Modalities and Parametric Adjoints

***Abstract*—Birkedal et al. recently introduced dependent right adjoints as an important class of (non-fibered) modalities in type theory. We observe that several aspects of their calculus are left underdeveloped, and that it cannot serve as an internal language. We resolve these problems by assuming that the modal context operator is a parametric right adjoint. We show that this hitherto unrecognized structure is common. Based on these discoveries we present a new well-behaved Fitch-style multimodal type theory, which can be used as an internal language. Finally, we apply this syntax to guarded recursion and parametricity.**

I. Introduction

When using Martin-Löf Type Theory (MLTT), we often wish to reason about structures present in specific classes of models. Many of these structures—such as notions of time, cohesion, truncation, proof-irrelevance, and globality—can be captured through the addition of unary *modal operators* on types. Unfortunately, the development of *modal type theories* is fraught with difficulties. The overwhelming majority of the modalities we are interested in are *non-fibered*: they send types in one context to types in a different context, disrupting the usual 'context-agnostic' structure of type theory. Thus, all but a few modal operators require extensive changes to the rules of type theory.

The alteration of the judgmental structure of type theory to account for new modal operators is no small task, and various methods have been used in the past. Here we focus on *Fitch-style modal type theories* [17, 21, 11]. In broad strokes, the modal operators of Fitch-style type theories are functors which are *right adjoints*. This criterion is frequently satisfied, so we might expect Fitch-style type theories to find particular use as internal languages. Unfortunately, while their theory has absorbed considerable effort, many technical aspects of the Fitch-style remain unsatisfactory. In particular, there seem to be some inexplicably delicate problems relating to substitution. The purpose of this paper is to research the origin of these problems, highlight a key property that is missing, and use it to completely resolve them.

A. On algebra and type theory

In order to simplify our technical development, for the rest of this paper we will systematically blur the distinction between a Martin-Löf type theory and the *generalized algebraic theory (GAT)* that presents it. GATs originate in the work of Cartmell [14], and are often used to present the semantics of type theory in the guise of *categories with families (CwFs)* [19, 26]. Our approach replaces the study of (variable-based) type-theoretic syntax with the study of the (variable-free) CwFs that support the appropriate connectives. The syntax itself can then be defined as the *free algebra* over the relevant CwF signature, and various theorems guarantee the existence and initiality of this object [27].

There are many technical benefits to this approach. Most importantly, it reifies substitutions as explicit parts of the calculus, which allows us to directly observe their structure rather than infer it as a series of admissibility results *ex post facto*. This is crucial in modal type theory, as the substitutions interact with the modalities in a highly nontrivial manner.

B. Type theory and substitution

The admissibility of substitution is a central property of type theory, and indeed of all logic. By way of example, suppose we have $\Gamma \vdash A \triangleq (x : A_0) \to A_1$ type, and a substitution $\sigma : \Delta \to \Gamma$. Consider the type $\Delta \vdash A[\sigma]$ type. At the very least, we expect that this is again a dependent product: there should exist σ_0 and σ_1 such that $A[\sigma] = (x : A_0[\sigma_0]) \to A_1[\sigma_1]$. In variable-based presentations of type theory, equations of this sort are part of the definition of substitution, which is then validated by proving that *substitution is admissible* in the type system. In variable-free presentations, such as CwFs, such equations are part of the definition of the generalized algebraic theory, which postulates a number of *naturality clauses* that allow pushing substitutions under connectives.

Each of the standard connectives of type theory is understood to satisfy a property of this sort. Together, they ensure that type theory behaves in a predictable and usable manner. This global property is variably referred to as *admissibility of substitution*, *naturality*, *associativity*, or *stability under substitution*.

C. Substitution and the Fitch-style

It hence comes as a surprise that proving the admissibility of substitution for Fitch-style calculi, such as DRA [11] or $\mathsf{MLTT}_{🔒}$ [21], is not an easy task. There is no obvious way to write down naturality equations for modal types akin to those for other connectives. Indeed, examining the proof of admissibility for both languages in *loc. cit.* we discover something surprising: substitution in a term is not defined by induction over the term, but over the substitution itself! In short, the naturality of modal rules depends on the precise structure of substitutions.

This might seem like a small technical point, but in fact it puts a spanner in the works. The ability to perform an induction on substitutions presupposes that we know exactly how they are generated. This is true in free algebra (syntax), where they consist entirely of definable terms. However, our concern lies with algebras in which not every substitution is generated in this way, where no such induction is possible.

In practice, the problem is encountered if we try to use Fitch-style type theories as internal languages. Suppose we begin with a category $\mathcal{C}$ with enough structure to interpret the modal types. We can then formulate the free type theory $\mathbb{T}_{\mathcal{C}}$ which includes the morphisms of $\mathcal{C}$ as substitutions. We call this type theory the *internal language* of $\mathcal{C}$: we can use it to talk about $\mathcal{C}$ in type-theoretic terms. If we try to adapt the admissibility proof to $\mathbb{T}_{\mathcal{C}}$ we find ourselves in a predicament: we can no longer induct on the substitution in order to commute it with the modal rules. It is therefore no longer evident that the theorems of the logic retain their 'shape' under substitution.

D. Rectifying the problem

The problems of Fitch-style systems are well-known, and previous work attempted to address them by replacing the elimination rule [23]. Nevertheless, the resultant type theory is weaker, and neither DRA nor $\mathsf{MLTT}_{\text{🔒}}$ can be embedded in it. The aim of this paper is to properly rectify the issue of substitution without weakening the intuitive and powerful Fitch-style elimination rules of [11, 21].

We begin by observing that the secret ingredient that makes substitution work in the free theory is that the modal context operator is a *parametric right adjoint (PRA)*. To illustrate why this works, we show that substitution commutes with the function space exactly because context extension itself is a PRA. It thus appears that PRAs underlie the stability of a wide variety of rules which modify the context by applying an operator. Thus, our observations compel us to introduce a new Fitch-style type theory which from the outset assumes that its modal context operator is a PRA. This new system, which we call $\mathsf{FitchTT}$, can readily be used as an internal language.

The type-theoretic rules we introduce for parametric right adjoints appear unusual at first sight, in part because they involve strange substitutions of type $\Gamma \longrightarrow F(\mathbf{1})$ whose codomain is a functor applied to a terminal object. Surprisingly, we show that a lot of previously introduced 'extra-logical' structure found in various type theories amounts to such substitutions. For example, we show that extending a context by an affine dimension variable, as used in internalizing parametricity [7, 15], forms a PRA. Furthermore, the 'tick variables' used in clocked type theory [4] can also be seen as arising from a PRA. We show that recognition of this fact can be used to provide a 'rational reconstruction' of the rule for the tick constant which is simpler and moreover evidently implementable.

E. Contributions

In summary we make the following contributions:

- We recognize parametric right adjoints as the key ingredient for validating substitution in modal type theories.
- We propose a new modal dependent type theory $\mathsf{FitchTT}$ which uses parametric right adjoints to generalize DRA to support multiple modes and modalities.
- We prove that an appropriate instance of $\mathsf{FitchTT}$ constitutes a conservative extension of DRA, and investigate its more complex relationship to $\mathsf{MLTT}_{\text{🔒}}$ [21].
- We show that instantiating $\mathsf{FitchTT}$ with appropriate mode theories yields the judgmental structure of prior type theories for internalized parametricity [7, 15], and guarded recursion [4, 5].

F. Notation

We will use standard notation for CwFs. We write Γ, Δ, etc. for contexts and σ, γ, δ for substitutions $\Delta \to \Gamma$. We also write $\mathbf{1}$ for the empty context and $\Gamma.A$ for the extension of Γ by $\Gamma \vdash A$ type. If $\sigma : \Delta \to \Gamma$ and $\Delta \vdash M : A[\sigma]$, we can extend σ to $\sigma.M : \Delta \to \Gamma.A$. There is a weakening substitution $\uparrow : \Gamma.A \to \Gamma$, and we write $\uparrow^n$ for the composite of n of them. The last element in a context is accessed by the term $\Gamma.A \vdash \mathbf{v}_0 : A[\uparrow]$. Finally, substitutions σ have an action on types and terms that is denoted by $A[\sigma]$ and $M[\sigma]$ respectively.

II. Modalities and Substitution

Suppose we have a type theory on a category $\mathcal{C}$, and some endofunctor $\Box : \mathcal{C} \to \mathcal{C}$ of interest. Our objective is to internalize $\Box$ in the type theory. Adopting the rule

$$\frac{\Gamma \vdash A \ \mathsf{type}}{\Gamma \vdash \Box A \ \mathsf{type}}$$

amounts to assuming that the functor $\Box$ is *fibered* [34, §2], i.e. has an action on types whose output remains in the same context. Most of the functors that we are interested in do not.

If we do not wish to assume that $\Box$ is fibered, we may formulate rules assuming only its functoriality, i.e.

$$\textsc{ty/functorial-form}\ \frac{\Gamma \vdash A \ \mathsf{type}}{\Box\Gamma \vdash \Box A \ \mathsf{type}} \qquad \textsc{ty/functorial-intro}\ \frac{\Gamma \vdash M : A}{\Box\Gamma \vdash \mathsf{mod}(M) : \Box A}$$

Unfortunately, these rules do not admit substitution. Suppose that $\Gamma \vdash M : A$ and $\sigma : \Delta \to \Box\Gamma$, so that $\Delta \vdash (\Box A)[\sigma]$ type. For $\Box$ to be natural, there should be a substitution σ' for which $\Delta \vdash (\Box A)[\sigma] = \Box(A[\sigma'])$ type. This is constitutionally impossible: the right hand side is typable only in a context of the form $\Box\Gamma'$, not a general Δ.

To obtain a usable type theory one must repair this. By and large, there are three standard solutions.

a) Delay substitutions: Instead of propagating substitutions under modal constructs, we may choose to absorb them. We can do so by building a *delayed substitution* into the modal introduction rules for both types and terms:

$$\frac{\sigma : \Gamma \to \Box\Delta \qquad \Delta \vdash M : A}{\Gamma \vdash \mathsf{mod}(M)^{\sigma} : \Box^{\sigma} A}$$

Substitution is then effected by absorbing a morphism into this cut: for any $\sigma' : \Gamma' \to \Gamma$ we have

$$\Gamma' \vdash \mathsf{mod}(M)^{\sigma}[\sigma'] = \mathsf{mod}(M)^{\sigma\circ\sigma'} : \Box^{\sigma\circ\sigma'} A$$

This method was pioneered by Bierman and de Paiva [8].

b) Split the contexts: Another approach, originally due to Davies and Pfenning [32], replaces the usual judgments by a form that involves two or more contexts. The *dual context* $\Delta; \Gamma$ stands for the object $\Box\Delta \times \Gamma$. The introduction rules are

$$\frac{\cdot; \Delta \vdash A \ \mathsf{type}}{\Delta; \Gamma \vdash \Box A \ \mathsf{type}} \qquad \frac{\cdot; \Delta \vdash M : A}{\Delta; \Gamma \vdash M : \Box A}$$

The semantics of these rules is clear: if $\cdot; \Delta \vdash A$ type is interpreted by a family $\pi_A : \Delta.A \longrightarrow \Delta$, then $\Box A$ is interpreted by the family $\Box\pi_A \times \mathsf{id}_\Gamma$ which is over $\Box\Delta \times \Gamma$. This rule is well-behaved under substitution, but with a caveat: we must change the notion to follow the structure of contexts. We must take our 'primitive' substitutions $(\delta;\gamma) : \Delta';\Gamma \to \Delta;\Gamma$ to be morphisms $F(\delta) \times \gamma : \Box\Delta' \times \Gamma' \to \Box\Delta \times \Gamma$ of $\mathcal{C}$.

c) Factorize the substitution: One way to push a substitution $\sigma : \Delta \to \Box\Gamma$ under a modality is to assume that it factorizes in a convenient way. For example, we may assume that for every Δ there is a *universal arrow* from Δ to $\Box$, i.e. an object $\Delta.\text{🔒}$ and a morphism $\eta_\Delta : \Delta \to \Box(\Delta.\text{🔒})$ through which every substitution into a modal context factorizes uniquely:

$$\begin{tikzcd} \Delta \arrow[r, "\eta_\Delta"] \arrow[dr, "\sigma"'] & \Box(\Delta.\text{🔒}) \arrow[d, "\Box\hat{\sigma}"] \\ & \Box\Gamma \end{tikzcd} \qquad \begin{tikzcd} \Delta.\text{🔒} \arrow[d, dashed, "\hat{\sigma}"] \\ \Gamma \end{tikzcd}$$

This does not quite solve the substitution problem for TY/FUNCTORIAL-FORM, but it simplifies it canonically: it allows us to find a 'maximal' substitution $\hat{\sigma}$ that we can push under the modality, so that $\Delta \vdash (\Box A)[\sigma] = \Box(A[\hat{\sigma}])[\eta_\Delta]$ type. In a sense, this is a case of carrying a 'canonical delayed substitution' η_Δ.

A simple observation allows us to make η invisible. It is a well-known fact from category theory that if such a universal arrow exists for every Δ, then $-.\text{🔒}$ extends to an endofunctor which is left adjoint to $\Box$. We can promote this to an additional *operator on contexts*, and replace the introduction rules with

$$\frac{\Gamma.\text{🔒} \vdash A \text{ type}}{\Gamma \vdash \Box A \text{ type}} \qquad \frac{\Gamma.\text{🔒} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \Box A}$$

These are called *Fitch-style rules* [17].

All three approaches have their strengths and weaknesses. The Bierman-de Paiva style of delayed substitutions is conceptually clear, but difficult to use and implement. Moreover, it does not readily adapt to support multiple modalities, at least not when they interact in a nontrivial way. On the other hand, the split-context approach has proven practical whenever the modalities interact in certain convenient ways (see e.g. [33]), but that is the exception rather than the rule.

In contrast, the Fitch-style approach is supported by a single *universal property* which fully determines the modality up to isomorphism—just as with standard connectives, like dependent products and sums. Thus, one might be led to believe that Fitch-style calculi are the preferred formalism. Alas, they suffer from a number of technical disadvantages. We illustrate these using a specific theory, viz. the calculus of dependent right adjoints.

A. The calculus of dependent right adjoints

The *calculus of dependent right adjoints* (DRA) [11] consists of standard Martin-Löf type theory extended with an operation on contexts—denoted by 🔒—and a single modality $\Box$ on types. In addition to the usual CwF structure, DRA postulates a *dependent adjunction*.

Definition 1. A *dependent adjunction* consists of

1) an endofunctor $-.\text{🔒}$ on the category of contexts
2) an assignment $\Box$ from types to types, such that

DRA/TY/MOD
$$\frac{\Gamma.\text{🔒} \vdash A \text{ type}}{\Gamma \vdash \Box A \text{ type}}$$

3) a bijection $\mathsf{mod}(-)/\mathsf{unmod}(-)$ on terms, such that

DRA/TM/MOD
$$\frac{\Gamma.\text{🔒} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \Box A}$$

DRA/TM/UNMOD
$$\frac{\Gamma \vdash M : \Box A}{\Gamma.\text{🔒} \vdash \mathsf{unmod}(M) : A}$$

All of $\Box$, $\mathsf{mod}(-)$, and $\mathsf{unmod}(-)$ must be natural in Γ.

While $-.\text{🔒}$ has an action on the entire category of contexts, the modality $\Box$ acts *only on types*, which are a distinct sort (depending on contexts).[1] The fact that $\mathsf{mod}(-)$ and $\mathsf{unmod}(-)$ form a bijection yields the following β and η laws for $\Box$.

$$\frac{\Gamma.\text{🔒} \vdash M : A}{\Gamma.\text{🔒} \vdash \mathsf{unmod}(\mathsf{mod}(M)) = M : A}$$

$$\frac{\Gamma \vdash M : \Box A}{\Gamma \vdash \mathsf{mod}(\mathsf{unmod}(M)) = M : \Box A}$$

Do these rules admit substitution? In the case of DRA/TM/MOD, the naturality required of $\Box$ and $\mathsf{mod}(-)$ solves the problem: for any $\Gamma.\text{🔒} \vdash M : A$ and $\sigma : \Delta \to \Gamma$ it implies that

$$\Delta \vdash \mathsf{mod}(M)[\sigma] = \mathsf{mod}(M[\sigma.\text{🔒}]) : \Box(A[\sigma])$$

where $\sigma.\text{🔒}$ is the action of $-.\text{🔒}$ on σ. The same cannot be said of the elimination rule DRA/TM/UNMOD: there is no evident way to commute a substitution with $\mathsf{unmod}(-)$. Indeed, we cannot use naturality, as a general substitution $\sigma : \Delta \to \Gamma.\text{🔒}$ need not be of the form $\gamma.\text{🔒}$.

In order to address this, the original paper on DRA replaces DRA/TM/UNMOD with a rule involving additional weakening:

DRA/TM/UNMOD*
$$\frac{\Gamma \vdash M : \langle \mu \mid A \rangle}{\Gamma.\text{🔒}.A_0.\cdots.A_{n-1} \vdash \mathsf{unmod}(M) : A[\uparrow^n]}$$

This rule has an 'exorbitant privilege': it is stable under substitution in the free theory. Every $\sigma : \Delta \to \Gamma$ in the free algebra is a substitution that is *definable* in the pure type theory with no constants. One can then prove that for every such $\sigma : \Delta \to \Gamma.\text{🔒}.A$ there is a $\sigma' : \Delta' \to \Gamma$ such that $\Delta = \Delta'.\text{🔒}.A$ and $\sigma = \sigma'.\text{🔒}.\mathbf{v}_0$. This enables us to extract σ' from σ, and push that under $\mathsf{unmod}(-)$.

This is all well and good if we just want a syntax for dependent adjunctions: we can write proofs in the free algebra and interpret them in any dependent adjunction [11, §3.1]. One can even implement this syntax, following an approach similar to that of [21] for $\mathsf{MLTT}_{\text{🔒}}$. Nevertheless, there is something unsatisfying about this state of affairs. The aforementioned factorization property of substitutions is proven by performing an *induction on the substitution* σ. As a consequence, it only

[1] This gap disappears if we can blend types and contexts. For example, if the CwF is *democratic*, i.e. if for every context Γ there is a $\vdash \tilde{\Gamma}$ type such that $\Gamma \cong \mathbf{1}.\tilde{\Gamma}$, then $\Box$ can be extended to a right adjoint of $-.\text{🔒}$ [11, §4.1].

works in the free algebra: it is not in general possible to decompose substitutions by induction in an arbitrary CwF. In other words, the stability of $\mathsf{unmod}(-)$ depends on the *absence* of certain substitutions.

This may seem like a small price to pay, but in fact this restriction has grave consequences: it prohibits the use of DRA as the internal language of an arbitrary dependent adjunction. In models of DRA, DRA/TM/UNMOD* may not respect substitution, leading to unwelcome surprises such as the truth of a lemma proved using the type theory depending on the precise context in which it is stated. Unfortunately, such models are not uncommon: for example, $\mathsf{MLTT}_{\faLock}$ [21] is a proper extension (and hence a model) of DRA, yet the $\mathsf{unmod}(-)$ form of DRA is not stable under substitution in $\mathsf{MLTT}_{\faLock}$. In short, Fitch-style type theories à la DRA cannot play the rôle of internal languages.

B. Parametric right adjoints

It is natural to wonder if there is a special property of the pure syntax which confers stability under substitution. If we were to identify and axiomatize it, we would have a chacterization of dependent adjunctions that support it.

To this end, it is instructive to consider a particular example. Suppose that $\vdash A$ type, i.e. that A is a closed type. Then context extension by A coincides with $- \times A$, and has a dependent right adjoint $A \to (-)$ [11, §5]. Writing out the rule DRA/TM/MOD yields the usual introduction rule for the function space. However, the elimination rule DRA/TM/UNMOD looks unfamiliar:

$$\frac{\Gamma \vdash M : A \to B}{\Gamma.A \vdash \mathsf{unlam}(M) : B}\ \text{TM/UNLAM}$$

This rule suffers from the same issues as the more general DRA/TM/UNMOD. Given a closed term $\vdash N : A$, there is no evident way to push the corresponding substitution $\mathbf{1} \to \mathbf{1}.A$ under $\mathsf{unlam}(-)$. In fact, the traditional elimination rule

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M(N) : B[\mathsf{id}.N]}\ \text{TM/APP}$$

is a kind of closure of $\mathsf{unlam}(-)$ under substitution: we may define $M(N) \triangleq \mathsf{unlam}(M)[\mathsf{id}.N]$. Conversely, using TM/APP we can define $\mathsf{unlam}(M) \triangleq M[\uparrow](\mathbf{v}_0)$. Thus, TM/UNLAM and TM/APP are interderivable rules, but only the latter preserves the admissibility of substitution.

On the surface, TM/APP does not seem to be the most general closure of TM/UNLAM under substitution: that would include an arbitrary $\sigma : \Delta \to \Gamma.A$ in the premise, which the conclusion would carry it in a delayed form. However, this is not necessary because every such σ is determined by a substitution $\Delta \to \Gamma$ and a term $\Delta \vdash N : A$.

Lemma 1. *Let* $\vdash A$ type*. Any substitution* $\sigma\colon \Delta \longrightarrow \Gamma.A$ *can be uniquely decomposed into a pair of substitutions*

$$\sigma_0 \triangleq {\uparrow} \circ \sigma : \Delta \to \Gamma \qquad r \triangleq (!_\Gamma)^+ \circ \sigma : \Delta \to \mathbf{1}.A$$

where $\gamma^+ \triangleq (\gamma \circ {\uparrow}).\mathbf{v}_0 : \Delta.A \to \Gamma.A$ *for any* $\gamma : \Delta \to \Gamma$.

Thus, in the presence of a chosen substitution $r : \Delta \to \mathbf{1}.A$ (i.e. a term), substitutions $\Delta \to \Gamma.A$ and $\Delta \to \Gamma$ correspond. This almost looks like the identity functor is left adjoint to $-.A$, but that is not quite right. To arrive at the right abstraction, we must see r as an object in the slice over $\mathbf{1}.A$. Its defining equation then shows that σ is a morphism $r \to (!_\Gamma)^+$ in the slice category. This is an instance of the following structure.

Definition 2. Let $\mathcal{C}$ have a terminal object $\mathbf{1}_\mathcal{C}$. A functor $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a *parametric right adjoint* if the induced functor $G/\mathbf{1}\colon \mathcal{C} \longrightarrow \mathcal{D}/G(\mathbf{1}_\mathcal{C})$ is a right adjoint.

See [13] and [37, §2] for the origins of PRAs.

In our case $(-.A)/\mathbf{1}$ maps $\Delta \mapsto (!_\Delta)^+ : \Delta.A \to \mathbf{1}.A$, and the left adjoint is given by $F(r : \Gamma \longrightarrow \mathbf{1}.A) \triangleq \Gamma$. The unit and counit have recognizable forms: the unit at $r\colon \Gamma \longrightarrow \mathbf{1}.A$ is the substitution $\eta[r] \triangleq \mathsf{id}.\mathbf{v}_0[r]\colon \Gamma \longrightarrow \Gamma.A$, and the counit at Γ is $\epsilon[\Gamma] \triangleq {\uparrow}\colon \Gamma.A \longrightarrow \Gamma$.

Using these insights we may restate the application rule TM/APP without actually changing any of its ingredients:

$$\frac{r\colon \Gamma \longrightarrow \mathbf{1}.A \qquad F(r) \vdash M : A \to B}{\Gamma \vdash M\langle r\rangle : B[\eta[r]]}\ \text{TM/PRA-APP}$$

This restatement uses only two facts: that $A \to (-)$ is a dependent right adjoint to $(-).A$, and that $(-).A$ is itself a parametric right adjoint with left adjoint F. One naturally wonders whether we can adapt this maneuver to a general dependent adjunction: can an ill-behaved elimination rule (like TM/UNLAM) always be replaced by an equivalent well-behaved rule (like TM/APP) if we assume that the modal context operator is a parametric right adjoint? The answer is positive: we will in fact show that the adjunctions automatically guarantee the admissibility of substitution!

Indeed, suppose $-.\faLock$ has a dependent right adjoint $\Box$. Suppose furthermore that $-.\faLock$ is a parametric right adjoint, and write Γ/r for the application of the left adjoint to $r : \Gamma \to \mathbf{1}.\faLock$. Recalling that $\eta[r] : \Gamma \to (\Gamma/r).\faLock$, we can write down a rule

$$\frac{r\colon \Gamma \longrightarrow \mathbf{1}.\faLock \qquad \Gamma/r \vdash M : \Box B}{\Gamma \vdash M \mathbin{@} r : B[\eta[r]]}\ \text{DRA/TM/PRA-UNMOD}$$

This rule can be derived from DRA/TM/UNMOD: we define the conclusion by $M \mathbin{@} r \triangleq \mathsf{unmod}(M)[\eta[r]]$. In fact, it is equivalent to DRA/TM/UNMOD. Given $\Gamma \vdash M : \Box A$, we recall that $\epsilon[\Gamma] : \Gamma.\faLock/(!_\Gamma.\faLock) \to \Gamma$, so we have

$$\Gamma.\faLock/(!_\Gamma.\faLock) \vdash M[\epsilon[\Gamma]] : (\Box A)[\epsilon[\Gamma]]$$

By naturality, this type is equal to $\Box(A[\epsilon[\Gamma].\faLock])$. Hence, we can define $\mathsf{unmod}(M)$ as

$$\Gamma.\faLock \vdash M[\epsilon] \mathbin{@} (!_\Gamma.\faLock) : A[\epsilon[\Gamma].\faLock][\eta[!_\Gamma.\faLock]]$$

The type of this term is equal to A, as $\epsilon[\Gamma].\faLock \circ \eta[!_\Gamma.\faLock]$ is the identity by one of the triangle laws of the adjunction.

Therefore, the PRA structure allows us to equivalently restate DRA/TM/UNMOD as DRA/TM/PRA-UNMOD. It remains to prove that, unlike the former rule, the latter can be made to admit substitution. Given any $\sigma : \Delta \to \Gamma$ we may see it as an arrow $r \circ \sigma \to r$ in the slice category over $\mathbf{1}.\text{🔒}$. Applying the left adjoint gives $\sigma/\text{🔒} : \Delta/r \circ \sigma \to \Gamma/r$. This substitution acts on M to yield $\Delta/r \circ \sigma \vdash M[\sigma/\text{🔒}] : (\Box B)[\sigma/\text{🔒}]$. But $(\Box B)[\sigma/\text{🔒}] = \Box(B[(\sigma/\text{🔒}).\text{🔒}])$, so we have

$$\Delta \vdash M[\sigma/\text{🔒}] \mathbin{@} (r \circ \sigma) : B[(\sigma/\text{🔒}).\text{🔒}][\eta[r \circ \sigma]]$$

This type is equal to $B[\eta[r]][\sigma]$ by the naturality of η. Hence, we can postulate that $(M \mathbin{@} r)[\sigma] = M[\sigma/\text{🔒}] \mathbin{@} (r \circ \sigma)$. In fact, this equation can be derived from $M \mathbin{@} r \triangleq \mathsf{unmod}(M)[\eta[r]]$ by the naturality of η and of $\mathsf{unmod}(-)$.

Some version of Lemma 1 has been shown *en passant* for all prior Fitch-style calculi in the process of proving the admissibility of substitution [4, 11, 21, 15]. In each case the modal elimination rules can be derived by unfolding the components of the parametric adjunction in the general rule DRA/TM/PRA-UNMOD. We choose the notation $M \mathbin{@} r$ to emphasize the connection with the application rule.

To recap: we have addressed the issue of admissibility of substitution. In the case of connectives which modify the context—like context extension and (Fitch-style) modalities—we have found that the structure that essentially underlies the admissibility of substitution is that of a parametric right adjoint. We continue by introducing a type theory which requires that each of its modal context operators is a PRA.

III. A MULTIMODAL FITCH-STYLE TYPE THEORY

In this section we introduce a modal type theory for dependent adjunctions whose left adjoints are also parametric right adjoints, which we call FitchTT. By the reasoning explored in Section II, this type theory will readily admit substitution.

A. Multimode and multimodal aspects

While our definition of dependent adjunction involved only a single category, adjunctions in general connect two possibly distinct categories. In order to obtain the most expressive theory, we therefore allow for multiple categories in FitchTT. We call each such category a *mode*, making FitchTT a *multimode* type theory. Each judgment of FitchTT is annotated by the mode it lives in. We denote modes by m, n, o, etc.

Accordingly, the modalities of FitchTT are no longer operators on types in a single category, but map types across categories. Each modality $\mu\colon n \longrightarrow m$ induces an operation $\langle \mu \mid - \rangle$ from types at mode n to types at mode m. As we allow many modalities between each pair of modes, FitchTT is a *multimodal* type theory. We denote modalities by μ, ν, ξ, etc.

Viewing modalities as functors suggests that modes and modalities should form a category: there should be a composite modality $\mu \circ \nu\colon o \longrightarrow m$ for every $\mu\colon n \longrightarrow m$ and $\nu\colon o \longrightarrow n$. To this structure we add one more layer, namely 2-cells *between* modalities. These induce natural transformations: a 2-cell $\alpha : \mu \Rightarrow \nu$ enables the definition of a function $\langle \nu \mid A \rangle \to \langle \mu \mid A \rangle$ for a type A. We denote 2-cells by α, β, γ, etc.

FITCH/CX/MEX
$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} \mathbin{@} m}{\Gamma.\{\mu\}\ \mathsf{cx} \mathbin{@} n}$$

FITCH/TY/MOD
$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma.\{\mu\} \vdash A\ \mathsf{type} \mathbin{@} n}{\Gamma \vdash \langle \mu \mid A \rangle\ \mathsf{type} \mathbin{@} m}$$

FITCH/TM/MOD
$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma.\{\mu\} \vdash M : A \mathbin{@} n}{\Gamma \vdash \mathsf{mod}_{\mu}(M) : \langle \mu \mid A \rangle \mathbin{@} m}$$

Fig. 1. Introduction and formation for modal types.

All in all, FitchTT follows prior modal type theories in organizing this data into a strict 2-category, a *mode theory* [28, 29, 23], for which we usually write $\mathcal{M}$. No rule changes the mode theory: it is a parameter to the type theory.

B. The mode-local fragment

Each judgment of FitchTT is indexed by a mode. For instance, we indicate that Γ is a well-formed context at mode m by writing $\Gamma\ \mathsf{cx} \mathbin{@} m$. Modes interact with each other only through modal types. In other words, if we do not include any modal rules, each typing derivation remains in a single mode. We call the collection of non-modal rules the *mode-local fragment* of FitchTT. This fragment is given parametrically in the mode m, and consists of the usual rules of MLTT with products, sums, and intensional identity types.

C. The modal fragment: formation and introduction

The modal rules of FitchTT mediate between the different modes of the type theory. They very closely follow the DRA calculus described in Section II, but incorporate slight generalizations to allow for the multimodal structure.

The formation and introduction rules are given in Fig. 1. For each modality $\mu\colon n \longrightarrow m$, there is both a modal context operator $-.\{\mu\}$ as well as an operator $\langle \mu \mid - \rangle$ on types. Like in DRA, the idea is that $-.\{\mu\}$ is the left adjoint and $\langle \mu \mid - \rangle$ is its dependent right adjoint. However, a modality may now cross between different modes. Thus, if $\Gamma\ \mathsf{cx} \mathbin{@} m$ is a context at mode m, and $\mu\colon n \longrightarrow m$, then we obtain a context $\Gamma.\{\mu\}\ \mathsf{cx} \mathbin{@} n$ at mode n. This action is *contravariant* by convention: the mode theory $\mathcal{M}$ covariantly specifies the structure of the modalities $\langle \mu \mid - \rangle$, so their left adjoints $-.\{\mu\}$ act with opposite variance.

The introduction rule for modal types, FITCH/TY/MOD, is a slight variation on DRA/TY/MOD which accounts for passing between modes. The same is true for the modal term introduction rule FITCH/TM/MOD: given M of the appropriate type at mode n, it ensures that $\mathsf{mod}_{\mu}(M)$ is a term at mode m.

D. The modal fragment: the elimination rule

Unlike the case of introduction, the well-behaved elimination rule of the DRA calculus does not readily adapt to a multimodal

$$\frac{\mu\colon n\longrightarrow m \qquad \Gamma.\{\mu\}\vdash A\ \mathsf{type} @ n \qquad r\colon \Gamma\longrightarrow\{\mu\} @ n \qquad \Gamma/(r:\mu)\vdash M:\langle\mu\mid A\rangle @ m}{\Gamma\vdash M @ r : A[\eta[r]] @ n}\ \textsc{fitch/tm/unmod}$$

$$\frac{\mu\colon n\longrightarrow m \qquad \Gamma.\{\mu\}\vdash A\ \mathsf{type} @ n \qquad r\colon \Gamma\longrightarrow\{\mu\} @ n \qquad \Gamma/(r:\mu).\{\mu\}\vdash M:A @ m}{\Gamma\vdash \mathsf{mod}_\mu(M) @ r = M[\eta[r]] : A[\eta[r]] @ n}\ \textsc{fitch/tm/unmod-mod}$$

$$\frac{\mu\colon n\longrightarrow m \qquad \Gamma.\{\mu\}\vdash A\ \mathsf{type} @ n \qquad \Gamma\vdash M:\langle\mu\mid A\rangle @ m}{\Gamma\vdash M = \mathsf{mod}_\mu(M[\epsilon] @ \mathbf{m}_\mu) : \langle\mu\mid A\rangle @ m}\ \textsc{fitch/tm/mod-unmod}$$

Fig. 2. The full elimination rule for modal types

$$\frac{\Gamma\ \mathsf{cx} @ m}{\Gamma.\{\mathsf{id}_m\} = \Gamma\ \mathsf{cx} @ m}\ \textsc{fitch/cx/id}$$

$$\frac{\nu\colon o\longrightarrow n \qquad \mu\colon n\longrightarrow m \qquad \Gamma\ \mathsf{cx} @ m}{\Gamma.\{\mu\circ\nu\} = \Gamma.\{\mu\}.\{\nu\}\ \mathsf{cx} @ o}\ \textsc{fitch/cx/comp}$$

$$\frac{\mu\colon n\longrightarrow m \qquad \delta\colon\Gamma\longrightarrow\Delta @ m}{\delta.\{\mu\}\colon \Gamma.\{\mu\}\longrightarrow\Delta.\{\mu\} @ n}\ \textsc{fitch/sb/mex}$$

$$\frac{\mu_0,\mu_1 : n\longrightarrow m \qquad \alpha:\mu_0\Rightarrow\mu_1 \qquad \Gamma\ \mathsf{cx} @ m}{\{\alpha\}_\Gamma\colon \Gamma.\{\mu_1\}\longrightarrow\Gamma.\{\mu_0\} @ n}\ \textsc{fitch/sb/coe}$$

Fig. 3. Selected rules for multiple modalities and modal substitutions.

setting. We must hence design an elimination rule anew, using the insights we developed in Section II.

We first introduce some notation. Whenever $\mu\colon n\longrightarrow m$, we write $\{\mu\}$ for the context $\mathbf{1}.\{\mu\}\ \mathsf{cx} @ n$. We will also write $\mathbf{m}_\mu \triangleq\, !_\Gamma.\{\mu\}\colon \Gamma.\{\mu\}\longrightarrow\{\mu\} @ n$

The place to begin is the multimodal analogue of the elimination rule of the dependent adjunction, viz.

$$\frac{\mu\colon n\longrightarrow m \qquad \Gamma\vdash M:\langle\mu\mid A\rangle @ m}{\Gamma.\{\mu\}\vdash \mathsf{unmod}_\mu(M) : A @ n}\ \textsc{fitch/tm/unmod-dra}$$

Now suppose that $-.\{\mu\}$ is a parametric right adjoint. This means that for each modality $\mu\colon n\longrightarrow m$, we are given a modal context operator which maps a $\Gamma\ \mathsf{cx} @ n$ and a substitution $r\colon\Gamma\longrightarrow\{\mu\} @ n$ to a new context $\Gamma/(r:\mu)\ \mathsf{cx} @ m$. The unit for this adjunction gives for each such r a substitution $\eta[r]:\Gamma\to\Gamma/(r:\mu).\{\mu\} @ n$. Using FITCH/TM/UNMOD-DRA, we can derive a rule

$$\frac{r\colon\Gamma\longrightarrow\{\mu\} @ n \qquad \Gamma/(r:\mu)\vdash M:\langle\mu\mid A\rangle @ m}{\Gamma\vdash M @ r : A[\eta[r]] @ n}$$

by setting $M @ r \triangleq \mathsf{unmod}_\mu(M)[\eta[r]]$. Just as in Section II-B, this new rule is equivalent to FITCH/TM/UNMOD-DRA and admits substitution. We take it as the definitive elimination rule for modal types. The β and η principles are given in in Fig. 2.

The presence of multiple modalities does not complicate the elimination rule, unlike in other multimodal calculi, e.g. [23]. Instead, the interaction of modalities is governed by the substitution calculus of FitchTT.

E. The substitution calculus

The substitution calculus for FitchTT can be divided into the mode-local part—the standard substitution operations of MLTT in each mode—and the part concerning modal operations. For instance, at each mode m there are identities and compositions of substitutions, as well as a unique substitution $!_\Gamma\colon\Gamma\longrightarrow\mathbf{1} @ m$ for each $\Gamma\ \mathsf{cx} @ m$. Mode-local substitutions are thus standard [26], so we focus on the novel modal ones.

As we mentioned before, the mode theory $\mathcal{M}$ is a strict 2-category. We mirror this fact within the type theory by postulating that the assignment of modal context operators $-.\{\mu\}$ to modalities μ is *2-functorial* in the mode theory. This is established by the rules of Fig. 3.

Furthermore, each one of these operators $-.\{\mu\}$ is itself a functor between context categories. For each $\mu\colon n\longrightarrow m$ there is a functorial action on substitutions, which to $\delta\colon\Gamma\longrightarrow\Delta @ m$ assigns a substitution $\delta.\{\mu\}\colon\Gamma.\{\mu\}\longrightarrow\Delta.\{\mu\} @ n$. This assignment respects identity and composition. For example, $(\gamma_0\circ\gamma_1).\{\mu\} = \gamma_0.\{\mu\}\circ\gamma_1.\{\mu\}$. It is also functorial in $\mathcal{M}$, so we have $\gamma.\{\mu\circ\nu\} = \gamma.\{\mu\}.\{\nu\}$. In short, we have a functorial assignment of functors to modalities.

We previously also mentioned that $-.\{\mu\}$ sends a 2-cell $\alpha:\nu\Rightarrow\mu$ to a natural transformation. This is effected by postulating a natural transformation with components $\{\alpha\}_\Gamma\colon\Gamma.\{\mu\}\longrightarrow\Gamma.\{\nu\} @ n$ at $\Gamma\ \mathsf{cx} @ m$. Notice that the action on 2-cells is also contravariant, so that the substitution $\{\alpha\}_\Gamma$ induces a function $\langle\nu\mid A\rangle\to\langle\mu\mid A\rangle$ in the type theory.

Finally, FitchTT requires that each $-.\{\mu\}$ be a parametric right adjoint. This can be expressed directly by requiring a *modal restriction operator* $-/(-:\mu)$ on contexts, like the one we described in Section III-D. Within the substitution calculus we also require the appropriate unit and counit substitutions. The basic rules are presented in Fig. 4, while the full definition is contained in Appendix A. Note the rule FITCH/SB/MRES, which expresses the functoriality of the left adjoint of the parametric adjunction. This works as in Section II: we regard δ as a morphism from $r\circ\delta\to r$ in the slice category over $\{\mu\}$, which allows us to apply the left adjoint $-/\mu$.

F. Some simple examples

As an example of using the type theory, we show that we can construct type-theoretic equivalences [35, §4] that weakly mirror the structure of the mode theory $\mathcal{M}$ within FitchTT. In particular, we show that $\langle\mu\circ\nu\mid A\rangle\simeq\langle\mu\mid\langle\nu\mid A\rangle\rangle$ and $\langle\mathsf{id}_m\mid A\rangle\simeq A$ for appropriate modalities and modes μ,ν,m.

FITCH/CX/MRES
$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} \mathbin{@} n \qquad r : \Gamma \longrightarrow \{\mu\} \mathbin{@} m}{\Gamma/(r:\mu)\ \mathsf{cx} \mathbin{@} m}$$

FITCH/SB/COUNIT
$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} \mathbin{@} m}{\epsilon[\Gamma]\colon \Gamma.\{\mu\}/(\mathbf{m}_\mu : \mu) \longrightarrow \Gamma \mathbin{@} m}$$

FITCH/SB/UNIT
$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} \mathbin{@} n \qquad r : \Gamma \longrightarrow \{\mu\} \mathbin{@} m}{\eta[r]\colon \Gamma \longrightarrow \Gamma/(r:\mu).\{\mu\} \mathbin{@} m}$$

FITCH/SB/MRES
$$\frac{\mu\colon n \longrightarrow m \qquad \delta\colon \Gamma \longrightarrow \Delta \mathbin{@} n \qquad r\colon \Delta \longrightarrow \{\mu\} \mathbin{@} m}{\delta/\mu\colon \Gamma/(r\circ\delta : \mu) \longrightarrow \Delta/(r:\mu) \mathbin{@} m}$$

Fig. 4. Selected rules for modal restriction.

Finally, we show that each 2-cell $\alpha : \nu \Rightarrow \mu$ of $\mathcal{M}$ induces a natural transformation $\langle \nu \mid A \rangle \to \langle \mu \mid A \rangle$.

We can straightforwardly construct a function

$$\mathbf{comp}_{\mu,\nu}(-) : \langle \mu \circ \nu \mid A \rangle \to \langle \mu \mid \langle \nu \mid A \rangle \rangle$$

by using FITCH/TM/UNMOD-DRA and, crucially, FITCH/CX/COMP:

$$\mathbf{comp}_{\mu,\nu}(M) \triangleq \mathsf{mod}_\mu(\mathsf{mod}_\nu(\mathsf{unmod}_{\mu\circ\nu}(M))) : \langle \mu \mid \langle \nu \mid A \rangle \rangle$$

This can be shown to be an equivalence. Similary, $\langle \mathsf{id} \mid A \rangle \simeq A$.

To construct a natural transformation $\langle \mu \mid A \rangle \to \langle \nu \mid A \rangle$, we must use the 2-functorial features of the substitution calculus. We combine these into a derivable elimination rule. Just like DRA/TM/UNMOD, this rule will not be stable under substitution, but it will function as a useful shorthand.

Transposing a term $\Gamma \vdash M : \langle \mu \mid A \rangle \mathbin{@} m$ yields a term $\Gamma.\{\mu\} \vdash \mathsf{unmod}_\mu(M) : A \mathbin{@} n$. The presence of $-.\{\mu\}$ in the conclusion is overly restrictive, so we would like to generalize it. This is achieved through judicious use of a 2-cell.

FITCH/TM/FIRST-UNMOD
$$\frac{\begin{array}{c}\mu\colon o \longrightarrow m \qquad \nu\colon m \longrightarrow n \qquad \xi\colon o \longrightarrow n \\ \alpha : \nu\circ\mu \Rightarrow \xi \qquad \Gamma.\{\nu\} \vdash M : \langle \mu \mid A \rangle \mathbin{@} m\end{array}}{\Gamma.\{\xi\} \vdash \mathsf{unmod}_{\mu,\alpha}(M) : A \mathbin{@} o}$$

This rule is derivable. Applying FITCH/TM/UNMOD-DRA, we obtain $\Gamma.\{\nu\}.\{\mu\} \vdash \mathsf{unmod}_\mu(M) : A \mathbin{@} o$. Using FITCH/CX/COMP we contract the context to $\Gamma.\{\nu\circ\mu\}$. The 2-cell induces a substitution $\{\alpha\}_\Gamma\colon \Gamma.\{\xi\} \longrightarrow \Gamma.\{\nu\circ\mu\} \mathbin{@} o$, so we let

$$\mathsf{unmod}_{\mu,\alpha}(M) \triangleq \mathsf{unmod}_\mu(M)[\{\alpha\}_\Gamma]$$

The *modal coercion* is then given by

FITCH/COE
$$\frac{\alpha : \mu \Rightarrow \nu \qquad \Gamma \vdash M : \langle \mu \mid A \rangle \mathbin{@} m}{\Gamma \vdash \mathbf{coe}[\alpha](M) \triangleq \mathsf{mod}_\nu(\mathsf{unmod}_{\mu,\alpha}(M)) : \langle \nu \mid A \rangle \mathbin{@} m}$$

IV. Semantics

$\mathsf{FitchTT}$ is already given as a generalized algebraic theory and so automatically induces a category of models (algebras and strict homomorphisms). In this section, we aim to restructure that definition in terms of more malleable categorical gadgets. We immediately reap the rewards of this effort by showing how to construct models of $\mathsf{FitchTT}$ from adjunctions between presheaf categories, which we use to present various instances of the type theory in Sections V and VI. Finally, we relate $\mathsf{FitchTT}$ to previous Fitch-style type theories. More specifically, if we equip it with the mode theory generated by a single endomodality, $\mathsf{FitchTT}$ is a conservative extension of DRA. Surprisingly, we prove this in a syntax-free manner using only the algebraic and categorical structure of the model.

A. Natural models of type theory

Each mode of $\mathsf{FitchTT}$ includes a completely independent Martin-Löf type theory. There are many equivalent ways of presenting a model of MLTT, but for the purposes of this paper we use *natural models* [20, 3], which are a categorical reformulation of *categories with families* [19].[2]

Definition 3. A *representable natural transformation over* $\mathcal{C}$ is a morphism $u\colon \dot{U} \longrightarrow U : \mathbf{PSh}(\mathcal{C})$ such that the pullback of u along any morphism $\mathbf{y}(\Gamma) \longrightarrow U$ is representable.

A natural transformation of presheaves over a context category is a concise way of encoding the type and term families of CwFs. The fact that it is representable encodes context extension. Just as with CwFs, the various connectives may be specified independently on top of this definition: see [3]. In the rest of the section we focus on the novel modal types.

B. Natural models and dependent adjunctions

Dependent adjunctions can be phrased in the language of natural models [24, §7.1]. First, notice that the restriction to an endoadjunction in the original definition of dependent adjunctions is artificial: the same definition works between any two CwFs. Suppose then that we encode these CwFs as natural models, $u\colon \dot{U} \longrightarrow U$ in $\mathbf{PSh}(\mathcal{C})$ and $v\colon \dot{V} \longrightarrow V$ in $\mathbf{PSh}(\mathcal{D})$. The left adjoint of the dependent adjunction is a functor $L\colon \mathcal{D} \longrightarrow \mathcal{C}$. On the other hand, the dependent right adjoint from u to v has actions on types and terms which may be exactly encoded by a pullback square in $\mathbf{PSh}(\mathcal{D})$:

$$\begin{tikzcd} L^*\dot{U} \arrow[r, "\mathsf{mod}"] \arrow[d, "L^*u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{V} \arrow[d, "v"] \\ L^*U \arrow[r, "\mathsf{Mod}"'] & V \end{tikzcd} \tag{1}$$

The left adjoint L induces a functor $L^* : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ by precomposition. Applying this to the family u yields a family of types and terms in contexts of the form $\Gamma.\text{🔒} \triangleq L(\Gamma)$

[2] This choice has little essential impact on what follows.

for $\Gamma \in \mathcal{D}$. The formation rule and introduction rule, which are interpreted by Mod and mod respectively, map such types and terms to types and terms of the family υ. By naturality, the resultant types and terms are over the context Γ. The universal property of the pullback precisely corresponds to the elimination rule. We note that this is strongly reminiscent of Voevodsky's notion of universe morphism [36, §4].

C. Models of FitchTT

The definition of a model of FitchTT assembles mode-local models and modalities into a 2-functor: the 0-dimensional component selects the mode-local model, the 1-dimensional action selects the modal context operators, and the 2-dimensional action selects appropriate natural transformations. Each modal context operator comes with a dependent right adjoint and is required to be a parametric right adjoint.

Definition 4. A model of FitchTT over the mode theory $\mathcal{M}$ consists of a 2-functor $[\![-]\!] : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$ such that

- For each $m : \mathcal{M}$, there is a natural model $u_m : \dot{U}_m \longrightarrow U_m$ in $\mathbf{PSh}([\![m]\!])$ closed under dependent products, sums, identity types.
- For each $\mu : n \longrightarrow m$, there is a dependent right adjoint from u_n to u_m whose left adjoint is given by $[\![\mu]\!]$.
- Finally, each $[\![\mu]\!]$ is a parametric right adjoint.

The category of models of FitchTT with $\mathcal{M}$ has models for objects, and strict morphisms preserving all connectives and operations on-the-nose for morphisms.

As this definition of model is a repackaging of the standard notion of model given by the definition of FitchTT as a generalized algebraic theory, we know that

Example 5. The free GAT (hereafter the *syntax*) is a model of FitchTT. More precisely, $\mathbb{S}[m]$ is the category of contexts and substitutions at mode m, while $\mathbb{S}[\mu]$ and $\mathbb{S}[\alpha]$ respectively become $-.\{\mu\}$ and $\{\alpha\}_-$.

D. Relationships to other Fitch-style type theories

The initiality of syntax is a powerful tool for relating FitchTT with other type theories. More specifically, if we are able to show that another type theory $\mathbb{T}$ is a model of FitchTT with $\mathcal{M}$, then initiality induces a unique morphism from the syntax of FitchTT to that type theory. This morphism is then a *translation* of FitchTT into $\mathbb{T}$.

For example, we can relate the DRA calculus to FitchTT. First, generate the free mode theory of a single modality: start with a single mode m and a single morphism $\mu : m \longrightarrow m$ and generate the free (strict) 2-category. Then,

Theorem 2. FitchTT *with a single endomodality μ is a conservative extension of* DRA.

Proof. By definition, a model of DRA is a model of FitchTT if and only if the functor $-.🔒$ is a parametric right adjoint. Thus, every model of FitchTT is a model of DRA. Moreover, every morphism of FitchTT models is *a fortiori* a morphism of DRA models (as the latter is a weaker theory than the former).

Consider the free theory of DRA: we know that $-.🔒$ is a parametric right adjoint [11, Lemma 10]. Therefore, the syntax of DRA is a model of FitchTT with $-.\{\mu\} \triangleq -.🔒$. This induces a unique morphism F from the syntax $\mathbb{S}[-]$ of FitchTT into the syntax of DRA.

Conversely, there is a unique morphism G from the syntax of DRA into $\mathbb{S}[-]$. But by our previous observation, F is also a morphism of DRA models, so $F \circ G$ is a morphism of DRA models from the syntax of DRA to itself and therefore must be the identity. Hence, G faithfully embeds DRA into FitchTT. □

Consequently, the addition of the $-/(- : \mu)$ operator on contexts does not change the strength of the type theory in the case of a single endomodality.

This technique extends to other Fitch-style type theories. For example, consider the mode theory $\mathcal{M}_\Box$ consisting again of a single mode m and endomodality $\mu : m \longrightarrow m$, but force $\mu \circ \mu = \mu$ and include a 2-cell $\mu \Rightarrow \mathsf{id}$ such that $\mathcal{M}_\Box$ is the walking idempotent comonad. The exact same technique can be used with the type theory $\mathsf{MLTT}_{🔒}$ [21] to prove that

Theorem 3. FitchTT *with the mode theory $\mathcal{M}_\Box$ can be embedded into* $\mathsf{MLTT}_{🔒}$.

This again relies on the fact $-.🔒$ is a parametric right adjoint, which was once more a lemma of the metatheory [22, Lemma 1.2.11]. However, FitchTT with $\mathcal{M}_\Box$ is *not* a conservative extension of $\mathsf{MLTT}_{🔒}$, for the latter proves some nonstandard theorems of modal logic, e.g. $(A \to \Box B) \to \Box(A \to B)$.

E. Presheaf models

We now give a theorem for constructing the most important class of non-syntactic models of FitchTT, viz. presheaf categories with adjunctions between them.

First, we recall from [26] that any presheaf category $\mathbf{PSh}(\mathcal{C})$ supports a model of Martin-Löf type theory. A functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ induces $f^* : \mathbf{PSh}(\mathcal{D}) \longrightarrow \mathbf{PSh}(\mathcal{C})$ by precomposition. The latter functor has both a left adjoint $f_!$ and a right adjoint f_* by Kan extension [2, §9.6]. Both f^* and f_* extend to dependent right adjoints [24, §7]. In order to use them with FitchTT, we must show that their corresponding left adjoints $f_!$ and f^* are parametric right adjoints. This is trivial for the latter, as every right adjoint is a PRA. For the former, we show that

Lemma 4. *If $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a PRA then so is $f_!$.*

When putting these together into a model of FitchTT, there is a coherence problem. The definition requires $[\![\mu]\!] \circ [\![\nu]\!] = [\![\mu \circ \nu]\!]$, but in general we only have $f_! \circ g_! \cong (f \circ g)_!$. This strictness mismatch is addressed by a strictification theorem for MTT [25] adapted to FitchTT. Using this we deduce the following.

Theorem 5. *Fix a pseudofunctor $F : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$ such that $F(m) = \mathbf{PSh}(\mathcal{C}_m)$ for each $m : \mathcal{M}$, and for each $\mu : n \longrightarrow m$ the functor $F(\mu)$ satisfies one of the following two conditions:*

1) *$F(\mu) = f_!$ for a PRA $f : \mathcal{C}_m \longrightarrow \mathcal{C}_n$.*
2) *$F(\mu) = f^*$ for an arbitrary functor $f : \mathcal{C}_n \longrightarrow \mathcal{C}_m$.*

Then there exists a model of $\mathsf{FitchTT}$ *with mode theory* $\mathcal{M}$ *where each mode* m *is modelled by* $F(m) = \mathbf{PSh}(\mathcal{C}_m)$ *and each modality* μ *by the dependent right adjoint of* $F(\mu)$.

V. Parametric type theory and FitchTT

As we saw in Section II-B, a simple source of parametric right adjoints is cartesian product: given a closed type $\vdash A$ type, the context extension operator $-.A$ is a parametric right adjoint, and we have modal types in the form of the function type former $A \to (-)$. In this section, we see how this picture generalizes to *substructural* function types from a fixed object. Concretely, we examine Bernardy, Coquand, and Moulin's *parametric type theory* [7], which relies on affine variables—supporting weakening and exchange but not contraction. We find that their *parametricity types* can be read as modal types in an instantiation of FitchTT. Although completely capturing parametric type theory requires more than modal types, FitchTT neatly resolves the issues of substitution that arise from the new variables.

A. Parametric type theory

Bernardy, Coquand, and Moulin's parametric type theory [7] extends Martin-Löf type theory with new primitives that make parametricity theorems internally available. As an example, it becomes possible to show *internally* that any polymorphic function $(A : \mathsf{U}) \to A \to A$ is identified with the polymorphic identity function. Parametric type theory introduces a form of substructural variable, variously called a *color* or *dimension* variable. These variables are *affine*: they support weakening and exchange but not contraction.

Given a context, we may extend it by a new dimension variable $i : \mathbb{I}$. In any context, we have a dimension constant $\Gamma \vdash 0 : \mathbb{I}$. In a context of the form $(\Gamma, i : \mathbb{I})$, we think of the assumptions in Γ as being *separated from* i. In particular, we cannot use one dimension variable to instantiate two; there is no 'diagonal' substitution from $\Gamma, i : \mathbb{I} \longrightarrow \Gamma, j : \mathbb{I}, k : \mathbb{I}$.

A type $\Gamma, i : \mathbb{I} \vdash A$ type is to be thought of as a predicate on its 'endpoint' $A[0/i]$. Likewise, an element $\Gamma, i : \mathbb{I} \vdash M : A$ is a witness that its endpoint $M[0/i]$ satisfies the predicate A. Dimension quantification is internalized by *parametricity types*, whose elements are abstracted terms with a fixed endpoint. The formation and introduction rules for these types are given as follows.[3]

$$\frac{\Gamma, i : \mathbb{I} \vdash A \ \mathsf{type} \qquad \Gamma \vdash M : A[0/i]}{\Gamma \vdash \mathsf{Pred}(i.A, M) \ \mathsf{type}}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash M : A}{\Gamma \vdash \lambda i.\, M : \mathsf{Pred}(i.A, M[0/i])}$$

The idea is that an element of $\mathsf{Pred}(i.A, M)$ is a witness that M belongs to the predicate represented by $i.A$. The intuition that types over $\mathbb{I}$ correspond to predicates is realized by an equivalence $\mathsf{Pred}(i.\mathsf{U}, A) \simeq (A \to \mathsf{U})$, the existence of which relies on an additional *colored type pair* connective [7, Theorem 3.1].

The application rule given for parametricity types in [7] enforces the 'no-diagonal' restriction by assuming a fresh variable in the conclusion.

$$\frac{\Gamma \vdash P : \mathsf{Pred}(i.A, M)}{\Gamma, i : \mathbb{I} \vdash P \mathbin{@} i : A} \qquad \frac{\Gamma \vdash P : \mathsf{Pred}(i.A, M)}{\Gamma \vdash P \mathbin{@} 0 = M : A[0/i]}$$

As we have seen with the equivalent rule DRA/TM/UNMOD, this creates a theory where substitution is not admissible. Cavallo and Harper [15], in their cubical parametric type theory, therefore introduce a *dimension restriction* operator, following Cheney's approach to nominal type theory [16].

$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma/(r : \mathbb{I}) \vdash P : \mathsf{Pred}(i.A, M)}{\Gamma \vdash P \mathbin{@} r : A}$$

The restriction $\Gamma/(r : \mathbb{I})$ removes r and terms succeeding it from the context when r is a variable and is the identity on the constant: $\Gamma/(0 : \mathbb{I}) \triangleq \Gamma$. Admissibility of substitution then relies on the existence of a functorial action by restriction: given $\sigma : \Gamma \longrightarrow \Delta$ and $\Delta \vdash r : \mathbb{I}$, there is some $\sigma/\mathbb{I} \colon \Gamma/(r[\sigma] : \mathbb{I}) \longrightarrow \Delta/(r : \mathbb{I})$ calculated by induction on σ.

B. Recovering parametric type theory

We now show that the judgmental structure of parametric type theory—dimension variables and a parametricity type internalizing them—can be recovered as an instance of FitchTT. On its own, this instance is insufficient to reconstruct, e.g., the proof that all functions $(A : U) \to A \to A$ are equal to the identity. It does, however, provide the basis on which the necessary additional structure can be built by resolving the technical issues around substitution and affine dimension variables.

To cast the kernel of parametric type theory as an instance of FitchTT, we first decompose $\mathsf{Pred}(i.A, M)$ into a combination of an identity type and an affine function type, $(i : \mathbb{I}) \multimap A$, similar to $\mathsf{Pred}(i.A, M)$ but with no fixed endpoint:

$$\mathsf{Pred}(i.A, M) \triangleq (p : (i : \mathbb{I}) \multimap A) \times \mathsf{Id}_{A[0/i]}(p \mathbin{@} 0, M)$$

This encoding will not satisfy the definitional η-principle enjoyed by primitive parametricity types, but it suffices for proving parametricity theorems. The $(i : \mathbb{I}) \multimap -$ connective is specified by the following rules:

PTT/TY/AFF-FORM
$$\frac{\Gamma, i : \mathbb{I} \vdash A \ \mathsf{type} \qquad \Gamma \vdash M : A[0/i]}{\Gamma \vdash (i : \mathbb{I}) \multimap A \ \mathsf{type}}$$

PTT/TM/AFF-INTRO
$$\frac{\Gamma, i : \mathbb{I} \vdash M : A}{\Gamma \vdash \lambda i.\, M : (i : \mathbb{I}) \multimap A}$$

PTT/TM/AFF-ELIM
$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma/(r : \mathbb{I}) \vdash P : (i : \mathbb{I}) \multimap A}{\Gamma \vdash P \mathbin{@} r : A}$$

We formulate $(i : \mathbb{I}) \multimap A$ as a modal type in an instance of FitchTT specialized with the mode theory $\mathcal{M}_{\mathsf{aff}}$, for which see

[3] In Bernardy, Coquand, and Moulin's notation, the type $\mathsf{Pred}(i.A, M)$ below is written $A \ni_i M$.

$$\mu\colon m \longrightarrow m$$

$$w : \mathsf{id} \Rightarrow \mu \qquad e : \mu \circ \mu \Rightarrow \mu \circ \mu \qquad f : \mu \Rightarrow \mathsf{id}$$

$$e \circ (\mathsf{id} \star w) = w \star \mathsf{id} \qquad e \circ e = \mathsf{id}$$

$$(e \star \mathsf{id}) \circ (\mathsf{id} \star e) \circ (e \star \mathsf{id}) = (\mathsf{id} \star e) \circ (e \star \mathsf{id}) \circ (\mathsf{id} \star e)$$

$$f \circ w = \mathsf{id} \qquad (\mathsf{id} \star f) \circ e = f \star \mathsf{id}$$

Fig. 5. $\mathcal{M}_{\mathsf{aff}}$: a mode theory for affine functions

Fig. 5. We use a single mode m with a modality $\mu\colon m \longrightarrow m$, with the intent to replace context extension by $\mathbb{I}$ with $-.\{\mu\}$. The choice of 2-cells and equations corresponds to the structural rules supported by dimension variables, i.e. weakening and exchange but not contraction. Note that the presentation in the mode theory is 'backwards' of what one might first expect, because we axiomatize the behavior of affine *functions*. For instance, weakening is a 2-cell $w : \mathsf{id} \Rightarrow \mu$ and not $w : \mu \Rightarrow \mathsf{id}$. Finally, we add a 'face map' f to represent the dimension constant 0, which is obtained as $0 \triangleq \{f\}_{\mathbf{1}} \circ !_{\Gamma} \colon \Gamma \longrightarrow \{\mu\} \mathbin{@} m$.

The affine function type is then given by $\mathbb{I} \multimap A \triangleq \langle \mu \mid A \rangle$. With this definition, all of the operations and equations for the affine line $\mathbb{I} \multimap A$ introduced in Section V-A can be recovered directly from the rules of modal types in FitchTT. The context restriction operation from parametric type theory is the modal restriction operation induced by the PRA structure on $-.\{\mu\}$. More precisely, we obtain the following correspondences: PTT/TY/AFF-FORM becomes FITCH/TY/MOD, PTT/TM/AFF-INTRO becomes FITCH/TM/MOD and PTT/TM/AFF-ELIM becomes FITCH/TM/UNMOD.

Note that the correspondence between $\Gamma/(r : \mathbb{I})$ and the modal restriction operation of FitchTT is not exact. In Parametric FitchTT, we can only show that Γ is a retract of $\Gamma/(0 : \mu)$, which is weaker than the equation $\Gamma/(0 : \mathbb{I}) = \Gamma$ of [15]. Nevertheless, this is not an obstacle in practice.

C. Models

While FitchTT with this mode theory sufficiently accounts for dimension variables, more is required to prove parametricity results. In [7], these theorems rely on the fact that the canonical map $\mathsf{Pred}(-,-)\colon \mathsf{Pred}(\_.\mathsf{U}, A) \longrightarrow (A \to \mathsf{U})$ is an equivalence. In fact, however, it is sufficient in most cases to work with a 'weak inverse' that only cancels it up to equivalence. This weakened inverse cannot be obtained by modifying the mode theory, but it can be added as an axiom and is supported by a specific model. Unlike affine functions and dimension variables, this addition does not disrupt substitution.

The model of interest is a variant of that in [7], but by requiring only a weak inverse considerable simplifications are possible. Most notably, contexts may be taken to be presheaves over the following category, rather than the 'refined presheaves' used there.

Definition 6. Define $\mathbf{pI}$ to be the category whose objects are finite sets and whose morphisms $S \to T$ are functions $f : T \to S + \mathbf{1}$ which, when restricted to the preimage of S, are injective.

The empty set $\emptyset$ is a zero object of $\mathbf{pI}$: it is both initial and terminal. There is a functor $F\colon \mathbf{pI} \longrightarrow \mathbf{pI}$ which takes a set S to $S+\mathbf{1}$; by extending this to a functor on the presheaf category $F_!\colon \mathbf{PSh}(\mathbf{pI}) \longrightarrow \mathbf{PSh}(\mathbf{pI})$ we can interpret extension by a dimension variable as $[\![\Gamma.\{\mu\}]\!] \triangleq F_![\![\Gamma]\!]$. To apply Theorem 5, it remains to show that (1) the 2-cells and their equations exist and (2) F is a PRA, so that $F_!$ is as well. The first is a routine computation, while the second follows by defining a left adjoint $G\colon \mathbf{pI}/F(\emptyset) \longrightarrow \mathbf{pI}$ as follows:

$$G(s : S \to F(\emptyset)) \triangleq \begin{cases} S \setminus s(\star) & \text{if } s(\star) \in S \\ S & \text{if } s(\star) \in \mathbf{1}_{\mathbf{Set}} \end{cases}$$

Note that s is a set-theoretic function $\mathbf{1}_{\mathbf{Set}} \longrightarrow S + \mathbf{1}_{\mathbf{Set}}$.

Now we can apply Theorem 5 together with the results of [7, 15] to obtain:

Theorem 6. *There is a model of Parametric* FitchTT *in* $\mathbf{PSh}(\mathbf{pI})$ *which interprets* $-.\{\mu\}$ *by* $F_!$*. Moreover, in this model there is a weak inverse to the canonical map* $\mathsf{Pred}(-,-)\colon \mathsf{Pred}(\_.\mathsf{U}, A) \longrightarrow (A \to \mathsf{U})$.

Summarizing, this model ensures that one may soundly postulate the inverse to $\mathsf{Pred}(\_.\mathsf{U}, A) \longrightarrow (A \to \mathsf{U})$ in FitchTT and, with this in hand, reproduce examples from [7] in Parametric FitchTT.

VI. Guarded Type Theory and FitchTT

One of the motivations for modal type theories is to obtain a syntax for *guarded recursion* [31, 10]. In this section we show not only that FitchTT can be a flexible guarded type theory, but that the extra structure of parametric right adjoints gives rise to a rationalization of the *tick variables* introduced in *Clocked Type Theory (*CloTT*)* [4].

Guarded type theories support *guarded recursive definitions*. This is achieved by using modalities that explicitly control productivity, such as the *later modality* ($\triangleright$). Intuitively, $\triangleright A$ classifies data which can only be accessed after 'one step of computation' has taken place. This fine control serves a similar purpose to the syntactic productivity checks used in coinductive definitions. In dependent guarded type theory, both recursive types and functions follow from a single principle, viz. *Löb induction*, an axiom of type $(\triangleright A \to A) \to A$ [9]. For instance, we can define the type of *guarded streams* $\mathsf{gStr}_A \cong A \times \triangleright \mathsf{gStr}_A$ by using Löb induction on the universe.

The $\triangleright$ modality and Löb induction comprise a useful framework for guarded definitions. However, the functions definable in this setting are *causal*, in that they proceed in lockstep with time. For example, the guarded type gStr_A does not admit a function $\mathsf{tail}_A : \mathsf{gStr}_A \to \mathsf{gStr}_A$: we can always project out the tail of a guarded stream, but it will have type $\triangleright \mathsf{gStr}_A$ instead, and we can only access that in the next step. The need to obtain fully defined, total objects (i.e. perform a

$$\ell\colon m \longrightarrow m \qquad\qquad b\colon m \longrightarrow m$$

$$b \leq \mathsf{id} \qquad b \circ b = b \qquad \mathsf{id} \leq \ell \qquad b \circ \ell \leq b$$

Fig. 6. A mode theory for guarded recursion

definition by *coinduction*) dictates the introduction of a second modality, the *always* modality $\Box$. Intuitively, $\Box A$ classifies fully defined coinductive data (i.e. global sections). The usual type of streams is given by $\mathsf{Str}_A \triangleq \Box\mathsf{gStr}_A$. Moreover, we expect an equivalence $\Box A \simeq \Box \rhd A$.

This combination of modalities has been explored previously [18], but a simple syntax that combines them had proved elusive until recently [23, §9]. In the meantime a number of papers focussed on generalizing $\rhd$ to a system of *ticks and clocks* [1, 12, 4, 30]. These systems are flexible, but have complicated semantics [30]. On the other hand, CloTT [4] presents an enticing syntax for guarded recursion, where the $\rhd$ operator behaves like a kind of function. These approaches are far from a parsimonious setting of two interacting modalities.

Here we show that instantiating FitchTT with a mode theory for guarded recursion gives rise to another practicable guarded type theory. Moreover, we observe that the extra structure of parametric right adjoints is precisely what is required to account for tick variables and the functional presentation of $\rhd$. In fact, the *tick constant* introduced in [4] emerges naturally from the 2-cell inducing the equivalence $\Box \rhd A \simeq \Box A$. Hence, we obtain the first purely algebraic presentation of CloTT (though limited to a single clock) and give a semantics that is simpler than that of [30]. In order to focus on the purely modal aspects of guarded type theories, we will set aside considerations of Löb induction. We only mention that it cannot be recovered through modal machinery in any known framework, so must be added axiomatically and justified externally.

A. *Guarded type theory in* FitchTT

In Fig. 6 we present a mode theory for guarded recursion in FitchTT. The mode theory is similar to that used with MTT in [23, §9], but it only uses one mode to facilitate comparison with CloTT. Note also that it is only poset-enriched: there is at most one 2-cell between any pair of modalities.

Instantiating FitchTT with this mode theory yields a modal type theory with modalities $\rhd A \triangleq \langle \ell \mid A\rangle$, and $\Box A \triangleq \langle b \mid A\rangle$. When used with the (in)equations of the mode theory, the combinators of Section III-F induce standard operations. The most important is the 'cancellation' of $\rhd$ by $\Box$:

$$\mathsf{now} \triangleq \mathbf{comp}^{-1}_{b,\ell}(-) : \Box \rhd A \to \Box A$$

The standard model of guarded recursion in $\mathbf{PSh}(\omega)$ [10] is also a model of FitchTT with this mode theory.

Theorem 7. FitchTT *with the guarded mode theory is soundly modelled by $\mathbf{PSh}(\omega)$, where the modality b is interpreted by the global sections comonad, and ℓ by the $\rhd$ endofunctor.*

As both $\rhd$ and $\Box$ have left adjoints given by precomposition [24, §9.2], the result follows from Theorem 5(2).

B. *Tick variables*

Clocked type theory alters the context structure of MLTT to introduce *tick variables*. A tick variable provides the capability to discard a $\rhd$ modality. We begin by considering a simplified clocked type theory, the *Ticked Type Theory (TTT)* of [30]. TTT extends MLTT with the following rules:

$$\frac{\Gamma.\checkmark \vdash A\ \mathsf{type}}{\Gamma \vdash \rhd A\ \mathsf{type}}\ \text{CTT/LATER-FORM} \qquad \frac{\Gamma.\checkmark \vdash M : A}{\Gamma \vdash \lambda(M) : \rhd A}\ \text{CTT/LATER-INTRO}$$

$$\frac{|\Gamma_2| = k \qquad \Gamma_1 \vdash M : \rhd A}{\Gamma_1.\checkmark.\Gamma_2 \vdash M(\boldsymbol{\alpha}_k) : A[\uparrow^{\Gamma_2}]}\ \text{CTT/LATER-ELIM}$$

The first two rules insinuate that $\rhd$ is a dependent right adjoint to a tick. The elimination rule CTT/LATER-ELIM allows us to eliminate a $\rhd$ by consuming a tick. We write $\boldsymbol{\alpha}_k$ to refer to the tick variable at the kth position in the context. This rule weakens the context by some additional assumptions Γ_2, which may contain additional tick variables. Consequently, CTT/LATER-ELIM enforces an *affine* discipline on tick variables.

We can embed TTT into guarded FitchTT. First, we interpret $\Gamma.\checkmark$ as $\Gamma.\{\ell\}$. CTT/LATER-FORM and CTT/LATER-INTRO are just FITCH/TY/MOD and FITCH/TM/MOD respectively. The elimination rule is less immediate: CTT/LATER-ELIM is not exactly FITCH/TM/UNMOD, but it is very similar to the elimination rule DRA/TM/UNMOD* of the DRA calculus. We may thus obtain it as FITCH/TM/UNMOD followed by weakening:

$$M(\boldsymbol{\alpha}_k) \triangleq \mathsf{unmod}_\ell(M)[\uparrow^{\Gamma_2}]$$

There is one important qualitative difference with DRA: the weakening Γ_2 may also include tick variables, while in DRA the rest of the context may not include further locks. Thus in defining $\uparrow^{\Gamma_2}$ we may have to use the substitution $\Gamma.\{\ell\} \longrightarrow \Gamma$ induced by the inequality $1 \leq \ell$ to eliminate ticks.

We have therefore established that

Theorem 8. *Ticked Type Theory can be embedded in* FitchTT.

C. *Tick constants*

As mentioned previously, the combination of $\rhd$ and Löb induction is not sufficiently expressive. We thus need some way of obtaining totalized, coinductive objects. Rather than introducing a second modality such as $\Box$, the clocked type theory CloTT parameterizes $\rhd$ by a *clock symbol* κ. Clock symbols may be quantified over with *clock quantification*, denoted $\forall\kappa.A$. Intuitively, each clock represents a distinct stream of time, and $\rhd^\kappa$ only affects the clock κ. The clock quantifier is then used to 'cancel a $\rhd$', much like $\Box$ does:

$$\forall\kappa.\rhd^\kappa A \simeq \forall\kappa.A \tag{2}$$

The pivotal insight behind CloTT is this: clocks allow us to recast a semantic check ('this context is constant in time') into a syntactic check ('this context does not mention a clock').

This check is performed in the rule for the *tick constant*, which in turn induces Eq. (2):

$$\frac{\Delta, \kappa; \Gamma \vdash M : \triangleright^{\kappa} A \qquad \kappa \notin \Gamma \qquad \kappa' \in \Delta}{\Delta; \Gamma \vdash M(\diamond)[\kappa'/\kappa] : A[\mathsf{id}.\diamond][\kappa'/\kappa]}\ \textsc{ctt/tm/now}$$

The syntactic check $\kappa \notin \Gamma$ ensures that nothing in Γ depends upon the clock κ. Hence, it is safe to eliminate $\triangleright^{\kappa}$, as the ticking of κ will not interfere with the term M. While this rule is sound, it is difficult to implement. Notice that κ does not appear at all in the conclusion of the rule. Accordingly, it is difficult to see how one might write down an algorithmic version of it: we would in fact need to conjure κ, M and A from just $M[\kappa'/\kappa]$ and $A[\kappa'/\kappa]$.

The same result can be achieved in guarded FitchTT in a more direct manner. Just as the $\triangleright$ modality replaces syntactic productivity checks, the $\Box$ modality can be used to supplant syntactic constancy checks. In particular, a context of the form $\Gamma.\{b\}$ is 'semantically constant'. A term depending on $\Gamma.\{b\}$ cannot depend on any temporal aspects of data in Γ: the $-.\{b\}$ prohibits access to anything which may change over time.

Moreover, the unique 2-cell $\alpha : b \circ \ell \Rightarrow b$ induces a substitution $\{\alpha\}_{\Gamma} : \Gamma.\{b\} \longrightarrow \Gamma.\{b\}.\{\ell\} \mathbin{@} m$, which allows us to absorb any occurrences of ℓ following a b. This substitution and term now replace $\diamond$ and Eq. (2) respectively. Using this encoding of $\diamond$ we obtain a 'rationalization' of CTT/TM/NOW:

$$\frac{\Gamma.\{b\} \vdash M : \triangleright A \mathbin{@} m}{\Gamma.\{b\} \vdash M(\diamond) \triangleq \mathsf{unmod}_{\ell,\alpha}(M) : A[\{\alpha\}_{\Gamma}] \mathbin{@} m}$$

The encoding reconstructs a 'single-clock' variant of CloTT. It is rich enough to allow definition by coinduction inside guarded type theory while also retaining the convenient functional syntax of CloTT. Moreover, the ingredients used to simulate CTT/TM/NOW do not suffer from the same issues as the original rule in CloTT, so that an algorithmic version of this syntax now seems achievable.

Using the primitives of FitchTT, we have shown that the more convenient syntax of (single-clock) CloTT can be systematically elaborated into semantically well-understood and well-behaved modal combinators. This elaboration also provides a model in the standard semantics of guarded recursion and avoids the need for more complex clock categories. Finally, we note that non-dependent variants of (single-clock) CloTT have proven useful for modeling reactive programming [5, 6]; these calculi can also be encoded in Guarded FitchTT.

VII. Related work

As it was designed to be a unifying Fitch-style modal type theory [17], FitchTT is closely related to many prior modal type theories.

The Fitch-style approach to modal types begins with the simply-typed system of Clouston [17], which was quickly adapted to the dependent type theory DRA [11]. The other two dependent systems in existence, namely MLTT🔒 [21] and CloTT [4], have already been discussed at length. FitchTT serves as either a rationalization or a generalization of each of these type theories: the PRA structure and the induced 'functional' syntax given in this paper is entirely novel.

Other Fitch-style type theories, which were crafted for more specialized applications, have a weaker relationship with FitchTT. For example, RaTT [5, 6] can be encoded in FitchTT, but this encoding would fail to capture many restrictions placed on modalities in order to ensure domain-specific theorems about RaTT (e.g. freedom from space leaks). We believe that, while FitchTT does not directly capture these restrictions, it can be manually adapted to give a dependent generalization of RaTT. As with Löb induction in guarded type theory, it would be necessary to extend FitchTT with specific constants.

By recognizing the central rôle of PRAs, the relationship between nominal type theory [16] and Fitch-style type theories that is suggested in [11] can be made more precise and extended to include parametric type theories [7, 15]. In particular, the discussion in Section V adapts *mutatis mutandis* to show that nominal type theory can be encoded in FitchTT.

Recently, MTT [23] also attempted to generalize DRA to support multiple modes and modalities, but without recognizing the PRA structure. As a result, MTT could not generalize DRA/TM/UNMOD. Instead, it adopted a 'pattern-matching' modal elimination rule, which is strictly weaker than DRA/TM/UNMOD and thus the DRA calculus. Note that the pattern-matching elimination rule of MTT can be expressed in FitchTT, so MTT can be embedded in it.

VIII. Conclusions and future work

In this paper we have introduced the notion of *parametric right adjoints* as a desirable universal property for context-modifying operations in type theory. We have shown that this extra property is essential for obtaining workable calculi based around dependent right adjoints. Through this observation we have generalized DRA to FitchTT, which supports multiple modes and modalities. Finally, we have shown that FitchTT can be instantiated to recover existing type theories for parametricity and guarded recursion. In the latter case, we provide a conceptual explanation and well-behaved syntax for ticks and the tick constant. In the future, we plan to develop these applications further.

Normalization and decidability of type-checking in FitchTT also offer interesting avenues for future work, and would possibly aid with implementing single-clock CloTT.

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Appendix A
Complete definition of FitchTT

Below, in Fig. 7 we include the new rules of FitchTT. We have elided rules for dependent products, dependent sums, (intensional) identity types, because these are unchanged from MLTT.

$$\boxed{\Gamma\ \mathsf{cx} @ m}$$

$$\frac{}{\cdot\ \mathsf{cx} @ m} \qquad \frac{\Gamma\ \mathsf{cx} @ m}{\Gamma.A\ \mathsf{cx} @ m} \qquad \frac{\Gamma\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m}{\Gamma.\{\mu\}\ \mathsf{cx} @ n} \qquad \frac{\Gamma\ \mathsf{cx} @ n \qquad \mu\colon n \longrightarrow m \qquad r\colon \Gamma \longrightarrow \{\mu\} @ m}{\Gamma/(r : \mu)\ \mathsf{cx} @ m}$$

$$\frac{\Gamma\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m \qquad \nu\colon o \longrightarrow n}{\Gamma.\{\mu\}.\{\nu\} = \Gamma.\{\mu \circ \nu\}\ \mathsf{cx} @ o} \qquad \frac{\Gamma\ \mathsf{cx} @ m}{\Gamma.\{1\} = \Gamma\ \mathsf{cx} @ m}$$

$$\boxed{\Gamma \vdash A\ \mathsf{type} @ m \qquad \Gamma \vdash M : A @ m}$$

$$\frac{\Gamma.\{\mu\} \vdash A\ \mathsf{type} @ n \qquad \mu\colon n \longrightarrow m}{\Gamma \vdash \langle \mu \mid A \rangle\ \mathsf{type} @ m} \qquad \frac{\Gamma.\{\mu\} \vdash M : A @ n}{\Gamma \vdash \mathsf{mod}_\mu(M) : \langle \mu \mid A \rangle @ m}$$

$$\frac{\Gamma/(r : \mu) \vdash M : \langle \mu \mid A \rangle @ m \qquad r\colon \Gamma \longrightarrow \{\mu\} @ n}{\Gamma \vdash M @ r : A[\eta[r]] @ n}$$

$$\frac{\Gamma/(r : \mu).\{\mu\} \vdash M : A @ n \qquad r\colon \Gamma \longrightarrow \{\mu\} @ n}{\Gamma \vdash \mathsf{mod}_\mu(M) @ r = M[\eta[r]] : A[\eta[r]] @ n} \qquad \frac{\Gamma \vdash M : \langle \mu \mid A \rangle @ m}{\Gamma \vdash M = \mathsf{mod}_\mu(M[\epsilon] @ \cdot.\{\mu\}) : \langle \mu \mid A \rangle @ m}$$

$$\boxed{\delta\colon \Gamma \longrightarrow \Delta @ m}$$

$$\frac{\delta\colon \Gamma \longrightarrow \Delta @ n \qquad \mu\colon n \longrightarrow m}{\delta.\{\mu\}\colon \Gamma.\{\mu\} \longrightarrow \Delta.\{\mu\} @ m} \qquad \frac{\delta\colon \Gamma \longrightarrow \Delta @ n \qquad \mu\colon n \longrightarrow m \qquad r\colon \Delta \longrightarrow \{\mu\} @ m}{\delta/\mu\colon \Gamma/(r \circ \delta : \mu) \longrightarrow \Delta/(r : \mu) @ m} \qquad \frac{\Gamma\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m}{\epsilon[\Gamma]\colon \Gamma.\{\mu\}/\mu \longrightarrow \Gamma @ m}$$

$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} @ n \qquad r\colon \Gamma \longrightarrow \{\mu\} @ m}{\eta[r]\colon \Gamma \longrightarrow \Gamma/(r : \mu).\{\mu\} @ n} \qquad \frac{\Gamma\ \mathsf{cx} @ m \qquad \mu, \nu\colon n \longrightarrow m \qquad \alpha : \nu \Rightarrow \mu}{\{\alpha\}_\Gamma\colon \Gamma.\{\mu\} \longrightarrow \Gamma.\{\nu\} @ n}$$

$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} @ m}{\mathsf{id}.\{\mu\} = \mathsf{id}\colon \Gamma.\{\mu\} \longrightarrow \Gamma.\{\mu\} @ n} \qquad \frac{\Gamma, \Delta, \Xi\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m \qquad \delta\colon \Gamma \longrightarrow \Delta @ m \qquad \xi\colon \Delta \longrightarrow \Xi @ m}{(\xi \circ \delta).\{\mu\} = \xi.\{\mu\} \circ \delta.\{\mu\}\colon \Gamma.\{\mu\} \longrightarrow \Xi.\{\mu\} @ n}$$

$$\frac{\Gamma, \Delta, \Xi\ \mathsf{cx} @ n \qquad \mu\colon n \longrightarrow m \qquad r\colon \Xi \longrightarrow \{\mu\} @ n \qquad \delta\colon \Gamma \longrightarrow \Delta @ n \qquad \xi\colon \Delta \longrightarrow \Xi @ n}{(\xi \circ \delta)/\mu = \xi/\mu \circ \delta/\mu\colon \Gamma/(r \circ \xi \circ \delta : \mu) \longrightarrow \Xi/(r : \mu) @ m}$$

$$\frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} @ n \qquad r\colon \Gamma \longrightarrow \{\mu\} @ n}{\mathsf{id}/\mu = \mathsf{id}\colon \Gamma/(r : \mu) \longrightarrow \Gamma/(r : \mu) @ m} \qquad \frac{\Gamma, \Delta\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m \qquad \mu\colon o \longrightarrow n \qquad \delta\colon \Gamma \longrightarrow \Delta @ m}{\delta.\{\nu \circ \mu\} = \delta.\{\nu\}.\{\mu\}\colon \Gamma.\{\nu \circ \mu\} \longrightarrow \Delta.\{\nu \circ \mu\} @ o}$$

$$\frac{\Gamma, \Delta\ \mathsf{cx} @ m \qquad \delta\colon \Gamma \longrightarrow \Delta @ m}{\delta.\{1\} = \delta\colon \Gamma \longrightarrow \Delta @ m} \qquad \frac{\mu\colon n \longrightarrow m \qquad \Gamma\ \mathsf{cx} @ n \qquad r\colon \Gamma \longrightarrow \{\mu\} @ m}{\cdot.\{\mu\} \circ \eta[r] = r\colon \Gamma \longrightarrow \{\mu\} @ n}$$

$$\frac{\Gamma, \Delta\ \mathsf{cx} @ n \qquad \mu\colon n \longrightarrow m \qquad \delta\colon \Gamma \longrightarrow \Delta @ n \qquad r\colon \Gamma \longrightarrow \{\mu\} @ n}{\eta[r] \circ \delta = \delta/\mu.\{\mu\} \circ \eta[r \circ \delta]\colon \Gamma \longrightarrow \Delta/(r : \mu).\{\mu\} @ m} \qquad \frac{\Gamma, \Delta\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m \qquad \delta\colon \Gamma \longrightarrow \Delta @ m}{\delta \circ \epsilon[\Gamma] = \epsilon[\Delta] \circ \delta.\{\mu\}/\mu\colon \Gamma.\{\mu\}/\mu \longrightarrow \Delta @ m}$$

$$\frac{\Gamma\ \mathsf{cx} @ n \qquad \mu\colon n \longrightarrow m \qquad r\colon \Gamma \longrightarrow \{\mu\} @ n}{\epsilon[\Gamma/(r : \mu)] \circ \eta[r]/\mu = \mathsf{id}\colon \Gamma/(r : \mu) \longrightarrow \Gamma/(r : \mu) @ m} \qquad \frac{\Gamma\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m}{\epsilon[\Gamma].\{\mu\} \circ \eta[\cdot.\{\mu\}] = \mathsf{id}\colon \Gamma.\{\mu\} \longrightarrow \Gamma.\{\mu\} @ m}$$

$$\frac{\Gamma\ \mathsf{cx} @ m \qquad \mu\colon n \longrightarrow m}{\mathsf{id} = \{1_\mu\}_\Gamma\colon \Gamma.\{\mu\} \longrightarrow \Gamma.\{\mu\} @ n} \qquad \frac{\Gamma, \Delta\ \mathsf{cx} @ m \qquad \mu, \nu\colon n \longrightarrow m \qquad \delta\colon \Gamma \longrightarrow \Delta @ m \qquad \alpha : \nu \Rightarrow \mu}{\{\alpha\}_\Gamma \circ (\delta.\{\mu\}) = (\delta.\{\nu\}) \circ \{\alpha\}_\Delta\colon \Gamma.\{\mu\} \longrightarrow \Delta.\{\nu\} @ n}$$

$$\frac{\Gamma\ \mathsf{cx} @ m \qquad \mu_0, \mu_1, \mu_2\colon n \longrightarrow m \qquad \alpha_0 : \mu_0 \Rightarrow \mu_1 \qquad \alpha_1 : \mu_1 \Rightarrow \mu_2}{\{\alpha_1 \circ \alpha_0\}_\Gamma = \{\alpha_0\}_\Gamma \circ \{\alpha_1\}_\Gamma : \Gamma.\{\mu_2\} \longrightarrow \Gamma.\{\mu_0\} @ n}$$

$$\frac{\Gamma\ \mathsf{cx} @ m \qquad \nu_0, \nu_1 : o \longrightarrow n \qquad \mu_0, \mu_1 : n \longrightarrow m \qquad \beta : \nu_0 \Rightarrow \nu_1 \qquad \alpha : \mu_0 \Rightarrow \mu_1}{\{\alpha \star \beta\}_\Gamma = \{\alpha\}_\Gamma.\{\nu_1\} \circ \{\beta\}_{\Gamma.\{\mu_0\}} : \Gamma.\{\mu_0 \circ \nu_0\} \longrightarrow \Gamma.\{\mu_1 \circ \nu_1\} @ o}$$

Fig. 7. Novel rules in FitchTT