Math 414: HW1

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Exercise 4.7

Given an S-formula that can be uniquely determined. suppose the removal ignoring rule F4 also results in a unique decomposition.

For The formula:

$$\mathcal{X} := \exists v_0 P v_0 \wedge Q v_1$$

$$SF_1(\mathcal{X}) = \{\exists v_0 P v_0 \wedge Q v_1\} \cup Sf(P v_0 \wedge Q v_1)$$
$$SF(P v_0 \wedge Q v_1) = \{P v_0 \wedge Q v_1\} \cup SF(P v_0) \cup SF(Q v_1)\}$$
$$SF(P v_0) = \{P v_0\}$$
$$SF(Q v_1) = \{Q v_1\}$$

Then...

$$SF_1(\mathcal{X}) = \{\exists v_0 P v_0 \wedge Q v_1, P v_0 \wedge Q v_1, P v_0, Q v_1\}$$

or

$$SF_1(\mathcal{X}) = \{\mathcal{X}, P v_0 \wedge Q v_1, P v_0, Q v_1\}$$

Another decomposition yealeds...

$$SF_2(\mathcal{X}) = \{\exists v_0 P v_0 \wedge Q v_1\} \cup Sf(\exists v_0 P v_0) \cup Sf(Q v_1)$$
$$SF(\exists v_0 P v_0) = \{\exists v_0 P v_0\} \cup SF(P v_0)$$
$$SF(Q v_1) = \{Q v_1\}$$
$$SF(P v_0) = \{P v_0\}$$

Then...

$$SF_2(\mathcal{X}) = \{\exists v_0 P v_0 \wedge Q v_1, \exists v_0 P v, P v_0, Q v_1\}$$

or

$$SF_2(\mathcal{X}) = \{\mathcal{X}, \exists v_0 P v, P v_0, Q v_1\}$$

clearly

$$SF_2(\mathcal{X}) = \{\mathcal{X}, \exists v_0 P v, P v_0, Q v_1\} \neq SF_1(\mathcal{X}) = \{\mathcal{X}, P v_0 \wedge Q v_1, P v_0, Q v_1\}$$

This is a contradiction, therefore removing the parenthesis rule F4 can result in S-formulas without a unique decomposition.

Exercise 4.8

Definition:

F1) If t_1 and t_2 are S-terms, then $t_1 \equiv t_2$ is an S-P-formula.

F2) If $t_1, ..., t_n$ are S-terms and R is an n-ary relation symbol in S, then $Rt_1, ...t_n$ is an S-P-formula.

F3) If φ is an S-formula, then $\neg\varphi$ is also an S-formula.

F4) If φ and ψ are S-P-formulas, then $\wedge\varphi\psi$, $\vee\varphi\psi$, $\rightarrow \varphi\psi$, and $\leftrightarrow \varphi\psi$ are also S-P-formulas.

F5) If φ is an S-P-formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are also S-P-formulas.

Sets

$A = \{v_1, v_2, ...\}$ (variables)

$B = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$

$C = \{\forall, \exists\}$

$D = \{\equiv\}$

$E = \{(,)\}$

$\mathcal{R}$ = set of n-array relation symbols

$\mathcal{F}$ = set of function symbols

$\mathcal{C}$ = set of constants

$\mathcal{A}' = \mathcal{C} \cup \mathcal{F} \cup \mathcal{R} \cup a \cup b \cup c \cup d \cup e$

S = is a symbol set

$\mathcal{A}'_{\mathcal{S}} = \mathcal{A}' \cup S$

SF

Let SF be a function that takes a formula and assigns it to sub-formulas, φ & ψ be formulas, and let $a \in A$, $b \in B$, $c \in C$.

SF is defined by:

$$
\begin{array}{ccc}
(I) & SF(v_1 \equiv v_2) := \{v_1 \equiv v_2\} & \text{(F1)} \\
(II) & SF(Rt_1...t_2) := \{Rt_1...t_2\} & \text{(F2)} \\
(III) & SF(\neg\varphi) := \{\neg\varphi\} \cup SF(\varphi) & \text{(F3)} \\
(IV) & SF(z\varphi\psi) := \{b\varphi\psi\} \cup SF(\varphi) \cup SF(\psi) & \text{(F4)} \\
(V) & SF(zx\varphi) := \{ca\varphi\} \cup SF(\varphi) & \text{(F5)}
\end{array}
$$

4.8.1

Given a formula φ, $SF(\varphi)$ is unique.

Proof

Base cases:

$$\begin{array}{ll} SF(v_1 \equiv v_2) := \{v_1 \equiv v_2\} & \text{(I)} \\ SF(Rt_1...t_2) := \{Rt_1...t_2\} & \text{(II)} \end{array}$$

Inductive Hypothisis:

For all formulas $\mathcal{X}$, where n ¿ 1 and $|SF(\mathcal{X})| < n$, $SF(\mathcal{X})$ is unique.

Inductive step:

Let $\mathcal{X}^*$ be such that $|SF(\mathcal{X}^*)| = n$

Case 1:

$$\mathcal{X}^* = \neg\varphi$$

$$SF(\mathcal{X}^*) = \{\neq \varphi\} \cup SF(\varphi) \quad \text{(F3)}$$

Since $SF(\varphi)$ falls under the inductive hypothesis $SF(\mathcal{X}^*)$ is unique.

Case 2:

$$\mathcal{X}^* = b\varphi\psi$$

$$SF(\mathcal{X}^*) = \{b\varphi\psi\} \cup SF(\varphi) \cup SF(\psi) \quad \text{(F4)}$$

Since $SF(\varphi)$ and $SF(\psi)$ fall under the inductive hypothesis $SF(\mathcal{X}^*)$ is unique.

Case 3:

$$\mathcal{X}^* = ca\psi$$

$$SF(\mathcal{X}^*) = \{ca\psi\} \cup SF(\psi) \quad \text{(F4)}$$

Since $SF(\psi)$ falls under the inductive hypothesis $SF(\mathcal{X}^*)$ is unique.

Lemma 4.8.2

Given two groups of formulas, $\varphi_1, ..., \varphi_n$ and $\varphi'_1, ..., \varphi'_m$, with $\varphi_1...\varphi_n = \varphi'_1...\varphi'_m$, then $n = m$ and for every i $\varphi_i = \varphi'_i$ for all $0 < i \leq n$.

Proof

Suppose we have $\varphi_1, ..., \varphi_n$ and $\varphi'_1, ..., \varphi'_m$ wich are all formulas and $\varphi_1 ... \varphi_n = \varphi'_1 ... \varphi'_m$.

By theorem 4.8.1 φ_1 and φ'_1 has a unique set of sub formulas. suppose $\varphi_1 \neq \varphi'_1$, then $\varphi_1 = \varphi'_1 \zeta$ or $\varphi'_1 = \varphi_1 \zeta$ which contradics theorem 4.8.1 therefore $\varphi_1 = \varphi'_1$.

A similar argument can be applied to φ_2 through φ_n. Suppose n != m, and WOLOG assume n¡m, then = $\varphi'_{n+1} ... \varphi'_m$ which doesnt make sence therefor n = m.