Directed univalence in simplicial homotopy type theory

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Simplicial type theory extends homotopy type theory with a directed path type which internalizes the notion of a homomorphism within a type. This concept has significant applications both within mathematics—where it allows for synthetic (higher) category theory—and programming languages—where it leads to a *directed* version of the structure identity principle. In this work, we construct the first types in simplicial type theory with non-trivial homomorphisms. We extend simplicial type theory with modalities and new reasoning principles to obtain *triangulated type theory* in order to construct the universe of discrete types $\mathcal{S}$. We prove that homomorphisms in this type correspond to ordinary functions of types i.e., that $\mathcal{S}$ is directed univalent.

The construction of $\mathcal{S}$ is foundational for both of the aforementioned applications of simplicial type theory. We are able to define several crucial examples of categories and to recover important results from category theory. Using $\mathcal{S}$, we are also able to define various types whose usage is guaranteed to be functorial. These provide the first complete examples of the proposed *directed structure identity principle*.

1 Introduction

Homotopy type theory (HoTT) is a type theory for synthetic ∞-groupoid theory; it shapes every type into an object equipped with a proof-relevant coherent equivalence relation which is silently manipulated and respected by every construction in type theory. While the motivation for this interpretation comes squarely from homotopy theory, HoTT [5, 6, 63] have proven to be useful even for those interested only in type theory and, especially, in proof assistants.

Proof assistants are well-tuned to support replacing equal elements by equal elements, where equality is reified by the intensional identity type within type theory. Accordingly, if two distinct terms can be identified, they can be swapped out for each other in large proofs without further effort. In HoTT, the identity type becomes far richer and, in particular, elements of the universe become identified whenever they are equivalent. Accordingly, users of proof assistants based on HoTT can swap out e.g., an implementation of the integers well-suited for reasoning with an equivalent version tuned for efficient computation without additional effort. This offers the same convenience to types that function extensionality grants functions. Angiuli et al. [4], for instance, show that this can be used to internalize some applications of parametricity but, crucially, without eliminating standard models which do not support the full apparatus of parametricity.

A type theory for groupoids makes it far easier to manipulate equality, but what about formalization challenges which are fundamentally asymmetric? For a toy example, consider an algorithm traversing a list to sum its elements $\mathsf{sum} : (A : \mathsf{Monoid}) \to \mathsf{List}\, A \to A$. Univalence and one of its important consequences, the structure identity principle, tell us that sum must respect monoid isomorphisms. But far more is true: sum commutes with all monoid homomorphisms. To prove this we must (1) formulate how a monoid homomorphism $f : A \to B$ induces a map $\mathsf{List}\, f : \mathsf{List}\, A \to \mathsf{List}\, B$ and (2) show that $\mathsf{sum} \circ \mathsf{List}\, f = f \circ \mathsf{sum}$. Neither task follows from univalence as f need not be invertible and univalence handles only symmetric relations.

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1.1 A type theory for categories

The above example would be possible in a version of type theory where types encoded not just groupoids but *categories*: a directed type theory. That is, each type would come equipped with a notion of homomorphism (along with composition, etc.) and each term in the type theory would be bound to automatically respect homomorphisms e.g., be functorial. Aside from the benefits to formalization, it is particularly desirable to find a directed version of HoTT where types would encode ∞[1]-categories [13, 23, 33, 47]; ∞-category theory is an important area of mathematics but whose foundations are well-known to be cumbersome. It is conjectured that directed homotopy theory could serve as the basis for a more usable and formalizable foundation of this field. Many such theories (both homotopical and not) have been studied over the years [3, 26, 27, 30, 39–42, 66, 67].

A key obstruction to this program is that (∞-)categories do not behave well enough to support a model of type theory where every type is a category. For instance, Π-types do not always exist because the category of categories is not locally cartesian closed. Most directed type theories therefore change how type theory works to e.g., allow only certain kinds of Π-types and dependence. We will focus on a different approach introduced by Riehl and Shulman [46]: *simplicial type theory (STT)*. The key insight is to not require that every type is an ∞-category, but instead a more flexible object from which we can carve out genuine ∞-categories using two definable predicates.

STT extends HoTT with a new type to probe the implicit categorical structure each type possesses: the *directed interval* $\mathbb{I}$. Riehl and Shulman [46] further equip it with the structure of a bounded linear order $(\wedge, \vee, 0, 1)$. One can then use $\mathbb{I}$ to access e.g., the morphisms a to b in A by studying ordinary functions within type theory $f : \mathbb{I} \to A$ such that $f(0) = a$ and $f(1) = b$.

Early evidence [9, 10, 35, 45, 46, 69–71] suggests that simplicial type theory approaches the desired usable foundations for ∞-category theory. A number of definitions and theorems from classical ∞-category theory have been ported to STT and the proofs are shorter and more conceptual. Even better, Kudasov's experimental proof assistant Rzk [28] for STT has shown that the arguments for e.g., the Yoneda lemma are simple enough to be formalized and machine-checked [29].

Convention 1.1. For the remainder of this paper, we shall be concerned only with ∞-categories and constructions upon them. Accordingly, hereafter we largely drop the "∞-" prefix and speak simply of categories, groupoids, etc. except in those few situations where it would cause ambiguity.

Simplicial type theory, a reprise. A brief description of simplicial type theory is in order. As mentioned, every type A in STT has a notion of homomorphism: functions $\mathbb{I} \to A$. However, in arbitrary types these do not really behave like homomorphisms e.g., they need not compose.

Suppose we are given $f, g : \mathbb{I} \to A$ such that $f\,1 = g\,0$. A composite h ought to be a homomorphism such that $h\,0 = f\,0$ and $h\,1 = g\,1$, but not every such h satisfying just these conditions ought to be a composite. In particular, further data is required to connect h with f and g. Classically, all of this is encoded by a 2-simplex H (see the left diagram in Fig. 1). Inside simplicial type theory, we represent such 2-simplices as maps $\Delta^2 \to A$ where $\Delta^2 = \{(i, j) : \mathbb{I} \times \mathbb{I} \mid i \geq j\}$ (c.f., Fig. 1).

In particular, a 2-simplex $H : \Delta^2 \to A$ witnesses that $H(-, 0)$ and $H(1, -)$ can be composed to obtain $\lambda i.\, H(i, i)$. It is convenient to isolate the subtype $\Lambda^2_1 = \{(i, j) \mid i = 1 \vee j = 0\} \subseteq \Delta^2 \subseteq \mathbb{I} \times \mathbb{I}$. Unfolding, a map $\Lambda^2_1 \to A$ corresponds to a pair of composable arrows f, g. Accordingly, every pair of composable arrows in A admits a unique composite i.e., A is *Segal* if $(\Delta^2 \to A) \simeq (\Lambda^2_1 \to A)$.

Segal types already possess enough structure to behave like categories e.g. it follows that composition is associative and unital just from the Segal condition. Unfortunately, they may suffer from an excess of data: they come equipped with two notions of sameness. Namely, $a, a' : A$ may be regarded as the same when $a = a'$ *or* when there is an invertible homomorphism from a to

[1]Specifically, $(\infty, 1)$-categories: categories whose morphisms form an ∞-groupoid.

$$
\begin{tikzcd}
 & f(1) \arrow[dr, "g"] \arrow[d, phantom, "H"] & \\
f(0) \arrow[ur, "f"] \arrow[rr, "h"'] & {} & g(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] \arrow[r] \arrow[ur] & (1,0) \arrow[u]
\end{tikzcd}
$$

Fig. 1. Illustrations of simplices

a'. In keeping with our pursuit of the directed structure identity, we shall be interested in types where these coincide i.e. where $A \to \sum_{f:A^{\mathbb{I}}} \mathsf{isIso}(f)$ is an equivalence. We say such a type is *Rezk* if it is Segal and satisfies this condition. An important result of Riehl and Shulman [46] is that Rezk types adequately model the standard notion of ∞-category [43].[2]

Directed univalence. However, simplicial type theory is not a panacea for replacing classical ∞-category theory. Presently, it is really only suitable for studying "formal" questions and, surprisingly, it is not possible to construct a non-trivial closed Rezk type within STT. Crucially, STT lacks an equivalent to the category of groupoids (the ∞-categorical version of the category of sets). Not only does this mean that STT faces severe limitations on what theorems can be *stated*, it is presently impossible to exploit directed path types when formalizing. Returning to our original example with sum, STT would automatically handle (1) and (2) if there was a type of monoids Monoid where directed paths were monoid homomorphisms, but such a definition is presently out of reach.

Our central contribution is to overcome these challenges by extending STT with new reasoning principles and constructing a Rezk type $\mathcal{S}$ whose objects correspond to groupoids (i.e., Rezk types where every homomorphism is invertible) and whose homomorphisms are functions. This last requirement is termed *directed univalence*:

Definition 1.2. A universe $\mathcal{S}$ is *directed univalent* if $\mathbb{I} \to \mathcal{S}$ is isomorphic to $\sum_{A\,B:\mathcal{S}} A \to B$.

Before discussing our approach, we survey a few consequences of this result. Once $\mathcal{S}$ is available, a number of applications of STT snap into focus. For instance, one can isolate subcategories of $\mathcal{S}$ such as the category of sets $\mathcal{S}_{\leq 0}$ and the category of propositions $\mathcal{S}_{\leq -1}$. Using the ordinary constructions of type theory, one can parlay these into our aforementioned category of monoids:

$$\mathsf{Monoid} = \textstyle\sum_{A:\mathcal{S}_{\leq 0}} \sum_{\epsilon:A} \sum_{\cdot:A\times A\to A} \mathsf{isAssociative}(\cdot) \times \mathsf{isUnit}(\cdot, \epsilon)$$

The only difference in this definition from the standard one seen in ordinary type theory is the replacement of $\mathcal{U}$ by $\mathcal{S}_{\leq 0}$. However, with just this change we are able to prove the following result:

Theorem 6.11. *If F, G : Monoid $\to \mathcal{S}$ and $\alpha : (A : \mathsf{Monoid}) \to F(A) \to G(A)$ then α is natural i.e. if $f : A \to B$ is a monoid homomorphism, $\alpha(B) \circ f = f \circ \alpha(A)$.*

In particular, choosing $F = \mathsf{List}$, $G = \mathsf{id}$, and $\alpha = \mathsf{sum}$ yields our desired earlier example.

Replacing Monoid with Ring, one could derive a similar theorem to argue that given numerical algorithm $f : (R : \mathsf{Ring}) \to R^n \to R$ then the parity of its output (when applied to $\mathbb{Z}$) depends on the parity of its inputs, as f commutes with the map $\mathbb{Z} \to \mathbb{Z}/2$. These are instances of a *directed* version of the structure identity principle (SIP) [2, 17, 63, 67]: if C is a type of algebraic structures, its homomorphisms coincide with classical morphisms of those structures. Consequently every term and type using C is therefore automatically bound to be functorial and respect these classical morphisms. It was observed by e.g., Weaver and Licata [67] that directed SIP could be used to ease

[2]In fact, combined with general results on HoTT [59, 68] they model *internal* ∞-categories in an ∞-topos [14, 36, 37].

formalization efforts and we provide the first complete examples of this and by proving directed SIP occurs for a wide class of structures.

More broadly, just as HoTT allowed us to internalize parametricity results based on equivalence relations, STT allows us to internalize parametricity arguments based on naturality. From this, we can also recover a classic result:

Theorem 6.2. *If* $f : (A : \mathcal{S}) \to A \to A$ *then* $f = \lambda A\, a.\, a$.

We may summarize these results by the slogan "$\mathcal{S}$ is a type which must be used *covariantly*." In particular, any type depending on $\mathcal{S}$ (or types derived from it) must be functorial in this argument.

Recreating parametricity arguments, however, is far from the only use of $\mathcal{S}$. Just as we defined Monoid, we can define various categories critical for ∞-category theory, such as the category of partial orders, the simplex category, the category of finite sets, etc. Using these, we present the first steps towards formalizing *higher algebra* (one of the main applications of ∞-category theory) within type theory. Higher algebra is most often encountered by type theorists in the form of the *coherence problem* and, from this point of view, using $\mathcal{S}$ we are able to give definitions of infinitely coherent monoids, groups, etc. Fundamentally, having just $\mathcal{S}$ available throws open the door to defining a wide variety of derived categories and all the applications this entails.

1.2 Constructing $\mathcal{S}$

In a certain sense, the difficulty with $\mathcal{S}$ to STT is not so much in its addition—we could always postulate a type $X : \mathcal{U}$ along with terms for the Segal and Rezk axioms, declare it to be $\mathcal{S}$, and call it a day! The challenge comes in finding a complete API for $\mathcal{S}$ within STT that, when established, allows us to prove all expected results and completely specifies $\mathcal{S}$. This is where ∞-categories prove substantially more complex than 1-categories. It no longer suffices to specify objects and morphisms to define $\mathcal{S}$, we must also specify the higher simplices needed for coherent composition. Thus, even if we set aside the distasteful nature of simply adding axioms to construct $\mathcal{S}$, we would be left with the task of adding an *infinite* number of axioms on top of e.g., directed univalence to fully specify its behavior. This is a famous problem of ∞-category theory where nearly all constructions must be carried out indirectly through heavy machinery.

Our main theorem therefore is to construct $\mathcal{S}$ internally and thereby provide a complete API for its use. We do this by adapting the methods of Licata et al. [32] to prove one of the most widely-used results in ∞-category theory, the straightening–unstraightening equivalence [13, 15, 22, 33], inside of type theory. Roughly, we define $\mathcal{S}$ and prove that the type $X \to \mathcal{S}$ is equivalent to the subtype of $X \to \mathcal{U}$ spanned by *amazingly covariant families*. That is, a map $X \to \mathcal{S}$ corresponds to a type family over X which is covariant in X as well as the context ie, *amazingly* covariant [50].

We show that all the central properties of $\mathcal{S}$ follow from this description. For instance, we are able to show that $\mathcal{S}$ is closed under the expected operations (limits, colimits, dependent sums, and certain dependent products) and, most importantly, we prove the directed univalence axiom.

1.3 Extending simplicial type theory to triangulated type theory

The central challenge is giving an adequate definition of *amazingly covariant* families: types $\Gamma \vdash A : X \to \mathcal{U}$ which are covariant not only in X, but the entire context Γ. This second condition, however, cannot be expressed inside of simplicial type theory. Similar situations have arisen in many contexts within HoTT [38, 53, 54, 58] and, as in prior work, we address this lack of expressivity by extending simplicial type theory by a collection of *modalities* to capture amazing covariance.

In fact, even without amazing covariance we are led to modal simplicial type theory or indeed, modal versions of any type theory seeking to internalize directed univalence. The reason why can be summed up in a single word: contravariance. It is all well and good to have a type whose use is automatically covariant, but many important operations (e.g., $X \mapsto X \to \mathsf{Bool}$) are simply not covariant, and some (e.g. $X \mapsto X \to X$) are neither co- nor contravariant. As it stands, $\mathcal{S}$ can only be used covariantly and so we cannot express these important and natural operations. To rectify this, we extend STT with modalities which allow us to express *contravariant* dependence on $\mathcal{S}$ as well as *invariant* dependence. Both of these modalities have central positions within synthetic category theory: the first sends a category to its opposite and the second sends a category to its underlying groupoid of objects. While neither operation can be realized as a function $\mathcal{U} \to \mathcal{U}$ [58], both of these operations can be included as modalities [21].

Having accepted that some modalities are necessary for simplicial type theory, it is then natural to ask what other modalities must be added in order to internally define amazing covariance and $\mathcal{S}$. Following Licata et al. [32], we would like to include a modality which behaves like the right adjoint to $A \mapsto (\mathbb{I} \to A)$; the so-called amazing right adjoint to $\mathbb{I} \to -$. In op. cit., the intended model (cubical sets) had such a modality but in the standard model of simplicial type theory, no such right adjoint exists. Accordingly, we could add such a modality to simplicial type theory, but we would have no means by which to justify it. In order to address this, we must also weaken the standard model of simplicial type theory and, with it, the assumed structure on $\mathbb{I}$. Rather than postulating a totally ordered $\mathbb{I}$, we only ask that $\mathbb{I}$ be a bounded distributive lattice where $0 \neq 1$. Semantically, this corresponds to shifting from simplicial spaces—the standard model—to the larger category of cubical[3] spaces. Within this new category, the necessary right adjoint exists and we can justify the addition of the necessary modality. In order to manipulate these new modalities and relaxed interval, we also axiomatize several general facts from the cubical spaces model. All told, we work within a version of MTT [21] (to account for modalities) and with a less structured interval $\mathbb{I}$. We term the result *triangulated type theory* $\mathsf{TT}_{\boxslash}$.

Within $\mathsf{TT}_{\boxslash}$, we isolate *simplicial types*, those which *believe* the interval to be totally ordered. Simplicial types "embed" STT into $\mathsf{TT}_{\boxslash}$ and we are eventually interested only in these types (in fact, mostly in simplicial Rezk types). However, the presence of non-simplicial types is crucial to allow for the constructions needed to define $\mathcal{S}$—even though $\mathcal{S}$ will itself turn out to be simplicial Rezk.

Finally, we note that while MTT enjoys canonicity [19], adding axioms (univalence, $\mathbb{I}$, etc.) obstructs computation and so canonicity does not hold for $\mathsf{TT}_{\boxslash}$. Accordingly, $\mathsf{TT}_{\boxslash}$ is closer to "book HoTT" [63] than cubical type theory [16]. We leave it to future work to develop computational versions of our new axioms and integrate existing computational accounts of univalence in MTT [1].

1.4 Contributions

We contribute $\mathsf{TT}_{\boxslash}$, a modal extension of simplicial type theory, and use it to construct a directed univalent universe of groupoids $\mathcal{S}$. In so doing, we construct the first non-trivial examples of categories within simplicial type theory. More specifically:

- We identify several general and reusable reasoning principles with which to extend STT.
- We prove that $\mathcal{S}$ satisfies (directed) univalence, as well as the Segal and Rezk conditions.
- We construct *full subcategories* purely internally and isolate important subcategories of $\mathcal{S}$.
- We build numerous important classical examples of categories e.g., presheaves, spectra, partial orders, and other (higher) algebraic categories from $\mathcal{S}$.

Finally, we crystallize a conjectured *directed structure identity principle* which can be used to recover various parametricity arguments as well as automatically discharge functoriality goals and proof obligations. We give the first complete example applications of this principle.

We have endeavored throughout this paper to provide as many proofs as space allows. This is not only for the sake of rigor, but because a major contribution of our synthetic approach with both STT

[3]Technically, we work within the category of Dedekind cubical spaces. See Section 3.4

and $\mathsf{TT}_{\boxslash}$ is the comparative simplicity of the proofs. Crucially, no knowledge of ∞-categories or the semantics of homotopy type theory is required by our key arguments. Even the most complex arguments in Section 5 take up only half of page and are possible to follow to those experienced with (modal) type theory. Ideally, we would substantiate this claim by formalizing our arguments in a proof assistant, but there is presently no suitably general implementation of modal type theory.

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2 A primer on simplicial and modal type theory

Before diving into the construction of the universe of groupoids, we recall some of the details of simplicial type theory from Riehl and Shulman [46] and its modal extension. In both cases, these can be understood as extensions of homotopy type theory.

Accordingly, we begin by recalling "book HoTT" as described by the Univalent Foundations Program [63]. Briefly, we will be extending intensional Martin-Löf type theory with a cumulative hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \dots$. We will follow op. cit. and write $a =_A b$ (or simply $a = b$) for the identity type. Given $p : a = b$, we will write $p_* : B(a) \to B(b)$ for the transport function induced by p in any type family $B : A \to \mathcal{U}$. We will hereafter assume that all universes $\mathcal{U}_i$ satisfy Voevodsky's univalence axiom. That is, the following canonical map is an equivalence:

$$\mathsf{ua} : (A\, B : \mathcal{U}_i) \to (A = B) \to (A \simeq B)$$

We also recall three definitions from the HoTT book [63] which we shall use repeatedly:

$$\mathsf{isContr}, \mathsf{isHProp}, \mathsf{isHSet} : \mathcal{U} \to \mathcal{U}; \quad \mathsf{isContr}\, A = \textstyle\sum_{a:A} \prod_{b:B} a = b,$$
$$\mathsf{isHProp}\, A = \textstyle\prod_{a\, b:A} \mathsf{isContr}(a = b), \quad \mathsf{isHSet}\, A = \prod_{a\, b:A} \mathsf{isHProp}(a = b)$$

These predicates respectively isolate (1) types which behave like Unit, (2) types which behave like a proposition, and (3) types which behave like a set by satisfying the unicity of identity proofs. Each of these induce subtypes of the universe e.g., $\mathsf{HProp} = \sum_{A:\mathcal{U}} \mathsf{isHProp}\, A$. More generally, these are cases $0, 1, 2$ of the predicate $\mathsf{hasHLevel} : \mathsf{Nat} \to \mathcal{U} \to \mathsf{HProp}$ which measures to what extent a type has non-trivial identity types between identity types between identity types etc.

2.1 Simplicial type theory and basic category theory

The basic definition of simplicial type theory is a straightforward extension of HoTT:

Definition 2.1. Core simplicial type theory STT extends homotopy type theory with the following:

(1) A *directed interval* type $\mathbb{I} : \mathsf{HSet}$
(2) The operations and equations shaping $\mathbb{I}$ into a bounded distributive lattice $(\wedge, \vee, 0, 1)$.

In order to simplify our discussion as well as the eventual definition of *triangulated type theory* (Section 3), we present a slight variation on STT compared with its normal definition. Accordingly, we have added $\mathbb{I}$ as an axiom rather than a piece of judgmental structure as in Riehl and Shulman [46]. This results in fewer definitional equalities than op. cit., but requires no additional rules over homotopy type theory to specify. We leave it to future work to incorporate all the convenience features of Riehl and Shulman [46] alongside the extensions we present below.

Using the lattice structure on $\mathbb{I}$, we can now specify the common simplicial shapes used to model composition in ∞-category theory e.g. Δ^n:

$$\Delta^0 \coloneqq \mathsf{Unit} \qquad \Delta^{n+1} \coloneqq \{(i_1, \ldots, i_{n+1}) : \mathbb{I}^{n+1} \mid i_1 \geq i_2 \geq \ldots \geq i_{n+1}\}$$

One can also give general descriptions of the boundaries $\partial\Delta^n$ and the (n,k)-horns Λ^n_k, for $n \geq 0$ and $0 \leq k \leq n$ [46, Section 3]. We use these to define categories and related structures.

Definition 2.2. Given $a, b : A$, the type of *homomorphisms* or *arrows* from a to b is given by

$$\hom_A(a,b) \coloneqq (a \to_A b) \coloneqq (a \to b) \coloneqq \textstyle\sum_{f:\mathbb{I}\to A} f\,0 = a \times f\,1 = b$$

For convenience, we suppress the forgetful map $\hom_A(a,b) \to (\mathbb{I} \to A)$.

We can relativize the notion of homomorphisms to dependent types:

Definition 2.3. Given $a, b : A$ and $f : \hom_A(a,b)$, for a type family $P : A \to \mathcal{U}$, a *dependent homomorphism* from $x : P\,a$ and $y : P\,b$ over f is given by

$$\hom^P_f(x,y) \coloneqq (x \to^P_f y) \coloneqq (x \to_f y) \coloneqq \textstyle\sum_{\varphi:(i:\mathbb{I})\to P(f\,i)} ((\pi_2 f)_*(\tilde{f}\,0) = x) \times ((\pi_3 f)_*(\tilde{f}\,1) = y)$$

Note that we must transport by the identifications $\pi_2 f : f(0) = a$ and $\pi_3 f : f(0) = b$.

We recall the following definition of *pre-categories* i.e. *Segal types* from the introduction:

Definition 2.4. $A : \mathcal{U}$ is *Segal* if the canonical map $i : (\Delta^2 \to A) \to (\Lambda^2_1 \to A)$ is an equivalence.

Notation 2.5. If A is Segal and $f : a \to_A b$, $g : b \to_A c$, we write $g \circ f$ for $\lambda t.\,(i^{-1}(f,g))\,(t,t)$ i.e. the long edge of the triangle obtained by extending $(f,g) : \Lambda^2_1 \to A$ to $\Delta^2 \to A$. This operation is automatically associative and constant functions $\mathbb{I} \to A$ (identity homomorphisms) are units for $\circ$.

Definition 2.6. We say an arrow $f : a \to_A b$ in a Segal type A is an *isomorphism* if the following type is inhabited: $\mathsf{isIso}(f) \coloneqq \sum_{g\,h:b\to a} (g \circ f = \mathsf{id}_a) \times (f \circ h = \mathsf{id}_b)$.

Note that $\mathsf{isIso}(f)$ is a proposition and we denote the induced subtype of $a \to_A b$ by $a \cong_A b$. With the definition of isomorphism to hand, we can properly define *categories* and *groupoids*:

Definition 2.7. If A is Segal, we say A is *category/Rezk-complete* if the following map (defined by path induction) is an equivalence: $\mathsf{IdToIso} : (a, b : A) \to (a = b) \to (a \cong b)$

Definition 2.8. A type A is *groupoid* or a *space* or $\mathbb{I}$-*null* if $(a = b) \to (a \to b)$ is an equivalence.[4]

Lemma 2.9. *A type is a groupoid if and only if it is a category where every arrow is an isomorphism.*

Intuitively, a type is Rezk when it satisfies a kind of univalence condition: isomorphism is identity. In the intended model of STT, they correspond to complete Segal spaces, in turn, model $(\infty,1)$-categories semantically. Op. cit. further show that maps between Segal types are automatically *functors* i.e. they preserve composition and identities.

[4]The terminology $\mathbb{I}$-null stems from Rijke et al. [49]; it is equivalent to requiring $\mathsf{isEquiv}(A \to (\mathbb{I} \to A))$.

2.2 Multimodal type theory

As mentioned in Section 1, we must extend type theory with various *modalities* in order to define $\mathcal{S}$. We shall do this by "rebasing" simplicial type theory atop MTT [21], a general framework for modal type theory. Prior to discussing the union of these type theories, we give a brief overview of MTT in this section and refer the reader to Gratzer et al. [21] or Gratzer [20, Chapter 6]. We will give an explanation of MTT as it is used in this paper: in the informal pen-and-paper style that we use type theory generally. We therefore avoid getting bogged down in the discussion of the various ways MTT modifies contexts, substitutions, and other building blocks of type theory.

First, MTT is a *framework* for modal type theories: a user picks a mode theory—a 2-category describing their modalities—and MTT produces a syntax for working with this collection of modalities. For this exposition, let us fix $\mathcal{M}$ an arbitrary 2-category where we think of objects m, n as type theories connected by the 1-cells/modalities μ, ν. The 2-cells α, β encode transformations between modalities enabling us to control e.g., whether μ is a comonad. MTT includes a modal type for each μ in the mode theory and these modal types are *2-functorial.* For $\mathsf{TT}_{\boxslash}$, we use a mode theory with only a single mode and so we restrict our attention to modalities and 2-cells.

The basic modification MTT makes to the type theory is to change the form of variables in the context. A context is no longer simply a telescope of bindings $x : A$. Instead, each declaration is annotated by a pair of modalities $x :_{\mu/\nu} A$. The annotation μ/ν signifies that x was constructed under the μ modality and, presently, we are working to construct an element of the ν modality. Both halves of this annotation restrict how variables are used to prevent terms from illegally escaping or entering modalities and, roughly, we are allowed to use a variable when they cancel.

$$\frac{x :_{\mu/\mu} A \in \Gamma}{\Gamma \vdash x : A} \qquad\qquad \frac{\Gamma/\mu \vdash a : A \qquad \Gamma, x :_{\mu/\mathsf{id}} A \vdash b(x) : B(x)}{\Gamma \vdash b[a/x] : B[b/x]}$$

Note we have presented only the relevant and simpler substitution rule allowing us to discharge an assumption with the "denominator" of an annotation is the identity. There is no alteration the actual definition of substitution from ordinary capture-avoiding substitution in type theory.

Notation 2.10. In a declaration $x :_{\mu/\nu} A$ we shall often omit μ or ν if they are the identity e.g., $x :_{\mu} A$ or $y : B$ rather than $x :_{\mu/\mathsf{id}}$ or $y :_{\mathsf{id}/\mathsf{id}} B$.

These annotations are also used to introduce the modal types associated with each μ. For instance, suppose we have a modality μ and we intend to form the modal type $\langle \mu \mid A \rangle$. This is well-formed in context Γ just when A is well-formed in the context Γ/μ i.e., the context formed by replacing each declaration $x :_{\xi/\nu} A$ in Γ with a new declaration $x :_{\xi/\nu\circ\mu} A$. Similarly, we can form an element of the modal type $\mathsf{mod}_{\mu}(a) : \langle \mu \mid A \rangle$ in context Γ just when $a : A$ in the context Γ/μ:

$$\frac{\Gamma/\mu \vdash A}{\Gamma \vdash \langle \mu \mid A \rangle} \qquad\qquad \frac{\Gamma/\mu \vdash a : A}{\Gamma \vdash \mathsf{mod}_{\mu}(a) : \langle \mu \mid A \rangle}$$

The elimination rule for $\langle \mu \mid - \rangle$ papers over the difference between $a :_{\nu\circ\mu/\mathsf{id}} A$ and $a' :_{\nu/\mathsf{id}} \langle \mu \mid A \rangle$:

$$\frac{\begin{array}{c}\Gamma/\nu \circ \mu \vdash A \qquad \Gamma, x :_{\nu/\mathsf{id}} \langle \mu \mid A \rangle \vdash B(a) \\ \Gamma, x :_{\nu\circ\mu/\mathsf{id}} A \vdash b(x) : B[\mathsf{mod}_{\mu}(x)/a] \qquad \Gamma/\nu \vdash a : \langle \mu \mid A \rangle\end{array}}{\Gamma \vdash \mathsf{let}_{\nu}\ \mathsf{mod}_{\mu}(x) \leftarrow a \ \mathsf{in}\ b(x) : B[a/x]}$$

Already, these rules are sufficient to prove the following facts that we shall use ubiquitously:

Lemma 2.11. *If* $\Gamma/\nu \circ \mu \vdash A$ *then* $\langle \nu \mid \langle \mu \mid A \rangle \rangle \simeq \langle \nu \circ \mu \mid A \rangle$ *and if* $\Gamma \vdash B$ *then* $\langle \mathsf{id} \mid B \rangle \simeq B$.

It remains to discuss how 2-cells $\alpha : \mu \longrightarrow \nu$ integrate into MTT. Roughly, each α introduces a function $\langle \mu \mid - \rangle \to \langle \nu \mid - \rangle$. In order to make this well-formed, MTT includes an operator $-^{\alpha}$ acting on both types and terms (formally realized as a special substitution):

$$\frac{\alpha : \mu \longrightarrow \nu \qquad \Gamma/\mu \vdash a : A}{\Gamma/\nu \vdash a^{\alpha} : A^{\alpha}}$$

This operator is really a substitution and so it commutes past all connectives to accumulate at variables—akin to weakening in systems designed with De Bruijn indices.[5] As a result, variables are often written x^{α}, signifying that they have been "shifted" from a $/\mu$-annotation to a $/\nu$-annotation.

Lemma 2.12. *If $\alpha : \mu \longrightarrow \nu$ and $\Gamma/\mu \vdash A$ then there is a map $\mathsf{coe}^{\alpha} : \langle \mu \mid A \rangle \to \langle \nu \mid A^{\alpha} \rangle$.*

In the formal definition of MTT, one takes Γ/μ as a new primitive form of context $\Gamma.\{\mu\}$ and annotates variables only with the μ of μ/ν; the denominator is encoded by composing all the $\{\nu_i\}$ that follow the occurrence of a variable in Γ. This distinction is largely unimportant for using MTT though it influences how we structure one of the axioms of $\mathsf{TT}_{\boxslash}$ in Section 3.1.

Notation 2.13. We shall also have occasion to use the convenience feature of modalized dependent products $(a :_{\mu} A) \to B(a)$ which abstract over $a :_{\mu} A$ directly rather than $a :_{\mathsf{id}} \langle \mu \mid A \rangle$ to allow us to avoid immediately pattern-matching on a. In particular, $(a :_{\mu} A) \to B(a)$ is equivalent to $(a :_{\mathsf{id}} \langle \mu \mid A \rangle) \to (\mathsf{let}\ \mathsf{mod}_{\mu}(a_0) \leftarrow a\ \mathsf{in}\ B(a_0))$.

3 Triangulated type theory

Our goal is to combine multimodal and simplicial type theory in order to follow Licata et al. [32] and build a directed univalent universe of groupoids. As mentioned in the introduction, however, the crucial modality (the amazing right adjoint to $\mathbb{I} \to -$) simply does not exist in the standard model of simplicial type theory. Thus, while we could perfectly well combine these two systems, the resulting combination of reasoning principles could not be justified.

In order to explain our eventual solution and crystallize the problem, it is helpful to recall the intended semantics of simplicial type theory: simplicial spaces. Up to the complexity needed to model homotopy type theory, these are simplicial sets i.e., presheaves on Δ, the category of finite, inhabited total linear orders. Already within this category, we can observe the problem: the interval is realized by $\mathbf{y}([1])$, the presheaf representing the total order $0 \le 1$, and exponentiation by $\mathbf{y}([1])$ does not have a right adjoint.[6] However, there is a closely related category to Δ which has also received a great deal of attention by type theorists interested in cubical type theory: the category of (Dedekind) cubes $\square$ or the full subcategory of partial orders spanned by $\{0 \le 1\}^n$ for all n.

Crucially, $\square$ enjoys two properties which make it interesting for simplicial type theorists: there is a fully faithful functor $\mathrm{PSh}(\Delta) \to \mathrm{PSh}(\square)$ which preserves the interval and admits both left and right adjoints [25, 52, 62] and within $\mathrm{PSh}(\square)$ exponentiating by the interval is a left adjoint.

Accordingly, we introduce a relaxation of simplicial type theory intended to capture (the homotopical version of) $\mathrm{PSh}(\square)$. Within this type theory, we can recover simplicial type theory by studying those types which are in the image of the aforementioned embedding alongside the amazing right adjoint necessary for constructing our sought-after universe [31].

Concretely, we work within a version of MTT instantiated with several modalities, further extended by a bounded distributive lattice $\mathbb{I} : \mathsf{HSet}$ which serves as our weakened version of the interval, and a handful of axioms. In particular, we no longer assume that $\mathbb{I}$ is totally ordered, this is our central deviation from simplicial type theory.

[5] Just as substitutions must account capture, $-^{\alpha}$ must account for the addition of Γ/μ so that e.g. $\mathsf{mod}_{\mu}(a)^{\alpha} = \mathsf{mod}_{\mu}(a^{\alpha \star \mu})$.
[6] This is easiest to check by observing that it does not commute with pushouts.

3.1 The definition of triangulated type theory

We now define *triangulated type theory*, beginning with the underlying instance of MTT.

3.1.1 The mode theory. As mentioned in Section 2.2, MTT must be instantiated by a mode theory. In our case, we shall require only one mode m which we shall think of as cubical spaces $\mathrm{PSh}_{\mathrm{sSet}}(\square)$. We shall then add the following modalities

- A pair of modalities $\flat, \sharp$ internalizing the global sections comonad and its right adjoint.
- A pair of modalities $\mathfrak{p}, \bar{\mathfrak{p}}$ internalizing the path space $(\mathbf{y}(0 \leq 1) \to -)$ and its right adjoint.
- A modality $\mathfrak{o}$ internalizing the "opposite."

Intuitively, $\langle \flat \mid - \rangle$ deletes all (higher) homomorphisms from a type, leaving only the underlying type of objects (its *groupoid core*). This modality recovers only the discrete categories and so it is often referred to as the *discrete* modality [38, 58]. On the other hand, $\langle \sharp \mid - \rangle$ deletes all homomorphisms, but then adds in a unique (higher) homomorphism between every pair of objects. We shall eventually force $\langle \mathfrak{p} \mid - \rangle$ to behave like $\mathbb{I} \to -$ and accordingly $\langle \bar{\mathfrak{p}} \mid - \rangle$ is the *amazing right adjoint* to this operation. Finally, $\langle \mathfrak{o} \mid - \rangle$ reverses all (higher) homomorphisms in a type.

We require a number of equations and natural transformations to force these modalities to behave as expected. In particular, we require the following 2-cells and equations on modalities:

$$\flat \circ \flat = \flat \circ \mathfrak{o} = \flat \circ \bar{\mathfrak{p}} = \flat \qquad \sharp \circ \flat = \sharp \qquad \sharp \circ \sharp = \sharp \qquad \mathfrak{o} \circ \mathfrak{o} = \mathsf{id}$$

$$\epsilon_{\flat \dashv \sharp} : \flat \circ \sharp \longrightarrow \mathsf{id} \qquad \eta_{\flat \dashv \sharp} : \mathsf{id} \longrightarrow \sharp \circ \flat \qquad \epsilon_{\mathfrak{p} \dashv \bar{\mathfrak{p}}} : \mathfrak{p} \circ \bar{\mathfrak{p}} \longrightarrow \mathsf{id} \qquad \eta_{\mathfrak{p} \dashv \bar{\mathfrak{p}}} : \mathsf{id} \longrightarrow \bar{\mathfrak{p}} \circ \mathfrak{p}$$

Finally, we impose equations on these 2-cells to ensure that (1) $(\flat, \sharp, \epsilon_{\flat \dashv \sharp}, \eta_{\flat \dashv \sharp})$ $(\mathfrak{p}, \bar{\mathfrak{p}}, \epsilon_{\mathfrak{p} \dashv \bar{\mathfrak{p}}}, \eta_{\mathfrak{p} \dashv \bar{\mathfrak{p}}})$ form adjunctions, and (2) the (co)join of the (co)monad induced by $\flat \dashv \sharp$ is the identity.

Notation 3.1. We will frequently omit the subscripts on the four generating 2-cells. We will often entirely suppress $\epsilon_{\flat \dashv \sharp} : \flat \longrightarrow \mathsf{id}$ as this causes no ambiguity.

3.1.2 The interval. As mentioned previously, we require an interval in order to capture the simplicial (or, in our case, cubical) structure. In order to marry this interval with the amazing right adjoint modality $\bar{\mathfrak{p}}$, we do this by adding an axiom along with a new rule to MTT:

Axiom 1 (The interval)**.** *There exists a bounded distributive lattice* $(\mathbb{I} : \mathsf{HSet}, \wedge, \vee, 0, 1)$.

Axiom 2 (Relating $\mathfrak{p}$ and $\mathbb{I}$)**.** *We extend MTT with the following rule:*

$$\frac{\vdash \Gamma \ \mathsf{cx}}{\vdash \Gamma/\mathfrak{p} = \Gamma, i : \mathbb{I} \ \mathsf{cx}}$$

Axiom 2 is a more substantive change than merely adding a constant to type theory as it imposes a genuinely new definitional equality on contexts. It is, however, less invasive than it may appear at first glance. Recall that in the formal definition of MTT, $\Gamma/\mathfrak{p}$ is realized not as an admissible operation on contexts but a genuinely new context former $\Gamma.\{\mathfrak{p}\}$ and variables are encoded in a locally-nameless style. Therefore, both the mysterious appearance of i and the appearance of an admissible operation in a definitional equality is merely an artifact of presenting a rule of our formal system with the more informal notation adopted throughout this paper. We note that while this definitional equality is very convenient to work with, it can be replaced with weaker typal equalities constraining $\langle \mathfrak{p} \mid - \rangle$ directly. We use Axiom 2 only to obtain the following pair of results:

Lemma 3.2. *If* $\Gamma/\mathfrak{p} \vdash A$ *and* $\Gamma \vdash i : \mathbb{I}$*, there is a substitution* $\Gamma \longrightarrow \Gamma/\mathfrak{p}$ *inducing a type* $\Gamma \vdash A \cdot i$.

Lemma 3.3. *There is an isomorphism between* $\langle \mathfrak{p} \mid A \rangle$ *and* $(i : \mathbb{I}) \to A \cdot i$.

Our next axiom controls the behavior of the opposite modality on $\mathbb{I}$:

Axiom 3 (Opposite of $\mathbb{I}$)**.** *There is an equivalence* $\neg : \langle \mathfrak{o} \mid \mathbb{I} \rangle \to \mathbb{I}$ *which swaps* 0 *for* 1 *and* $\vee$ *for* $\wedge$*.*

3.1.3 The simplicial monad. Before moving on to the list of additional axioms that form $\mathsf{TT}_{\boxslash}$, we must take a moment to discuss an additional construct: the simplicial monad. As motivation, while we have already noted that the interval is not totally ordered, there is a large number of types which "act as though it is." The simplicial monad isolates and classifies these types.

More precisely, a type is *simplicial* if it satisfies the following predicate:

$$\mathsf{isSimp}(A) = (i\, j : \mathbb{I}) \to \mathsf{isEquiv}(\lambda a, z.\, a : A \to (i \le j \vee j \le i \to A))$$

If a type A satisfies isSimp, this acts as a license to totally order elements of the interval whenever we are constructing an element of A. Furthermore, as the name suggests, simplicial types are those which come from simplicial rather than cubical sets (see Section 3.4).

Theorem 3.4 (Rijke et al. [49])**.** *There is an idempotent lex monad* $(\boxslash : \mathcal{U} \to \mathcal{U}, \eta, \mu)$ *such that:*

- *For every* $A : \mathcal{U}$*,* $\mathsf{isSimp}(\boxslash A)$ *holds.*
- *If* B *is simplicial, then* $\eta^* : (\boxslash A \to B) \to (A \to B)$ *is an equivalence.*
- $\boxslash$ *commutes with dependent sums and the identity type.*

We refer to $\boxslash$ as the *simplicial monad*[7] and write $\mathcal{U}_{\boxslash}$ for the subtype $\sum_{A:\mathcal{U}} \mathsf{isSimp}(A)$.

Convention 3.5. We reserve the words "category" and "groupoid" for types which are simplicial in addition to satisfying the Segal/Rezk conditions from STT. Accordingly, e.g. *category* signifies a type which is simplicial, Segal, and Rezk complete.

3.1.4 Additional axioms. Finally, we require a handful of additional axioms which either improve the behavior of modalities generally or form a more tight correspondence between our system and our intended model. We offer some intuition for each axiom and note that each is validated by the intended model described in Section 3.4.

Our first axiom is the famous univalence axiom giving us access to homotopy type theory [63]:

Axiom 4 (Univalence)**.** *We assume that each universe* $\mathcal{U}_i$ *is univalent.*

Next, we require that modalities commute with identity types:

Axiom 5 (Crisp induction)**.** *For every* μ*, the canonical map* $\mathsf{mod}_\mu(a) = \mathsf{mod}_\mu(b) \to \langle \mu \mid a = b \rangle$ *is an equivalence.*

After these fairly general reasoning principles, we now have a sequence of more simplicial-specific axioms. The first of these links the global sections modality to the interval. In particular, it states that the global sections of a type always form a groupoid.

Axiom 6 ($\mathbb{I}$ detects discreteness)**.** *If* $A :_\flat \mathcal{U}$ *then* $\langle \flat \mid A \rangle \to A$ *is an equivalence (*A *is* discrete*) if and only if* $A \to (\mathbb{I} \to A)$ *is an equivalence (*A *is* $\mathbb{I}$*-null).*

The next axiom states that the global points of $\mathbb{I}$ itself are just 0 and 1 and that $0 \neq 1$:

Axiom 7 (Global points of $\mathbb{I}$)**.** *The canonical map* $\mathsf{Bool} \to \mathbb{I}$ *is injective and* $\mathsf{Bool} \simeq \langle \flat \mid \mathbb{I} \rangle$*.*

In our intended model, various properties can be proven by "testing" them at the representable presheaves $\mathbf{y}(\{0 \le 1\}^n)$. We include a version of this idea as an axiom in our theory. Namely, we assert that maps between global types can be tested for invertibility at $\mathbb{I}^n$:

[7] The notation $\boxslash$ is chosen deliberately: simplicial types are those which believe the square $\mathbb{I} \times \mathbb{I}$ (along with all hypercubes) comes from gluing together a pair of triangles $\Delta^2 \sqcup_{\mathbb{I}} \Delta^2$.

Axiom 8 (Cubes separate). *A map $f :_\flat A \to B$ is an equivalence if and only if the following holds:*

$$(n :_\flat \mathsf{Nat}) \to \mathsf{isEquiv}(f_* : \langle \flat \mid \mathbb{I}^n \to A\rangle \to \langle \flat \mid \mathbb{I}^n \to B\rangle)$$

This follows from another possible axiom, *cubes detect continuity*, following Myers and Riley [38]. Note that if A and B are simplicial, one can derive a version of Axiom 8 which replaces $\mathbb{I}^n$ with Δ^n.

It is relatively easy to characterize maps out of $\boxslash A$ as they are closely related to maps out of A itself. It is much harder, however, to characterize $X \to \boxslash A$. Our next axiom states that in certain favorable cases these, too, coincide with the corresponding situation for A:

Axiom 9 (Simplicial stability). *If $A :_\flat \mathcal{U}$ then the following map is an equivalence for all $n :_\flat \mathsf{Nat}$:*

$$\eta_* : \langle \flat \mid \Delta^n \to A\rangle \to \langle \flat \mid \Delta^n \to \boxslash A\rangle$$

Finally, while simplicial type theory allows us to prove many interesting facts about maps out of the interval, it is far more difficult to prove properties about $X \to \mathbb{I}$. In order to balance the scales, we follow Cherubini et al. [11] and add an axiom characterizing these maps in certain special cases. Prior to stating this principle, we require the following definition:

Definition 3.6. A map $\mathbb{I} \to A$ of bounded distributive lattices is a *finitely presented (fp) $\mathbb{I}$-algebra* if it is merely equivalent to the canonical map $\mathbb{I} \to \mathbb{I}[x_1, \dots, x_n]/\langle t_1 = s_1, \dots, t_m = s_m\rangle$ for some n, m.

The definition of a homomorphism of bounded distributive lattices (a map which commutes with $0, 1, \wedge, \vee$) extends to a notion of homomorphism between fp $\mathbb{I}$-algebras $\hom_{\mathbb{I}}(A, B)$ by further requiring the underlying map to commute with the maps $\mathbb{I} \to A$ and $\mathbb{I} \to B$.

Axiom 10 (Duality). *Given an fp $\mathbb{I}$-algebra $f : \mathbb{I} \to A$ the following map is an equivalence:*

$$\lambda a\, f.\, f(a) : A \to (\hom_{\mathbb{I}}(A, \mathbb{I}) \to \mathbb{I})$$

Definition 3.7. Triangulated type theory $\mathsf{TT}_{\boxslash}$ is MTT with mode $\mathcal{M}$ extended by Axioms 1 to 10.

3.2 Duality and Δ^n

Axiom 10 has a number of remarkable consequences for $\mathbb{I}$. While these are not specific to directed univalent universes, they allow us to construct the first non-trivial categories inside $\mathsf{TT}_{\boxslash}$.

Lemma 3.8 (Phoa's principle). *Evaluation at $0, 1$ is an embedding $(\mathbb{I} \to \mathbb{I}) \to \mathbb{I} \times \mathbb{I}$ with image Δ^2.*

Proof. We first will argue via Axiom 10 that $\mathbb{I}[x]$ is equivalent to $\mathbb{I} \to \mathbb{I}$ via the evaluation map. To see this, let us note that $\mathbb{I} \to \mathbb{I}[x]$ is an $\mathbb{I}$-algebra by definition, and $\hom_{\mathbb{I}}(\mathbb{I}[x], \mathbb{I}) \simeq \mathbb{I}$. Accordingly, by Axiom 10, the map $\mathsf{eval} : \mathbb{I}[x] \to (\mathbb{I} \to \mathbb{I})$ is an equivalence.

By the 2-for-3 principle of equivalences, it then suffices to show that evaluating a polynomial at 0 and 1 induces an embedding $\mathbb{I}[x] \to \mathbb{I} \times \mathbb{I}$ whose image is Δ^2. An inductive argument allows us to conclude that $\mathsf{eval}(p, -)$ is a monotone map from $\mathbb{I} \to \mathbb{I}$ and so evaluation of polynomials at endpoints factors through Δ^2. We therefore are reduced to showing that this map is an equivalence. To see this, we observe that any polynomial in one variable can be placed in the following normal form: $p = \mathsf{eval}(p, 0) \vee x \wedge \mathsf{eval}(p, 1)$ whereby the conclusion is immediate. □

Notation 3.9. In light of the equivalence used in the proof of Phoa's principle, we will no longer distinguish between polynomials in one variable $\mathbb{I}[x]$ and functions $\mathbb{I} \to \mathbb{I}$.

Lemma 3.10 (Generalized Phoa's principle).

- *The evaluation map from $\mathbb{I}^n \to \mathbb{I}$ to monotone maps $\mathsf{Bool}^n \to \mathbb{I}$ is an equivalence.*
- *The evaluation map from $\Delta^n \to \mathbb{I}$ to monotone maps $[0 \le \cdots \le n] \to \mathbb{I}$ is an equivalence.*

In the above, we have regarded Bool as a 2-element partial order $\mathsf{ff} \le \mathsf{tt}$.

Both claims follow from induction on n and repeated application of Phoa's principle.

Remark 3.11. Our cube category is equivalent by Birkhoff duality to the category of *flat* finite bounded distributive lattices [61]. Lemma 3.10 is in some sense a manifestation of this fact.

Theorem 3.12. $\mathbb{I}$ *is simplicial.*

Proof. To show that $\mathbb{I} \to ((i \le j \vee j \le i) \to \mathbb{I})$ is an equivalence, it suffices, by Axiom 8, to consider $f, g :_\flat \mathbb{I}^n \to \mathbb{I}$ and show that the following is an equivalence:

$$\langle \flat \mid \mathbb{I}^n \to \mathbb{I} \rangle \to \langle \flat \mid \{\vec{x} : \mathbb{I}^n \mid f(\vec{x}) \le g(\vec{x}) \vee g(\vec{x}) \le f(\vec{x})\} \to \mathbb{I} \rangle$$

Using Lemma 3.10, we can extend an element of the codomain to a total function $\mathbb{I}^n \to \mathbb{I}$ provided we can specify its behavior on $\vec{x} : \mathsf{Bool}^n$. The proposition $f(\vec{x}) \le g(\vec{x}) \vee g(\vec{x}) \le f(\vec{x})$ holds for all $\vec{x} : \mathsf{Bool}^n$ using Axiom 7 and so such an extension always exists and is necessarily unique. $\square$

Corollary 3.13. Δ^n *is a category.*

Proof. Since there are no invertible morphisms in Δ^n, it is trivially Rezk-complete and, as a retract of $\mathbb{I}^n$, it is simplicial. Therefore, it suffices to show that Δ^n is Segal.

To this end, let us consider $\Lambda^2_1 \to \Delta^n$. This is equivalent to a pair of maps $f, g : \mathbb{I} \to \Delta^n$ such that $f(1) = g(0)$. Next, by the Phoa principle $f, g : \mathbb{I} \to \Delta^n \to \mathbb{I}^n$ are fully determined by n-tuples of pairs e.g., $(\pi_k(f(0)) \le \pi_k(f(1)))_{k \le n}$. In total then, we are given n-many 3-tuples:

$$(\pi_k(f(0)) \le \pi_k(f(1)) = \pi_k(g(0)) \le \pi_k(g(1)))_{k \le n}$$

By Lemma 3.10, these are 2-simplices in Δ^n and so every horn has a unique extension as required. $\square$

We note that Corollary 3.13 is already a significant step forward for STT: it is the first result constructing an explicit example of a non-discrete category within the system.

3.3 Reasoning with modalities in $\mathsf{TT}_\boxslash$

A number of useful results in $\mathsf{TT}_\boxslash$ are immediate corollaries of standard results from MTT combined with one of the axioms. We record some of the most important results in this section for future use and to give a flavor for how modalities can be used to enhance simplicial reasoning.

By general results about adjoint modalities from MTT [21], we obtain the following:

Lemma 3.14.

- *If* $A :_{\mathsf{id}} \mathcal{U}$, $B :_{\bar{\mathfrak{p}}} \mathcal{U}$ *there is an equivalence* $\langle \bar{\mathfrak{p}} \mid ((i : \mathbb{I}) \to A^\eta \cdot i) \to B \rangle \simeq (A \to \langle \bar{\mathfrak{p}} \mid B \rangle)$.
- *If* $A :_{\mathsf{id}} \mathcal{U}$, $B :_{\mathfrak{o}} \mathcal{U}$ *there is an equivalence* $\langle \mathfrak{o} \mid \langle \mathfrak{o} \mid A \rangle \to B \rangle \simeq (A \to \langle \mathfrak{o} \mid B \rangle)$.
- *If* $A :_{\mathsf{id}} \mathcal{U}$, $B :_\sharp \mathcal{U}$ *there is an equivalence* $\langle \sharp \mid \langle \flat \mid A^\eta \rangle \to B \rangle \simeq (A \to \langle \sharp \mid B \rangle)$.

There are also a dependent versions where e.g., $B :_{\bar{\mathfrak{p}}} (\mathfrak{p} \mid A) \to \mathcal{U}$.

This result is, in some sense, the entire purpose of $\mathsf{TT}_\boxslash$: it is the rule that allows transposing across the adjunction $(\mathbb{I} \to -) \dashv \langle \bar{\mathfrak{p}} \mid - \rangle$. This rule, in turn, is the key ingredient required to carry out the construction of $\mathcal{S}$ following Licata et al. [32].

We record two useful consequences of the transposition principle for $\mathfrak{o} \dashv \mathfrak{o}$ and $\sharp \dashv \flat$:

Lemma 3.15 (Gratzer [20])**.** $\langle \mathfrak{o} \mid - \rangle$ *commutes with colimits.*

Lemma 3.16. *Evaluation at endpoints* $(\mathbb{I} \to \langle \sharp \mid A \rangle) \to (\mathsf{Bool} \to \langle \sharp \mid A \rangle)$ *is an equivalence.*

To give an example of how these reasoning principles can be used, we show how they can be used to enhance our stock of simplicial types.

Theorem 3.17. *Given* $A :_{\mathfrak{o}} \mathcal{U}$ *if* $\langle \mathfrak{o} \mid \mathsf{isSimp}(A) \rangle$ *then* $\mathsf{isSimp}(\langle \mathfrak{o} \mid A \rangle)$.

Proof. Fix $i, j : \mathbb{I}$ such that we must show $\langle \mathfrak{o} \mid A \rangle \to (i \leq j \vee j \leq i \to \langle \mathfrak{o} \mid A \rangle)$ is an equivalence. Using Lemma 3.14, the codomain is equivalent to $\langle \mathfrak{o} \mid \langle \mathfrak{o} \mid i \leq j \vee j \leq i \rangle \to A \rangle$. By Axiom 3 and Lemma 3.15, $\langle \mathfrak{o} \mid i \leq j \vee j \leq i \rangle$ is $\neg\mathrm{mod}_o(i) \geq \neg\mathrm{mod}_o(j) \vee \neg\mathrm{mod}_o(j) \geq \neg\mathrm{mod}_o(i)$ and the conclusion follows immediately from our assumption $\langle \mathfrak{o} \mid \mathsf{isSimp}(A) \rangle$. □

Theorem 3.18. *If $A :_\flat \mathcal{U}$ is discrete then A is simplicial.*

Proof. Assume A is discrete, i.e., $A \to A^{\mathbb{I}}$ is an equivalence. Since cubes separate by Axiom 8, it suffices to show for all polynomials $p, q : \mathbb{I}[\vec{x}]$ in n variables $\vec{x}$ that the map $A \to (\varphi(\vec{x}) \to A)$ is an equivalence, where $\varphi(\vec{x}) := p(\vec{x}) \leq q(\vec{x}) \vee q(\vec{x}) \leq p(\vec{x})$.

In turn, it suffices to give an $\mathbb{I}$-homotopy h connecting the constant map at 0 to the identity on $\varphi(\vec{x})$, for each $\vec{x} : \mathbb{I}^n$. We notice that the straight-line homotopy $h(\vec{x}, t) = \vec{x} \wedge t$ from 0 to $\vec{x}$ works: We have to show for each $\vec{x}$ with $\varphi(\vec{x})$ that $\varphi(\vec{x} \wedge t)$ holds, for each t. But notice that $\varphi(0)$ is true, as any pair of constants among 0, 1 are comparable. By Lemma 3.8, $\varphi(\vec{x} \wedge t)$ then holds for all t. □

Using the adjunction $\flat \dashv \sharp$, we can prove that e.g., $\mathsf{Nat} \simeq \langle \flat \mid \mathsf{Nat} \rangle$ [20]. Accordingly, by Axiom 6:

Corollary 3.19. Nat *and* Bool *are both simplicial and $\mathbb{I}$-null i.e. groupoids.*

Remark 3.20. The result analogous to Theorem 3.18 for Rezk-complete Segal types does not hold, falsifying a conjecture of Weaver and Licata [67]. In particular, $\Delta^2 \sqcup_{\mathbb{I}} \Delta^2$ can be shown to be Rezk-complete and Segal, but is not simplicial. This same example shows that the requirement that A be annotated with $\flat$ is necessary: as a family over $\mathbb{I} \times \mathbb{I}$ the type $\Delta^2 \sqcup_{\mathbb{I}} \Delta^2$ is fiberwise a proposition—explicitly, it is $\lambda i\, j.\, i \leq j \vee j \leq i$—and therefore it is fiberwise $\mathbb{I}$-null. If we could apply Theorem 3.18 without the $\flat$-annotation we could conclude that each fiber $i \leq j \vee j \leq i$ was simplicial. Combined with the fact that $\mathbb{I} \times \mathbb{I}$ is simplicial, this leads again to the false conclusion that $\Delta^2 \sqcup_{\mathbb{I}} \Delta^2$ is simplicial.

3.4 The cubical spaces model

$\mathsf{TT}_{\boxslash}$ is intended to be an internal language for cubical spaces i.e. $\mathrm{PSh}(\square)$ (or rather its ∞-categorical engagement). In order to make this precise, we construct a model of $\mathsf{TT}_{\boxslash}$ inside a *model category* which presents the relevant ∞-category in order to link type theory (a 1-categorical object) with an ∞-category [24, 57, 59]. For $\mathsf{TT}_{\boxslash}$, this model category is the *injective model structure on simplicial presheaves* $\mathrm{PSh}_{\mathrm{sSet}}(\square)$. That is, types in $\mathsf{TT}_{\boxslash}$ are interpreted as certain families of presheaves over $\square$ valued in $\mathrm{sSet} = \mathrm{PSh}(\square)$. It is helpful to view the simplicial sets layer as "mixing in" homotopy theory with ordinary presheaves over $\square$. This result largely follows from combining off-the-shelf results about models of HoTT and models of MTT. First, we require the following result:

Theorem 3.21 (Cisinski [12], Shulman [59])**.** *Homotopy type theory has a model in* $\mathrm{PSh}_{\mathrm{sSet}}(\square)$ *where types are interpreted as injective fibrations.*

We use the model theory of MTT [20, 21, 60] to extend this model to MTT:

Theorem 3.22. *MTT with mode theory $\mathcal{M}$ has a model in* $\mathrm{PSh}_{\mathrm{sSet}}(\square)$ *where*

- $\flat \dashv \sharp$ *is interpreted by the global sections and codiscrete functors on this category.*
- $\mathfrak{o}$ *is interpreted by precomposition with the involution* $\square \to \square$.
- $\mathfrak{p} \dashv \bar{\mathfrak{p}}$ *is interpreted by the adjunction* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

In order to apply these off-the-shelf results we must show that all the functors interpreting modalities appropriately preserve types e.g. are right Quillen. This is true for all but the global sections functor, which is not right Quillen for the injective model structure. Fortunately, it is right Quillen for the *projective* model structure, and an unpublished argument due to Shulman shows that this, combined with his *cobar* construction [59], suffices to interpret even this problematic modality. We omit the details of these computations for reasons of space.

Finally, we must show that this model of MTT validates the axioms necessary for $\mathsf{TT}_{\boxslash}$. Each verification is a routine computation, with many following immediately from the corresponding fact about ordinary presheaves $\mathrm{PSh}(\square)$. The only exception to this is Axiom 10, which is follows from its proof for ordinary presheaves by Blechschmidt [8]. All told, we conclude the following:

Theorem 3.23. *$\mathsf{TT}_{\boxslash}$ has a model in $\mathrm{PSh}_{\mathrm{sSet}}(\square)$ where types are injective fibrations and modalities are interpreted as described above.*

Crucially, within this model simplicial types are precisely those belonging to the subtopos $\mathrm{PSh}_{\mathrm{sSet}}(\Delta)$. Consequently, the adequacy result from Riehl and Shulman [46] applies and we conclude that this model shows that any fact proven about categories and groupoids inside of $\mathsf{TT}_{\boxslash}$ is a valid proof for the standard definition of ∞-categories.

Theorem 3.24. *Categories in $\mathsf{TT}_{\boxslash}$ adequately model ∞-categories.*

4 Covariant and amazingly covariant families

In Section 2, we saw how groupoids were defined internally as those types satisfying $\mathrm{isGroupoid}\,A = \mathsf{isEquiv}(f A \to A^{\mathbb{I}})$ was an equivalence. We might hope this induces a directed univalent universe of groupoids directly, by considering $\mathcal{U}_{\mathrm{grp}} = \sum_{A:\mathcal{U}} \mathrm{isGroupoid}\,A$. However, this is far from our desired universe. Most glaringly, while $F : A \to \mathcal{U}_{\mathrm{grp}}$ is a family of groupoids over A, this family is not required to respect the category structure of A in any way. In fact, one may show that a map $F : \mathbb{I} \to \mathcal{U}_{\mathrm{grp}}$ is akin to an unstructured relation between $F(0)$ and $F(1)$ and nothing like the function required for directed univalence. In order to rectify this and define $\mathcal{S}$, we shall require a theory of families of groupoids where a morphism $f : a \to a'$ in A induces a functor of groupoids $F(a) \to F(a')$. Riehl and Shulman [46] termed these *covariant* families and they are further studied by Buchholtz and Weinberger [10]. As mentioned in the introduction, we shall also require a modal version of covariant families $F : A \to \mathcal{U}$ which are covariant not only in A but also in the entire context.

4.1 Covariant families and transport

We begin by recalling the definition of a covariant family from Riehl and Shulman [46].

Definition 4.1. A family $A : X \to \mathcal{U}$ is *covariant* if the following proposition holds:

$$\mathsf{isCov}(F) = \prod_{x:\mathbb{I}\to X} \prod_{a_0:X(x\,0)} \mathsf{isContr}\big(\textstyle\sum_{a_1:A(x\,1)} (a_0 \to_x a_1)\big)$$

Convention 4.2. While not strictly necessary, we will assume that the base of a covariant family A is a Segal type unless explicitly noted otherwise.

Definition 4.3. Given a type family $A : X \to \mathcal{U}$, we shall write $\tilde{A}$ for the *total type* $\sum_{x:X} A(x)$.

Lemma 4.4. *Given $\phi : \mathsf{isCov}(A : X \to \mathcal{U})$ and $f : x_0 \to x_1$ then there is an induced transport map $A(f) : A(x_0) \to A(x_1)$. Moreover, transport maps respect composition and identities.*

Proof Sketch. Roughly, one defines $A(f)(a_0) = \pi_1\pi_1(\phi\, f\, a_0)$. We leave it to the reader to check that this has the appropriate type and that the expected identities are satisfied. □

It is often helpful to rephrase covariant families in terms of orthogonality conditions:

Lemma 4.5. *A family $A : X \to \mathcal{U}$ is covariant if and only if the projection map $\tilde{A} \to X$ is right orthogonal to $\{0\} \hookrightarrow \mathbb{I}$ i.e., if $(\tilde{A})^{\mathbb{I}} \to (\tilde{A})^{\{0\}} \times_{X^{\{0\}}} X^{\mathbb{I}}$ is an equivalence.*

To cultivate intuition for this definition, we recall another result from Riehl and Shulman [46].

Lemma 4.6. *Given $\phi : \mathsf{isCov}(A : X \to \mathcal{U})$ and $x : X$, the fiber $A(x)$ is a groupoid.*

4.2 Amazing covariance

We now refine our search from a universe of groupoids to a universe of *covariant fibrations*. That is, we wish to define some universe $\mathcal{S}$ such that a map $A \to \mathcal{S}$ corresponds (in some sense) to a covariant fibration over A. Let us leave this correspondence imprecise for now and consider the behavior $\mathcal{S}$.

In light of Lemma 4.6, the points of $\mathcal{S}$ will be covariant over $\mathbf{1}$ i.e. groupoids. However, elements $f : \mathbb{I} \to \mathcal{S}$ will become richer: they are covariant fibrations $B \to \mathbb{I}$ therefore consist not only of a pair of groupoids B_0, B_1 over 0 and 1, but also include a transport function $B_0 \to B_1$ (Lemma 4.4). Phrased differently, a homomorphism $F : \mathbb{I} \to \mathcal{S}$ contains an ordinary function $F(0) \to F(1)$.

Clearly this is a step towards directed univalence over $\sum_{A:\mathcal{U}} \mathsf{isGroupoid}\, A$, but it is far from obvious how to define such a type $\mathcal{S}$. In particular, while we have sketched how behavior ought to differ between elements of $\mathcal{S}$ compared with functions $\mathbb{I} \to \mathcal{S}$ and so on, we cannot really cleanly divide elements of $\mathcal{S}$ from functions into $\mathcal{S}$ within type theory! An element of $\mathcal{S}$ is formed in a context Γ and if that contexts contains a variable $i : \mathbb{I}$, then this is the same as a function $\mathbb{I} \to \mathcal{S}$.

There is an even more straightforward way to see why this causes a problem. Suppose we attempt to define another subtype of $\mathcal{U}$ to isolate this universe of covariant fibrations $\sum_{A:\mathcal{U}} \mathsf{isCov}(A)$. A cursory inspection reveals this to be nonsense: being covariant is not a property of A, it is a property of a family of types $A : X \to \mathcal{U}$. So in this 'definition', what exactly is A covariant over?

It is here that modalities are vital: A should be covariant with respect to the entire ambient context. This is not something that can be expressed in standard type theory, but with the amazing right adjoint to $\mathbb{I} \to -$ we are able to define such a subtype.

Types covariant over Γ. We define a predicate on types $\mathsf{isACov} : \mathcal{U} \to \mathsf{HProp}$ which encodes whether a type is covariant over the entire context following Riley [50]. We note that this predicate is a refinement of Licata et al. [31] which capitalizes on the existence of the amazing right adjoint as a proper modality. The construction of this predicate proceeds in three steps:

(1) We begin by observing that, when specialized, isCov has the type $(\mathbb{I} \to \mathcal{U}) \to \mathsf{HProp}$.
(2) As this is a closed term, we may apply Lemma 3.14 to obtain a function $\mathcal{U} \to \langle \bar{\mathfrak{p}} \mid \mathsf{HProp} \rangle$.
(3) Finally, we postcompose with the dependent $\bar{\mathfrak{p}}$ modality which sends $\langle \bar{\mathfrak{p}} \mid \mathsf{HProp} \rangle \to \mathsf{HProp}$.

All told, we obtain a predicate $\mathcal{U} \to \mathsf{HProp}$ which encodes whether a given type is covariant over the entire context. Unfolding, this predicate sends A to $\langle \bar{\mathfrak{p}} \mid \mathsf{isCov}(\lambda i.\, A^{\eta} \cdot i) \rangle$ where A^{η} moves A from the context Γ to $\Gamma.\{\bar{\mathfrak{p}}\}.\{\mathfrak{p}\} = \Gamma.\{\bar{\mathfrak{p}}\}.\mathbb{I}$ and $- \cdot i$ realizes the $\mathbb{I}$ hypothesis with i.

Definition 4.7. A type is said to be *amazingly covariant* when it satisfies the following predicate:

$$\mathsf{isACov}(A) = \langle \bar{\mathfrak{p}} \mid \mathsf{isCov}(\lambda i.\, A^{\eta} \cdot i) \rangle$$

We begin by substantiating the claim that $\mathsf{isACov}(A)$ implies that A is truly covariant over all variables in the context.

Theorem 4.8. *Given $F : X \to \sum_{A:\mathcal{U}} \mathsf{isACov}(A)$, the type family $F_0 = \pi_1 \circ F$ is a covariant.*

Notation 4.9. We will write $\mathcal{U}_{\mathsf{ACov}}$ for the subtype $\sum_{A:\mathcal{U}} \mathsf{isACov}(A)$.

Proof. We must show $\mathsf{isCov}(F_0)$. We begin by noting that $\phi = \pi_2 \circ F$ has the following type:

$$\phi : (x : X) \to \mathsf{isACov}(F_0(x))$$

Using a dependent version of Lemma 3.14, we therefore obtain an element of the following type:

$$\phi' : \langle \bar{\mathfrak{p}} \mid (x : (i : \mathbb{I}) \to X^\eta \cdot i) \to \mathsf{isCov}(\lambda i.\, (F_0^\eta \cdot i)\,(x\, i)) \rangle$$

By weakening, we may regard ϕ' as an element of the type $\mathbb{I} \to \langle \bar{\mathfrak{p}} \mid \dots \rangle$ or, equivalently, $\phi'' : \langle \mathfrak{p} \circ \bar{\mathfrak{p}} \mid \dots \rangle$. We use coe^ϵ to remove $\langle \mathfrak{p} \circ \bar{\mathfrak{p}} \mid - \rangle$ from this type, and a tedious if mechanical calculation of the action $-^\epsilon$ yields $\mathsf{coe}^\epsilon\, \phi'' : (x : \mathbb{I} \to X) \to \mathsf{isCov}(\lambda i.\, F_0(x\, i))$ as required. □

We emphasize that in the above $\mathcal{U}_{\mathsf{ACov}}$ does not "know about" X. In particular, this is a subtype of $\mathcal{U}$ such that any map into this subtype induces covariant families.

Finally, the additional burden of being covariant over the context does not apply when working under $\langle \flat \mid - \rangle$, a reflection of the fact of $\langle \flat \mid A \rangle$ is "a proof of A not depending on the context."

Lemma 4.10. *If $X :_\flat \mathcal{U}$ and $A :_\flat X \to \mathcal{U}$ then $\langle \flat \mid (x : X) \to \mathsf{isACov}(A(x)) \rangle = \langle \flat \mid \mathsf{isCov}(A) \rangle$.*

4.3 Closure properties of amazing covariance

Given the strength of isACov, the reader may wonder how one every proves that $\mathsf{isACov}(A)$ for any element $A : \mathcal{U}$. In this section, we give a partial answer by building up a stock of amazingly covariant types. We shall see in Section 5 that these results undergird the closure properties of our directed univalent universe. Our main is the result the following:

Theorem 4.11. *In what follow, let us assume that $A, A_0, A_1 : \mathcal{U}$ and $B : A \to \mathcal{U}$.*

(1) If $X :_\flat \mathcal{U}$ then $\mathsf{isACov}(\langle \flat \mid X \rangle)$.
(2) If $i : \mathbb{I}$ then $\mathsf{isACov}(i = 1)$.
(3) If $\mathsf{isACov}(A)$ and $a, b : A$ then $\mathsf{isACov}(a = b)$.
(4) If $\mathsf{isACov}(A)$ and $(a : A) \to \mathsf{isACov}(B(a))$ then $\mathsf{isACov}\big(\sum_{a:A} B(a)\big)$.
(5) If $\mathsf{isACov}(A_0)$, $\mathsf{isACov}(A_1)$ and $f, g : A_0 \to A_1$ then $\mathsf{isACov}(\mathsf{Coeq}(f, g))$.[8]

Moreover, isACov is closed under Π-types provided modalities are used to manage the variance swap:

(6) If $C :_{\mathfrak{o}} \mathcal{U}$ and $D : (\mathfrak{o} \mid A) \to \mathcal{U}$ such that $\langle \mathfrak{o} \mid \mathsf{isACov}(C) \rangle$ and $(c :_{\mathfrak{o}} C) \to \mathsf{isACov}(D(c))$ then $\mathsf{isACov}((c :_{\mathfrak{o}} C) \to D(c))$.

We record a useful special case of (5) which follows from the involative property of $\langle \mathfrak{o} \mid - \rangle$:

Corollary 4.12. *If $X :_\flat \mathcal{U}$, $B : X \to \mathcal{U}$ such that $\prod_{x:X} \mathsf{isACov}(B(x))$ then $\mathsf{isACov}\big(\prod_{x:X} B(x)\big)$.*

For reasons of space, we will not prove each closure property listed in the above theorem, but instead limit ourselves to two representative cases: (2) and (4).

Lemma 4.13. *If $i : \mathbb{I}$ then $\mathsf{isACov}(i = 1)$.*

Proof. To prove this result, we shall switch to a more general goal, $\langle \flat \mid (i : \mathbb{I}) \to \mathsf{isACov}(i = 1) \rangle$, which can then be specialized to yield the original result. Using Lemma 3.14, it suffices to construct an element of $\langle \flat \mid (f : \mathbb{I} \to \mathbb{I}) \to \mathsf{isCov}(\lambda j.\, f(j) = 1) \rangle$

Since we have no additional hypotheses in this proof, we may forget the $\langle \flat \mid - \rangle$ and assume $f : \mathbb{I} \to \mathbb{I}$. By Axiom 1, $\mathbb{I}$ is an h-set and so $\mathsf{isCov}(f(j) = 1)$ is equivalent to showing that $f(0) = 1$ implies that $f(1) = 1$ i.e. that f is monotone. This is an immediate consequence of Lemma 3.8. □

Lemma 4.14. *If $\mathsf{isACov}(A)$ and $(a : A) \to \mathsf{isACov}(B(a))$ then $\mathsf{isACov}\big(\sum_{a:A} B(a)\big)$.*

Proof. Let us begin by applying the dependent version of Lemma 3.14 to our assumption $(a : A) \to \mathsf{isACov}(B(a))$ such that it becomes the following:

$$\langle \bar{\mathfrak{p}} \mid (a : (i : \mathbb{I}) \to A^\eta \cdot i) \to \mathsf{isCov}(\lambda i.\, (B^\eta \cdot i)\,(a\, i)) \rangle$$

[8] Here $\mathsf{Coeq}(f, g)$ denotes the coequalizer of f, g realized as a higher-inductive type [63].

Applying the elimination and introduction rules for $\langle \bar{\mathfrak{p}} \mid - \rangle$, the following now suffices: if $A : \mathbb{I} \to \mathcal{U}$ and $B : (i : \mathbb{I}) \to A(i) \to \mathcal{U}$ such that $\mathsf{isCov}(A)$ and $\mathsf{isCov}(\lambda i.\, B\, i\, (a\, i))$ then $\mathsf{isCov}\big(\lambda i.\ \sum_{a:A\,i} B\, i\, a\big)$. This statement is proven by Buchholtz and Weinberger [10, Proposition 6.2.1]. □

Remark 4.15. Both proofs exhibit proof strategies that are common when working with isACov in $\mathsf{TT}_{\boxslash}$: either reducing to a generic global case where various modalities can be simplified or performing several small modal manipulations and then applying standard and non-modal arguments.

5 The directed univalent universe

With our preliminary work on amazing covariance in place, we are now in a position to define our directed univalent universe of groupoids $\mathcal{S}$ and establish its core properties. To a first approximation, $\mathcal{S}$ consists of types which are (1) amazingly covariant and (2) simplicial. The former condition is needed to ensure directed univalence, etc. while the latter is really only needed to ensure that $\mathcal{S}$ captures the *standard* category of groupoids.

Definition 5.1. We define $\mathcal{S}$ to be $\sum_{A:\mathcal{U}_{\boxslash}} \mathsf{isACov}(A)$.

In particular, $\mathcal{S}$ is the subtype of $\mathcal{U}$ formed by intersecting $\sum_{A:\mathcal{U}} \mathsf{isACov} A$ and $\sum_{A:\mathcal{U}} \mathsf{isSimp}(A)$. We note that $\mathcal{S}$ can be fully characterized without reference to ACov:[9]

Lemma 5.2. *If $A : X \to \mathcal{U}_{\boxslash}$ lifts to $\mathcal{S}$, it is covariant. If $A :_\flat X \to \mathcal{U}_{\boxslash}$, this is a bi-implication.*

Corollary 5.3. *$A :_\flat \mathcal{U}$ factors through $\mathcal{S}$ if and only if it is a groupoid.*

Moreover, by Theorem 4.11 along with the closure results from Rijke et al. [49], we conclude:

Theorem 5.4. *As a subtype of $\mathcal{U}$, $\mathcal{S}$ is (1) univalent (2) closed under dependent sums, equality, and $i = 1$ (3) closed under the two modalized forms of Π-types indicated by Theorem 4.11.*

Thus, we already have established that $\mathcal{S}$ is a subuniverse of $\mathcal{U}$ spanned by groupoids. What remains is to prove directed univalence i.e., to characterize $\mathbb{I} \to \mathcal{S}$. To this end, we will first prove two important lemmas for constructing elements of $\mathcal{S}$. With these in place, we shall show that $\mathcal{S}$ is not only closed under various connectives, but also simplicial, Segal, Rezk, and directed univalent. Our main result can be summed up as follows

Theorem 5.5. *$\mathcal{S}$ is a directed univalent category.*

5.1 The two key lemmas

Before we can prove that $\mathcal{S}$ is directed univalent, we require a better understanding of when two maps $\mathbb{I} \to \mathcal{S}$ are equivalent. In particular, suppose we are given $f, g : \mathbb{I} \to \mathcal{S}$. We already know that $\mathcal{S}$ is univalent and so f and g are equal when there is an equivalence $\alpha : (i : \mathbb{I}) \to f(i) \to g(i)$. Accordingly, it suffices to find conditions to establish that $\alpha(i)$ is an equivalence for each $i : \mathbb{I}$. Our first result shows that this holds everywhere if it holds at 0 and 1. In other words, to check that a natural transformation α is an equivalence, it suffices to check that it is an equivalence at each object. We prove a slight generalization of this result which applies to any Δ^ℓ rather than just Δ^1.

Notation 5.6. We denote $(1, \dots, 1, 0, \dots 0) : \Delta^\ell$ with k copies of 1 followed by $\ell - k$ copies of 0 by $\bar{k}$.

Lemma 5.7. *Fix $\ell :_\flat \mathsf{Nat}$ and suppose that $f, g : \Delta^\ell \longrightarrow \mathcal{S}$ and $\alpha : (\delta : \Delta^\ell) \to f\,\delta \to g\,\delta$ then α is invertible if and only if $\alpha\,\bar{k} : f\,\bar{k} \longrightarrow g\,\bar{k}$ is invertible for all $k \leq \ell$.*

[9] Theoretically, every result about $\mathcal{S}$ can be proven using this characterization. We will not endeavor to do so and instead optimize for more readable proofs.

Proof. We begin by generalizing to apply Axiom 8. To this end, fix the following global types:

$$X = \textstyle\sum_{F\,G:\Delta^\ell \to \mathcal{S}} \sum_{\alpha:(\delta:\Delta^\ell)\to F\,\delta \to G\,\delta} (k : \mathsf{Nat}_{\leq \ell}) \to \mathsf{isEquiv}(\alpha\,\bar{k})$$
$$Y = \textstyle\sum_{F\,G:\Delta^\ell \to \mathcal{S}} \sum_{\alpha:(\delta:\Delta^\ell)\to F\,\delta \to G\,\delta} \prod_{\delta:\Delta^\ell} \mathsf{isEquiv}(\alpha(\delta))$$

It suffices to show that the forgetful map $Y \to X$ is an equivalence and so, by Axiom 8, we must show that for each $n :_\flat \mathsf{Nat}$ the map $\langle \flat \mid \mathbb{I}^n \to Y\rangle \to \langle \flat \mid \mathbb{I}^n \to X\rangle$ is an equivalence. For clarity, we write $\Gamma = \mathbb{I}^n$ and $\Gamma' = \mathbb{I}^n \times \Delta^\ell$ in what follows.

We now unfold this slightly. Fix $F, G :_\flat \Gamma' \to \mathcal{S}$ along with $\alpha :_\flat ((v,\delta) : \Gamma') \to F(v,\delta) \to G(v,\delta)$ and $e :_\flat (v : \Gamma)(k : \mathsf{Nat}_{\leq \ell}) \to \mathsf{isEquiv}(\alpha(v, \bar{k}))$. We must show the following:

$$\langle \flat \mid ((v,\delta) : \Gamma') \to \mathsf{isEquiv}(\alpha(v,\delta))\rangle$$

We can reorient F, G as global families $\pi_F, \pi_G : \tilde{F}, \tilde{G} \longrightarrow \Gamma'$. That both F, G factor through space implies that they are both covariant fibrations and, therefore, orthogonal to the maps $\{0\} \to \mathbb{I}^m$ for any $m :_\flat \mathsf{Nat}$. Note, too, that from this viewpoint, α is a map $\tilde{\alpha} : \tilde{F} \to \tilde{G}$ over Γ' such that pulling back along $(\mathsf{id}, \bar{k}) : \Gamma \to \Gamma'$ induces an equivalence. We must show that $\tilde{\alpha}$ is an equivalence.

By another application Axiom 8, to show that $\tilde{\alpha}$ is an equivalence we must show it induces an equivalence $\langle \flat \mid \mathbb{I}^m \to \tilde{F}\rangle \simeq \langle \flat \mid \mathbb{I}^m \to \tilde{G}\rangle$. By orthogonality, we note that $\langle \flat \mid \mathbb{I}^m \to \tilde{F}\rangle \simeq \langle \flat \mid \tilde{F} \times_{\Gamma'} (\mathbb{I}^m \to \Gamma')\rangle$. Consequently, it suffices to show that the following map is an equivalence:

$$\langle \flat \mid \tilde{F} \times_{\Gamma'} (\mathbb{I}^m \to \Gamma')\rangle \to \langle \flat \mid \tilde{G} \times_{\Gamma'} (\mathbb{I}^m \to \Gamma')\rangle$$

We may refactor this using the various properties of $\langle \flat \mid - \rangle$ to obtain the following equivalent map:

$$\textstyle\sum_{v:_\flat \mathbb{I}^m \to \Gamma} \sum_{\theta:_\flat \mathbb{I}^m \to \Delta^\ell} \langle \flat \mid F(v(\vec{0}), \theta(\vec{0}))\rangle \to \sum_{v:_\flat \mathbb{I}^m \to \Gamma} \sum_{\theta:_\flat \mathbb{I}^m \to \Delta^\ell} \langle \flat \mid F(v(\vec{0}), \theta(\vec{0}))\rangle$$

Finally, $\theta(\vec{0})$ is an element of $\langle \flat \mid \Delta^\ell\rangle$ and is therefore equal to $\bar{k}$ for some k by Axiom 7. For any k, the map is an equivalence as it is derived from α and our conclusion follows. □

Remark 5.8. Weaver and Licata [67] axiomatized their *cobar modality* to formulate and postulate a special case of this lemma (their *equivalence axiom*). In our case, no such steps are required as this result follows from Axiom 8.

To ensure that elements of $\mathcal{S}$ are indeed groupoids, $A : \mathcal{U}$ lands in the subtype $\mathcal{S}$ only when it is A simplicial in addition to being amazingly covariant. Often, it is easiest to do this by proving that A is amazingly covariant and then applying $\boxslash$ to A to obtain a simplicial type. In order for this to be possible, however, we must know that applying $\boxslash$ to an amazingly covariant type results in an amazingly covariant type. The next lemma proves (a generalization of) this fact.

Let us note that the canonical maps $\mathcal{U}_{\mathrm{ACov}} \to \mathcal{U}$ and $\boxslash\mathcal{U} \to \mathcal{U}$ induce a map $\hat{\boxslash}\pi : \boxslash\mathcal{U}_{\mathrm{ACov}} \to \mathcal{U}$. Showing that $\boxslash A$ is amazingly covariant if A is amazingly covariant corresponds to showing that $\hat{\boxslash} \circ \eta$ factors through $\mathcal{S}$. We prove this by proving the following stronger result:

Lemma 5.9 (Simplicial exchange). *$\hat{\boxslash}\pi : \boxslash\mathcal{U}_{\mathrm{ACov}} \to \mathcal{U}$ factors through $\mathcal{S}$.*

Proof. Given that the composite $\hat{\boxslash}\pi : \boxslash\mathcal{U}_{\mathrm{ACov}} \to \mathcal{U}$ mentions no free variables and (by construction) factors through $\mathcal{U}_{\boxslash}$, it suffices by Lemma 4.10 to show that $\hat{\boxslash}\pi$ is covariant.

For concision, we write $X = \boxslash\mathcal{U}_{\mathrm{ACov}}$ and $\tilde{X}$ for $\sum_{A:X} \hat{\boxslash} A$. We must show that the map given by evaluating at 0 induces an equivalence between $\mathbb{I} \to \tilde{X}$ and $(\mathbb{I} \to X) \times_X \tilde{X}$. Using Axiom 8 along with the observation that these types are all simplicial, it suffices to show that the following map is an equivalence for all $n :_\flat \mathsf{Nat}$:

$$\langle \flat \mid \Delta^n \times \mathbb{I} \to \tilde{X}\rangle \to \langle \flat \mid \Delta^n \to (\mathbb{I} \to X) \times_X \tilde{X}\rangle$$

In other words, we must show that $\Delta^n \times \{0\} \to \Delta^n \times \mathbb{I}$ is *globally* orthogonal to $\tilde{X} \to X$. By standard simplicial combinatorics [43], $\Delta^n \times \mathbb{I}$ is the colimit of a collection of simplices Δ^{n+1}. Using this and the left-cancellation property of the class of left maps in an orthogonality problem, it suffices to show that for all $m :_\flat \mathsf{Nat}$ that is right orthogonal to $\{(0,\ldots,0)\} \to \Delta^m$. All told then, it suffices to show the following canonical map is an equivalence:

$$\langle \flat \mid \Delta^m \to \tilde{X} \rangle \to \langle \flat \mid (\Delta^m \to X) \times_X \tilde{X} \rangle$$

By Axiom 9, we may "remove the $\boxslash$" from X and $\tilde{X}$ and so this type is equivalent to the following:

$$\left\langle \flat \mid \Delta^m \to \textstyle\sum_{A:\mathcal{U}_{\mathsf{ACov}}} A \right\rangle \to \left\langle \flat \mid (\Delta^m \to \mathcal{U}_{\mathsf{ACov}}) \times_{\mathcal{U}_{\mathsf{ACov}}} \left(\textstyle\sum_{A:\mathcal{U}_{\mathsf{ACov}}} A\right) \right\rangle$$

This, finally, is an equivalence because $\left(\sum_{A:\mathcal{U}_{\mathsf{ACov}}} A\right) \to \mathcal{U}_{\mathsf{ACov}}$ is a covariant (Theorem 4.8). □

Corollary 5.10. *$\mathcal{S}$ is closed under coequalizers in $\mathcal{U}_{\boxslash}$.*

Proof. By Theorem 4.11, $\mathcal{U}_{\mathsf{ACov}}$ is closed under coequalizers $\mathsf{Coeq}(f,g)$ and so Lemma 5.9 ensures the $\boxslash\mathsf{Coeq}(f,g)$ lands in $\mathcal{S}$ as well. By Rijke et al. [49], this is the coequalizer in $\mathcal{U}_{\boxslash}$. □

5.2 $\mathcal{S}$ is directed univalent, Segal, Rezk, and simplicial

We are now able to show that $\mathcal{S}$ satisfies all the desired properties for a universe of groupoids. We begin by showing that we have, at last, constructed a directed univalent universe.

First, we note that Definition 1.2 merely states that there is some isomorphism between two types. We are already in a position to construct one of these two maps:

Lemma 5.11. *There is a function* mor2fun *from* $\mathbb{I} \to \mathcal{S}$ *to* $\sum_{A\,B:\mathcal{S}} A \to B$.

Proof. Given $F : \mathbb{I} \to \mathcal{S}$, by Theorem 4.8 this induces a covariant family $F_0 : \mathbb{I} \to \mathcal{U}$. We then define $\mathsf{mor2fun}(F) \coloneqq (F_0\,0, F_0\,1, F_0\,\mathsf{id})$ where the last component is induced by Lemma 4.4. □

Theorem 5.12 (Directed univalence). mor2fun *is an equivalence.*

Prior to proving this result, we will construct a putative inverse to mor2fun.

Definition 5.13. Given $A, B : \mathcal{S}$ and $f : A \to B$, $\mathsf{Gl}(A,B,f) : \mathbb{I} \to \mathcal{S}$ is $\lambda i.\ \sum_{b:B} i = 0 \to f^{-1}(b)$.

Gl is the directed version of the glue type from cubical type theory [16, 51]. We have no need to add it as a primitive in our setting: this was necessary in cubical type theory toachieve certain definitional equalities, but we are pervasively working up to equivalence. We note that $\mathsf{Gl}(f)$ factors through $\mathcal{S}$ by virtue of (2–4) of Theorem 4.11 along Axiom 3 which ensures that $\langle \mathfrak{o} \mid \neg j = 1 \rangle = (j = 0)$.

Lemma 5.14. *Given A, B, f as above,* $\mathsf{Gl}(A,B,f)\,0 = A$, $\mathsf{Gl}(A,B,f)\,1 = B$, *and* $\mathsf{Gl}(A,B,f)\,\mathsf{id} = f$.

Proof of Theorem 5.12. We will prove that Gl forms a quasi-inverse to mor2fun and thereby conclude that mor2fun is an equivalence. We must therefore prove (1) $\mathsf{mor2fun} \circ \mathsf{Gl} = \mathsf{id}$ and (2) $\mathsf{Gl} \circ \mathsf{mor2fun} = \mathsf{id}$. (1) follows from direct calculation and Lemma 5.14, so we will detail only (2).

Suppose we are given $F : \mathbb{I} \to \mathcal{S}$. We must show that $F = \mathsf{Gl}(\mathsf{mor2fun}(F))$ or equivalently, using the fact that $\mathcal{S}$ is univalent, that there is an equivalence $\alpha : (i : \mathbb{I}) \to F(i) \simeq \mathsf{Gl}(\mathsf{mor2fun}(F))\,i$. To prove this, we will begin by constructing α and then use Lemma 5.7 to reduce to checking that α is an equivalence at 0 and 1. It is helpful to do this in stages and so we begin by supposing $i : \mathbb{I}$ and $f : F(i)$ and define α as follows for some X and Y to be determined:

$$\alpha\,i\,f = (X : F(1), Y : i = 0 \to F(\mathsf{id})^{-1}(X))$$

We will construct X and Y separately.

We can substantiate X immediately: $F(\lambda j.\, i \vee j) : F(i) \to F(1)$ and so we choose $X \coloneqq F(- \vee i)\,f$. This refines the type of Y to $i = 0 \to F(\mathsf{id})^{-1}(F(- \vee i)\,f)$. Assume $\phi : i = 0$ so that it suffices to define $Y.1 : F(0)$ and $Y.2 : F(\mathsf{id})\,Y.1 = F(- \vee i)\,f$. Using ϕ, we may suppose that $f : F(0)$ and that the type of $Y.2$ is $F(\mathsf{id})\,Y.1 = F(\mathsf{id})\,f$ (since $0 \vee - = \mathsf{id}$). After this, $Y.1 \coloneqq f$ and $Y.2 \coloneqq \mathsf{refl}$ suffices.

Finally, it is now straightforward to check that $\alpha\,0$ and $\alpha\,1$ are equivalences using Lemma 5.14. □

The proof that $\mathcal{S}$ is Segal is very similar to the proof of directed univalence, though not quite a consequence of it. Since the proof is similar to Theorem 5.12, we provide only a sketch.

Theorem 5.15. *$\mathcal{S}$ is Segal.*

Proof sketch. We must show that $(\Delta^2 \to \mathcal{S}) \to (\Lambda^2_1 \to \mathcal{S})$ is an equivalence. We begin by noting that the codomain can be rewritten with Theorem 5.12 as $T = \sum_{A\,B\,C:\mathcal{S}} A \to B \times B \to C$. We only need to show that the forgetful map from $(\Delta^2 \to \mathcal{S}) \to T$ is an equivalence.

This argument proceeds along the same lines as Theorem 5.12 where we replace $\mathbb{I}$ with Δ^2: we introduce a variant of Gl which glues together three spaces along two maps and show that this procedure induces a quasi-inverse to the forgetful map $(\Delta^2 \to \mathcal{S}) \to T$. It is here that we require Lemma 5.7 with $\ell = 2$ rather than $\ell = 1$. ◻

Corollary 5.16. *Composition of the morphisms in $\mathcal{S}$ is realized by ordinary function composition.*

In particular, an invertible morphism corresponds via Theorem 5.12 to an equivalence. Combining this with ordinary univalence, we obtain:

Corollary 5.17. *$\mathcal{S}$ is Rezk.*

Our final result is that $\mathcal{S}$ lands in the subuniverse of simplicial types.

Theorem 5.18. *$\mathcal{S}$ is simplicial.*

Proof. By Rijke et al. [49, Lemma 1.20], it suffices to show that $\eta : \mathcal{S} \to \boxslash\mathcal{S}$ has a retraction. By univalence, the composite of $\eta : \mathcal{U}_{\boxslash} \to \boxslash\mathcal{U}_{\boxslash}$ followed by $\hat{\boxslash} : \boxslash\mathcal{U}_{\boxslash} \to \mathcal{U}_{\boxslash}$ is the identity and so it suffices to show that both these maps restrict to $\mathcal{S}$. That is, it suffices to show that $\hat{\boxslash} \circ \pi : \boxslash\mathcal{S} \to \mathcal{U}$ factors through $\mathcal{S}$. This is an immediate consequence of Lemma 5.9. ◻

We conclude by noting a few of the categorical properties $\mathcal{S}$ enjoys:

Theorem 5.19. *$\mathcal{S}$ is finitely complete and finitely cocomplete and satisfies* descent *[48, Chapter 2].*

Proof Sketch. Finite completeness and cocompleteness are an immediate consequence of Theorem 5.4 and Corollary 5.10 along with Theorem 5.12 which implies that a e.g., categorical limit in $\mathcal{S}$ is an ordinary HoTT limit of groupoids. To prove the descent properties, we must show that various limits and colimits commute appropriately. However, by Theorem 5.12 once more, this is an immediate consequence of the fact that limits and colimits in HoTT enjoy descent [48]. ◻

6 Consequences of a directed univalent universe

We now reap the rewards of our efforts in constructing $\mathcal{S}$ and give a brief tour of the consequences of this type. We show how directed univalence may be used to prove free theorems and substantiate the directed structure identity principle. We also use it to construct various foundational example categories and lay the groundwork for the development of *higher algebra* within $\mathsf{TT}_{\boxslash}$.

6.1 Free theorems from naturality

Directed univalence allows us to make a precise link between familiar parametricity arguments [65] with the categorical naturality arguments that helped motivate them. In particular, directed univalence implies that a function $\alpha : (A : \mathcal{S}) \to F(A) \to G(A)$ is natural:

Theorem 6.1. *If $F_0, F_1 : \mathcal{S} \to \mathcal{S}$ and $\alpha : (A : \mathcal{S}) \to F_0(A) \to F_1(A)$ then $\alpha(B) \circ f = f \circ \alpha(B)$ for any $f : A \to B$.*

Proof. Fix $A, B : \mathcal{S}$ along with $f : A \to B$ and denote the corresponding morphism $G : \mathbb{I} \to \mathcal{S}$. Note that $\alpha \circ G$ is then a function $(i : \mathbb{I}) \to F_0(i) \to F_1(i)$. Applying Theorem 5.12 once more, we note that $\alpha(G(i)) : F_0(i) \to F_1(i)$ is a morphism in $\mathcal{S}$ for every i. Accordingly, $\alpha \circ G$ is equivalent to some $s : (i\ j : \mathbb{I}) \to H\ i\ j$ for some H where $H\ i\ 0 = F_0\ i$ and $H\ i\ 1 = F\ i$. We visualize H as:

$$\begin{tikzcd} F_0\,0 \arrow[r, "F_0"] \arrow[d, "\alpha(G\,0)"'] & F_0\,1 \arrow[d, "\alpha(G\,1)"] \\ F_1\,0 \arrow[r, "F_1"'] & F_1\,1 \end{tikzcd}$$

This commuting square is equivalently an equality between the composites F_1 and $\alpha(G\,0)$ and $\alpha(G\,1)$ and F_0. The conclusion then follows from Corollary 5.16. ☐

Theorem 6.2. *If $f : (A : \mathcal{S}) \to A \to A$ then $f = \lambda A\, a.\, a$.*

Proof. Fix $A : \mathcal{S}$ and suppose we are given $a : A$. Applying Theorem 6.1 to f and $\lambda\_.\, a$, we conclude that $f\, A\, (a\, \star) = a(f\, \mathbf{1}\, \star)$. Since $f\, \mathbf{1}\, \star = \star$ by the η principle of $\mathbf{1}$, $f = \lambda A\, a.\, a$. ☐

Nothing limits us to considering only operations $\mathcal{S} \to \mathcal{S}$. The same techniques scale to multi-argument operations such as $\mathcal{S}\times\mathcal{S} \to \mathcal{S}$ or even mixed-variance operations such as $\langle \mathfrak{o} \mid \mathcal{S}\rangle\times\mathcal{S} \to \mathcal{S}$:

Lemma 6.3. *If $\alpha : (A\, B : \mathcal{S}) \to A \times B \to A$ then $\alpha = \pi_1$.*

Lemma 6.4. *If $A, B :_\flat \mathcal{S}$ and $\alpha : (C :_\mathfrak{o} \mathcal{S}) \to A^{\langle \mathfrak{o} \mid C\rangle} \to B^{\langle \mathfrak{o} \mid C\rangle}$ then $\alpha = \lambda\_g.f \circ g$ for some $f : A \to B$.*

This methodology highlights the *limitations* of naturality as a facsimile for parametricity: for operations whose parameters are not used strictly co- or contravariantly, directed univalence does not provide any free theorems. We leave it to future work to consider alternative universes of correspondences [7] and what parametricity arguments they might provide.

6.2 Full subcategories of $\mathcal{S}$

A large number of important categories can be described as a *full subcategories* of $\mathcal{S}$. To do this, we must first show how to obtain full subcategories inside of $\mathsf{TT}_{\boxslash}$. Recall that a full subcategory of C_0 of a category $C :_\flat \mathcal{U}$ is a category C_0 where objects are a subset of those in C but the morphisms and all the higher cells agree. In other words, a full subcategory is described by a predicate $(\flat \mid C) \to \mathsf{HProp}_{\boxslash}$ which picks out those objects which land in C_0.

Definition 6.5. Given $\phi :_\sharp (\flat \mid C) \to \mathsf{HProp}_{\boxslash}$, the resulting full subcategory C_ϕ is $\sum_{c:C} \langle \sharp \mid \phi(c^\eta)\rangle$.[10]

Here we for the first time have occasion to explicitly use the right adjoint $\sharp$ to $\flat$. Let us note that C_ϕ is a category because (1) categories are closed under dependent sums and (2) $\langle \sharp \mid \phi(c^\eta)\rangle$ is a groupoid. Furthermore, we can prove that C_ϕ is actually a full subcategory:

Theorem 6.6. *Given C and ϕ as above, if $a, b : C_\phi$ then* $\hom_{C_\phi}(a, b) \simeq \hom_C(\pi_1\, a, \pi_1\, b)$.

By choosing different predicates on $\mathcal{S}$ we obtain a number of familiar categories. For instance:

Definition 6.7. The category of n-truncated groupoids $\mathcal{S}_{\le n}$ is given by $\mathcal{S}_{\mathsf{hasHLevel}\,(n+2)}$.[11] In particular, the category of *propositions* is given by $\mathcal{S}_{\le -1}$, and the category of *sets* is given by $\mathcal{S}_{\le 0}$.

Definition 6.8. The category of finite sets $\mathcal{F}$ is given by $\mathcal{S}_\phi$ where $\phi(X) = \sum_{n:\mathsf{Nat}}(X = \mathsf{Nat}_{\le n})$.

[10] In practice, ϕ will be $\flat$-annotated.

[11] The correction +2 ensures that $\mathcal{S}_{\le n}$ comports with the standard indexing in homotopy theory which begins at -2, not 0.

Note that $\mathcal{F}$ is quite different than $\sum_{A:\mathcal{S}} \exists n.\, \mathsf{Nat}_{\leq n} = A$, which has only invertible morphisms. The definition of C_ϕ is necessary to ensure that ϕ is applied only to the objects of C, not its higher cells.

Theorem 6.6 implies that these examples inherit directed univalence from $\mathcal{S}$, the first instance of the *directed structure identity principle (DSIP)* [67]: homomorphisms in structured types coincide with their standard analytic formulations and, consequently, all terms and types are functorial for these analytic morphisms. For instance, a morphism in $\mathcal{F}$ corresponds to an ordinary function and, consequently, a family $F : \mathcal{F} \to \mathcal{S}$ has an action $F(A) \to F(B)$ for any ordinary function $A \to B$.

6.3 The directed structure identity principle

Not only full subcategories of $\mathcal{S}$ enjoy DSIP, in this section we survey other categories which satisfy it as well. As a prototypical example, we consider pointed spaces, $\mathcal{S}_* = \sum_{A:\mathcal{S}} A$:

Lemma 6.9. *Homomorphisms* $\hom_{\mathcal{S}_*}((A,a),(B,b))$ *are equal to pointed functions* $\sum_{f:A\to B} f(a) = b$.

Proof. By construction, the projection map $\mathcal{S}_* \to \mathcal{S}$ is covariant, giving, for any pair of pointed spaces (A, a_0) and (B, b_0), an equivalence between homomorphisms from a_0 to b_0 lying over a homomorphism $f : A \to B$ and identifications $f(a_0) = b_0$. □

This same methodology can be applied to more general algebraic structures to yield categories of e.g., monoids, groups, rings, etc. which all enjoy DSIP. Rather than dealing with this generality, we will focus on monoids to complete the example given in Section 1. We recall the type of monoids:

$$\mathsf{Monoid} = \textstyle\sum_{A:\mathcal{S}_{\leq 0}} \sum_{\epsilon:A} \sum_{\cdot:A\times A\to A} \mathsf{isAssociative}(\cdot) \times \mathsf{isUnit}(\cdot,\epsilon)$$

By repeated application of the closure of categories under dependent sums, functions, and equalities, we already conclude that Monoid is a category. More interesting, we can characterize its homomorphisms. For space, we omit the proof; it is a rehashing of Theorem 6.1 and Lemma 6.9.

Theorem 6.10. *A homomorphism* $\hom((A,\epsilon_A,\cdot_A,\alpha_A,\mu_A),(B,\epsilon_B,\cdot_B,\alpha_B,\mu_B))$ *is precisely a standard monoid homomorphism e.g. a function* $A \to B$ *commuting with multiplication and the unit.*

Substituting Theorem 6.10 within Theorem 6.1, we obtain the promised result:

Theorem 6.11. *If* $F, G : \mathsf{Monoid} \to \mathcal{S}$ *and* $\alpha : (A : \mathsf{Monoid}) \to F(A) \to G(A)$ *then* α *is natural i.e. if* $f : A \to B$ *is a monoid homomorphism,* $\alpha(B) \circ f = f \circ \alpha(A)$.

To complete our goal of proving sum natural automatically, it remains only to define List as an endomap of monoids where List A has pointwise multiplication. Remarkably, this straightforward consequence of our results. One need only write down the definition of this monoid in the ordinary way and conclude that it lifts to a functor because the carrier ($\mathsf{List} = \sum_{n:\mathsf{Nat}} -^n$) is already known to be a functor $\mathcal{S}_{\leq 0} \to \mathcal{S}_{\leq 0}$ using the closure under Σ and Nat; no special argument is required.

We can also apply directed univalence to non-algebraic structures using our ability to define n-presheaf categories $\mathrm{PSh}_n(C) = \langle \mathfrak{o} \mid C \rangle \to \mathcal{S}_{\leq n}$. We consider the representative example of partial orders, which we isolate as a full subcategory of a presheaf category. In particular, we begin with the category of reflexive graphs: $\mathsf{RGraph} = \mathrm{PSh}_0(\Delta_{\leq 1})$ where $\Delta_{\leq 1}$ is the "walking fork" given by the pushout $\Delta_2 \sqcup_{\mathbb{I}} \Delta_2$ adjoining a pair of retractions ∂_0, ∂_1 to a single arrow $r : 1 \to 0$. While we have not ensured $\Delta_{\leq 1}$ is a category, this does not matter as RGraph is a category regardless.

We use directed univalence to characterize this categories *objects* as well as its higher structure:

Theorem 6.12. *The category* RGraph *is equivalent to* $\sum_{G_0:\mathcal{S}_{\leq 0}} \sum_{G_1:G_0\times G_0\to\mathcal{S}_{\leq 0}} \prod_{x:G_0} G_1(x,x)$.

Proof. Using the universal property of a pushout, $\mathsf{RGraph} = \mathcal{S}_{\leq 0}^{\Delta^2} \times_{\mathcal{S}^{\mathbb{I}}} \mathcal{S}_{\leq 0}^{\Delta^2}$ and so repeated application of Theorems 5.12 and 5.15 proves $\mathsf{RGraph} = \sum_{G_0\, G_1:\mathcal{S}_{\leq 0}} \sum_{s\, t:G_1\to G_0} \sum_{r:G_0\to G_1} sr = st$ and the conclusion now follows from a standard argument. □

We isolate Pos $\subseteq$ RGraph as a full subcategory spanned by objects where G_1 is a partial order:

Definition 6.13. $\mathsf{Pos} = \mathsf{RGraph}_\phi$ where $\phi(G) \coloneqq \mathsf{isASym}(G_1)\times\mathsf{isTrans}(G_1)\times\prod_{x,y:G_0}\mathsf{isHProp}(G_1(x,y))$

Theorem 6.6 now proves that homomorphisms in Pos are precisely monotone maps:

Lemma 6.14. *If* $P, Q : \mathsf{Pos}$ *then* $\hom_{\mathsf{Pos}}(P,Q) \simeq \sum_{f:P_0\to Q_0}\prod_{x,y:P_0} P_1(x,y) \to Q_1(fx,fy)$.

Finally, for the next subsection we isolate a category which is foundational to ∞-category theory: the simplex category Δ is the full subcategory Pos_ϕ where $\phi(P) = \sum_{n:\mathsf{Nat}} P = \Delta^n$.

6.4 First steps in synthetic higher algebra

As homotopy (type) theorists like to quip: homotopy types are modern sets. Higher algebra seeks to take this slogan a step further by studying groups, rings, modules, etc. in a world where homotopy types have replaced sets. While higher algebra has numerous applications to algebraic topology, algebraic K-theory, and algebraic geometry, it is also a notoriously technical: even the simplest higher algebraic structure must account for an infinite tower of coherences for each imposed equation. For our final application of $\mathcal{S}$, we initiate the study of *higher algebra* [18, 34] in $\mathsf{TT}_{\boxslash}$ by defining some of the central objects of study. We begin by upgrading defining the category of (homotopy-coherent and untruncated) monoids following Segal [55].

Definition 6.15. The *category of coherent monoids* Monoid_∞ as the full subcategory of $\mathsf{PSh}(\Delta)$ carved out by the following predicate (the Segal condition):

$$\phi(X :_\flat \langle \mathfrak{o} \mid \Delta\rangle \to \mathcal{S}) = (n : \mathsf{Nat}) \to \mathsf{isEquiv}(\langle X(\iota_k)_{k<n}\rangle : X(\Delta^n) \to X(\Delta^1)^n)$$

In the above, $\iota_k : \Delta^1 \to \Delta^n$ is $\lambda i.\,(1,\dots,1,i,0,\dots)$ picking out k copies of 1.

In other words, a coherent monoid is a functor $X : \langle \mathfrak{o} \mid \Delta\rangle \to \mathcal{S}$ such that $X(\Delta^n)$ is the n-fold product of $X(\Delta^1)$. While somewhat indirect, these conditions encode all the necessary structure e.g., *multiplication* is given by the composite map $\mu_X : X(\Delta^1)^2 \simeq X(\Delta^2) \to X(\Delta^1)$.

As a small example of manipulating this definition, we prove the following:

Lemma 6.16. *The functor* $\mathsf{Monoid}_\infty \to \mathcal{S}$ *induced by evaluation at* Δ^1 *is conservative.*

Proof. Given $f : X \to Y$, by Riehl and Shulman [46] and Theorem 6.6, it suffices to show that if $f(\Delta^1)$ is an isomorphism so is $f(\Delta^n)$ for any n. By the Segal condition and naturality, $f(\Delta^n)$ is equivalent to $(f(\Delta^1))_{i\leq n}$ which is invertible if $f(\Delta^1)$ is an isomorphism. □

We can also define the category of coherent *groups*:

Definition 6.17. The *category of coherent groups* Grp_∞ is the full subcategory of Monoid_∞ carved out by the predicate $\phi(X :_\flat \mathsf{Monoid}_\infty) = \mathsf{isEquiv}(\lambda x\,y.(x,\mu(x,y))) : X(\Delta^1)^2 \to X(\Delta^1)^2)$.

These concepts and many others can be unified through the formalism of (∞-)operads but we leave it to future work to develop this apparatus in $\mathsf{TT}_{\boxslash}$. An application of such a formalism would be the ability to develop higher algebra not just in $\mathcal{S}$, but in *spectra*, another fundamental category in modern homotopy theory. We conclude this section by constructing this category.

Suppose C is a pointed category with pullbacks, i.e., C has pullbacks and comes with an element $0 : C$ which is simultaneously initial terminal and initial. Within C, we define the *loop functor* $\Omega : C \to C$ by $\Omega \coloneqq \lambda x.0 \times_x 0$. We have already encountered such a pointed category: $\mathcal{S}_*$.

Definition 6.18. The *category of spectra* Sp is defined as $\lim_{n:\mathsf{Nat}}(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$.

Here lim refers to the ordinary definition of a limit from HoTT and we note that as the limit of categories, Sp is itself automatically a category. Using directed univalence, we can easily show that objects of Sp are infinite deloopings of a groupoid as expected [56, 64].

7 Conclusions and related work

We have introduced $\mathsf{TT}_{\boxslash}$, an enhancement of simplicial type theory featuring modalities and a relaxed interval type. We have used $\mathsf{TT}_{\boxslash}$ as a framework to construct a directed univalent universe of groupoids $\mathcal{S}$ which we have further proven to be a well-behaved category. Finally, we have used $\mathcal{S}$ as a jumping off point to construct numerous examples of categories and categorical reasoning in $\mathsf{TT}_{\boxslash}$ relevant both to ∞-category theory and mechanized verification. In order to do so, we have shown how our same modal operators can be used to e.g., construct full subcategories.

7.1 Related work

While directed type theory generally and simplicial type theory specifically are relatively new areas, there is already substantial work exploring the impact of a "type theory where types are categories." Much of this work focuses on either constructing such type theories [3, 26, 30, 39–42, 46, 66] or studying "formal" category theory within them [10, 35, 46, 70, 71] i.e., statements which do not use particular closed non-trivial categories but instead quantify over arbitrary categories. This is distinct from our focus, which has been to combine essentially off-the-shelf type theories [21, 46] and to use this combination to prove facts about the concrete type $\mathcal{S}$ and types derived thereof. Closely related to this is the work by Cavallo, Riehl, and Sattler [44] and Weaver and Licata [67], who both study directed univalence, in respectively simplicial and *bicubical type theory* (BCTT).

Cavallo, Riehl, and Sattler give an alternative construction of $\mathcal{S}$ in the intended model of STT, similar to the classical proof due to Cisinski [13]. They have argued externally that this subuniverse satisfies directed univalence and a version of Lemma 5.2. However, their work is strictly external and does not consider how one might integrate $\mathcal{S}$ within STT. Since they do not consider a modal extension of STT, they cannot even formulate e.g., Lemma 5.2 within their target type theory nor capitalize on internal arguments to simplify their proofs. However, given that both our universe and theirs satisfy Lemma 5.2, they are weakly equivalent and so our results further show that their universe is e.g., a finitely (co)complete category and closed under various connectives.

On the other hand, Weaver and Licata [67] consider a variant of STT based on two layers of cubical type theory: one to account for homotopy type theory and a further layer for the directed interval. Bicubical type theory is to STT as cubical type theory is to HoTT: it is conjectured that BCTT can be formally presented[12] so as to enjoy canonicity and normalization, but bicubical categories and groupoids are not expected to be adequate for ordinary ∞-categories or ∞-groupoids. Accordingly, Weaver and Licata make a different trade-off than us with a system with better expected computation, but which cannot be used to properly reason about ∞-category theory. We believe both approaches to directed type theory warrant further consideration to (1) study our results on top of base cubical type theory rather than HoTT and (2) to translate our new results to their setting. In particular, op. cit. proves only that $\mathcal{S}$ is directed univalent and does not prove e.g. Theorem 5.15 but we believe our proof, along with those results in Section 6, can be translated.

More fundamentally, while they also work within an internal language and we draw on their overal strategy in Section 5, theirs is the internal *extensional* type theory of $\mathrm{PSh}(\square_{\mathsf{undirected}} \times \square_{\mathsf{directed}})$ and so they must not only constructing $\mathcal{S}$ but also the base HoTT around it. This substantially complicates some of their constructions; their versions of e.g., covariance, Gl and so on include details that are automatically handled when working pervasively with HoTT. This model falsifies Axiom 8 and so they must introduce an additional set of axioms (the cobar modality) work around this. Finally, op. cit. observes the utility of $\mathcal{S}$ for formalization and we are able to provide complete examples of this having proven that $\mathcal{S}$ is a directed univalent category.

[12] Weaver and Licata [67] do not give a definition of BCTT but instead describe the intended model for any such situation. Their model is, however, constructive and so it is conjectured that such a definition would satisfy canonicity.

While not about directed type theory, Myers and Riley [38] also consider a HoTT for simplicial spaces. We drew inspiration for some of our axioms (e.g. Axiom 8) from them and expect their other principles will prove useful to STT. Furthermore, Cherubini et al. [11] formulated a version of Axiom 10 to study synthetic algebraic geometry which led us to its inclusion in $\mathsf{TT}_{\boxslash}$. Finally, Riley [50] presents a type theory with a single amazing right adjoint whose syntax is well-adapted for this situation. We hope that op. cit. can be generalized for $\mathsf{TT}_{\boxslash}$ to yield more usable syntax.

7.2 Future work

We isolate three key directions for future work. First, we wish to extend the experimental proof assistant Rzk [28] with the minimum level of modal reasoning (e.g., at least $\langle \flat \mid - \rangle$, $\langle \sharp \mid - \rangle$ and $\langle \mathfrak{o} \mid - \rangle$) to properly axiomatize and work with $\mathcal{S}$ as constructed in this paper. We hope to then use this to mechanize Section 6. Related to this, we hope to give a constructive model of $\mathsf{TT}_{\boxslash}$ to give a computational justification of our axioms. We expect this to contribute to a version of $\mathsf{TT}_{\boxslash}$ with canonicity and normalization [1, 19].

Second, we intend to generalize our construction of $\mathcal{S}$ to construct the category of (small) categories Cat and prove that it is suitably directed univalent [15]. While modalities were required to construct $\mathcal{S}$, they will be required to *state* the properties of Cat; directed univalence will become $\langle \flat \mid \mathbb{I} \to \mathsf{Cat} \rangle \simeq \left\langle \flat \mid \sum_{A:\mathsf{Cat}} \sum_{B:\mathsf{Cat}} A \to B \right\rangle$ because homomorphisms from A to B must be the *groupoid* of the category of functors $A \to B$, not the category. Aside from this, we believe our results will scale to this more general setting.

Finally, while we discussed presheaf categories in Section 6, we avoided describing the Yoneda embedding $C \times \langle \mathfrak{o} \mid C \rangle \to \mathcal{S}$. While it is possible to construct this operation, it requires one additional modality (the twisted arrow construction) and, for reasons of space, we have regretfully chosen to omit it in the present work. In forthcoming work, we will detail this additional modality along with the resulting definition of the Yoneda embedding. Using this in conjunction with our work on full subcategories, we are able to prove various important results e.g. that $\mathcal{S}$ is cocomplete.

References

[1] Frederik Lerbjerg Aagaard, Magnus Baunsgaard Kristensen, Daniel Gratzer, and Lars Birkedal. 2022. Unifying cubical and multimodal type theory. arXiv:2203.13000 [cs.LO]

[2] Benedikt Ahrens, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. 2022. The Univalence Principle. arXiv:2102.06275 [math.CT] https://arxiv.org/abs/2102.06275

[3] Benedikt Ahrens, Paige Randall North, and Niels van der Weide. 2023. Bicategorical type theory: semantics and syntax. *Mathematical Structures in Computer Science* 33, 10 (10 2023), 868–912. https://doi.org/10.1017/s0960129523000312

[4] Carlo Angiuli, Evan Cavallo, Anders Mörtberg, and Max Zeuner. 2021. Internalizing representation independence with univalence. *Proceedings of the ACM on Programming Languages* 5, POPL (01 2021), 1–30. https://doi.org/10.1145/3434293

[5] Steve Awodey, Álvaro Pelayo, and Michael A Warren. 2013. Voevodsky's univalence axiom in homotopy type theory. *Notices of the AMS* 60, 9 (2013), 1164–1167.

[6] Steve Awodey and Michael A Warren. 2009. Homotopy theoretic models of identity types. In *Mathematical proceedings of the cambridge philosophical society*, Vol. 146. Cambridge University Press, 45–55.

[7] David Ayala and John Francis. 2020. Fibrations of ∞-categories. *Higher Structures* 4, 1 (02 2020), 168–265. https://doi.org/10.21136/hs.2020.05

[8] Ingo Blechschmidt. 2023. A general Nullstellensatz for generalized spaces. https://rawgit.com/iblech/internal-methods/master/paper-qcoh.pdf Draft.

[9] Ulrik Buchholtz. 2019. Higher structures in homotopy type theory. *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts* (2019), 151–172.

[10] Ulrik Buchholtz and Jonathan Weinberger. 2023. Synthetic fibered $(\infty, 1)$-category theory. *Higher Structures* 7 (2023), 74–165. Issue 1. https://doi.org/10.21136/HS.2023.04

[11] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. 2023. A Foundation for Synthetic Algebraic Geometry. arXiv:2307.00073

[12] Denis-Charles Cisinski. 2014. Univalent universes for elegant models of homotopy types. (2014). arXiv:1406.0058

[13] Denis-Charles Cisinski. 2019. *Higher Categories and Homotopical Algebra*. Cambridge University Press. https://doi.org/10.1017/9781108588737

[14] Denis-Charles Cisinski, Bastiaan Cnossen, Kim Nguyen, and Tashi Walde. 2024. Formalization of Higher Categories. https://drive.google.com/file/d/1lKaq7watGGl3xvjqw9qHjm6SDPFJ2-0o/view Notes in progress.

[15] Denis-Charles Cisinski and Hoang Kim Nguyen. 2022. The universal coCartesian fibration. arXiv:2210.08945 [math.CT]

[16] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. 2017. Cubical Type Theory: a constructive interpretation of the univalence axiom. 4, 10 (2017), 3127–3169. arXiv:1611.02108 [cs.LO]

[17] Thierry Coquand and Nils Anders Danielsson. 2013. Isomorphism is equality. *Indagationes Mathematicae* 24, 4 (Nov. 2013), 1105–1120. https://doi.org/10.1016/j.indag.2013.09.002

[18] David Gepner. 2020. An introduction to higher categorical algebra. In *Handbook of homotopy theory*. Chapman and Hall/CRC, 487–548.

[19] Daniel Gratzer. 2022. Normalization for Multimodal Type Theory. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science* (Haifa, Israel) *(LICS '22)*. Association for Computing Machinery, New York, NY, USA, Article 2, 13 pages. https://doi.org/10.1145/3531130.3532398

[20] Daniel Gratzer. 2023. *Syntax and semantics of modal type theory*. Ph. D. Dissertation. Aarhus University.

[21] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. 2021. Multimodal Dependent Type Theory. *Logical Methods in Computer Science* Volume 17, Issue 3 (07 2021). https://doi.org/10.46298/lmcs-17(3:11)2021

[22] Gijs Heuts and Ieke Moerdijk. 2016. Left fibrations and homotopy colimits II. arXiv:1602.01274 [math.AT] https://arxiv.org/abs/1602.01274

[23] André Joyal. 2008. The Theory of Quasi-Categories. Online. https://math.uchicago.edu/~may/PEOPLE/JOYAL/0newqcategories.pdf

[24] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. 2021. The simplicial model of Univalent Foundations (after Voevodsky). *Journal of the European Mathematical Society* 6 (2021), 2071–2126. https://doi.org/10.4171/JEMS/1050

[25] Krzysztof Kapulkin and Vladimir Voevodsky. 2020. A cubical approach to straightening. *Journal of Topology* 13, 4 (2020), 1682–1700. https://doi.org/10.1112/topo.12173

[26] G.A. Kavvos. 2019. A quantum of direction. Online. https://seis.bristol.ac.uk/~tz20861/papers/meio.pdf

[27] Astra Kolomatskaia and Michael Shulman. 2023. Displayed Type Theory and Semi-Simplicial Types. (2023). arXiv:2311.18781 [math.CT]

[28] Nikolai Kudasov. 2023. Rzk. https://github.com/rzk-lang/rzk An experimental proof assistant based on a type theory for synthetic ∞-categories.

[29] Nikolai Kudasov, Emily Riehl, and Jonathan Weinberger. 2004. Formalizing the ∞-Categorical Yoneda Lemma. In *Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs*. 274–290. https://doi.org/10.1145/3636501.3636945

[30] Daniel R. Licata and Robert Harper. 2011. 2-Dimensional Directed Type Theory. *Electronic Notes in Theoretical Computer Science* 276 (09 2011), 263–289. https://doi.org/10.1016/j.entcs.2011.09.026

[31] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. 2018. Internal Universes in Models of Homotopy Type Theory. In *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 108)*, Hélène Kirchner (Ed.). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 22:1–22:17. https://doi.org/10.4230/LIPIcs.FSCD.2018.22

[32] Daniel R. Licata, Michael Shulman, and Mitchell Riley. 2017. A Fibrational Framework for Substructural and Modal Logics. In *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 84)*, Dale Miller (Ed.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 25:1–25:22. https://doi.org/10.4230/LIPIcs.FSCD.2017.25

[33] Jacob Lurie. 2009. *Higher Topos Theory*. Princeton University Press.

[34] Jacob Lurie. 2017. Higher Algebra. https://www.math.ias.edu/~lurie/papers/HA.pdf Book draft.

[35] César Bardomiano Martínez. 2023. Limits and exponentiable functors in simplicial homotopy type theory. arXiv:2202.12386 [math.CT] https://arxiv.org/abs/2202.12386

[36] Louis Martini. 2022. Yoneda's lemma for internal higher categories. arXiv:2103.17141 [math.CT]

[37] Louis Martini and Sebastian Wolf. 2023. Internal higher topos theory. arXiv:2303.06437

[38] David Jaz Myers and Mitchell Riley. 2023. Commuting Cohesions. arXiv:2301.13780 [math.CT]

[39] Jacob Neumann and Thorsten Altenkirch. 2024. The Category Interpretation of Directed Type Theory. Online. https://jacobneu.github.io/research/preprints/catModel-2024.pdf

[40] Paige Randall North. 2018. Towards a directed homotopy type theory. arXiv:1807.10566 [cs.LO] https://arxiv.org/abs/1807.10566

[41] Andreas Nuyts. 2015. *Towards a Directed Homotopy Type Theory based on 4 Kinds of Variance*. Master's thesis. KU Leuven.

[42] Andreas Nuyts. 2020. A Vision for Natural Type Theory. https://anuyts.github.io/files/nattt-vision.pdf

[43] Charles Rezk. 2001. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.* 353, 3 (2001), 973–1007. https://doi.org/10.1090/S0002-9947-00-02653-2
[44] Emily Riehl. 2018. On the directed univalence axiom. Online. https://math.jhu.edu/~eriehl/JMM2018-directed-univalence.pdf Talk on work joint with Evan Cavallo and Christian Sattler..
[45] Emily Riehl. 2023. Could ∞-Category Theory Be Taught to Undergraduates? *Notices of the American Mathematical Society* 70, 5 (2023). https://doi.org/10.1090/noti2692
[46] Emily Riehl and Michael Shulman. 2017. A type theory for synthetic ∞-categories. *Higher Structures* 1 (2017), 147–224. Issue 1. https://doi.org/10.21136/HS.2017.06
[47] Emily Riehl and Dominic Verity. 2022. *Elements of ∞-Category Theory.* Cambridge University Press. https://doi.org/10.1017/9781108936880
[48] Egbert Rijke. 2018. *Classifying Types.* Ph. D. Dissertation. Carnegie Mellon University.
[49] Egbert Rijke, Michael Shulman, and Bas Spitters. 2020. Modalities in homotopy type theory. *Logical Methods in Computer Science* 16, 1 (2020). https://doi.org/10.23638/LMCS-16(1:2)2020
[50] Mitchell Riley. 2024. A Type Theory with a Tiny Object. arXiv:2403.01939 [math.CT]
[51] Christian Sattler. 2017. The Equivalence Extension Property and Model Structures. arXiv:1704.06911 [math.CT] https://arxiv.org/abs/1704.06911
[52] Christian Sattler. 2019. Idempotent completion of cubes in posets. arXiv:1805.04126 [math.CT]
[53] Urs Schreiber. 2013. Differential cohomology in a cohesive infinity-topos. (2013). arXiv:1310.7930 https://ncatlab.org/schreiber/files/dcct170811.pdf
[54] Urs Schreiber and Michael Shulman. 2012. Quantum Gauge Field Theory in Cohesive Homotopy Type Theory. In *Proceedings 9th Workshop on Quantum Physics and Logic, QPL 2012, Brussels, Belgium, 10-12 October 2012 (EPTCS, Vol. 158)*, Ross Duncan and Prakash Panangaden (Eds.). 109–126. https://doi.org/10.4204/EPTCS.158.8
[55] Graeme Segal. 1974. Categories and cohomology theories. *Topology* 13, 3 (09 1974), 293–312. https://doi.org/10.1016/0040-9383(74)90022-6
[56] Michael Shulman. 2013. Cohomology. Online. https://homotopytypetheory.org/2013/07/24/cohomology/
[57] Michael Shulman. 2015. The Univalence Axiom for Elegant Reedy Presheaves. *Homology, Homotopy and Applications* 17 (2015), 81–106. Issue 2. https://doi.org/10.4310/HHA.2015.v17.n2.a6
[58] Michael Shulman. 2018. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *Mathematical Structures in Computer Science* 28, 6 (2018), 856–941. https://doi.org/10.1017/S0960129517000147
[59] Michael Shulman. 2019. All $(\infty, 1)$-toposes have strict univalent universes. arXiv:1904.07004 [math.AT]
[60] Michael Shulman. 2023. Semantics of multimodal adjoint type theory. *Electronic Notes in Theoretical Informatics and Computer Science* Volume 3-Proceedings of MFPS XXXIX (11 2023). https://doi.org/10.46298/entics.12300
[61] Bas Spitters. 2016. Cubical sets and the topological topos. arXiv:1610.05270 [cs.LO]
[62] Thomas Streicher and Jonathan Weinberger. 2021. Simplicial sets inside cubical sets. *Theory and Application of Categories* 37, 10 (2021), 276–286.
[63] The Univalent Foundations Program. 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics.* Institute for Advanced Study. https://homotopytypetheory.org/book
[64] Floris van Doorn. 2018. *On the Formalization of Higher Inductive Types and Synthetic Homotopy Theory.* Ph. D. Dissertation. Carnegie Mellon University.
[65] Philip Wadler. 1989. Theorems for free!. In *Functional Programming Languages and Computer Architecture.* ACM Press, 347–359.
[66] Michael Warren. 2013. Directed type theory. Online. https://www.ias.edu/video/univalent/1213/0410-MichaelWarren Seminar talk.
[67] Matthew Z. Weaver and Daniel R. Licata. 2020. A Constructive Model of Directed Univalence in Bicubical Sets. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS '20).* ACM. https://doi.org/10.1145/3373718.3394794
[68] Jonathan Weinberger. 2022. Strict stability of extension types. https://arxiv.org/abs/2203.07194
[69] Jonathan Weinberger. 2024. Generalized Chevalley criteria in simplicial homotopy type theory. arXiv:2403.08190 [math.CT]
[70] Jonathan Weinberger. 2024. Internal sums for synthetic fibered $(\infty, 1)$-categories. *Journal of Pure and Applied Algebra* 228, 9 (09 2024), 107659. https://doi.org/10.1016/j.jpaa.2024.107659
[71] Jonathan Weinberger. 2024. Two-sided cartesian fibrations of synthetic $(\infty, 1)$-categories. *Journal of Homotopy and Related Structures* 19, 2 (06 2024). https://doi.org/10.1007/s40062-024-00348-3