B \times \Delta N$. First we note that $\Delta B \times \Delta N = \Delta(B \times N)$. Therefore, we can define A as

$$A(p) = \{(b, n) \mid p(b, n) = 1\}$$

This subobject is designed to pick out the graph of the function we have currently available to us. It, crucially, will vary as we move from condition to condition allowing A to better and better approximate our hypothetical monomorphism. Moreover, A happens to be a closed object of $\Delta B \times \Delta N$.

Lemma 3.3. *A is a closed subobject of $\Delta B \times \Delta N$ under $\neg\neg$.*

Proof. It suffices to show that $\neg\neg A \leqslant A$ as the reverse holds for any subobject. Suppose that $(b, n) \in (\neg\neg A)(p)$. This indicates that for all $q \leqslant p$ there is an $r \leqslant q$ for which we have $(b, n) \in A(r)$ which is to say, $r(b, n) = 1$. This in particular implies that $r(b, n) = 1$.

Now if $(b, n) \notin A(p)$. This implies that either $p(b, n) = 0$ or that $p(b, n) \uparrow$. If the former is the case then we have an immediate contradiction: $r \leqslant p$ so $p(b, n) = r(b, n)$ must hold. If the latter, then we have $p' = p[(b, n) \mapsto 0] \leqslant p$ and for all $r \leqslant p'$, we know that $(b, n) \notin A(r)$. This contradicts the assumption that $(b, n) \in (\neg\neg A)(p)$ so we're done. $\square$

Having established that A is a closed subobject, we know that $\mathrm{char}(A) : \Delta B \times \Delta N \to \Omega$ must factor through $\Omega_{\neg\neg} \rightarrowtail \Omega$ by lemma 1.10. Now this means we have the following diagram

$$\begin{tikzcd}
\Delta N \times \Delta B \arrow[rr, "\mathrm{char}(A)"] \arrow[dr, dashed, "f"'] & & \Omega \\
& \Omega_{\neg\neg} \arrow[ur, tail] &
\end{tikzcd}$$

Now we're going to show that this map $g = \Lambda f : \Delta B \to \Omega_{\neg\neg}^{\Delta N}$ is actually the desired monomorphism.

Lemma 3.4. *g is a monomorphism in $\widehat{P}$.*

Proof. It suffices to show that this is a monomorphism in **Set** for all $p \in P$. Now unfolding definitions, we know that

$$g_p : B \to (\Delta N \times \mathbf{y}(p) \to \Omega_{\neg\neg})$$

Moreover, if we apply g_p to $b \in B$ and $(n, \star) : \Delta(N)(q) \times \mathbf{y}(p)(q)$ we get

$$(g_p(b))_q(n, \star) = \{r \mid r \leqslant q \wedge r(b, n) = 1\}$$

Suppose that $b \neq c \in B$, we wish to show that there is an n so that

$$(g_p(b))_q(n, \star) \neq (g_p(c))_q(n, \star)$$

However, since q is finite, there is an n so that $q(b, n_0) \uparrow$ and $q(c, n_0) \uparrow$. We then consider $(g_p(b))_q(n_0, \star)$ and $(g_p(c))_q(n_0, \star)$. Since $q(b, n_0) \uparrow$, $(g_p(b))_q(n_0, \star)$ must contain q' where $q' = q[(b, n_0 \mapsto 1)]$. However, since $q'(c, n_0) \uparrow$ it cannot be that $q' \in (g_p(c))_q(n_0, \star)$ so these are distinct as required. Therefore, g is a mono. □

Corollary 3.5. $m = \mathbf{a}(g)$ *is a monomorphism in* $\mathbf{Sh}_{\neg\neg}(P)$ *from* $\mathbf{a}(\Delta B) \to \Omega_{\neg\neg}^{\mathbf{a}(\Delta N)}$.

Proof. Since $\mathbf{a}$ is left exact, we immediately have that m is a mono. It remains to show that

$$\mathbf{a}(\Omega_{\neg\neg}^{\Delta N}) \cong \Omega_{\neg\neg}^{\mathbf{a}(\Delta N)}$$

This follows because

$$\begin{aligned}
\hom(X, \Omega_{\neg\neg}^{\Delta N}) &= \hom(\Delta N \times X, \Omega_{\neg\neg}) \\
&= \hom(\mathbf{a}(\Delta N \times X), \Omega_{\neg\neg}) \\
&= \hom(\mathbf{a}(\Delta N) \times \mathbf{a}(X), \Omega_{\neg\neg}) \\
&= \hom(\mathbf{a}(X), \Omega_{\neg\neg}^{\mathbf{a}(\Delta N)}) \\
&= \hom(X, \Omega_{\neg\neg}^{\mathbf{a}(\Delta N)})
\end{aligned}$$

so this follows immediately from yoneda. Above we have made use of the fact that

$$\hom(X, Y) = \hom(\mathbf{a}(X), Y)$$

when Y is a sheaf and lemma 1.13, an immediate result of the fact that sheaves form a reflective subcategory with $\mathbf{a}$. □

Now let us stop and take stock. At this point, we have the

$$\mathbf{a}(\Delta N) \overset{\mathbf{a}(\Delta(\iota))}{\rightarrowtail} \mathbf{a}(\Delta B) \overset{m}{\rightarrowtail} \Omega_{\neg\neg}^{\mathbf{a}(\Delta B)}$$

So we've completed the task of "forcing" $\Omega_{\neg\neg}^{\mathbf{a}(\Delta N)}$ to become quite large. In fact we can insert $\mathbf{a}(\Delta 2^N)$ into the above diagram just using the monomorphism we have present in $\widehat{P}$. Let us factor $\iota : N \to B$ as $\iota_1 : N \to 2^N$ and $\iota_2 : 2^N \to B$. Then,

$$\mathbf{a}(\Delta N) \overset{\mathbf{a}(\Delta(\iota_1))}{\rightarrowtail} \mathbf{a}(\Delta 2^N) \overset{\mathbf{a}(\Delta(\iota_2))}{\rightarrowtail} \mathbf{a}(\Delta B) \overset{m}{\rightarrowtail} \Omega_{\neg\neg}^{\mathbf{a}(\Delta B)}$$

What remains then is to prove that no new epimorphisms have been introduced in this new topos and that all these first two inclusions are still strict.

4. The Preservation of Strictness

Proving the strictness of the cardinal inequalities turns out to be a rather technical endeavor. To begin with, we need to define what it means categorically for a monomorphism to be strict. In order to do this, we will use the internal language of the topos to define a subobject $\mathrm{Epi}(X, Y) \rightarrowtail Y^X$.

$$\mathrm{Epi}(X, Y) = \{f \mid \forall y \in Y.\ \exists x \in X.\ f(x) = y\}$$

First we show that this actually corresponds to the epimorphisms from X to Y. For this we will make heavy use of the Kripke-Joyal semantics explained in MacLane and Moerdijk [16, Chapter 6].

Lemma 4.1. $\langle \pi_1, e \rangle : P \times X \twoheadrightarrow P \times Y$ *if and only if* $\Lambda e : P \to Y^X$ *factors through* $\mathrm{Epi}(X, Y)$.

Proof. It suffices to show that $P \Vdash \forall x.\ \exists y.\ f(x) = y$ if and only if $\hat{f} : P \times X \to Y$ is epi. Now by the Kripke-Joyal semantics, $P \Vdash \forall x.\ \exists y.\ f(x) = y$ if and only if for all V,

$$P \times V \Vdash \exists y.\ (f\pi_1)(x) = \pi_2$$

Next, this holds if and only if for some $p : U \twoheadrightarrow P \times V$ and some $b : U \to X$

$$U \Vdash f\pi_1 p(b) = \pi_2 p$$

Now the rule for equality simply states that this holds if the interpretation of these two maps are equal. That is, that $\varepsilon \circ \langle f\pi_1 p, b \rangle = \pi_2 p$. However, then we know that

$$\varepsilon \circ \langle f\pi_1 p, b \rangle = \hat{f} \circ \langle \pi_1 p, b \rangle$$

here $\hat{f}$ is just the transpose of f. Finally, this gives us that $p = \langle \pi_1, \hat{f} \rangle \circ \langle \pi_1 p, b \rangle$. This tell us that $\langle \pi_1, \hat{f} \rangle$ is epi and this is true if and only if $\hat{f}$ is epi as required. $\square$

Now that we have established an internal representation of the existence of epimormorphisms we can define strict inequality.

Definition 4.2. $X < Y$ *if and only if* $X \rightarrowtail Y$ *and* $\mathrm{Epi}(X, Y) \cong 0$.

Having internalized this, we are at least now in a position to state the theorem that we want to prove: $X < Y \implies \mathbf{a}(\Delta X) < \mathbf{a}(\Delta Y)$. This property will rely crucially on the structure of P. In particular we shall show that P has the Souslin property.

Definition 4.3. *A partial order Q satisfies the Souslin property if any set of objects which are pairwise disjoint ($a \wedge b = 0$ for all a and b) is at most countable.*

Definition 4.4. *A topos $\mathcal{E}$ satisfies the Souslin property if it is generated by objects for whom* $\mathrm{Sub}(-)$ *has satisfies the Souslin property 4.3*

In fact the Souslin property is precisely what we need in order to get this fact to go through as the following lemma shows.

Lemma 4.5. *If $X < Y$ in* **Set** *and X and Y are infinite. Grothendieck topos $\mathcal{E}$ which satisfies the Souslin property then $\mathbf{a}(\Delta X) < \mathbf{a}(\Delta Y)$.*

Proof. It is clear that if $X \leqslant Y$, then $\mathbf{a}(\Delta X) \leqslant \mathbf{a}(\Delta Y)$. Therefore, it suffices to show that if $\mathrm{Epi}(X, Y) = 0$ then $\mathrm{Epi}(\mathbf{a}(\Delta X), \mathbf{a}(\Delta Y)) = 0$ as well. Let us suppose not. Then there must be a nonzero object U which satisfies the Souslin property so that $U \to \mathrm{Epi}(X, Y)$. Therefore, by lemma 4.1 there must be an epimorphism $g = \langle \pi_1, f \rangle : U \times X \twoheadrightarrow U \times Y$.

Take $x \in X$ and $y \in Y$, have two points $\mathbf{a}(\Delta x) : 1 \to \mathbf{a}(\Delta X)$ and $\mathbf{a}(\Delta y) : 1 \to \mathbf{a}(\Delta Y)$. Using these, we can form the two pullback squares

$$\begin{tikzcd}
V_{x,y} \arrow[r, tail] \arrow[d, tail] & P_y \arrow[r, two heads, "h"] \arrow[d, tail] & U \cong U \times 1 \arrow[d, "(1,\mathbf{a}(\Delta y))"] \\
U \cong U \times 1 \arrow[r, "(1,\mathbf{a}(\Delta x))"'] & U \times \mathbf{a}(\Delta X) \arrow[r, two heads, "g"'] & U \times \mathbf{a}(\Delta Y)
\end{tikzcd}$$

Now let us define $W = \{(x, y) \in X \times Y \mid V_{x,y} \neq 0\}$. First note that $S \cong \amalg_{x \in X} 1$. Therefore, $\amalg_{x \in X} 1 \times U \cong \mathbf{a}(\Delta X) \times U$. Moreover, this colimit exists because $\mathcal{E}$ is a Grothendieck topos. Since pullbacks have a right adjoint, we know that pulling back along $P_y \to U \times \mathbf{a}(\Delta X)$ gives us that $\amalg_x V_{x,y} \to P_y$. However, since $\amalg_x U \times 1 \cong U \times \mathbf{a}(\Delta X)$ we know that this isomorphism is preserved by pullback so in fact $\amalg_x V_{x,y} \cong P_y$. Now since U is known to be nonzero and we have an epimorphism $P_y \twoheadrightarrow U$, it must be that P_y is also nonzero. Therefore, we know that there is some x, y so that $V_{x,y}$ is nonempty since $\amalg_x V_{x,y} \cong P_y$. This tells us for every $y \in Y$ there exists an $x \in X$ so that $(x, y) \in W$.

This tells us that $\pi_2 : W \to Y$ is a surjection of sets so it suffices to show that $X \twoheadrightarrow W$ in order to show our contradiction that $X \twoheadrightarrow Y$. Now, in order to this, let us first note that $V_{x,y} \rightarrowtail U$. Moreover, if $y \neq y'$, then it must be that $V_{x,y} \wedge V_{x,y'} = 0$. This is because $y : 1 \rightarrowtail Y$ and $y' : 1 \rightarrowtail Y$ clearly are disjoint subobjects of Y in **Set**. However, this pullback diagram is preserved by Δ since limits are computed pointwise and then by $\mathbf{a}$ since it is left exact. Therefore, $\mathbf{a}(\Delta(y)) \wedge \mathbf{a}(\Delta(y')) = 0$. Finally, since meets are preserved by pullback and so is 0, this gives us that $V_{x,y} \wedge V_{x,y'} = 0$.

Now it is time to make use of this Souslin property. We know that $W_x = \{y \mid (x, y) \in W\}$ must be at most countable as they are necessarily disjoint. Since S is assumed to be infinite, since we know that

$$|W| = |S| \times \omega = |S|$$

Therefore, $S \cong W$ and we have our desired surjection. ☐

Having proving all of this, all that remains is to show that our P does in fact satisfy the Souslin property. This turns out to be a fun[4] exercise in order theory.

Lemma 4.6. $\mathbf{Sh}_{\neg\neg}(P)$ *satisfies the Souslin property.*

Proof. We know that $\mathbf{y}(p)$ generates $\mathbf{Sh}_{\neg\neg}(P)$. It is clear that if $A \rightarrowtail B$ and B has the Souslin property, then so does A (as $\mathrm{Sub}(A) \subseteq \mathrm{Sub}(B)$). Therefore, since $\mathbf{y}(p) \rightarrowtail 1$, it suffices to show that 1 has the Souslin property. Therefore, suppose that $(U_i)_i$ is a family of pairwise disjoint nonzero subterminals. We wish to show that it is at most countable. Now we know that $\mathbf{y}(p_i) \leqslant U_i$ and that $U_i \wedge U_j = 0$ so $\mathbf{y}(p_i) \wedge \mathbf{y}(p_j) = 0$. Therefore, $(U_i)_i$ really represents a set of pairwise incompatible conditions. We wish to show that this is at most countable.

Let us define

$$A_i = \{p_i \mid p_i \text{ is defined for } n \text{ entries}\}$$

[4]Boring.

We wish to show by induction on i that each A_i is countable. Since $\bigcup_i A_i = (p_i)_i$ this shows our original goal. Suppose that A_i is countable for all $i < j$, we wish to show that A_j is countable. To show that A_j is countable, it suffices to show that $A_{j,n}$ is countable where

$$A_{j,n} = \{p_i \mid p_i \in A_j \wedge \exists b.\ p_i(b,n) \downarrow\}$$

this is because $\bigcup_{i\in\mathbb{N}} A_{j,i} = A_j$. Now we can divide each $A_{j,n}$ into two sets, $A_{j,n,0}$ and $A_{j,n1}$ where

$$A_{j,n,i} = \{p_i \mid p_i \in A_j \wedge \exists b.\ p_i(b,n) = i\}$$

However, we note that $A_{j,n,i}$ must be comprised of pairwise incompatible counditions still. Since we know that for any $p, q \in A_{j,ni}$ that $p(b_p, n) = q(q_p, n)$, it must be that there is some other b', n' so that $p(b', n') \neq q(b', n')$. Therefore, the set

$$R_{j,n,i} = \{p_i \setminus \{(b_{p_i}, n, i)\} \mid p_i \in A_{j,n,i}\}$$

is pairwise incompatible. Since it is comprised of conditions of length $j-1$, it must be that $R_{j,n,i} \subseteq A_{j-1}$ so it is countable. Furthermore, then $A_{j,n,i}$ is countable and so is $A_{j,n}$ as we required. Therefore, A_j is countable and we are done by induction. $\square$

Now all told, this gives us that there is no epimorphism $\mathbf{a}(\Delta\Omega^N) \twoheadrightarrow \mathbf{a}(\Delta B)$ nor an epimorphism $\mathbf{a}(\Delta N) \twoheadrightarrow \mathbf{a}(\Delta\Omega^N)$ so that

$$\mathbf{a}(\Delta N) \xrightarrow{\mathbf{a}(\Delta(\iota_1))} \mathbf{a}(\Delta\Omega^N) \xrightarrow{\mathbf{a}(\Delta(\iota_2))} \mathbf{a}(\Delta B) \xrightarrow{m} \Omega_{\neg\neg}^{\mathbf{a}(\Delta B)}$$

indeed has strict inclusions for the first two maps. Thus, we have established an object which lies strictly between $\mathbf{a}(\Delta N)$ and $\Omega_{\neg\neg}^{\mathbf{a}(\Delta N)}$. This, combined with the result of Fourman [10] is sufficient to establish the independence of the continuum hypothesis from ZFC.

5. Forcing in a More General Context

Having proven the independence of the continuum hypothesis, I wanted to take a little time to discuss how this proof fits into a broader context. In general forcing is incredibly useful for establishing independence results in both set theory and type theory. This proof shows how syntactic forcing proofs can be smoothly translated into a proof about toposes. I am not capable of speaking of interesting results established in set theory using forcing but several developments in type theory have used a topos-theoretic forcing technique. Specifically, Coquand and Jaber [4] represents a coherent introduction to some of the developments done in Coquand [3] and Coquand and Mannaa [5]. This work was developed in the Jaber [13]. While not directly using topos-theoretic forcing, Escardó [9] and Sterling [18] are both results about type theory done by a similar technique.

The alternative characterization of forcing in terms of boolean valued models was explored by Scott and others during the 60s and 70s. These are remarkably topos-theoretic semantics in which the validity of a statement isn't a simple boolean but given in terms of an element of a complete boolean lattice. By quotienting this lattice by a particular ultrafilter one can translate forcing proofs into this framework. I am not well positioned to recommend literature on this but Jech [14] contains an approachable introduction and our own Clive Newstead has produced notes on this [17].

More familiar to logicians will be Kripke semantics and Beth semantics. In these we index the $\Vdash$ relation with a "world" at which we consider it. These worlds are assumed to form a preorder which is intended to represent time with $w_1 \leqslant w_2$ implying that w_2 is a possible future of w_1. Kripke semantics correspond closely to presheaf semantics on the poset of worlds. Beth semantics add the local character sheaves enjoy to the semantics. Accordingly then, Kripke semantics, step-indexed logical relations, and Beth semantics in general can be translated as a special case of the Kripke-Joyal semantics for the internal logic of some presheaf or sheaf topos. This idea for intuitionistic logic was discussed in Fourman [11] and Fourman [12]. Kripke and Beth semantics are given a lengthy consideration in Dummett [8] and Troelstra and van Dalen [19]. Recently, the more topos-theoretic approach that has been present in developments like Dreyer et al. [7] has been made explicit in the work of Birkedal et al. [1].

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