ADJOINT MODALITIES IN MTT

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ABSTRACT. We record several results about the behavior of adjoint modalities in MTT. In particular, we show that internal adjunctions can be used to recover stronger rules, similar to Birkedal et al. [Bir+20].

We explore the relationship between dependent right adjoints and a weak dependent right adjoint whose left adjoint also internalizes as a modality. We argue that these *internal right adjoints* exhibit many of the nice properties of dependent right adjoints. Together with recent results of Gratzer et al. [Gra+22], we argue that restricting to weak dependent right adjoints poses little issue in practice.

1. INTERNAL ADJOINTS

Let us consider the mode theory $\mathcal{M}$ which contains two modalities $\mu : n \longrightarrow m$ and $\nu : m \longrightarrow n$ together with 2-cells witnessing $\nu \dashv \mu$. Explicitly, there are 2-cells $\eta : \mathsf{id}_m \longrightarrow \mu \circ \nu$ and $\epsilon : \nu \circ \mu \longrightarrow \mathsf{id}_n$ satisfying the triangle equations:

$$\begin{tikzcd} \mu \arrow[r, "\eta \star \mathsf{id}_\mu"] \arrow[dr, "\mathsf{id}_\mu"'] & \mu \circ \nu \circ \mu \arrow[d, "\mathsf{id}_\mu \star \epsilon"] \\ & \mu \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} \nu \arrow[r, "\mathsf{id}_\nu \star \eta"] \arrow[dr, "\mathsf{id}_\nu"'] & \nu \circ \mu \circ \nu \arrow[d, "\epsilon \star \mathsf{id}_\nu"] \\ & \nu \end{tikzcd} \tag{2}$$

Mode theories of this shape were considered to some extent in Gratzer et al. [Gra+20] and they have shown that ν behaves like a left adjoint internal to MTT and that e.g. it preserves certain colimits.

We consider the behavior of the right adjoint μ. We first observe that the action of μ on context can be encoded to through ν:

Theorem 1. *For any context Γ cx, modality $\xi : o \longrightarrow m$, and $\Gamma.\{\xi\} \vdash A$:*

$$\Gamma.(\xi \mid A).\{\mu\} \cong \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \xi})$$

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Proof. First, we observe that because $-.\{-\}$ is a 2-functor, it preserves adjoints. Therefore, the $-.\{\nu\} \dashv -.\{\mu\}$ as functors on categories of contexts.

We will first argue that $\Gamma.(\xi \mid A).\{\mu\}$ and $\Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \xi})$ are isomorphic as they represent the same functor. To this end, we make use of universal property of context extension in MTT: a substitution $\Delta_0 \longrightarrow \Delta_1.(\xi \mid A)$ is determined by (1) a substitution $\delta : \Delta_0 \longrightarrow \Delta_1$ and (2) a term $\Delta_0 \vdash M : A[\delta.\{\xi\}]$ [Gra+20].

Fix a context Δ in mode n. Using the above universal property along with transposition, a substitution $\Delta \longrightarrow \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \xi})$ is determined by (1) a substitution $\gamma : \Delta.\{\nu\} \longrightarrow \Gamma$ and (2) a term $\Delta.\{\nu \circ \xi\} \vdash M : A^{\eta \star \xi}[\widehat{\gamma}.\{\nu \circ \xi\}]$ naturally Δ. Unfolding the definition of transposition, $A^{\eta \star \xi}[\widehat{\gamma}.\{\nu \circ \xi\}]$ is simply $A[\gamma.\{\xi\}]$.

Next, a substitution $\Delta \longrightarrow \Gamma.(\xi \mid A).\{\mu\}$ is determined by (1) a substitution $\gamma : \Delta.\{\nu\} \longrightarrow \Gamma$ and (2) a term $\Delta.\{\nu \circ \xi\} \vdash M : A[\gamma.\{\xi\}]$ naturally in Δ.

The two contexts are therefore isomorphic by the Yoneda lemma. $\square$

Theorem 2. *Given any context* $\Gamma\ \mathsf{cx}$ *and* $\Gamma.\{\mu\} \vdash A$ *when* $\nu \dashv \mu$*, there is a pair of substitutions*

$$\gamma^{\rightarrow} : \Gamma.(\mu \mid A) \longrightarrow \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle)$$
$$\gamma^{\leftarrow} : \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \longrightarrow \Gamma.(\mu \mid A)$$

Moreover, $\gamma^{\leftarrow} \circ \gamma^{\rightarrow} = \mathsf{id}$ *and, if one assumes extensional equality,* $\gamma^{\rightarrow} \circ \gamma^{\leftarrow} = \mathsf{id}$.

Proof. One direction of this isomorphism holds regardless of the precise properties of μ:

$$\gamma^{\rightarrow} \triangleq {\uparrow}.\mathsf{mod}_\mu(\mathbf{v}_0) : \Gamma.(\mu \mid A) \longrightarrow \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \tag{3}$$

The inverse direction is more subtle:

$$\gamma^{\leftarrow} \triangleq {\uparrow}.M : \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \longrightarrow \Gamma.(\mu \mid A) \tag{4}$$

Here, M must be a term of the following type:

$$\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\} \vdash M : A[{\uparrow}.\{\mu\}]$$

In order to define this, consider the following term:

$$\frac{\begin{array}{c}\Gamma.(\mathsf{id}_n \mid \langle \mu \mid A \rangle).\{\mu \circ \nu\} \vdash \mathbf{v}_0^\eta : \langle \mu \mid A[\{\eta \star \mathsf{id}_\mu\}] \rangle \\ \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\}.(\nu \circ \mu \mid A) \vdash \mathbf{v}_0^\epsilon : A[{\uparrow}.\{\nu\}]\end{array}}{\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\} \vdash M \triangleq \mathsf{let}_\nu\ \mathsf{mod}_\mu(\_) \leftarrow \mathbf{v}_0^\eta\ \mathsf{in}\ \mathbf{v}_0^\epsilon : A[{\uparrow}.\{\mu\}]}$$

By computation, we immediately have $\gamma^{\leftarrow} \circ \gamma^{\rightarrow} = \mathsf{id}$. In the reverse direction, we must show that the following terms are definitionally equivalent

$$\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \vdash \mathbf{v}_0 = \mathsf{mod}_\mu(\mathsf{let}_\nu\ \mathsf{mod}_\mu(\_) \leftarrow \mathbf{v}_0^\eta\ \mathsf{in}\ \mathbf{v}_0^\epsilon) : \langle \mu \mid A[{\uparrow}.\{\mu\}] \rangle \tag{5}$$

This equation is true *propositionally*, by performing induction on $\mathbf{v}_0$. Therefore, in the presence of extensional equality this holds definitionally as well. $\square$

With this result to hand, we define $\mathsf{unmod}_\mu(M)$ as follows:

$$\frac{\Gamma \vdash M : \langle \mu \mid A \rangle}{\Gamma.\{\mu\} \vdash \mathsf{unmod}_\mu(M) = \mathbf{v}[\gamma^{\leftarrow}.\{\mu\} \circ \mathsf{id}.M.\{\mu\}] : A}$$

Remark 1. We could alternatively formulate $\mathsf{unmod}_\mu(-)$ with the following rule:

$$\frac{\Gamma.\{\nu\} \vdash M : \langle \mu \mid A \rangle}{\Gamma \vdash \mathsf{unmod}_\mu(M) : A[\{\epsilon\}]}$$

The two formulations are inter-derivable. The one we gave above is more commonly found in the literature [GSB19; Bir+20], but this alternative can be taken as primitive without disrupting substitution.

Lemma 3. *If* $\Gamma.\{\mu\} \vdash M : A$ *then* $\mathsf{unmod}_\mu(\mathsf{mod}_\mu(M)) = M$

Proof. We must show the following:

$$\mathbf{v}[(\gamma^{\leftarrow} \circ \mathsf{id}.\mathsf{mod}_\mu(M)).\{\mu\}] = M$$

To this end, let us first rewrite $\mathsf{id}.\mathsf{mod}_\mu(M)$ as $\uparrow.\mathbf{v} \circ \mathsf{id}.M$. We then observe that this is precisely $\gamma^{\rightarrow} \circ \mathsf{id}.M$ whence we have the following:

$$\begin{aligned}
&\mathbf{v}[(\gamma^{\leftarrow} \circ \mathsf{id}.\mathsf{mod}_\mu(M)).\{\mu\}] \\
&= \mathbf{v}[(\gamma^{\leftarrow} \circ \gamma^{\rightarrow} \circ \mathsf{id}.M).\{\mu\}] \\
&= M
\end{aligned}$$

□

Lemma 4. *There is a propositional equality:*

$$(x : \langle \mu \mid A \rangle) \to \mathsf{Id}_{\langle \mu | A \rangle}(\mathsf{mod}_\mu(\mathsf{unmod}_\mu(x)), x)$$

Proof. Modal induction on x reduces this to Lemma 3. □

Remark 2. Note that Theorem 2 and Lemmas 3 and 4 only requires a fraction of the full elimination rule MTT provides. In particular, it is only necessary to use id or ν as a framing modality.

Remark 3. Note that, in particular, if we consider a mode theory with a single self-adjoint modality $\mu = \nu$, these results ensure that (extensional) MTT coincides with the type theory proposed by Riley, Finster, and Licata [RFL21]. In standard MTT, there is a slight distinction with *op. cit.* providing a definitional η law for modalities where Lemma 4 is merely propositional.

References

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