Układ dyskretny

Dany jest układ opisany równaniami:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0,2 & 0,6 \\ 0,6 & 0,2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Wielomian charakterystyczny:

$$M(z) = \det(zI - A) = \begin{vmatrix} z-0,2 & -0,6 \\ -0,6 & z-0,2 \end{vmatrix} =$$

$$= (z-0,2)^2 - 0,6^2 = (z+0,4)(z-0,8)$$

Wartości własne:

$$z_1 = -0,4 \quad z_2 = 0,8$$

Wektory własne:

$$(z_i I - A)v_i = 0$$

$$\begin{bmatrix} -0,6 & -0,6 \\ -0,6 & -0,6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0,6 & -0,6 \\ -0,6 & 0,6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Układ ciągły

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$M(s) = \det(sI - A) = \begin{vmatrix} s+4 & -2 \\ -2 & s+4 \end{vmatrix} =$$

$$= (s+4)^2 - 2^2 = (s+2)(s+6)$$

$$s_1 = -2 \quad s_2 = -6$$

$$(s_i I - A)v_i = 0$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Macierz przekształcenia liniowego do przestrzeni modalnej:

$$V=\begin{bmatrix} v_1 & v_2 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$W=V^{-1}=\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$$

$$V=\begin{bmatrix} v_1 & v_2 \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$W=V^{-1}=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}=\begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$$

Macierz tranzycyjna:

$$A^k=\sum_{i=1}^{n}(z_i)^k v_i w_i^T$$

$$A^k=(z_1)^k v_1 w_1^T+(z_2)^k v_2 w_2^T=$$

$$=(-0{,}4)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}+(0{,}8)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}=$$

$$=\frac{1}{2}\begin{bmatrix} (-0{,}4)^k+(0{,}8)^k & -(-0{,}4)^k+(0{,}8)^k \\ -(-0{,}4)^k+(0{,}8)^k & (-0{,}4)^k+(0{,}8)^k \end{bmatrix}$$

$$\Phi(t)=\sum_{i=1}^{n}e^{s_i t} v_i w_i^T$$

$$\Phi(t)=e^{s_1 t} v_1 w_1^T+e^{s_2 t} v_2 w_2^T=$$

$$=e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}+e^{-6t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}=$$

$$=\frac{1}{2}\begin{bmatrix} e^{-2t}+e^{-6t} & e^{-2t}-e^{-6t} \\ e^{-2t}-e^{-6t} & e^{-2t}+e^{-6t} \end{bmatrix}$$

Odpowiedź swobodna wektora stanu dla warunków początkowych

$$x_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$x_s(k) = \sum_{i=1}^{n} (z_i)^k v_i w_i^T x_0$$

$$x_s(k) = (z_1)^k v_1 w_1^T x_0 + (z_2)^k v_2 w_2^T x_0 =$$

$$= (-0{,}4)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} +$$

$$+ (0{,}8)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} =$$

$$= \boxed{-\begin{bmatrix} 1 \\ -1 \end{bmatrix}} (-0{,}4)^k + \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} (0{,}8)^k =$$

$$= \begin{bmatrix} -(-0{,}4)^k + (0{,}8)^k \\ (-0{,}4)^k + (0{,}8)^k \end{bmatrix}$$

$$x_s(t) = \sum_{i=1}^{n} e^{s_i t} v_i w_i^T x_0$$

$$x_s(t) = e^{s_1 t} v_1 w_1^T x_0 + e^{s_2 t} v_2 w_2^T x_0 =$$

$$= e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} +$$

$$+ e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} =$$

$$= \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} e^{-2t} + \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} e^{-6t} =$$

$$= \begin{bmatrix} e^{-2t} - e^{-6t} \\ e^{-2t} + e^{-6t} \end{bmatrix}$$

Warunek początkowy jest kombinacją liniową wektorów własnych:

$$x_0 = \boxed{-v_1} + \boxed{v_2}$$

$$x_0 = \boxed{v_1} + \boxed{v_2}$$

Czyli w odpowiedzi swobodnej wektora stanu występują oba mody.

Odpowiedź swobodna układu:

$$y_s(k) = [0 \quad 1] x_s(k) = (-0{,}4)^k + (0{,}8)^k$$

$$y_s(t) = [0 \quad 1] x_s(t) = e^{-2t} + e^{-6t}$$

Odpowiedź wymuszona wektora stanu dla skoku jednostkowego

$$x_w(k)=\sum_{j=1}^{n}\underline{v}_j\,\underline{w}_j^T B\sum_{i=1}^{k}(z_j)^{i-1}u(k-i)=$$

$$=v_1w_1^T B\sum_{i=1}^{k}(z_1)^{i-1}u(k-i)+v_2w_2^T B\sum_{i=1}^{k}(z_2)^{i-1}u(k-i)=$$

$$=\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}\sum_{i=1}^{k}(-0{,}4)^{i-1}+$$

$$+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}\sum_{i=1}^{k}(0{,}8)^{i-1}=$$

$$=\begin{bmatrix}3\\-3\end{bmatrix}\frac{1-(-0{,}4)^k}{1-(-0{,}4)}+\begin{bmatrix}1\\1\end{bmatrix}\frac{1-(0{,}8)^k}{1-0{,}8}=$$

$$=\begin{bmatrix}\frac{15}{7}\left\{1-(-0{,}4)^k\right\}+5\left\{1-(0{,}8)^k\right\}\\-\frac{15}{7}\left\{1-(-0{,}4)^k\right\}+5\left\{1-(0{,}8)^k\right\}\end{bmatrix}$$

$$x(t)=\sum_{j=1}^{n}\underline{v}_j\,\underline{w}_j^T Be^{s_jt}\int_0^t e^{-s_j\tau}u(\tau)d\tau=$$

$$=v_1w_1^T Be^{s_1t}\int_0^t e^{-s_1\tau}u(\tau)d\tau+v_2w_2^T Be^{s_2t}\int_0^t e^{-s_2\tau}u(\tau)d\tau=$$

$$=\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}e^{-2t}\int_0^t e^{2\tau}d\tau+$$

$$+\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}e^{-6t}\int_0^t e^{2\tau}d\tau=$$

$$=\begin{bmatrix}1\\1\end{bmatrix}e^{-2t}\left(\frac{1}{2}e^{2\tau}\Big|_0^t\right)+\begin{bmatrix}3\\-3\end{bmatrix}e^{-6t}\left(\frac{1}{6}e^{6\tau}\Big|_0^t\right)=$$

$$=\begin{bmatrix}\frac{1}{2}\left(1-e^{-2t}\right)+\frac{1}{2}\left(1-e^{-6t}\right)\\\frac{1}{2}\left(1-e^{-2t}\right)-\frac{1}{2}\left(1-e^{-6t}\right)\end{bmatrix}$$

Odpowiedź wymuszona układu:

$$y_w(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x_w(k) = -\frac{15}{7}\left\{1-(-0{,}4)^k\right\} + 5\left\{1-(0{,}8)^k\right\} =$$

$$= \frac{20}{7} + \frac{15}{7}(-0{,}4)^k - 5(0{,}8)^k$$

$$y_w(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x_w(t) = \frac{1}{2}\left(1-e^{-2t}\right) - \frac{1}{2}\left(1-e^{-6t}\right) =$$

$$= -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-6t}$$

Struktura układu w przestrzeni modalnej:

$$\Lambda = WAV = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0{,}2 & 0{,}6 \\ 0{,}6 & 0{,}2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -0{,}4 & 0{,}8 \\ 0{,}4 & 0{,}8 \end{bmatrix} = \begin{bmatrix} -0{,}4 & 0 \\ 0 & 0{,}8 \end{bmatrix}$$

$$B_\Lambda = WB = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$C_\Lambda = CV = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$S = WAV = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 6 \\ -2 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}$$

$$B_S = WB = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$C_S = CV = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Równania w postaci rozwiniętej:

$$\boxed{q_1(k+1) = -0{,}4q_1(k) + 3u(k)}$$

$$\boxed{q_2(k+1) = 0{,}8q_2(k) + u(k)}$$

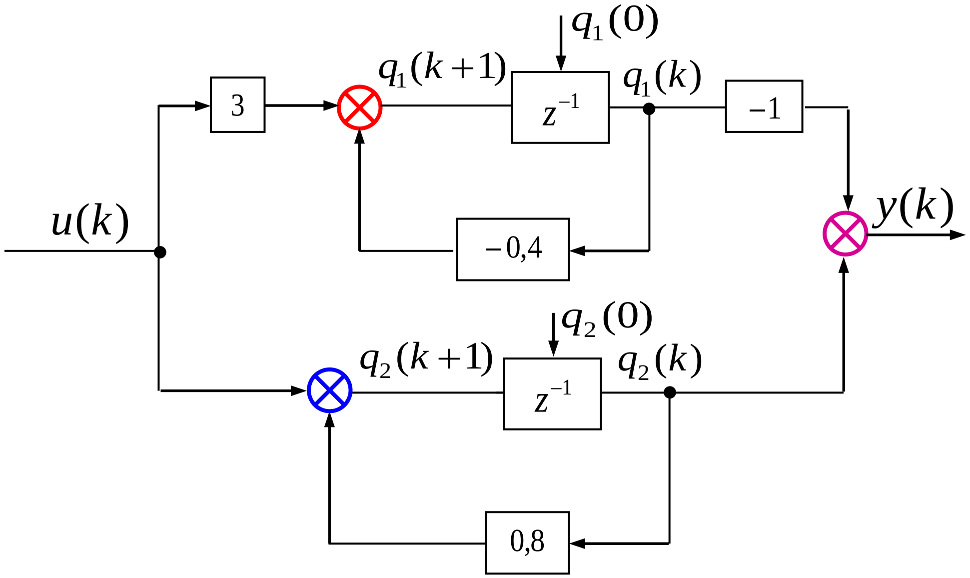
$$\boxed{y(k) = -q_1(k) + q_2(k)}$$

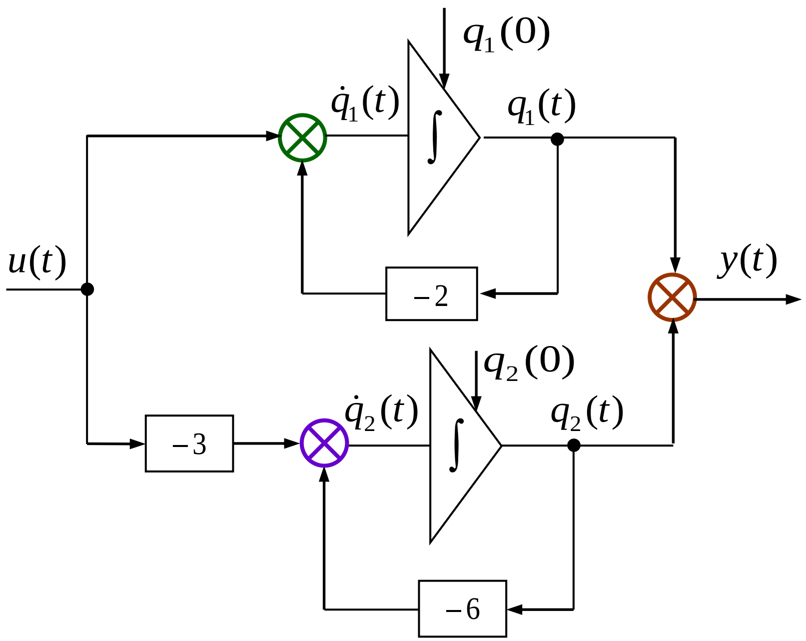
$$\boxed{\frac{d}{dt}q_1(t) = -2q_1(t) + u(t)}$$

$$\boxed{\frac{d}{dt}q_2(t) = -6q_2(t) - 3u(t)}$$

$$\boxed{y(t) = q_1(t) + q_2(t)}$$

Schematy blokowe:





Transmitancja ze schematów:

$$G(z) = -3\frac{z^{-1}}{1+0{,}4z^{-1}} + \frac{z^{-1}}{1-0{,}8z^{-1}} =$$

$$= \frac{-3}{z+0{,}4} + \frac{1}{z-0{,}8} = \frac{-2z+2{,}8}{(z+0{,}4)(z-0{,}8)}$$

$$G(s) = \frac{\frac{1}{s}}{1+\frac{2}{s}} - 3\frac{\frac{1}{s}}{1+\frac{6}{s}} = \frac{1}{s+2} - \frac{3}{s+6} =$$

$$= \frac{-2s}{(s+2)(s+6)}$$

Odpowiedź jednostkowa liczona z transmitacji:

$$H(z) = \frac{z}{z-1}G(z) = z\frac{-2z+2{,}8}{(z-1)(z+0{,}4)(z-0{,}8)} =$$

$$= z\left(\frac{A_1}{z-1} + \frac{A_2}{z+0{,}4} + \frac{A_3}{z-0{,}8}\right) =$$

$$= z\frac{A_1(z+0{,}4)(z-0{,}8) + A_2(z-1)(z-0{,}8) + A_3(z-1)(z+0{,}4)}{(z-1)(z+0{,}4)(z-0{,}8)} =$$

$$z = 1 \quad \Rightarrow \quad 0{,}8 = 0{,}28A_1 \quad \Rightarrow \quad A_1 = \frac{20}{7}$$

$$z = -0{,}4 \quad \Rightarrow \quad 3{,}6 = 1{,}68A_2 \quad \Rightarrow \quad A_2 = \frac{15}{7}$$

$$z = 0{,}8 \quad \Rightarrow \quad 1{,}2 = -0{,}24A_3 \quad \Rightarrow \quad A_3 = -5$$

$$H(z)=\frac{20}{7}\frac{z}{z-1}+\frac{15}{7}\frac{z}{z+0{,}4}-5\frac{z}{z-0{,}8}$$

$$h(k)=\frac{20}{7}+\frac{15}{7}(-0{,}4)^{k}-5(0{,}8)^{k}$$

$$H(s)=\frac{1}{s(s+2)}+\frac{3}{s(s+6)}=$$

$$=\frac{1}{2}\frac{1}{s\left(\frac{1}{2}s+1\right)}-\frac{1}{2}\frac{1}{s\left(\frac{1}{6}s+1\right)}$$

$$h(t)=\frac{1}{2}\left(1-e^{-2t}\right)-\frac{1}{2}\left(1-e^{-6t}\right)=-\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-6t}$$

Układ dyskretny

Dany jest układ opisany równaniami:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0,6 \\ -0,4 & -0,2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Wielomian charakterystyczny:

$$M(z) = \det(zI - A) = \begin{vmatrix} z & 0,6 \\ 0,4 & z+0,2 \end{vmatrix} =$$

$$= z(z+0,2) - 0,24 = (z-0,4)(z+0,6)$$

Wartości własne:

$$z_1 = 0,4 \quad z_2 = -0,6$$

Wektory własne:

$$(z_i I - A)v_i = 0$$

$$\begin{bmatrix} 0,4 & 0,6 \\ 0,4 & 0,6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} -0,6 & 0,6 \\ 0,4 & -0,4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Układ ciągły

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$M(s) = \det(sI - A) = \begin{vmatrix} s & 6 \\ -2 & s+8 \end{vmatrix} =$$

$$= s(s+8) + 12 = s^2 + 8s + 12 = (s+2)(s+6)$$

$$s_1 = -2 \quad s_2 = -6$$

$$(s_i I - A)v_i = 0$$

$$\begin{bmatrix} -2 & 6 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 6 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Macierz przekształcenia liniowego do przestrzeni modalnej:

$$V = [v_1 \quad v_2] = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

$$W = V^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$$

$$V = [v_1 \quad v_2] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$W = V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$$

Macierz tranzycyjna:

$$A^k = \sum_{i=1}^{n} (z_i)^k v_i w_i^T$$

$$A^k = (z_1)^k v_1 w_1^T + (z_2)^k v_2 w_2^T =$$

$$= (0{,}4)^k \begin{bmatrix} 3 \\ -2 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \end{bmatrix} + (-0{,}6)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix} =$$

$$= \frac{1}{5}\begin{bmatrix} 3(0{,}4)^k + 2(-0{,}6)^k & -3(0{,}4)^k + 3(-0{,}6)^k \\ -2(0{,}4)^k + 2(-0{,}6)^k & 2(0{,}4)^k + 3(-0{,}6)^k \end{bmatrix}$$

$$\Phi(t) = \sum_{i=1}^{n} e^{s_i t} v_i w_i^T$$

$$\Phi(t) = e^{s_1 t} v_1 w_1^T + e^{s_2 t} v_2 w_2^T =$$

$$= e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix} + e^{-6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} 3e^{-2t} - e^{-6t} & -3e^{-2t} + 3e^{-6t} \\ e^{-2t} - e^{-6t} & -e^{-2t} + 3e^{-6t} \end{bmatrix}$$

Odpowiedź swobodna wektora stanu dla warunków początkowych

$$x_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$x_s(k) = \sum_{i=1}^{n} (z_i)^k v_i w_i^T x_0$$

$$x_s(k) = (z_1)^k v_1 w_1^T x_0 + (z_2)^k v_2 w_2^T x_0 =$$

$$= (0{,}4)^k \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} +$$

$$+ (-0{,}6)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} =$$

$$= \boxed{-\frac{2}{5} \begin{bmatrix} 3 \\ -2 \end{bmatrix}} (0{,}4)^k + \boxed{\frac{6}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} (-0{,}6)^k =$$

$$= \frac{2}{5} \begin{bmatrix} -3(0{,}4)^k + 3(-0{,}6)^k \\ 2(0{,}4)^k + 3(-0{,}6)^k \end{bmatrix}$$

$$x_s(t) = \sum_{i=1}^{n} e^{s_i t} v_i w_i^T x_0$$

$$x_s(t) = e^{s_1 t} v_1 w_1^T x_0 + e^{s_2 t} v_2 w_2^T x_0 =$$

$$= e^{-2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} +$$

$$+ e^{-6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} =$$

$$= \boxed{-\begin{bmatrix} 3 \\ 1 \end{bmatrix}} e^{-2t} + \boxed{3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}} e^{-6t} =$$

$$= \begin{bmatrix} -3e^{-2t} + 3e^{-6t} \\ -e^{-2t} + 3e^{-6t} \end{bmatrix}$$

Warunek początkowy jest kombinacją liniową wektorów własnych:

$$x_0 = \boxed{-\frac{2}{5} v_1} + \boxed{\frac{6}{5} v_2}$$

$$x_0 = \boxed{-v_1} + \boxed{3v_2}$$

Czyli w odpowiedzi swobodnej wektora stanu występują oba mody.

Odpowiedź swobodna układu:

$$y_s(k)=\begin{bmatrix}0 & 1\end{bmatrix}x_s(k)=2(0{,}4)^{k+1}-2(-0{,}6)^{k+1}$$

$$y_s(t)=\begin{bmatrix}0 & 1\end{bmatrix}x_s(t)=-e^{-2t}+3e^{-6t}$$

Odpowiedź wymuszona wektora stanu dla skoku jednostkowego

$$x_w(k)=\sum_{j=1}^{n}\underline{v}_j\,\underline{w}_j^T B\sum_{i=1}^{k}(z_j)^{i-1}u(k-i)=$$

$$=v_1w_1^T B\sum_{i=1}^{k}(z_1)^{i-1}u(k-i)+v_2w_2^T B\sum_{i=1}^{k}(z_2)^{i-1}u(k-i)=$$

$$=\begin{bmatrix}3\\-2\end{bmatrix}\begin{bmatrix}\frac{1}{5} & -\frac{1}{5}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}\sum_{i=1}^{k}(0{,}4)^{i-1}+$$

$$+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\frac{2}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}\sum_{i=1}^{k}(-0{,}6)^{i-1}=$$

$$=\frac{6}{5}\begin{bmatrix}3\\-2\end{bmatrix}\frac{1-(0{,}4)^k}{1-0{,}4}+\frac{2}{5}\begin{bmatrix}1\\1\end{bmatrix}\frac{1-(-0{,}6)^k}{1+0{,}6}=$$

$$=\begin{bmatrix}6\left\{1-(0{,}4)^k\right\}+\frac{1}{4}\left\{1-(-0{,}6)^k\right\}\\-4\left\{1-(0{,}4)^k\right\}+\frac{1}{4}\left\{1-(-0{,}6)^k\right\}\end{bmatrix}$$

$$x(t)=\sum_{j=1}^{n}\underline{v}_j\,\underline{w}_j^T Be^{s_jt}\int_0^t e^{-s_j\tau}u(\tau)d\tau=$$

$$=v_1w_1^T Be^{s_1t}\int_0^t e^{-s_1\tau}u(\tau)d\tau+v_2w_2^T Be^{s_2t}\int_0^t e^{-s_2\tau}u(\tau)d\tau=$$

$$=\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}e^{-2t}\int_0^t e^{2\tau}d\tau+$$

$$+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{3}{2}\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}e^{-6t}\int_0^t e^{2\tau}d\tau=$$

$$=\begin{bmatrix}9\\3\end{bmatrix}e^{-2t}\left(\frac{1}{2}e^{2\tau}\Big|_0^t\right)+\begin{bmatrix}5\\5\end{bmatrix}e^{-6t}\left(\frac{1}{6}e^{6\tau}\Big|_0^t\right)=$$

$$=\begin{bmatrix}\frac{9}{2}\left(1-e^{-2t}\right)+\frac{5}{6}\left(1-e^{-6t}\right)\\\frac{3}{2}\left(1-e^{-2t}\right)-\frac{5}{6}\left(1-e^{-6t}\right)\end{bmatrix}$$

Odpowiedź wymuszona układu:

$$y_w(k)=[0 \quad 1]x_w(k)=-4\left\{1-(0{,}4)^k\right\}+\frac{1}{4}\left\{1-(-0{,}6)^k\right\}=$$

$$=-\frac{15}{4}+4(0{,}4)^k-\frac{1}{4}(-0{,}6)^k$$

$$y_w(t)=[0 \quad 1]x_w(t)=\frac{3}{2}\left(1-e^{-2t}\right)-\frac{5}{6}\left(1-e^{-6t}\right)=$$

$$=\frac{2}{3}-\frac{3}{2}e^{-2t}+\frac{5}{6}e^{-6t}$$

Struktura układu w przestrzeni modalnej:

$$\Lambda=WAV=\frac{1}{5}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & -0{,}6\\ -0{,}4 & -0{,}2\end{bmatrix}\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}=$$

$$=\frac{1}{5}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1{,}2 & -0{,}6\\ -0{,}8 & -0{,}6\end{bmatrix}=\begin{bmatrix}0{,}4 & 0\\ 0 & -0{,}6\end{bmatrix}$$

$$B_\Lambda=WB=\frac{1}{5}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}4\\ -2\end{bmatrix}=\begin{bmatrix}1{,}2\\ 0{,}4\end{bmatrix}$$

$$C_\Lambda=CV=[0 \quad 1]\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}=[-2 \quad 1]$$

$$S=WAV=\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\begin{bmatrix}0 & -6\\ 2 & -8\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}=$$

$$=\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\begin{bmatrix}-6 & -6\\ -2 & -6\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -6\end{bmatrix}$$

$$B_S=WB=\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\begin{bmatrix}4\\ -2\end{bmatrix}=\begin{bmatrix}3\\ -5\end{bmatrix}$$

$$C_S=CV=[0 \quad 1]\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}=[1 \quad 1]$$

Równania w postaci rozwiniętej:

$$q_1(k+1) = 0{,}4q_1(k) + 1{,}2u(k)$$

$$q_2(k+1) = -0{,}6q_2(k) + 0{,}4u(k)$$

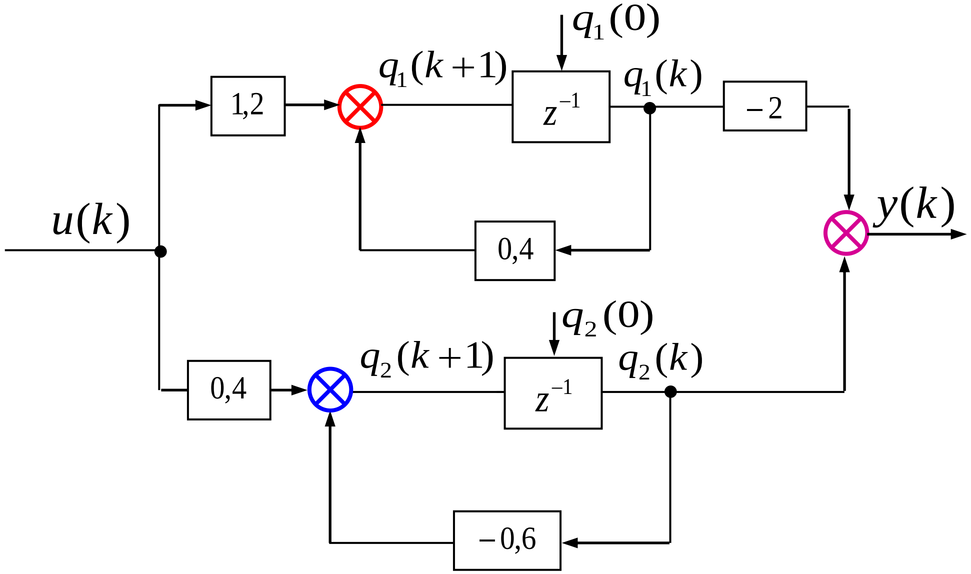
$$y(k) = -2q_1(k) + q_2(k)$$

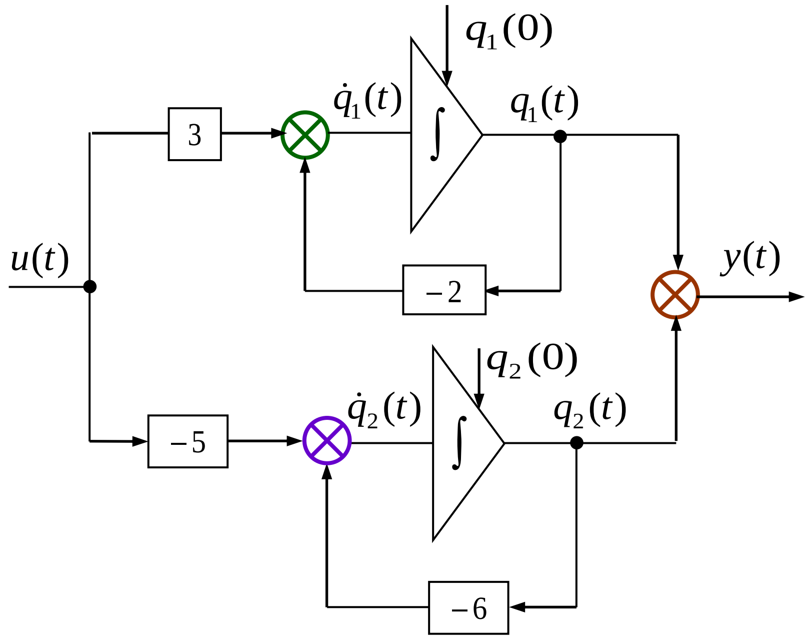
$$\frac{d}{dt}q_1(t) = -2q_1(t) + 3u(t)$$

$$\frac{d}{dt}q_2(t) = -6q_2(t) - 5u(t)$$

$$y(t) = q_1(t) + q_2(t)$$

Schematy blokowe:





Transmitancja ze schematów:

$$G(z) = -2{,}4\frac{z^{-1}}{1-0{,}4z^{-1}} + 0{,}4\frac{z^{-1}}{1+0{,}6z^{-1}} =$$

$$= \frac{-2{,}4}{z-0{,}4} + \frac{0{,}4}{z+0{,}6} = \frac{-2z-1{,}6}{(z-0{,}4)(z+0{,}6)}$$

$$G(s) = 3\frac{\frac{1}{s}}{1+\frac{2}{s}} - 5\frac{\frac{1}{s}}{1+\frac{6}{s}} = \frac{3}{s+2} - \frac{5}{s+6} =$$

$$= \frac{-2s+8}{(s+2)(s+6)}$$

Odpowiedź jednostkowa liczona z transmitacji:

$$H(z) = \frac{z}{z-1}G(z) = z\frac{-2z-1{,}6}{(z-1)(z-0{,}4)(z+0{,}6)} =$$

$$= z\left(\frac{A_1}{z-1} + \frac{A_2}{z-0{,}4} + \frac{A_3}{z+0{,}6}\right) =$$

$$= z\frac{A_1(z-0{,}4)(z+0{,}6) + A_2(z-1)(z+0{,}6) + A_3(z-1)(z-0{,}4)}{(z-1)(z-0{,}4)(z+0{,}6)} =$$

$$z = -1 \quad \Rightarrow \quad -3{,}6 = 0{,}96A_1 \quad \Rightarrow \quad A_1 = -\frac{15}{4}$$

$$z = 0{,}4 \quad \Rightarrow \quad -2{,}4 = -0{,}6A_2 \quad \Rightarrow \quad A_2 = 4$$

$$z = -0{,}6 \quad \Rightarrow \quad -0{,}4 = 1{,}6A_3 \quad \Rightarrow \quad A_3 = -\frac{1}{4}$$

$$H(z) = -\frac{15}{4}\frac{z}{z-1} + 4\frac{z}{z-0{,}4} - \frac{1}{4}\frac{z}{z+0{,}6}$$

$$h(k) = -\frac{15}{4} + 4(0{,}4)^k - \frac{1}{4}(-0{,}6)^k$$

$$H(s) = \frac{3}{s(s+2)} - \frac{5}{s(s+6)} =$$

$$= \frac{3}{2}\frac{1}{s\left(\frac{1}{2}s+1\right)} - \frac{5}{6}\frac{1}{s\left(\frac{1}{6}s+1\right)}$$

$$h(t) = \frac{3}{2}\left(1-e^{-2t}\right) - \frac{5}{6}\left(1-e^{-6t}\right) = \frac{2}{3} - \frac{3}{2}e^{-2t} + \frac{5}{6}e^{-6t}$$