Principles of Dependent Type Theory

Carlo Angiuli
Indiana University

Daniel Gratzer
Aarhus University

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1 *Introduction*

In these lecture notes, we aim to introduce the reader to a modern research perspective on the design of "full-spectrum" dependent type theories. At the end of this course, readers should be prepared to engage with contemporary research papers on dependent type theory, and to understand the motivations behind recent extensions of Martin-Löf's dependent type theory [ML84], including observational type theory [AMS07], homotopy type theory [UF13], and cubical type theory [CCHM18; Ang+21].

These lecture notes are in an early draft form and are missing many relevant citations. The authors welcome any feedback.

Dependent type theory (henceforth just *type theory*) often appears arcane to outside observers for a handful of reasons. First, as in the parable of the elephant, there are myriad perspectives on type theory. The language presented in these lecture notes, *mutatis mutandis*, can be accurately described as:

- the core language of assertions and proofs in *proof assistants* like Agda [Agda], Coq [Coq], Lean [MU21], and Nuprl [Con+85];
- a richly-typed *functional programming language*, as in Idris [Bra13] and Pie [FC18], as well as in the aforementioned proof assistants Agda and Lean [Chr23].
- an *axiom system* for reasoning synthetically in a number of mathematical settings, including locally cartesian closed 1-categories [Hof95], homotopy types [Shu21], and Grothendieck ∞-topoi [Shu19];
- a structural [Tse17], constructive [ML82] *foundation for mathematics* as an alternative to ZFC set theory [Alt23].

A second difficulty is that it is quite complex to even *define* type theory in a precise fashion, for reasons we shall discuss in Section 2.2, and the relative merits of different styles of definition—and even which ones satisfactorily define any object whatsoever—have been the subject of great debate among experts over the years.

Finally, much of the literature on type theory is highly technical—involving either lengthy proofs by induction or advanced mathematical machinery—in order to account for its complex definition and applications. In these lecture notes we attempt to split the difference by presenting a mathematically-informed viewpoint on type theory while avoiding advanced mathematical prerequisites.

Goals of the course As researchers who work on designing new type theories, our goal in this course is to pose and begin to answer the following questions: *What makes a good type theory, and why are there so many?* We will focus on *notions of equality in Martin-Löf type theory* as a microcosm of this broader question, studying how extensional [ML82], intensional [ML75], observational [AMS07; SAG22; PT22], homotopy [UF13], and cubical type theories [CCHM18; Ang+21] have provided increasingly sophisticated answers to this deceptively simple question.

In this chapter In Section 1.1 we introduce and motivate the concepts of type and term dependency, definitional equality, and propositional equality through the lens of typed functional programming. Note that Chapter 2 is self-contained albeit lacking in motivation, so readers unfamiliar with functional programming can safely skip ahead.

Goals of the chapter By the end of this chapter, you will be able to:

- Give examples of full-spectrum dependency.
- Explain the role of definitional equality in type-checking, and how and why it differs from ordinary closed-term evaluation.
- Explain the role of propositional equality in type-checking.

1.1 Dependent types for functional programmers

The reader is forewarned that the following section assumes some familiarity with functional programming, unlike the remainder of the lecture notes.

Types in programming languages For the purposes of this course, one should regard a programming language's (static) type system as its *grammar*, not as one of many potential static analyses that might be enabled or disabled.[1] Indeed, just as a parser may reject as nonsense a program whose parentheses are mismatched, or an untyped language's interpreter may reject as nonsense a program containing unbound identifiers, a type-checker may reject as nonsense the program `1 + "hi"` on the grounds that—much like the previous two examples—there is no way to successfully evaluate it.

Concretely, a type system divides a language's well-parenthesized, well-scoped expressions into a collection of sets: the *expressions of type* **Nat** are those that "clearly" compute natural numbers, such as literal natural numbers (0, 1, 120), arithmetic expressions (1 + 1),

[1] The latter perspective is valid, but we wish to draw a sharp distinction between types *qua* (structural) grammar, and static analyses that may be non-local, non-structural, or non-substitutive in nature.

and fully-applied functions that return natural numbers (fact `5`, atoi `"120"`). Similarly, the expressions of type **String** are those that clearly compute strings (`"hi"`, itoa `5`), and for any types A and B, the expressions of type $A \to B$ are those that clearly compute functions that, when passed an input of type A, clearly compute an output of type B.

What do we mean by "clearly"? One typically insists that type-checking be fully automated, much like parsing and identifier resolution. Given that determining the result of a program is in general undecidable, any automated type-checking process will necessarily compute a conservative underapproximation of the set of programs that compute (e.g.) natural numbers. (Likewise, languages may complain about unbound identifiers even in programs that can be evaluated without a runtime error!)

The goal of a type system is thus to rule out as many undesirable programs as possible without ruling out too many desirable ones, where both of these notions are subjective depending on which runtime errors one wants to rule out and which programming idioms one wants to support. Language designers engage in the neverending process of refining their type systems to rule out more errors and accept more correct code; full-spectrum dependent types can be seen as an extreme point in this design space.

1.1.1 *Uniform dependency: length-indexed vectors*

Every introduction to dependent types starts with the example of vectors, or lists with specified length. We start one step earlier by considering lists with a specified type of elements, a type which already exhibits a basic form of dependency.

Parameterizing by types One of the most basic data structures in functional programming languages is the *list*, which is either empty (written []) or consists of an element x adjoined to a list xs (written $x :: xs$). In typed languages, we typically require that a list's elements all have the same type so that we know what operations they support.

The simplest way to record this information is to have a separate type of lists for each type of element: a **ListOfNats** is either empty or a **Nat** and a **ListOfNats**, a **ListOfStrings** is either empty or a **String** and a **ListOfStrings**, etc. This strategy clearly results in repetition at the level of the type system, but it also causes code duplication because operations that work uniformly for any type of elements (e.g., reversing a list) must be defined twice for the two apparently unrelated types **ListOfNats** and **ListOfStrings**.

In much the same way that functions—terms indexed by terms—promote code reuse by allowing programmers to write a series of operations once and perform them on many different inputs, we can solve both problems described above by allowing types and terms to be uniformly parameterized by types. Thus the types **ListOfNats** and **ListOfStrings** become two instances (**List Nat** and **List String**) of a single family of types **List**:[2]

[2]For the time being, the reader should understand A : **Set** as notation meaning "A is a type."

```
data List (A : Set) : Set where
  [] : List A
  _::_ : A → List A → List A
```

and any operation that works for all element types A, such as returning the first (or all but first) element of a list, can be written as a family of operations:

```
head : (A : Set) → List A → A
head A [] = error "List must be non-empty."
head A (x :: xs) = x

tail : (A : Set) → List A → List A
tail A [] = error "List must be non-empty."
tail A (x :: xs) = xs
```

By partially applying head to its type argument, we see that head **Nat** has type **List Nat** → **Nat** and head **String** has type **List String** → **String**, and the expression 1 + (head **Nat** (1 :: [])) has type **Nat** whereas 1 + (head **String** ("hi" :: [])) is ill-typed because the second input to + has type **String**.

Parameterizing types by terms The perfectionist reader may find the **List** A type unsatisfactory because it does not prevent runtime errors caused by applying head and tail to the empty list []. We cannot simply augment our types to track which lists are empty, because 2 :: 1 :: [] and 1 :: [] are both nonempty but we can apply tail **Nat** twice to the former before encountering an error, but only once to the latter.

Instead, we parameterize the type of lists not only by their type of elements as before but also by their length—a *term* of type **Nat**—producing the following family of types:[3]

```
data Vec (A : Set) : Nat → Set where
  [] : Vec A 0
  _::_ : {n : Nat} → A → Vec A n → Vec A (suc n)
```

Types parameterized by terms are known as *dependent types*.

Now the types of concrete lists are more informative—(2 :: 1 :: []) : **Vec Int** 2 and (1 :: []) : **Vec Int** 1—but more importantly, we can give head and tail more informative types which rule out the runtime error of applying them to empty lists. We do so by revising their input type to **Vec** A (suc n) for some n : **Nat**, which is to say that the vector has length at least one, hence is nonempty:

[3] Curly braces {n : **Nat**} indicate *implicit* arguments automatically inferred by the type-checker.

```
head : {A : Set} {n : Nat} → Vec A (suc n) → A
-- head [] is impossible
head (x :: xs) = x

tail : {A : Set} {n : Nat} → Vec A (suc n) → Vec A n
-- tail [] is impossible
tail (x :: xs) = xs
```

Consider now the operation that concatenates two vectors:

```
append : {A : Set} {n : Nat} {m : Nat} → Vec A n → Vec A m → Vec A (n + m)
```

Unlike our previous examples, the output type of this function is indexed not by a variable A or n, nor a constant **Nat** or 0, nor even a constructor suc $-$, but by an *expression* $n + m$. This introduces a further complication, namely that we would like this expression to be simplified as soon as n and m are known. For example, if we apply append to two vectors of length one ($n = m = 1$), then the result will be a vector of length two ($n + m = 1 + 1 = 2$), and we would like the type system to be aware of this fact in the sense of accepting as well-typed the expression head (tail (append l l')) for l and l' of type **Vec Nat** 1.

Because head (tail x) is only well-typed when x has type **Vec** A (suc (suc n)) for some n : **Nat**, this condition amounts to requiring that the expression append l l' not only has type **Vec** A ((suc 0) + (suc 0)) as implied by the type of append, but also type **Vec** A (suc (suc 0)) as implied by its runtime behavior. In short, we would like the two type expressions **Vec** A $(1 + 1)$ and **Vec** A 2 to *denote the same type* by virtue of the fact that $1 + 1$ and 2 *denote the same value*. In practice, we achieve this by allowing the type-checker to *evaluate expressions in types during type-checking*.

In fact, the length of a vector can be any expression whatsoever of type **Nat**. Consider filter, which takes a function $A \to$ **Bool** and a list and returns the sublist for which the function returns true. If the input list has length n, what is the length of the output?

```
filter : {A : Set} {n : Nat} → (A → Bool) → Vec A n → Vec A ?
```

After a moment's thought we realize the length is not a function of n at all, but rather a recursive function of the input function and list:

```
filter : {A : Set} {n : Nat} → (f : A → Bool) → (l : Vec A n) → Vec A (filterLen f l)

filterLen : {A : Set} {n : Nat} → (A → Bool) → Vec A n → Nat
filterLen f [] = 0
filterLen f (x :: xs) = if f(x) then suc (filterLen f xs) else filterLen f xs
```

As before, once f and l are known the type of filter $f\ l$: **Vec** A (filterLen $f\ l$) will simplify by evaluating filterLen $f\ l$, but as long as either remains a variable we cannot learn much by computation. Nevertheless, filterLen has many properties of interest: filterLen $f\ l$ is at most the length of l, filterLen $(\lambda x \to \mathsf{false})\ l$ is always 0 regardless of l, etc. We will revisit this point in Section 1.1.3.

Remark 1.1.1. If we regard **Nat** and + as a user-defined data type and recursive function, as type theorists are wont to do, then filter's type using filterLen is entirely analogous to append's type using +. We wish to emphasize that, whereas one could easily imagine arithmetic being a privileged component of the type system, filter demonstrates that type indices may need to contain arbitrary user-defined recursive functions. ◇

Another approach? If our only goal was to eliminate runtime errors from head and tail, we might reasonably feel that dependent types have overcomplicated the situation—we needed to introduce a new function just to write the type of filter! And indeed there are simpler ways of keeping track of the length of lists, which we describe briefly here.

First let us observe that a lower bound on a list's length is sufficient to guarantee it is nonempty and thus that an application of head or tail will succeed; this allows us to trade precision for simplicity by restricting type indices to be arithmetic expressions. Secondly, in the above examples we can perform type-checking and "length-checking" in two separate phases, where the first phase replaces every occurrence of **Vec** $A\ n$ with **List** A before applying a standard non-dependent type-checking algorithm. This is possible because we can regard the dependency in **Vec** $A\ n$ as expressing a computable *refinement*—or subset—of the non-dependent type of lists, namely $\{l : \mathbf{List}\ A \mid \mathsf{length}\ l = n\}$.

Combining these insights, we can by and large automate length-checking by recasting the type dependency of **Vec** in terms of arithmetic inequality constraints over an ML-style type system, and checking these constraints with SMT solvers and other external tools. At a very high level, this is the approach taken by systems such as Dependent ML [Xi07] and Liquid Haskell [Vaz+14]. Dependent ML, for instance, type-checks the usual definition of filter at the following type, without any auxiliary filterLen definition:

$$\mathsf{filter} : \mathbf{Vec}\ A\ m \to (\{n : \mathbf{Nat} \mid n \leq m\} \times \mathbf{Vec}\ A\ n)$$

Refinement type systems like these have proven very useful in practice and continue to be actively developed, but we will not discuss them any further for the simple reason that, although they are a good solution to head/tail and many other examples, they cannot handle full-spectrum dependency as discussed below.

1.1.2 *Non-uniform dependency: computing arities*

Thus far, all our examples of (type- or term-) parameterized types are *uniformly* parameterized, in the sense that the functions **List** : **Set** → **Set** and **Vec** A : **Nat** → **Set** do not inspect their arguments; in contrast, ordinary term-level functions out of **Nat** such as fact : **Nat** → **Nat** can and usually do perform case-splits on their inputs. In particular, we have not yet considered any families of types in which the head, or top-level, type constructor (→, **Vec**, **Nat**, etc.) differs between indices.

A type theory is said to have full-spectrum dependency if it permits *non-uniformly term-indexed* families of types, as in the following example:

```
nary : Set → Nat → Set
nary A 0 = A
nary A (suc n) = A → nary A n
```

Although **Vec Nat** and nary **Nat** are both functions **Nat** → **Set**, the latter's head type constructor varies between indices: nary **Nat** 0 = **Nat** but nary **Nat** 1 = **Nat** → **Nat**.

Using nary to compute the type of n-ary functions, we can now define not only varadic functions but even higher-order functions taking variadic functions as input, such as apply which applies an n-ary function to a vector of length n:

```
apply : {A : Set} {n : Nat} → nary A n → Vec A n → A
apply x [] = x
apply f (x :: xs) = apply (f x) xs
```

For A = **Nat** and n = 1, apply applies a unary function **Nat** → **Nat** to the head element of a **Vec Nat** 1; for A = **Nat** and n = 3, it applies a ternary function **Nat** → **Nat** → **Nat** → **Nat** to the elements of a **Vec Nat** 3:

```
apply suc (1 :: []) : Nat    -- evaluates to 2
apply _+_ : Vec Nat 2 → Nat
apply _+_ (1 :: 2 :: []) : Nat    -- evaluates to 3
apply (λx y z → x + y + z) (1 :: 2 :: 3 :: []) : Nat    -- evaluates to 6
```

Although apply is not the first time we have seen a function whose type involves a different recursive function—we saw this already with filter—this is our first example of a function that cannot be straightforwardly typed in an ML-style type system. Another way to put it is that nary A n → **Vec** A n → A is not the refinement of an ML type because nary A n is sometimes but not always a function type.

Remark 1.1.2. For the sake of completeness, it is also possible to consider *non-uniformly type-indexed* families of types, which go by a variety of names including non-parametric polymorphism, intensional type analysis, and typecase [HM95]. These often serve as

optimized implementations of uniformly type-indexed families of types; a classic non-type-theoretic example is the C++ family of types `std::vector` for dynamically-sized arrays, whose `std::vector<bool>` instance may be compactly implemented using bitfields. ◇

To understand the practical ramifications of non-uniform dependency, we will turn our attention to a more complex example: a basic implementation of sprintf in Agda (Figure 1.1). This function takes as input a **String** containing format specifiers such as `%u` (indicating a **Nat**) or `%s` (indicating a **String**), as well as additional arguments of the appropriate type for each format specifier present, and returns a **String** in which each format specifier has been replaced by the corresponding argument rendered as a **String**.

```
sprintf "%s %u" "hi" 2 : String   -- evaluates to "hi 2"
sprintf "%s" : String → String
sprintf "nat %u then int %d then char %c" : Nat → Int → Char → String
sprintf "%u" 5 : String   -- evaluates to "5"
sprintf "%u%% of %s%c" 3 "GD" 'P' : String   -- evaluates to "3% of GDP"
```

Our implementation uses various types and functions imported from Agda's standard library, notably toList : **String** → **List Char** which converts a string to a list of characters (length-one strings `'x'`). It consists of four main components:

- a data type Token which enumerates all relevant components of the input **String**, namely format specifiers (such as natTok : Token for `%u` and strTok : Token for `%s`) and literal characters (char `'x'` : Token);
- a function lex which tokenizes the input string, represented as a **List Char**, from left to right into a **List** Token for further processing;
- a function args which converts a **List** Token into a function type containing the additional arguments that sprintf must take; and
- the sprintf function itself.

Let us begin by convincing ourselves that our first example type-checks:

```
sprintf "%s %u" "hi" 2 : String   -- evaluates to "hi 2"
```

Because sprintf : (*s* : **String**) → printfType *s*, the partial application sprintf `"%s %u"` has type printfType `"%s %u"`. By evaluation, the type-checker can see printfType `"%s %u"` = args (strTok :: char `' '` :: natTok :: []) = **String** → **Nat** → **String**. Thus sprintf `"%s %u"` : **String** → **Nat** → **String**, and the remainder of the expression type-checks easily.

Now let us consider the definition of sprintf, which uses a helper function loop : (*toks* : **List** Token) → **String** → args *toks* whose first argument stores the Tokens yet to

```
data Token : Set where
  char : Char → Token
  intTok : Token
  natTok : Token
  chrTok : Token
  strTok : Token

lex : List Char → List Token
lex [] = []
lex ('%' :: '%' :: cs) = char '%' :: lex cs
lex ('%' :: 'd' :: cs) = intTok :: lex cs
lex ('%' :: 'u' :: cs) = natTok :: lex cs
lex ('%' :: 'c' :: cs) = chrTok :: lex cs
lex ('%' :: 's' :: cs) = strTok :: lex cs
lex (c :: cs) = char c :: lex cs

args : List Token → Set
args [] = String
args (char _ :: toks) = args toks
args (intTok :: toks) = Int → args toks
args (natTok :: toks) = Nat → args toks
args (chrTok :: toks) = Char → args toks
args (strTok :: toks) = String → args toks

printfType : String → Set
printfType s = args (lex (toList s))

sprintf : (s : String) → printfType s
sprintf s = loop (lex (toList s)) ""
  where
  loop : (toks : List Token) → String → args toks
  loop [] acc = acc
  loop (char c :: toks) acc = loop toks (acc ++ fromList (c :: []))
  loop (intTok :: toks) acc = λi → loop toks (acc ++ showInt i)
  loop (natTok :: toks) acc = λn → loop toks (acc ++ showNat n)
  loop (chrTok :: toks) acc = λc → loop toks (acc ++ fromList (c :: []))
  loop (strTok :: toks) acc = λs → loop toks (acc ++ s)
```

Figure 1.1: A basic Agda implementation of sprintf.

be processed, and whose second argument is the **String** accumulated from printing the already-processed Tokens. What is needed to type-check the definition of loop? We can examine a representative case in which the next Token is natTok:

$$\mathsf{loop}\ (\mathsf{natTok} :: \mathit{toks})\ \mathit{acc} = \lambda n \longrightarrow \mathsf{loop}\ \mathit{toks}\ (\mathit{acc} \mathbin{++} \mathsf{showNat}\ n)$$

Note that *toks* : **List** Token and *acc* : **String** are (pattern) variables, and the right-hand side ought to have type args (natTok :: *toks*). We can type-check the right-hand side—given that _++_ : **String** → **String** → **String** is string concatenation and showNat : **Nat** → **String** prints a natural number—and observe that it has type **Nat** → args *toks* by the type of loop.

Type-checking this clause thus requires us to reconcile the right-hand side's expected type args (natTok :: *toks*) with its actual type **Nat** → args *toks*. Although these type expressions are quite dissimilar—one is a function type and the other is not—the definition of args contains a promising clause:

$$\mathsf{args}\ (\mathsf{natTok} :: \mathit{toks}) = \mathbf{Nat} \longrightarrow \mathsf{args}\ \mathit{toks}$$

As in our earlier example of **Vec** *A* (1+1) and **Vec** *A* 2 we would like the type expressions args (natTok :: *toks*) and **Nat** → args *toks* to denote the same type, but unlike the equation 1 + 1 = 2, here both sides contain a free variable *toks* so we cannot appeal to evaluation, which is a relation on *closed* terms (ones with no free variables).

One can nevertheless imagine some form of *symbolic evaluation* relation that extends evaluation to open terms and *can* equate these two expressions. In this particular case, this step of closed evaluation is syntactically indifferent to the value of *toks* and thus can be safely applied even when *toks* is a variable. (Likewise, to revisit an earlier example, the equation filterLen f [] = 0 should hold even for variable f.)

Thus we would like the type expressions args (natTok :: *toks*) and **Nat** → args *toks* to denote the same type by virtue of the fact that they *symbolically evaluate to the same symbolic value*, and to facilitate this we must allow the type-checker to *symbolically evaluate* expressions in types during type-checking. The congruence relation on expressions so induced is known as *definitional equality* because it contains defining clauses like this one.

Remark 1.1.3. Semantically we can justify this equation by observing that for any closed instantiation toks of *toks*, args (natTok :: toks) and **Nat** → args toks will evaluate to the same type expression—at least, once we have defined evaluation of type expressions—and thus this equation always holds at runtime. But just as (for reasons of decidability) the condition "when this expression is applied to a natural number it evaluates to a natural number" is a necessary but not sufficient condition for type-checking at **Nat** → **Nat**, we do not want to take this semantic condition as the definition of definitional equality. It is however a necessary condition assuming that the type system is sound for the given evaluation semantics. ◇

add forward ref to Ch 3

Definitional equality is the central concept in full-spectrum dependent type theory because it determines which types are equal and thus which terms have which types. In practice, it is typically defined as the congruence closure of the β-like reductions (also known as $\beta\delta\zeta\iota$-reductions) plus η-equivalence at some types; see Chapter 2 for details.

1.1.3 *Proving type equations*

Unfortunately, in light of Remark 1.1.3, there are many examples of type equations that are not direct consequences of ordinary or even symbolic evaluation. On occasion these equations are of such importance that researchers may attempt to make them definitional—that is, to include them in the definitional equality relation and adjust the type-checking algorithm accordingly [AMB13]. But such projects are often major research undertakings, and there are even examples of equations that can be definitional but are in practice best omitted due to efficiency or usability issues [Alt+01].

Let us turn once again to the example of filter from Section 1.1.2.

$$\mathsf{filter} : \{A : \mathbf{Set}\}\ \{n : \mathbf{Nat}\} \to (f : A \to \mathbf{Bool}) \to (l : \mathbf{Vec}\ A\ n) \to \mathbf{Vec}\ A\ (\mathsf{filterLen}\ f\ l)$$

$$\begin{aligned}&\mathsf{filterLen} : \{A : \mathbf{Set}\}\ \{n : \mathbf{Nat}\} \to (A \to \mathbf{Bool}) \to \mathbf{Vec}\ A\ n \to \mathbf{Nat}\\ &\mathsf{filterLen}\ f\ [\,] = 0\\ &\mathsf{filterLen}\ f\ (x :: xs) = \mathbf{if}\ f(x)\ \mathbf{then}\ \mathsf{suc}\ (\mathsf{filterLen}\ f\ xs)\ \mathbf{else}\ \mathsf{filterLen}\ f\ xs\end{aligned}$$

Suppose for the sake of argument that we want the operation of filtering an arbitrary vector by the constantly false predicate to return a $\mathbf{Vec}\ A\ 0$:

$$\begin{aligned}&\mathsf{filterAll} : \{A : \mathbf{Set}\}\ \{n : \mathbf{Nat}\} \to \mathbf{Vec}\ A\ n \to \mathbf{Vec}\ A\ 0\\ &\mathsf{filterAll}\ l = \mathsf{filter}\ (\lambda x \to \mathsf{false})\ l \quad \text{-- does not type-check}\end{aligned}$$

The right-hand side above has type $\mathbf{Vec}\ A\ (\mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ l)$ rather than $\mathbf{Vec}\ A\ 0$ as desired, and in this case the expression $\mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ l$ cannot be simplified by (symbolic) evaluation because filterLen computes by recursion on l which is a variable. However, by induction on the possible instantiations of $l : \mathbf{Vec}\ A\ n$, either:

- $l = [\,]$, in which case $\mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ [\,]$ is definitionally equal (in fact, evaluates) to 0; or
- $l = x :: xs$, in which case we have the definitional equalities

$$\begin{aligned}&\mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ (x :: xs)\\ &= \mathbf{if}\ \mathsf{false}\ \mathbf{then}\ \mathsf{suc}\ (\mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ xs)\ \mathbf{else}\ \mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ xs\\ &= \mathsf{filterLen}\ (\lambda x \to \mathsf{false})\ xs\end{aligned}$$

for any x and xs. By the inductive hypothesis on xs, filterLen $(\lambda x \to \text{false})$ $xs = 0$ and thus filterLen $(\lambda x \to \text{false})$ $(x :: xs) = 0$ as well.

By adding a type of *provable equations* $a \equiv b$ to our language, we can compactly encode this inductive proof as a recursive function computing filterLen $(\lambda x \to \text{false})$ $l \equiv 0$:

```
_≡_ : {A : Set} → A → A → Set
refl : {A : Set} {x : A} → x ≡ x

lemma : {A : Set} {n : Nat} → (l : Vec A n) → filterLen (λl → false) l ≡ 0
lemma [] = refl
lemma (x :: xs) = lemma xs
```

The [] clause of lemma ought to have type filterLen $(\lambda l \to \text{false})$ [] $\equiv 0$, which is definitionally equal to the type $0 \equiv 0$ and thus **refl** type-checks. The $(x :: xs)$ clause must have type filterLen $(\lambda l \to \text{false})$ $(x :: xs) \equiv 0$, which is definitionally equal to filterLen $(\lambda l \to \text{false})$ $xs \equiv 0$, the expected type of the recursive call lemma xs.

Now armed with a function lemma that constructs for any l : **Vec** A n a proof that filterLen $(\lambda l \to \text{false})$ $l \equiv 0$, we can justify *casting* from the type **Vec** A (filterLen $(\lambda l \to$ false) l) to **Vec** A 0. The dependent casting operation that passes between provably equal indices of a dependent type (in this case **Vec** A : **Nat** $\to$ **Set**) is typically called **subst**:

```
subst : {A : Set} {x y : A} → (P : A → Set) → x ≡ y → P(x) → P(y)

filterAll : {A : Set} {n : Nat} → Vec A n → Vec A 0
filterAll {A} l = subst (Vec A) (lemma l) (filter (λx → false) l)
```

Remark 1.1.4. The **subst** operation above is a special case of a much stronger principle stating that the two types $P(x)$ and $P(y)$ are *isomorphic* whenever $x \equiv y$: we can not only cast $P(x) \to P(y)$ but also $P(y) \to P(x)$ by symmetry of equality, and both round trips cancel. So although a proof $x \equiv y$ does not make $P(x)$ and $P(y)$ definitionally equal, they are nevertheless equal in the sense of having the same elements up to isomorphism. ◇

Uses of **subst** are very common in dependent type theory; because dependently-typed functions can both require and ensure complex invariants, one must frequently prove that the output of some function is a valid input to another.[4] Crucially, although **subst** is an "escape hatch" that compensates for the shortcomings of definitional equality, it cannot result in runtime errors—unlike explicit casts in most programming languages—because casting from $P(x)$ to $P(y)$ requires a machine-checked proof that $x \equiv y$. We can ask

[4] A more realistic variant of our lemma might account for any predicate that returns false on all the elements of the given list, not just the constantly false predicate. Alternatively, one might prove that for any s : **String**, the final return type of sprintf s is **String**.

for such proofs because dependent type theory is not only a functional programming language but also a higher-order intuitionistic logic that can express inductive proofs of type equality, and as we saw with filterAll, its type-checker serves also as a proof-checker.

The dependent type $x \equiv y$ is known as *propositional equality*, and it is perhaps the second most important concept in dependent type theory because it is the source of all non-definitional type equations visible within the theory. There are many formulations of propositional equality; they all implement _≡_, **refl**, and **subst** but differ in many other respects, and each has unique benefits and drawbacks. We will discuss propositional equality at length in Chapters 3 and 5.

To foreshadow the design space of propositional equality, consider that the **subst** operator may itself be subject to various definitional equalities. If we apply filterAll to a closed list *ls*, then lemma *ls* will evaluate to **refl**, so filterAll *ls* is definitionally equal to **subst** (**Vec** A) **refl** (filter ($\lambda x \to$ false) *ls*). At this point, filter ($\lambda x \to$ false) *ls* already has the desired type **Vec** A 0 because filterLen ($\lambda x \to$ false) *ls* evaluates to 0, and thus the two types involved in the cast are now definitionally equal. Ideally the **subst** term would now disappear having completed its job, and indeed the corresponding definitional equality **subst** P **refl** $x = x$ does hold for many versions of propositional equality.

Further reading

Our four categories of dependency—types/terms depending on types/terms—are reminiscent of the λ-*cube* of generalized type systems in which one augments the simply-typed λ-calculus (whose functions exhibit term-on-term dependency) with any combination of the remaining three forms of dependency [Bar91]; adding all three yields the full-spectrum dependent type theory known as the calculus of constructions [CH88]. However, the technical details of this line of work differ significantly from our presentation in Chapter 2.

The remarkable fact that type theory is both a functional programming language and a logic is known by many names including *the Curry–Howard correspondence* and *propositions as types*. It is a very broad topic with many treatments; book-length expositions include *Proofs and Types* [GLT89] and *PROGRAM = PROOF* [Mim20].

The code in this chapter is written in Agda syntax [Agda]. For more on dependently-typed programming in Agda, see *Verified Functional Programming in Agda* [Stu16]; for a more engineering-oriented perspective on dependent types, see *Type-Driven Development with Idris* [Bra17]. The sprintf example in Section 1.1.2 is inspired by the paper *Cayenne — A Language with Dependent Types* [Aug99]. Conversely, to learn about using Agda as a proof assistant for programming language theory, see *Programming Language Foundations in Agda* [WKS22].

2 *Extensional type theory*

In order to understand the subtle differences between modern dependent type theories, we must first study the formal definition of a dependent type theory as a mathematical object. We will then be prepared for Chapter 3, in which we study mathematical properties of type theory—and particularly of definitional and propositional equality—and their connection to computer implementations of type theory. In this chapter we therefore present the judgmental theory of extensional Martin-Löf type theory [ML82], one of the canonical variants of dependent type theory. We strongly suggest following the exposition rather than simply reading the rules, but the rules are collected for convenience in Appendix A.

Given the time constraints of this course, we do not attempt to give a comprehensive account of the syntax of type theories, nor do we present any of the many alternative methods of defining type theory, some of which are more efficient (but more technical) than the one we present here. These questions lead to the fascinating and deep area of *logical frameworks* which we must regrettably leave for a different course.

In this chapter In Section 2.1 we recall the concepts of judgments and inference rules in the setting of the simply-typed lambda calculus. In Section 2.2 we consider how to adapt these methods to the dependent setting, and develop the basic judgmental structure of dependent type theory. In particular, we observe that substitution plays a more crucial role in this setting, and introduce some technology needed to better accommodate it.

TODO reflect restructuring

In Section 2.4 we extend the basic rules of type theory with rules governing dependent products, dependent sums, extensional equality, and unit types. We argue that these connectives can be understood as *internalizations of judgmental structure*, a perspective which provides a conceptual justification of these connectives' rules. In Section 2.5 we define several inductive types—the empty type, booleans, and natural numbers—and explain how and why these types do not fit the pattern of the previous section. Finally, in Section 2.6 we discuss large elimination, which is implicit in our examples of full-spectrum dependency from Section 1.1, and its internalization via universe types.

Goals of the chapter By the end of this chapter, you will be able to:

- Define the core judgments of dependent type theory, and explain how and why they differ from the judgments of simple type theory.
- Explain the role of substitutions in the syntax of dependent type theory.
- Define and justify the rules of the core connectives of type theory.

2.1 The simply-typed lambda calculus

The theory of typed functional programming is built on extensions of a core language known as the *simply-typed lambda calculus*, which supports two types of data:

- functions of type $A \to B$ (for any types A, B): we write $\lambda x.b$ for the function that sends any input x of type A to an output b of type B, and write $f\ a$ for the application of a function f of type $A \to B$ to an input a of type A; and
- ordered pairs of type $A \times B$ (for any types A, B): we write (a, b) for the pair of a term a of type A with a term b of type B, and write $\mathbf{fst}(p)$ and $\mathbf{snd}(p)$ respectively for the first and second projections of a pair p of type $A \times B$.

It can also be seen as the implication–conjunction fragment of intuitionistic propositional logic, or as an axiom system for cartesian closed categories.

In this section we formally define the simply-typed lambda calculus as a collection of judgments presented by inference rules, in order to prepare ourselves for the analogous—but considerably more complex—definition of dependent type theory in the remainder of this chapter. Our goal is thus not to give a textbook account of the simply-typed lambda calculus but to draw the reader's attention to issues that will arise in the dependent setting.

Readers familiar with the simply-typed lambda calculus should be aware that our definition does not reference the untyped lambda calculus (as discussed in Remark 2.1.2) and considers terms modulo $\beta\eta$-equivalence (Section 2.1.2).

2.1.1 Contexts, types, and terms

The simply-typed lambda calculus is made up of two *sorts*, or grammatical categories, namely types and terms. We present these sorts by two well-formedness *judgments*:

- the judgment A type stating that A is a well-formed type, and
- for any well-formed type A, the judgment $a : A$ stating that a is a well-formed term of that type.

By comprehension these judgments determine respectively the collection of well-formed types and, for every element of that collection, the collection of well-formed terms of that type. (From now on we will stop writing "well-formed" because we do not consider any other kind of types or terms; see Remark 2.1.2.)

Remark 2.1.1. A judgment is simply a proposition in our ambient mathematics, one which takes part in the definition of a logical theory; we use this terminology to distinguish such meta-propositions from the propositions of the logic that is being defined [ML87].

Similarly, a sort is a type in the ambient mathematics, as distinguished from the types of the theory being defined. We refer to the ambient mathematics (in which our definition is being carried out) as the *metatheory* and the logic being defined as the *object theory*.

In this course we will be relatively agnostic about our metatheory, which the reader can imagine as "ordinary mathematics." However, one can often simplify matters by adopting a domain-specific metatheory (a *logical framework*) well-suited to defining languages/logics, as an additional level of indirection within the ambient metatheory. ◇

Types We can easily define the types as the expressions generated by the following context-free grammar:

$$\textit{Types} \quad A, B ::= \quad \mathbf{b} \mid A \times B \mid A \to B$$

We say that the judgment $A\,\mathsf{type}$ ("A is a type") holds when A is a type in the above sense. Note that in addition to function and product types we have included a base type $\mathbf{b}$; without $\mathbf{b}$ the grammar would have no terminal symbols and would thus be empty.

Equivalently, we could define the $A\,\mathsf{type}$ judgment by three *inference rules* corresponding to the three production rules in the grammar of types:

$$\frac{}{\mathbf{b}\,\mathsf{type}} \qquad \frac{A\,\mathsf{type} \qquad B\,\mathsf{type}}{A \times B\,\mathsf{type}} \qquad \frac{A\,\mathsf{type} \qquad B\,\mathsf{type}}{A \to B\,\mathsf{type}}$$

Each inference rule has some number of premises (here, zero or two) above the line and a single conclusion below the line; by combining these rules into trees whose leaves all have no premises, we can produce *derivations* of judgments (here, the well-formedness of a type) at the root of the tree. The tree below is a proof that $(\mathbf{b} \times \mathbf{b}) \to \mathbf{b}$ is a type:

$$\dfrac{\dfrac{\dfrac{}{\mathbf{b}\,\mathsf{type}} \qquad \dfrac{}{\mathbf{b}\,\mathsf{type}}}{\mathbf{b} \times \mathbf{b}\,\mathsf{type}} \qquad \dfrac{}{\mathbf{b}\,\mathsf{type}}}{(\mathbf{b} \times \mathbf{b}) \to \mathbf{b}\,\mathsf{type}}$$

Terms Terms are considerably more complex than types, so before attempting a formal definition we will briefly summarize our intentions. For the remainder of this section, fix a finite set I. The well-formed terms are as follows:

- for any $i \in I$, the base term $\mathbf{c}_i$ has type $\mathbf{b}$;
- pairing (a, b) has type $A \times B$ when $a : A$ and $b : B$;
- first projection $\mathbf{fst}(p)$ has type A when $p : A \times B$;

- second projection $\mathbf{snd}(p)$ has type B when $p : A \times B$;
- a function $\lambda x.b$ has type $A \to B$ when $b : B$ where b can contain (in addition to the usual term formers) the variable term $x : A$ standing for the function's input; and
- a function application $f\ a$ has type B when $f : A \to B$ and $a : A$.

The first difficulty we encounter is that unlike types, which are a single sort, there are infinitely many sorts of terms (one for each type) many of which refer to one another. A more significant issue is to make sense of the clause for functions: the body b of a function $\lambda x.b : A \to B$ is a term of type B according to our original grammar *extended by* a new constant $x : A$ representing an indeterminate term of type A. Because b can again be or contain a function $\lambda y.c$, we must account for finitely many extensions $x : A, y : B, \ldots$.

To account for these extensions we introduce an auxiliary sort of *contexts*, or lists of variables paired with types, representing local extensions of our theory by variable terms.

Contexts The judgment $\vdash \Gamma\ \mathsf{cx}$ ("Γ is a context") expresses that Γ is a list of pairs of term variables with types. We write $\mathbf{1}$ for the empty context and $\Gamma, x : A$ for the extension of Γ by a term variable x of type A. As a context-free grammar, we might write:

$$\begin{array}{lrl} \textit{Variables} & x, y \coloneqq & x \mid y \mid z \mid \cdots \\ \textit{Contexts} & \Gamma \coloneqq & \mathbf{1} \mid \Gamma, x : A \end{array}$$

Equivalently, in inference rule notation:

$$\frac{}{\vdash \mathbf{1}\ \mathsf{cx}} \qquad \frac{\vdash \Gamma\ \mathsf{cx} \qquad A\ \mathsf{type}}{\vdash \Gamma, x : A\ \mathsf{cx}}$$

We will not spend time discussing variables or binding in these lecture notes because variables will, perhaps surprisingly, not be a part of our definition of dependent type theory. For the purposes of this section we will simply assume that there is an infinite set of variables $x, y, z \ldots$, and that all the variables in any given context or term are distinct.

Terms revisited With contexts in hand we are now ready to define the term judgment, which we revise to be relative to a context Γ. The judgment $\Gamma \vdash a : A$ ("a has type A in context Γ") is defined by the following inference rules:

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{i \in I}{\Gamma \vdash \mathbf{c}_i : \mathbf{b}} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \mathbf{fst}(p) : A}$$

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \mathbf{snd}(p) : B} \qquad \frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : A \to B} \qquad \frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

The rules for $\mathbf{c}_i$, pairing, projections, and application straightforwardly render our text into inference rule form, framed by a context Γ that is unchanged from premises to conclusion. The lambda rule explains how contexts are changed: the body of a lambda is typed in an extended context; and the variable rule explains how contexts are used: in context Γ, the variables of type A in Γ serve as additional terminal symbols of type A.

Rules such as pairing or lambda that describe how to create terms of a given type former are known as *introduction* rules, and rules describing how to use terms of a given type former, like projection and application, are known as *elimination* rules.

Remark 2.1.2. An alternative approach that is perhaps more familiar to programming languages researchers is to define a collection of *preterms*

$$\textit{Terms} \quad a, b \coloneqq \quad \mathbf{c}_i \mid x \mid (a, b) \mid \mathbf{fst}(a) \mid \mathbf{snd}(a) \mid \lambda x.a \mid a\ b$$

which includes ill-formed (typeless) terms like $\mathbf{fst}(\lambda x.x)$ in addition to the well-formed (typed) ones captured by our grammar above, and the inference rules are regarded as carving out various subsets of well-formed terms [Har16]. In fact, one often gives computational meaning to *all* preterms (as an extension of the untyped lambda calculus) and then proves that the well-typed ones are in some sense computationally well-behaved.

This is *not* the approach we are taking here; to us the term expression $\mathbf{fst}(\lambda x.x)$ does not exist any more than the type expression $\to \times \to$.[1] In fact, in light of Section 2.1.2, there will not even exist a "forgetful" map from our collections of terms to these preterms. ⋄

2.1.2 *Equational rules*

One shortcoming of our definition thus far is that our projections don't actually project anything and our function applications don't actually apply functions—there is no sense yet in which $\mathbf{fst}((a, b)) : A$ or $(\lambda x.x)\ a : A$ "are" $a : A$. Rather than equip our terms with operational meaning, we will *quotient* our terms by equations that capture a notion of sameness including these examples. The reader can imagine this process as analogous to the presentation of algebras by *generators and relations*, in which our terms thus far are the generators of a "free algebra" of (well-formed but) uninterpreted expressions.

Our true motivation for this quotient is to anticipate the definitional equality of dependent type theory, but there are certainly intrinsic reasons as well, perhaps most notably that the quotiented terms of the simply-typed lambda calculus serve as an axiom system for reasoning about cartesian closed categories [Cro94, Chapter 4].

[1] Perhaps one's definition of context-free grammar carves out the grammatical expressions out of arbitrary strings over an alphabet, but this process occurs at a different level of abstraction. The reader should banish such thoughts along with their thoughts about terms with mismatched parentheses.

We quotient by the congruence relation generated by the following rules:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathbf{fst}((a,b)) = a : A} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathbf{snd}((a,b)) = b : B} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash p = (\mathbf{fst}(p), \mathbf{snd}(p)) : A \times B}$$

$$\frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x.b)\ a = b[a/x] : B} \qquad \frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f = \lambda x.(f\ x) : A \to B}$$

The equations pertaining to elimination after introduction (projection from pairs and application of lambdas) are called *β-equivalences*; the equations pertaining to introduction after elimination (pairs of projections and lambdas of applications) are *η-equivalences*.

We emphasize that these equations are not *a priori* directed, and are not restricted to the "top level" of terms; we genuinely take the quotient of the collection of terms at each type by these equations, automatically inducing equations such as $\lambda x.x = \lambda x.\mathbf{fst}((x,x))$.

The first two rules explain that projecting from a pair has the evident effect. The third rule states that every term of type $A \times B$ can be written as a pair (of its projections), in effect transforming the introduction rule for products from merely a sufficient condition to a necessary one as well. Similarly, the fifth rule states that every $f : A \to B$ can be written as a lambda (of its application).

The fourth rule explains that applying a lambda function $\lambda x.b$ to an argument a is equal to the body b of that lambda with all occurrences of the placeholder variable x replaced by the term a. However, this equation makes reference to a *substitution* operation $b[a/x]$ ("substitute a for x in b") that we have not yet defined.

Substitution We can define substitution $b[a/x]$ by structural recursion on b:

$$\begin{aligned}
\mathbf{c}_i[c/x] &:= \mathbf{c}_i \\
x[c/x] &:= c \\
y[c/x] &:= y && \text{(for } x \neq y\text{)} \\
(a,b)[c/x] &:= (a[c/x], b[c/x]) \\
\mathbf{fst}(p)[c/x] &:= \mathbf{fst}(p[c/x]) \\
\mathbf{snd}(p)[c/x] &:= \mathbf{snd}(p[c/x]) \\
(\lambda y.b)[c/x] &:= \lambda y.b[c/x] && \text{(for } x \neq y\text{)} \\
(f\ a)[c/x] &:= f[c/x]\ a[c/x]
\end{aligned}$$

In the case of substituting into a lambda $(\lambda y.b)[c/x]$, we assume that the bound variable y introduced by the lambda is different from the variable x being substituted away. In practice they may coincide, in which case one must rename y (and all references to y in b)

before applying this rule. In any case, we intend this substitution to be *capture-avoiding* in the sense of not inadvertently changing the referent of bound variables.

However, because we have quotiented our collection of terms by $\beta\eta$-equivalence, it is not obvious that substitution is well-defined as a function out of the collection of terms; in order to map out of the quotient, we must check that substitution behaves equally on equal terms. (It is also not obvious that substitution is a function *into* the collection of terms, in the sense of producing well-formed terms, as we will discuss shortly.)

Consider the equation $\mathbf{fst}((a, b)) = a$. To see that substitution respects this equation, we can substitute into the left-hand side, yielding:

$$(\mathbf{fst}((a, b)))[c/x] = \mathbf{fst}((a, b)[c/x]) = \mathbf{fst}((a[c/x], b[c/x]))$$

which is β-equivalent to $a[c/x]$, the result of substituting into the right-hand side. We can check the remaining equations in a similar fashion; the $x \neq y$ condition on substitution into lambdas is necessary for substitution to respect β-equivalence of functions.

2.1.3 *Who type-checks the typing rules?*

Our stated goal in Section 2.1.1 was to define a collection of well-formed types (written $A\,\mathsf{type}$), and for each of these a collection of well-formed terms (written $a : A$). Have we succeeded? First of all, our definition of terms is now indexed by contexts Γ and written $\Gamma \vdash a : A$, to account for variables introduced by lambdas. This is no problem: we recover the original notion of (closed) term by considering the empty context $\mathbf{1}$. Nor is there any issue defining the collections of types $\mathsf{Ty} = \{A \mid A\,\mathsf{type}\}$ and contexts $\mathsf{Cx} = \{\Gamma \mid \vdash \Gamma\,\mathsf{cx}\}$ as presented by the grammars or inference rules in Section 2.1.1.

It is less clear that the collections of *terms* are well-defined. We would like to say that the collection of terms of type A in context Γ, $\mathsf{Tm}(\Gamma, A)$, is the set of a for which there exists a derivation of $\Gamma \vdash a : A$, modulo the relation $a \sim b \iff$ there exists a derivation of $\Gamma \vdash a = b : A$. Several questions arise immediately; for instance, is it the case that whenever $\Gamma \vdash a : A$ is derivable, Γ is a context and A is a type? If not, then we have some "junk" judgments that should not correspond to elements of some $\mathsf{Tm}(\Gamma, A)$.

Lemma 2.1.3. *If* $\Gamma \vdash a : A$ *then* $\vdash \Gamma\,\mathsf{cx}$ *and* $A\,\mathsf{type}$.

To prove such a statement, one proceeds by induction on derivations of $\Gamma \vdash a : A$. If, say, the derivation ends as follows:

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash p : A \times B}}{\Gamma \vdash \mathbf{fst}(p) : A}$$

then the inductive hypothesis applied to the derivation of $\Gamma \vdash p : A \times B$ tells us that $\vdash \Gamma\ \mathsf{cx}$ and $A \times B$ type. The former is exactly one of the two statements we are trying to prove. The other, A type, follows from an "inversion lemma" (proven by cases on the $-$ type judgment) that A type is not only a sufficient but also a necessary condition for $A \times B$ type.

Unfortunately our proof runs into an issue at the base cases, or at least it is not clear over what Γ the following rules range:

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad\qquad \frac{i \in I}{\Gamma \vdash \mathbf{c}_i : \mathbf{b}}$$

We must either add premises to these rules stating $\vdash \Gamma\ \mathsf{cx}$, or else clarify that Γ always ranges only over contexts (which will be our strategy moving forward; see Notation 2.2.1).

Another question is the well-definedness of our quotient:

Lemma 2.1.4. *If* $\Gamma \vdash a = b : A$ *then* $\Gamma \vdash a : A$ *and* $\Gamma \vdash b : A$.

But because β-equivalence refers to substitution, proving this lemma requires:

Lemma 2.1.5 (Substitution)**.** *If* $\Gamma, x : A \vdash b : B$ *and* $\Gamma \vdash a : A$ *then* $\Gamma \vdash b[a/x] : B$.

We already saw that we must check that substitution $b[a/x]$ respects equality of b, but we must also check that it produces well-formed terms, again by induction on b. Note that substitution changes a term's context because it eliminates one of its free variables.

If we resume our attempt to prove Lemma 2.1.4, we will notice that substitution is not the only time that the context of a term changes; in the right-hand side of the η-rule of functions, f is in context $\Gamma, x : A$, whereas in the premise and left-hand side it is in Γ:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f = \lambda x.(f\ x) : A \to B}$$

And thus we need yet another lemma.

Lemma 2.1.6 (Weakening)**.** *If* $\Gamma \vdash b : B$ *and* $\Gamma \vdash A\ \mathsf{type}$ *then* $\Gamma, x : A \vdash b : B$.

We will not belabor the point any further; eventually one proves enough lemmas to conclude that we have a set of contexts Cx, a set of types Ty, and for every $\Gamma \in \mathsf{Cx}$ and $A \in \mathsf{Ty}$ a set of terms $\mathsf{Tm}(\Gamma, A)$. The complexity of each result is proportional to the complexity of that sort's definition: we define types outright, contexts by simple reference to types, and terms by more complex reference to both types and contexts. The judgments of dependent type theory are both more complex and more intertwined; rather than enduring proportionally more suffering, we will adopt a slightly different approach.

Finally, whereas all the metatheorems mentioned in this section serve only to establish that our definition is mathematically sensible, there are more genuinely interesting and

contentful metatheorems one might wish to prove, including *canonicity*, the statement that (up to equality) the only closed terms of $\mathbf{b}$ are of the form $\mathbf{c}_i$ (i.e., $\mathsf{Tm}(\mathbf{1}, \mathbf{b}) = \{\mathbf{c}_i\}_{i \in I}$), and *decidability of equality*, the statement that for any $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$ we can write a program which determines whether or not $\Gamma \vdash a = b : A$.

2.2 Towards the syntax of dependent type theory

The reader is forewarned that the rules in this section serve to bridge the gap between Section 2.1 and our "official" rules for extensional type theory, which start in Section 2.3.

As we discussed in Section 1.1, the defining distinction between dependent and simple type theory is that in the former, types can contain term expressions and even term variables. Thus, whereas in Section 2.1 a simple context-free grammar sufficed to define the collection of types and we needed a context-sensitive system of inference rules to define the well-typed terms, in dependent type theory we will find that both the types and terms are context-sensitive because they refer to one another.

Types and contexts When is the dependent function type $(x : A) \to B$ well-formed? Certainly A and B must be well-formed types, but B is allowed to contain the term variable $x : A$ whereas A is not. In the case of $(n : \mathbf{Nat}) \to \mathbf{Vec}\ \mathbf{String}\ (\mathsf{suc}\ n)$, the well-formedness of the codomain depends on the fact that $\mathsf{suc}\ n$ is a well-formed term of type $\mathbf{Nat}$ (the indexing type of $\mathbf{Vec}\ \mathbf{String}$), which in turn depends on the fact that n is known to be an expression (in particular, a variable) of type $\mathbf{Nat}$.

Thus as with the *term* judgment of Section 2.1, the *type* judgment of dependent type theory must have access to the context of term variables, so we replace the A type judgment ("A is a type") of the simply-typed lambda calculus with a judgment $\Gamma \vdash A\ \mathsf{type}$ ("A is a type in context Γ"). This innocuous change has many downstream implications, so we will be fastidious about the context in which a type is well-formed.

The first consequence of this change is that contexts of term variables, which we previously defined simply as lists of well-formed types, must now also take into account *in what context* each type is well-formed. Informally we say that each type can depend on all the variables before it in the context; formally, one might define the judgment $\vdash \Gamma\ \mathsf{cx}$ by the following pair of rules:

$$\frac{}{\vdash \mathbf{1}\ \mathsf{cx}} \qquad\qquad \frac{\vdash \Gamma\ \mathsf{cx} \qquad \Gamma \vdash A\ \mathsf{type}}{\vdash \Gamma, x : A\ \mathsf{cx}}$$

Notice that the rules defining the judgment $\vdash \Gamma\ \mathsf{cx}$ refer to the judgment $\Gamma \vdash A\ \mathsf{type}$, which in turn depends on our notion of context. This kind of mutual dependence will continue to crop up throughout the rules of dependent type theory.

Notation 2.2.1 (Presuppositions)**.** With a more complex notion of context, it is more important than ever for us to decide over what Γ the judgment $\Gamma \vdash A\,\mathsf{type}$ ranges. We will say that the judgment $\Gamma \vdash A\,\mathsf{type}$ is only well-formed when $\vdash \Gamma\,\mathsf{cx}$ holds, as a matter of "meta-type discipline," and similarly that the judgment $\Gamma \vdash a : A$ is only well-formed when $\Gamma \vdash A\,\mathsf{type}$ (and thus also $\vdash \Gamma\,\mathsf{cx}$).

One often says that $\vdash \Gamma\,\mathsf{cx}$ is a *presupposition* of the judgment $\Gamma \vdash A\,\mathsf{type}$, and that the judgments $\vdash \Gamma\,\mathsf{cx}$ and $\Gamma \vdash A\,\mathsf{type}$ are presuppositions of $\Gamma \vdash a : A$. We will globally adopt the convention that whenever we assert the truth of some judgment in prose or as the premise of a rule, we also implicitly assert that its presuppositions hold. Dually, we will be careful to check that none of our rules have meta-ill-typed conclusions.

Now that we have added a term variable context to the type well-formedness judgment, we can explain when $(x : A) \to B$ is a type: it is a (well-formed) type in Γ when A is a type in Γ and B is a type in $\Gamma, x : A$, as follows.

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma, x : A \vdash B\,\mathsf{type}}{\Gamma \vdash (x : A) \to B\,\mathsf{type}}$$

Rules like this describing how to create a type are known as *formation rules*, to parallel the terminology of introduction and elimination rules.

We can now sketch the formation rules for many of the types we encountered in Chapter 1. Dependent types like _≡_ and **Vec** are particularly interesting because they entangle the $\Gamma \vdash A\,\mathsf{type}$ judgment with the term well-formedness judgment $\Gamma \vdash a : A$.

$$\frac{\vdash \Gamma\,\mathsf{cx}}{\Gamma \vdash \mathbf{Nat}\,\mathsf{type}} \qquad \frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma \vdash n : \mathbf{Nat}}{\Gamma \vdash \mathbf{Vec}\ A\ n\,\mathsf{type}} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a \equiv b\,\mathsf{type}}$$

Note that the convention of presuppositions outlined in Notation 2.2.1 means that the second and third rules have an implicit $\vdash \Gamma\,\mathsf{cx}$ premise, and the third rule also has an implicit $\Gamma \vdash A\,\mathsf{type}$ premise. To see that the conclusions of these rules are meta-well-typed, we must check that $\vdash \Gamma\,\mathsf{cx}$ holds in each case; this is an explicit premise of the first rule and a presupposition of the premises of the second and third rules.

The formation rule for propositional equality _≡_ in particular is a major source of dependency because it singlehandledly allows arbitrary terms of arbitrary type to occur within types. In fact, this rule by itself causes the inference rules of all three judgments $\vdash \Gamma\,\mathsf{cx}$, $\Gamma \vdash A\,\mathsf{type}$, and $\Gamma \vdash a : A$ to all depend on one another pairwise.

Exercise 2.1. *Attempt to derive that* $(n : \mathbf{Nat}) \to \mathbf{Vec}\ \mathbf{String}\ (\mathsf{suc}\ n)$ *is a well-formed type in the empty context* $\mathbf{1}$*, using the rules introduced in this section thus far. Several rules are missing; which judgments can you not yet derive?*

The variable rule Let us turn now to the term judgment $\Gamma \vdash a : A$, and in particular the rule stating that term variables in the context are well-formed terms. For simplicity, imagine the special case where the last variable is the one under consideration:

$$\frac{}{\Gamma, x : A \vdash x : A}\ !?$$

This rule needs considerable work, as neither of the conclusion's presuppositions, $\vdash (\Gamma, x : A)\ \mathsf{cx}$ and $\Gamma, x : A \vdash A\ \mathsf{type}$, currently hold. We can address the former by adding premises $\vdash \Gamma\ \mathsf{cx}$ and $\Gamma \vdash A\ \mathsf{type}$ to the rule, from which it follows that $\vdash (\Gamma, x : A)\ \mathsf{cx}$.[2] As for the latter, note that $\Gamma \vdash A\ \mathsf{type}$ does not actually imply $\Gamma, x : A \vdash A\ \mathsf{type}$—this would require proving a *weakening lemma* (see Lemma 2.1.6) for types! (Conversely, if the rule has the premise $\Gamma \vdash A\ \mathsf{type}$, then we cannot establish well-formedness of the context.)

There are several ways to proceed. One is to prove a weakening lemma, but given that the well-formedness of the variable rule requires weakening, it is necessary to prove all our well-formedness, weakening, and substitution lemmas by a rather heavy simultaneous induction. A second approach would be to add a silent weakening *rule* stating that $\Gamma, x : A \vdash B\ \mathsf{type}$ whenever $\Gamma \vdash B\ \mathsf{type}$; however, this introduces ambiguity into our rules regarding the context(s) in which a type or term is well-formed.

We opt for a third option, which is to add *explicit* weakening rules asserting the existence of an operation sending types and terms in context Γ to types and terms in context $\Gamma, x : A$, both written $-[\mathbf{p}]$. (This notation will become less mysterious later.)

$$\frac{\Gamma \vdash B\ \mathsf{type} \qquad \Gamma \vdash A\ \mathsf{type}}{\Gamma, x : A \vdash B[\mathbf{p}]\ \mathsf{type}} \qquad\qquad \frac{\Gamma \vdash b : B \qquad \Gamma \vdash A\ \mathsf{type}}{\Gamma, x : A \vdash b[\mathbf{p}] : B[\mathbf{p}]}$$

Note that the type weakening rule is needed to make sense of the term weakening rule.

We can now fix the variable rule we wrote above: using $-[\mathbf{p}]$ to weaken A by itself, we move A from context Γ to $\Gamma, x : A$ as required in the conclusion of the rule.

$$\frac{\vdash \Gamma\ \mathsf{cx} \qquad \Gamma \vdash A\ \mathsf{type}}{\Gamma, x : A \vdash x : A[\mathbf{p}]}$$

To use variables that occur earlier in the context, we can apply weakening repeatedly until they are the last variable. Suppose that $\mathbf{1} \vdash A\ \mathsf{type}$ and $x : A \vdash B\ \mathsf{type}$, and in the context $x : A, y : B$ we want to use the variable x. Ignoring the $y : B$ in the context for a moment, we know that $x : A \vdash x : A[\mathbf{p}]$ by the last variable rule; thus by weakening we

[2] Of course one could just directly add the premise $\vdash (\Gamma, x : A)\ \mathsf{cx}$, but our short-term memory is robust enough to recall that our next task is to ensure that A is a type.

have $x : A, y : B \vdash x[\mathbf{p}] : A[\mathbf{p}][\mathbf{p}]$. In general, we can derive the following principle:

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma, x : A \vdash B_1\,\mathsf{type} \qquad \dots \qquad \Gamma, x : A, y_1 : B_1, \dots \vdash B_n\,\mathsf{type}}{\Gamma, x : A, y_1 : B_1, \dots, y_n : B_n \vdash x\underbrace{[\mathbf{p}]\dots[\mathbf{p}]}_{n\text{ times}} : A\underbrace{[\mathbf{p}]\dots[\mathbf{p}]}_{n+1\text{ times}}}$$

This approach to variables is elegant in that it breaks the standard variable rule into two simpler primitives: a rule for the last variable, and rules for type and term weakening. However, it introduces a redundancy in our notation, because the term $x[\mathbf{p}]^n$ encodes in two different ways the variable to which it refers: by the name x as well as positionally by the number of weakenings n.

A happy accident of our presentation of the variable rule is thus that we can delete variable names altogether; in Section 2.3 we will present contexts simply as lists of types $A.B.C$ with no variable names, and adopt a single notation for "the last variable in the context," an encoding of the lambda calculus known as *de Bruijn indexing* [Bru72]. Conceptual elegance notwithstanding, this notation is very unfriendly to the reader in larger examples[3] so we will continue to use named variables outside of the rules themselves; translating between the two notations is purely mechanical.

Remark 2.2.2. The first author wishes to mention another approach to maintaining readability, which is to continue using both named variables and explicit weakenings [Gra09]; this approach has the downside of requiring us to explain variable binding, but is simultaneously readable and precise about weakenings. ◇

[3] According to Conor McBride, "Bob Atkey once memorably described the capacity to put up with de Bruijn indices as a Cylon detector." (https://mazzo.li/epilogue/index.html%3Fp=773.html)

A

Martin-Löf type theory

This appendix presents a substitution calculus [ML92; Tas93; Dyb96] for Martin-Löf's extensional dependent type theory.

Judgments

Martin-Löf type theory has four basic judgments:

1. $\vdash \Gamma\ \mathsf{cx}$ asserts that Γ is a context.
2. $\Delta \vdash \gamma : \Gamma$, presupposing $\vdash \Delta\ \mathsf{cx}$ and $\vdash \Gamma\ \mathsf{cx}$, asserts that γ is a substitution from Δ to Γ (*i.e.*, assigns a term in Δ to each variable in Γ).
3. $\Gamma \vdash A\ \mathsf{type}$, presupposing $\vdash \Gamma\ \mathsf{cx}$, asserts that A is a type in context Γ.
4. $\Gamma \vdash a : A$, presupposing $\vdash \Gamma\ \mathsf{cx}$ and $\Gamma \vdash A\ \mathsf{type}$, asserts that a is an element/term of type A in context Γ.

The *presuppositions* of a judgment are its meta-implicit-arguments, so to speak. For instance, the judgment $\Gamma \vdash A\ \mathsf{type}$ is sensible to write (is meta-well-typed) only when the judgment $\vdash \Gamma\ \mathsf{cx}$ holds. We adopt the convention that asserting the truth of a judgment implicitly asserts its well-formedness; thus asserting $\Gamma \vdash A\ \mathsf{type}$ also asserts $\vdash \Gamma\ \mathsf{cx}$.

As we assert the existence of various contexts, substitutions, types, and terms, we will simultaneously need to assert that some of these (already introduced) objects are equal to other (already introduced) objects of the same kind.

1. $\Delta \vdash \gamma = \gamma' : \Gamma$, presupposing $\Delta \vdash \gamma : \Gamma$ and $\Delta \vdash \gamma' : \Gamma$, asserts that γ, γ' are equal substitutions from Δ to Γ.
2. $\Gamma \vdash A = A'\ \mathsf{type}$, presupposing $\Gamma \vdash A\ \mathsf{type}$ and $\Gamma \vdash A'\ \mathsf{type}$, asserts that A, A' are equal types in context Γ.
3. $\Gamma \vdash a = a' : A$, presupposing $\Gamma \vdash a : A$ and $\Gamma \vdash a' : A$, asserts that a, a' are equal elements of type A in context Γ.

Two types (*resp.*, contexts, substitutions, terms) being equal has the force that it does in standard mathematics: any expression can be replaced silently by an equal expression without affecting the meaning or truth of the statement in which it appears. One important example of this principle is the "conversion rule" which states that if $\Gamma \vdash A = A'\ \mathsf{type}$ and $\Gamma \vdash a : A$, then $\Gamma \vdash a : A'$.

In the rules that follow, some arguments of substitution, type, and term formers are typeset as gray subscripts; these are arguments that we will often omit because they can be inferred from context and are tedious and distracting to write.

Contexts and substitutions

$$\frac{}{\vdash \mathbf{1}\ \mathsf{cx}}\ \textsc{cx/emp} \qquad \frac{\vdash \Gamma\ \mathsf{cx} \qquad \Gamma \vdash A\ \mathsf{type}}{\vdash \Gamma.A\ \mathsf{cx}}\ \textsc{cx/ext}$$

$$\frac{\vdash \Gamma\ \mathsf{cx}}{\Gamma \vdash \mathbf{id}_\Gamma : \Gamma}\ \textsc{sb/id} \qquad \frac{\Gamma_2 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_0 : \Gamma_0}{\Gamma_2 \vdash \gamma_0 \circ_{\Gamma_2,\Gamma_1,\Gamma_0} \gamma_1 : \Gamma_0}\ \textsc{sb/comp}$$

$$\frac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathbf{id}_\Gamma \circ \gamma = \gamma : \Gamma} \qquad \frac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash \gamma \circ \mathbf{id}_\Delta = \gamma : \Gamma} \qquad \frac{\Gamma_3 \vdash \gamma_2 : \Gamma_2 \qquad \Gamma_2 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_0 : \Gamma_0}{\Gamma_3 \vdash \gamma_0 \circ (\gamma_1 \circ \gamma_2) = (\gamma_0 \circ \gamma_1) \circ \gamma_2 : \Gamma_0}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\ \mathsf{type}}{\Delta \vdash A[\gamma]_{\Delta,\Gamma}\ \mathsf{type}}\ \textsc{ty/sb} \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash a : A}{\Delta \vdash a[\gamma]_{\Delta,\Gamma} : A[\gamma]}\ \textsc{tm/sb}$$

$$\frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma \vdash A[\mathbf{id}_\Gamma] = A\ \mathsf{type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash a[\mathbf{id}_\Gamma] = a : A}$$

$$\frac{\Gamma_2 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_0 : \Gamma_0 \qquad \Gamma_0 \vdash A\ \mathsf{type}}{\Gamma_2 \vdash A[\gamma_0 \circ \gamma_1] = A[\gamma_0][\gamma_1]\ \mathsf{type}} \qquad \frac{\Gamma_2 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_0 : \Gamma_0 \qquad \Gamma_0 \vdash a : A}{\Gamma_2 \vdash a[\gamma_0 \circ \gamma_1] = a[\gamma_0][\gamma_1] : A[\gamma_0 \circ \gamma_1]}$$

$$\frac{\vdash \Gamma\ \mathsf{cx}}{\Gamma \vdash !_\Gamma : \mathbf{1}}\ \textsc{sb/emp} \qquad \frac{\Gamma \vdash \delta : \mathbf{1}}{\Gamma \vdash !_\Gamma = \delta : \mathbf{1}}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\ \mathsf{type} \qquad \Delta \vdash a : A[\gamma]}{\Delta \vdash \gamma._{\Delta,\Gamma,A} a : \Gamma.A}\ \textsc{sb/ext} \qquad \frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \mathbf{p}_{\Gamma,A} : \Gamma}\ \textsc{sb/wk}$$

$$\frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \mathbf{q}_{\Gamma,A} : A[\mathbf{p}_{\Gamma,A}]}\ \textsc{var} \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Delta \vdash a : A[\gamma]}{\Delta \vdash \mathbf{p}_{\Gamma,A} \circ_{\Gamma.A} (\gamma.a) = \gamma : \Gamma} \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Delta \vdash a : A[\gamma]}{\Delta \vdash \mathbf{q}_{\Gamma,A}[\gamma.a] = a : A[\gamma]}$$

$$\frac{\Delta \vdash \gamma : \Gamma.A}{\Delta \vdash \gamma = (\mathbf{p}_{\Gamma,A} \circ_{\Gamma.A} \gamma).(\mathbf{q}_{\Gamma,A}[\gamma]) : \Gamma.A}$$

Π-*types*

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type}}{\Gamma \vdash \mathbf{\Pi}_\Gamma(A,B)\,\mathsf{type}}\ \textsc{pi/form} \qquad \frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash b : B}{\Gamma \vdash \lambda_{\Gamma,A,B}(b) : \mathbf{\Pi}(A,B)}\ \textsc{pi/intro}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash f : \mathbf{\Pi}(A,B)}{\Gamma \vdash \mathbf{app}_{\Gamma,A,B}(f,a) : B[\mathbf{id}_\Gamma.a]}\ \textsc{pi/elim}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type}}{\Delta \vdash \mathbf{\Pi}_\Gamma(A,B)[\gamma] = \mathbf{\Pi}_\Delta(A[\gamma], B[(\gamma \circ \mathbf{p}_{\Delta,A[\gamma]}).\mathbf{q}_{\Delta,A[\gamma]}])\,\mathsf{type}}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash b : B}{\Delta \vdash \lambda(b)[\gamma] = \lambda(b[(\gamma \circ \mathbf{p}).\mathbf{q}]) : \mathbf{\Pi}(A,B)[\gamma]}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash a : A \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash f : \mathbf{\Pi}(A,B)}{\Delta \vdash \mathbf{app}(f,a)[\gamma] = \mathbf{app}(f[\gamma], a[\gamma]) : B[(\mathbf{id}_\Gamma.a) \circ \gamma]}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash b : B}{\Gamma \vdash \mathbf{app}(\lambda(b), a) = b[\mathbf{id}.a] : B[\mathbf{id}.a]} \qquad \frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash f : \mathbf{\Pi}(A,B)}{\Gamma \vdash f = \lambda(\mathbf{app}(f[\mathbf{p}_{\Gamma,A}], \mathbf{q}_{\Gamma,A})) : \mathbf{\Pi}(A,B)}$$

Σ-*types*

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type}}{\Gamma \vdash \Sigma_\Gamma(A,B)\,\mathsf{type}}\ \textsc{sigma/form}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash b : B[\mathbf{id}_\Gamma.a]}{\Gamma \vdash \mathbf{pair}_{\Gamma,A,B}(a,b) : \Sigma(A,B)}\ \textsc{sigma/intro}$$

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash p : \Sigma(A,B)}{\Gamma \vdash \mathbf{fst}_{\Gamma,A,B}(p) : A}\ \textsc{sigma/elim/fst}$$

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash p : \Sigma(A,B)}{\Gamma \vdash \mathbf{snd}_{\Gamma,A,B}(p) : B[\mathbf{id}_\Gamma.\mathbf{fst}(p)]}\ \textsc{sigma/elim/snd}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type}}{\Delta \vdash \Sigma_\Gamma(A,B)[\gamma] = \Sigma_\Delta(A[\gamma], B[(\gamma \circ \mathbf{p}).\mathbf{q}])\,\mathsf{type}}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash a : A \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash b : B[\mathbf{id}.a]}{\Gamma \vdash \mathbf{pair}(a,b)[\gamma] = \mathbf{pair}(a[\gamma], b[\gamma]) : \Sigma(A,B)[\gamma]}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash p : \Sigma(A,B)}{\Gamma \vdash \mathbf{fst}(p)[\gamma] = \mathbf{fst}(p[\gamma]) : A[\gamma]}$$

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash p : \Sigma(A,B)}{\Gamma \vdash \mathbf{snd}(p)[\gamma] = \mathbf{snd}(p[\gamma]) : B[(\mathbf{id}.\mathbf{fst}(p)) \circ \gamma]}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash b : B[\mathbf{id}.a]}{\Gamma \vdash \mathbf{fst}(\mathbf{pair}(a,b)) = a : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash b : B[\mathbf{id}.a]}{\Gamma \vdash \mathbf{snd}(\mathbf{pair}(a,b)) = b : B[\mathbf{id}.a]}$$

$$\frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma.A \vdash B\,\mathsf{type} \qquad \Gamma \vdash p : \Sigma(A,B)}{\Gamma \vdash p = \mathbf{pair}(\mathbf{fst}(p), \mathbf{snd}(p)) : \Sigma(A,B)}$$

Unit-types

$$\frac{\vdash \Gamma\,\mathsf{cx}}{\Gamma \vdash \mathbf{Unit}_\Gamma\,\mathsf{type}}\ \textsc{unit/form} \qquad \frac{\vdash \Gamma\,\mathsf{cx}}{\Gamma \vdash \mathbf{tt}_\Gamma : \mathbf{Unit}}\ \textsc{unit/intro}$$

$$\frac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathbf{Unit}_\Gamma[\gamma] = \mathbf{Unit}_\Delta\,\mathsf{type}} \qquad \frac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathbf{tt}_\Gamma[\gamma] = \mathbf{tt}_\Delta : \mathbf{Unit}} \qquad \frac{\Gamma \vdash a : \mathbf{Unit}}{\Gamma \vdash a = \mathbf{tt} : \mathbf{Unit}}$$

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