DOMAIN EQUATIONS OVER TRANSFINITE ORDINALS

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We show how to solve domain equations over ordinals larger than ω.

1. Basic Definitions

Definition 1.1. An OFE X over α is a carrier set $|X|$ along with a family of equivalence relations $(\overset{\beta}{=})_{\beta<\alpha}$ satisfying the following conditions:

- If $\beta_0 \leq \beta_1$, $\left(\overset{\beta_0}{=}\right) \supseteq \left(\overset{\beta_1}{=}\right)$.
- If $\beta = \lim_{\beta'<\beta} \beta'$, then $\left(\overset{\beta}{=}\right) = \bigcap_{\beta'<\beta}\left(\overset{\beta'}{=}\right)$.
- $\bigcap_{\beta<\alpha}\left(\overset{\beta}{=}\right) = (=)$

Observe that this definition means that $\left(\overset{0}{=}\right)$ is always the total relation.

Definition 1.2. A coherent family in an OFE X for an ordinal γ is a subset of X, $(x_\beta)_{\beta<\gamma}$, satisfying $x_{\beta_0} \overset{\beta_0}{=} x_{\beta_1}$ where $\beta_0 \leq \beta_1$.

Convention 1.3. In what follows we will fix a limit ordinal α and work with OFEs over α. We will use γ,β and other low Greek letters to refer to ordinals.

Definition 1.4. A COFE X is an OFE equipped with two functions picking out limits:

(1) First, for any matching family $(x_\beta)_{\beta<\alpha}$, $\lim_{\beta<\alpha} x_\beta$ is the chosen limit for the coherent family: $\lim_{\beta<\alpha} x_\beta \overset{\beta}{=} x_\beta$. It is unique with this property.
(2) Second, for any limit ordinal $\beta < \alpha$, given a coherent family $(x_\gamma)_{\gamma<\beta}$, $\lim_{\gamma<\beta} x_\gamma$ is the chosen limit for the matching family: $\lim_{\gamma<\beta} x_\gamma \overset{\gamma}{=} x_\gamma$. It is unique up to $\overset{\beta}{=}$ with this property.

Remark 1.5. We observe that this uniqueness implies that $\lim_{-<-}$ is "nonexpansive".

Definition 1.6. $\blacktriangleright X$ for a COFE X is defined as $|X|$ with the equivalence relation being "shifted by one":

$$\left(\overset{1+\beta}{=}\right)_{\blacktriangleright X} \triangleq \left(\overset{\beta}{=}_X\right)$$
$$\left(\overset{0}{=}\right)_{\blacktriangleright X} \triangleq \left(\overset{0}{=}_X\right)$$
$$\left(\overset{\lim_{\gamma\leq\beta}\gamma}{=}\right)_{\blacktriangleright X} \triangleq \left(\overset{\lim_{\gamma\leq\beta}\gamma}{=}\right)_X \qquad\qquad \lim_{\gamma<\beta}\gamma = \beta$$

Lemma 1.7. *$\blacktriangleright X$ is actually a COFE with limit functions induced by those in X*

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Proof. Let us define the limit functions for $\blacktriangleright X$ as follows:

$$\lim_{\beta<\alpha}(x_\beta : \blacktriangleright X) : \blacktriangleright X \triangleq \lim_{\beta<\alpha}(x_{1+\beta} : X)$$
$$\lim_{\gamma<\beta}(x_\gamma : \blacktriangleright X) : \blacktriangleright X \triangleq \lim_{\gamma<\beta}(x_{1+\gamma} : X) \qquad \beta < \alpha$$

The limit conditions as well as uniqueness follow from the respective properties on the limit functions on X. □

We will also have use for the following construction, though to my knowledge it was not developed in Iris previously.

Definition 1.8. Given a COFE X, the $\beta < \alpha$ truncation of X at β, $[X]_\beta$, is defined as follows: the carrier set of X is $X/\stackrel{\beta}{=}$ and the equivalence relations are those of X lifted to representatives of the quotient. In other words, $x \stackrel{\gamma}{=} x'$ if and only if $x \stackrel{\min(\gamma,\beta)}{=} x'$.

The limit functions for X lift to limit functions of $[X]_\beta$.

Lemma 1.9. *A nonexpansive function $f : A \to B$ descends to all truncations: $f : [A]_\gamma \to [B]_\gamma$. A contractive function also induces $f : [A]_\gamma \to [B]_{1+\gamma}$.*

2. Recursive Domain Equations in $\mathbf{COFE}^\alpha$

We now turn to constructing the solutions to particular recursive domain equations. To begin with, let us recall that a domain equation can be represented as a particular functor, $F : \mathbf{COFE}^{\alpha\,\mathsf{op}} \times \mathbf{COFE}^\alpha \to \mathbf{COFE}^\alpha$. For instance, the classic domain equation $D = D \to D$ can be represented with $E(D^-, D^+) = (D^+)^{(D^-)}$. The mixed-variance ($\mathbf{COFE}^{\alpha\,\mathsf{op}} \times \mathbf{COFE}^\alpha$ instead of just $\mathbf{COFE}^\alpha$) is crucial: without this, E would not be a functor because D^- occurs negatively.

Definition 2.1. A mixed-variance functor of COFEs is locally non-expansive if for every for every $X_0, X_1, Y_0, Y_1 : \mathbf{COFE}^\alpha$ and pairs of maps f, f' and g, g', if $f \stackrel{\beta}{=} f'$ and $g \stackrel{\beta}{=} g'$, then $F(f, g) \stackrel{\beta}{=} F(f', g')$.

Definition 2.2. A mixed-variance functor F is locally contractive if for every for every $X_0, X_1, Y_0, Y_1 : \mathbf{COFE}^\alpha$ and pairs of maps f, f' and g, g', if $f \stackrel{\beta}{=} f'$ and $g \stackrel{\beta}{=} g'$, then $F(f, g) \stackrel{1+\beta}{=} F(f', g')$.

Definition 2.3. We will say that f and f' are β-inverses if $f \circ f' \stackrel{\beta}{=} 1$ and $f' \circ f \stackrel{\beta}{=} 1$. By convention, we will write $\stackrel{\alpha}{=}$ to mean $=$ so that α-inverses are simply proper inverses.

Lemma 2.4. *If $f : X_1 \to X_0$ is γ-inverse to f' and $g : Y_0 \to Y_1$ is γ-inverse to g' for all $\gamma < \beta \leq \alpha$ and F is a locally-contractive functor, then $F(f, g)$ and $F(f', g')$ are β-inverses.*

Proof. We will proceed by case analysis on β.
Case:

$$\beta = 0$$

Trivial, all pairs of morphisms are β-inverses so $F(f, g)$ and $F(f', g')$ certainly are 0-inverses.

Case:
$$\beta = 1 + \beta'$$
For this, we observe that by assumption $f : X_1 \to X_0$ is β'-inverse to f' and $g : Y_0 \to Y_1$ is β'-inverse to g'.

By definition of local contractiveness then, $F(f,g)$ is $1+\beta'$-inverse to $F(f',g')$. So we are done with the observation that $\beta = 1+\beta'$.

Case:
$$\beta = \lim_{\gamma<\beta} \gamma$$
In this case we have $f \circ f' \stackrel{\gamma}{=} 1$ for all $\gamma < \beta$, and since β is a limit ordinal this implies that $f \circ f' \stackrel{\beta}{=} 1$. Similarly for $f' \circ f$, $g \circ g'$, and $g' \circ g$. Therefore, we have f is β-inverse to f', and g is β-inverse to g'. By the definition of local contractiveness, then, we have $F(f,g)$ is $1+\beta$-inverse to $F(f',g')$. Recall that if $x \stackrel{1+\beta}{=} x'$, then $x \stackrel{\beta}{=} x'$ so $F(f,g)$ is β-inverse to $F(f',g')$. □

Theorem 2.5. *Given a locally contractive functor* $F : \mathbf{COFE}^{\alpha^{\mathsf{op}}} \times \mathbf{COFE}^{\alpha} \to \mathbf{COFE}^{\alpha}$ *such that* $F(\emptyset, \{\star\}) \neq \emptyset$*, there exists a unique non-empty COFE* I *together with an isomorphism*
$$\iota : F(I,I) \cong I$$

Proof. The general strategy for constructing I is remarkably similar to what was to construct (unique) fixed points of maps $\blacktriangleright A \to A$. However, we must account for the "asymmetry" in F (which has 2 inputs, but only 1 output).

In order to do this we will define a chain of objects along with *embedding* and *projection* pairs between them, such that the limit of this chain is the fixed-point we're after. This chain for COFEs and presheaves over ω consisted of ω different objects, in our case, we need a far longer chain with $1+\alpha$ many objects.

We also cannot merely define embedding and projection operators $e_n : X_n \to X_{n+1}$, $p_n : X_{n+1} \to X_n$ because not every element is indexed by a successor ordinal. Instead, we will require the more general $e_{\gamma,\beta} : X_\gamma \to X_\beta$ and $p_{\gamma,\beta} : X_\beta \to X_\gamma$ for all $\gamma < \beta$. We will also require that these morphisms are *functorial*, that is $e_{\gamma',\beta} \circ e_{\gamma,\gamma'} = e_{\gamma,\beta}$ and likewise for p. Finally, we require that $p_{\gamma,\beta} \circ e_{\gamma,\beta} = 1$ and $e_{\gamma,\beta} \circ p_{\gamma,\beta} \stackrel{\gamma}{=} 1$.

All of these morphisms must be defined simultaneously during a transfinite induction along with the objects X_β themselves. Moreover, in order to handle the successor step properly we must include a special embedding-projection pair between X_β and $F(X_\beta, X_\beta)$. All together we have the following:

(1)	$e_{\gamma,\beta} : X_\gamma \to X_\beta,$	$\gamma < \beta$	(6)	$p_{\gamma_2,\gamma_0} = p_{\gamma_2,\gamma_1} \circ p_{\gamma_1,\gamma_0}$
(2)	$p_{\gamma,\beta} : X_\beta \to X_\gamma,$	$\gamma < \beta$	(7)	$\phi_\beta : X_\beta \to [F(X_\beta, X_\beta)]_{1+\beta}$
(3)	$1 = p_{\gamma,\beta} \circ e_{\gamma,\beta}$		(8)	$\psi_\beta : [F(X_\beta, X_\beta)]_{1+\beta} \to X_\beta$
(4)	$1 \stackrel{\gamma}{=} p_{\gamma,\beta} \circ e_{\gamma,\beta}$		(9)	$\psi_\beta \circ \phi_\beta = 1$
(5)	$e_{\gamma_2,\gamma_0} = e_{\gamma_2,\gamma_1} \circ e_{\gamma_1,\gamma_0}$		(10)	$\phi_\beta \circ \psi_\beta \stackrel{\beta}{=} 1$

Moreover, we will assume (as a form of induction hypothesis) that for all smaller successor ordinals $1+\gamma$, $X_{1+\gamma} = F(X_\gamma, X_\gamma)$ and that $X_\gamma = [X_\gamma]_\gamma$. Recall that this definition is sensible for all $\beta \leq 1+\alpha$ because we have interpreted the notation $\stackrel{\alpha}{=}$ as the intersection of all $\stackrel{\gamma}{=}$, $\gamma < \alpha$, *ie.* equality.

We now turn to actually carrying out the construction:

Case:

$$\beta = 0$$

We start by setting $X_0 = \{\star\}$. In this case, there are no embeddings or projections to define (Eqs. (1) and (2)) because there are no smaller Xs.

We must, however, define Eqs. (7) and (8). We define $\phi_0 : F(X_0, X_0) \to X_0$ as !, the unique map into $\{\star\}$. We use our assumption that $F(X_0, X_0) = F(1,1)$ is nonempty and set $\psi_0(\star)$ to be that chosen element. We immediately have $\psi_0 \circ \phi_0 = 1$ by calculation so Eq. (3) is satisfied. Eq. (4) is trivial because all pairs of morphisms are 0-inverses.

Moreover, we have that $X_0 = [X_0]_0$ because X_0 is already a singleton.

Case:

$$\beta = 1 + \beta'$$

We now consider the successor case. Suppose that we have defined X_γ for all $\gamma \leq \beta'$ and showed that they satisfy all the specified equations.

We now need to construct $X_{1+\beta'}$ and we choose $X_{1+\beta'} = [F(X_{\beta'}, X_{\beta'})]_{1+\beta'}$ (so X_β is automatically β-truncated). For Eq. (1) we define the following:

$$e_{\gamma,1+\beta'} = \phi_{\beta'} \circ e_{\gamma,\beta'}$$

For Eq. (2):

$$p_{\gamma,1+\beta'} = p_{\gamma,\beta'} \circ \psi_{\beta'}$$

For this, we observe that $\psi_{\beta'}$ lifts to a function $X_\beta \to X_{\beta'}$ because $X_{\beta'}$ is β'-truncated.

The functoriality of these definitions (Eqs. (5) and (6)) follows from the functoriality of e and p on smaller ordinals. We can also calculate that these two morphisms define an embedding-projection pair (Eq. (3),Eq. (4)):

$$\begin{aligned}
p_{\gamma,\beta} \circ e_{\gamma,\beta} &= p_{\gamma,\beta'} \circ \psi_{\beta'} \circ \phi_{\beta'} \circ e_{\gamma,\beta'} \\
&= p_{\gamma,1+\beta'} \circ e_{\gamma,1+\beta'} && \mathit{Eq.}\ (9) \\
&= 1 && \mathit{Eq.}\ (3) \\
e_{\gamma,1+\beta'} \circ p_{\gamma,1+\beta'} &= e_{\gamma,\beta'} \circ \phi_{\beta'} \circ \psi_{\beta'} \circ p_{\gamma,\beta'} \\
&\overset{\beta'}{=} e_{\gamma,1+\beta'} \circ e_{\gamma,1+\beta'} && \mathit{Eq.}\ (10) \\
&\overset{\gamma}{=} 1 && \mathit{Eq.}\ (4)
\end{aligned}$$

Finally, we must construct ϕ_β and ψ_β (Eqs. (7) and (8)). We can define these $F(\psi_{\beta'}, \phi_{\beta'})$ and $F(\phi_{\beta'}, \psi_{\beta'})$. In order to see that these satisfy Eq. (9) and Eq. (10) we apply Lemma 2.4.

Case:

$$\beta = \lim_{\gamma < \beta} \gamma$$

Now we consider the limit case, suppose that $\beta = \lim_{\beta' < \beta} \beta'$ and that we have defined all $X_{\beta'}$. We will set $X_\beta = \lim_{\beta'<\beta} [F(X_{\beta'}, X_{\beta'})]_{1+\beta'}$ with the full collection of the maps $F(e_{\gamma_0,\gamma_1}, p_{\gamma_0,\gamma_1})$ for all $\gamma_0, \gamma_1 < \beta$. That is, the limit of the following diagram:

$$[F(X_0,X_0)]_1 \longleftarrow \cdots \longleftarrow [F(X_\gamma, X_\gamma)]_{1+\gamma} \xleftarrow{F(e_{\gamma,1+\gamma}, p_{\gamma,1+\gamma})} [F(X_{1+\gamma}, X_{1+\gamma})]_{1+1+\gamma} \longleftarrow \cdots$$

We remark that these maps descend to the truncation because truncation is functorial (if $A \to B$ then $[A]_\gamma \to [B]_\gamma$) and there is a canonical projection ($[A]_{1+\gamma} \to [A]_\gamma$).

The limit in this case can also be concretely constructed, it is a subset of the COFE $\prod_\gamma F(X_\gamma, X_\gamma)$ defined as follows:

$$X_\beta = \{f : \prod_{\beta' < \beta} [F(X_{\beta'}, X_{\beta'})]_{1+\beta} \mid \forall \gamma_0 > \gamma_1.\ F(e_{\gamma_1,\gamma_0}, p_{\gamma_1,\gamma_0})(f(\gamma_0)) = f(\gamma_1)\}$$

Because X_β is defined as this limit, it comes equipped with $\pi_\gamma : X_\beta \to [F(X_\gamma, X_\gamma)]_{1+\gamma} = X_{1+\gamma}$. Moreover, these map satisfy $F(e_{\gamma',\gamma}, p_{\gamma',\gamma}) \circ \pi_\gamma = \pi_{\gamma'}$. Let us define $e_{\gamma,\beta}$ (Eq. (1)) as the unique map induced by the following diagram

$$\begin{tikzcd}
& & X_\gamma \arrow[dll, "p_{1,\gamma}"'] \arrow[d, "e_{\gamma,1+\gamma}"] \arrow[dr, "e_{\gamma,1+1+\gamma}"] & & \\
{[F(X_0, X_0)]_1} & \cdots \arrow[l] & {[F(X_\gamma, X_\gamma)]_{1+\gamma}} \arrow[l] & {[F(X_{1+\gamma}, X_{1+\gamma})]_{1+1+\gamma}} \arrow[l, "{F(e_{\gamma,1+\gamma}, p_{\gamma,1+\gamma})}"] & \cdots \arrow[l]
\end{tikzcd}$$

Even more concretely:

$$e_{\gamma,\beta}(x)(\gamma') = \begin{cases} p_{1+\gamma',\gamma}(x) & 1 + \gamma' < \gamma \\ x & 1 + \gamma' = \gamma \\ e_{\gamma,1+\gamma'}(x) & 1 + \gamma' > \gamma \end{cases}$$

This induces a map $e_{\gamma,\beta} : X_\gamma \to X_\beta$. For projection (Eq. (2)), we define $p_{\gamma,\beta} = p_{\gamma,1+\gamma} \circ \pi_\gamma$. We will now show that these operations satisfy Eqs. (3) to (6). For Eq. (5), we must show $e_{\gamma_1,\beta} \circ e_{\gamma_0,\gamma_1} = e_{\gamma_1,\beta}$. This can be shown by unfolding definitions and checking things pointwise. A more efficient method is to observe that since $e_{\gamma_1,\beta}$ and $e_{\gamma_0,\beta}$ are induced by universal properties it suffices to check that the following hold:

$$\begin{cases} e_{\gamma_0,1+\gamma} = e_{\gamma_1,1+\gamma} \circ e_{\gamma_0,\gamma_1} & \gamma_1 < 1 + \gamma \\ e_{\gamma_0,1+\gamma} = e_{\gamma_0,\gamma_1} & \gamma_1 = 1 + \gamma \\ e_{\gamma_0,1+\gamma} = p_{1+\gamma,\gamma_1} \circ e_{\gamma_0,\gamma_1} & \gamma_0 < 1 + \gamma < \gamma_1 \\ 1 = p_{1+\gamma,\gamma_1} \circ e_{\gamma_0,\gamma_1} & \gamma_0 = 1 + \gamma \\ p_{1+\gamma,\gamma_0} = p_{1+\gamma,\gamma_1} \circ e_{\gamma_0,\gamma_1} & 1 + \gamma < \gamma_0 \end{cases}$$

Each of these conditions can be checked using the properties we have already established for e and p. We leave the verification of Eq. (6) as an exercise; it follows by unfold the definition of $p_{\gamma,\beta}$. Let us next consider Eq. (4):

$$p_{\gamma,\beta} \circ e_{\gamma,\beta} = p_{\gamma,1+\gamma} \circ \pi_\gamma \circ e_{\gamma,\beta} = p_{\gamma,1+\gamma} \circ e_{\gamma,1+\gamma} = 1$$

Finally, Eq. (3) reduces to a similar calculation to functoriality for e, splitting into many cases.

We must check the following for all β':

$$\begin{cases} \pi_{\beta'} \overset{\gamma}{=} p_{\beta',\gamma} \circ p_{\gamma,1+\gamma} \circ \pi_\gamma & \beta' < \gamma \\ \pi_{\beta'} \overset{\gamma}{=} p_{\gamma,1+\gamma'} \circ p_{\gamma,1+\gamma'} \circ \pi_\gamma & \gamma = \beta' \\ \pi_{\beta'} \overset{\gamma}{=} e_{\gamma,\beta'} \circ p_{\gamma,1+\gamma'} \circ \pi_\gamma & \gamma < \beta' \end{cases}$$

The first two are strict equalities, and the last uses Eq. (4).

We also must define ϕ_β and ψ_β (Eqs. (7) and (8)). We start by defining $\psi_\beta : F(X_\beta, X_\beta) \to X_\beta$ as the map induced by the following diagram

$$\begin{tikzcd} & & F(X_\beta, X_\beta) \arrow[dll, "F(e,p)"'] \arrow[d, "F(e,p)"'] \arrow[dr, "F(e,p)"] & & \\ {[F(X_0, X_0)]_1} & \cdots \arrow[l] & {[F(X_\gamma, X_\gamma)]_{1+\gamma}} \arrow[l] & {[F(X_{1+\gamma}, X_{1+\gamma})]_{1+1+\gamma}} \arrow[l, "F(e_{\gamma,1+\gamma}, p_{\gamma,1+\gamma})"'] & \cdots \arrow[l] \end{tikzcd}$$

Defining $\phi_\beta : X_\beta \to F(X_\beta, X_\beta)$ is more troublesome. The issue is that we do not really have a mapping out property for X_β, which is a limit. We define ϕ_β as follows

$$\phi_\beta(x) = \lim_{\gamma<\beta} F(p_{\gamma,\beta}, e_{\gamma,\beta})(\pi_\gamma(x))$$

In order to show that this is well-defined, we must show that this is a coherent family, that is, for any $\gamma' < \gamma$

$$F(p_{\gamma,\beta}, e_{\gamma,\beta})(\pi_\gamma(x)) \overset{\gamma'}{=} F(p_{\gamma',\beta}, e_{\gamma',\beta})(\pi_{\gamma'}(x))$$

For this, observe

$$\begin{aligned} F(p_{\gamma',\beta}, e_{\gamma',\beta})(\pi_{\gamma'}(x)) &= F(p_{\gamma',\beta}, e_{\gamma',\beta})(F(e_{\gamma',\gamma}, p_{\gamma',\gamma})(\pi_\gamma(x))) \\ &= F(e_{\gamma',\gamma} \circ p_{\gamma',\beta}, e_{\gamma',\beta} \circ p_{\gamma',\gamma})(\pi_\gamma(x)) \\ &\overset{\gamma'}{=} F(p_{\gamma,\beta}, e_{\gamma,\beta})(\pi_\gamma(x)) && \text{Using the } \gamma'\text{-inverse property} \end{aligned}$$

We must also check that ϕ_β and ψ_β satisfy Eqs. (9) and (10). Happily this can be done by a routine point-wise calculation.

$$\begin{aligned} (\pi_\gamma \circ \psi_\beta \circ \phi_\beta)(x) &= (F(e_{\gamma,\beta}, p_{\gamma,\beta}) \circ \phi_\beta)(x) \\ &= F(e_{\gamma,\beta}, p_{\gamma,\beta})(\lim_{\beta'<\beta} F(p_{\beta',\beta}, e_{\beta',\beta})(\pi_{\beta'}(x))) \\ &= \lim_{\beta'<\beta} F(e_{\gamma,\beta}, p_{\gamma,\beta})(F(p_{\beta',\beta}, e_{\beta',\beta})(\pi_{\beta'}(x))) \\ &= \lim_{\beta'<\beta} F(p_{\beta',\beta} \circ e_{\gamma,\beta}, p_{\gamma,\beta} \circ e_{\beta',\beta})(\pi_{\beta'}(x)) \\ &= \lim_{\beta'<\beta} \pi_\gamma(x) && \text{Using truncation and functoriality} \\ &= \pi_\gamma(x) && \text{Using truncation} \end{aligned}$$

For the next calculation:

$$\begin{aligned}(\phi_\beta \circ \psi_\beta)(x) &= \lim_{\gamma<\beta} F(p_{\gamma,\beta}, e_{\gamma,\beta})(\pi_\gamma(\psi_\beta(x)))\\ &= \lim_{\gamma<\beta} F(p_{\gamma,\beta}, e_{\gamma,\beta})(F(e_{\gamma,\beta}, p_{\gamma,\beta})(x))\\ &= \lim_{\gamma<\beta} F(p_{\gamma,\beta}, e_{\gamma,\beta})(F(e_{\gamma,\beta}, p_{\gamma,\beta})(x))\\ &\overset{\beta}{=} \lim_{\gamma<\beta} x\\ &\overset{\beta}{=} x\end{aligned}$$

The Final Construction. At this point we have $X_{1+\alpha}$ available by transfinite induction and $e_{\alpha,1+\alpha} : X_\alpha \to F(X_\alpha, X_\alpha) = X_{1+\alpha}$ has an α-inverse, $p_{\alpha,1+\alpha}$. Since being an α-inverse is in fact equivalent to being an inverse we're done. □

Remark 2.6. At this point I have not proven the uniqueness of this fixed-point. This proof requires completeness (because it requires Löb induction). It does not seem to be used in Iris proper right now so I have chosen to omit it for the time being.