EC502 Macroeconomics
Week 14 Seminar: Growth Theory I

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Calendar

- Week 14: Growth Theory I
- Week 15: Growth Theory II
- Week 16: Borrowing, Lending and Intertemporal Budget Constraints
- Week 17: Consumption
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- Week 21: Monetary Policy
- Week 22: Fiscal Policy
- Week 23: Limits to demand management
- Week 24: ICT TEST

1. Two countries have the production function:

$$Y = F(K, L) = K^{0.5}L^{0.5}$$
$$Y = F(K, L) = AK^{0.5}L^{0.5}, \text{ where A } = 1$$

a) Explain why it has **constant returns to scale** (CRS)

- Definition:if all inputs increase by the same factor, output increases by the *same* amount
- A function has CRS if it is homogeneous of degree 1:

$$F(zK, zL) = zY$$

- Take the production function and multiply inputs by same factor z

$$\begin{aligned} F(zK, zL) &= (zK)^{0.5}(zL)^{0.5} = \\ &= z^{0.5}K^{0.5}z^{0.5}L^{0.5} = \\ &= z^{0.5+0.5}K^{0.5}L^{0.5} = \\ &= zK^{0.5}L^{0.5} = zY \qquad \text{Q.E.D.} \end{aligned}$$

1. Two countries have the production function:

a) Explain why it has **constant returns to scale** (CRS)

- CRS imply that there are diminishing returns to capital
- An increase of capital stock will increase output - successive increases have a lower impact
- As capital stock accumulates, the output gain is smaller

$$\frac{\partial Y}{\partial K} = \frac{\partial K^{0.5} L^{0.5}}{\partial K} = 0.5 K^{0.5-1} L^{0.5} \quad > 0$$

- First derivative gives as the slope of production function: how does output change when capital changes

$$\frac{\partial^2 Y}{\partial K^2} = \frac{\partial (0.5 K^{-0.5} L^{0.5})}{\partial K} = 0.5(-0.5) K^{-0.5-1} L^{0.5} \quad < 0$$

- Second derivative gives as the change in the slope of production function: how does the slope adjust for a change in capital stock

1. Two countries have the production function:

$$Y = F(K, L) = K^{0.5} L^{0.5}$$

b) Derive the **per worker** production function

- Define $z = 1/L$. Then by CRS property we should get:

$$F(zK, zL) = (zK)^{0.5}(zL)^{0.5} = zY$$

$$F\left(\frac{K}{L}, \frac{L}{L}\right) = \left(\frac{K}{L}\right)^{0.5} \left(\frac{L}{L}\right)^{0.5} = \frac{Y}{L}$$

- Define $y = Y/L$ and $k = K/L$. Then:

$$F(k, 1) \equiv f(k) = (k)^{0.5} (1)^{0.5} = y$$

or simply $\quad y = k^{0.5} \quad$ **PF in intensive form**

c) Find the steady-state levels of capital, consumption and output per worker for each country A and B

Assumptions:

$$n = 0 \qquad a = 0 \qquad \delta = 0.05 \qquad s_A = 0.1 \qquad s_B = 0.2$$

- Define growth rate of capital per worker $k = K/L$:

$$\frac{\Delta k}{k} = \frac{\Delta K}{K} - \frac{\Delta L}{L} = \frac{\Delta K}{K} - n$$

- Law of Motion of Capital / Fundamental Equation of Solow model:

Aggregate capital stock: $$\underbrace{\Delta K}_{\text{Capital accumulation}} = \underbrace{sY}_{\text{investment}} - \underbrace{\delta K}_{\text{depreciation}}$$

- Divide over K:

$$\frac{\Delta K}{K} = s\frac{Y}{K} - \delta$$

- Replace with growth rate of capital per worker:

$$\frac{\Delta k}{k} + n = s\frac{Y}{K} - \delta$$

- c) Find the steady-state levels of capital, consumption and output per worker for each country A and B

- Replace production function $Y = K^{0.5}L^{0.5}$ and simplify

$$\frac{\Delta k}{k} + n = s\frac{K^{0.5}L^{0.5}}{K} - \delta = sK^{-0.5}L^{0.5} - \delta = s\left(\frac{K}{L}\right)^{-0.5} - \delta = sk^{-0.5} - \delta$$

- Move n to other side of equation to obtain Law of Motion of Capital per worker

$$\Delta k = sk^{0.5} - (\delta + n)k$$

- Steady State (SS): zero per-worker economic growth. There's just enough new capital to offset depreciation, meaning we get no additions to the overall capital stock

$$\Delta k = 0 \quad \longleftrightarrow sk^{0.5} = (\delta + n)k$$

$$\frac{k}{k^{0.5}} = \frac{s}{\delta + n} \quad \longleftrightarrow \quad k^{1-0.5} = \frac{s}{\delta + n}$$

c) Find the steady-state levels of capital, consumption and output per worker for each country A and B

- Assumptions

$$n = 0 \qquad a = 0 \qquad \delta = 0.05 \qquad s_A = 0.1 \qquad s_B = 0.2$$

- Rearrange to obtain k^*:

$$k_A^* = \left(\frac{s_A}{\delta + n}\right)^{\frac{1}{1-0.5}} = \left(\frac{0.1}{0.05 + 0}\right)^2 = 4$$

- Substitute k^* into y to get steady state income per worker y^*:

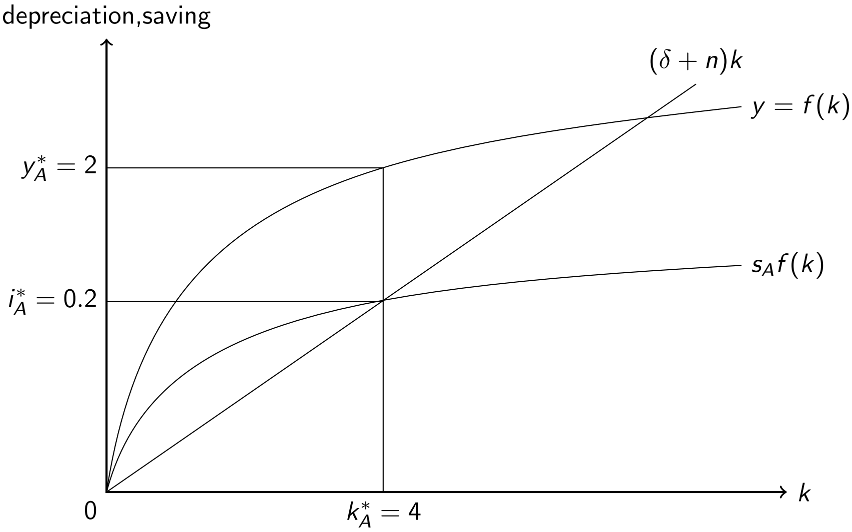
$$y = k^{0.5} \quad \longleftrightarrow \quad y_A^* = k^{*0.5} = 4^{0.5} = 2$$

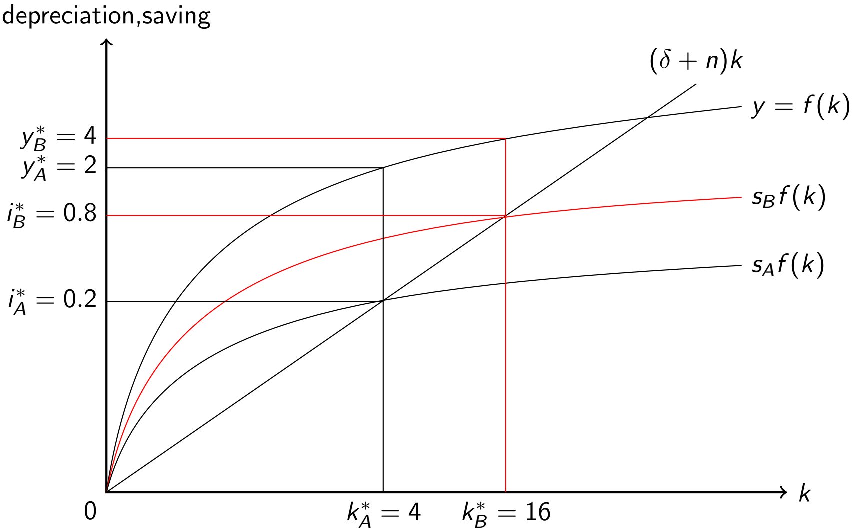
- Define consumption per worker $c = (1 - s)y$

$$c_A^* = (1 - s)y^* = (1 - 0.1) \cdot 2 = 1.8$$

- Country B:

$$k_B^* = (0.2/0.05)^2 = 16; \quad y_B^* = 4; \quad c_B^* = (1 - 0.2) \cdot 4 = 3.2$$


depreciation,saving
$(\delta + n)k$
$y = f(k)$
$y_A^* = 2$
$s_A f(k)$
$i_A^* = 0.2$
0
$k_A^* = 4$
k


depreciation,saving
$(\delta + n)k$
$y = f(k)$
$s_B f(k)$
$s_A f(k)$
$y_B^* = 4$
$y_A^* = 2$
$i_B^* = 0.8$
$i_A^* = 0.2$
0
$k_A^* = 4$
$k_B^* = 16$
k

d) Assume initial level of capital per worker $k = 2$ for both countries. What are the levels of income and consumption per worker? How will capital stock per worker evolve over time?

- Steady states:

$$k_A^* = 4; \quad y_A^* = 2; \quad c_A^* = 1.8$$
$$k_B^* = 16; \quad y_B^* = 4; \quad c_B^* = 3.2$$

- Initial capital per worker level $k_0 = 2$:

$$y_A = 2^{0.5} = 1.414; \quad c_A = (1 - s_A)y_A = 1.273$$
$$y_B = 2^{0.5} = 1.414; \quad c_B = (1 - s_B)y_B = 1.131$$

- Capital Law of Motion:

$$\Delta k = sk^{0.5} - (\delta + n)k \longleftrightarrow \Delta k = 0.1 \cdot 2^{0.5} - (0.05 + 0) \cdot 2 = 0.041 > 0$$
$$\Delta k = 0.2 \cdot 2^{0.5} - (0.05 + 0) \cdot 2 = 0.183 > 0$$

2. The Solow Growth Model shows that only technological progress can explain persistently rising living standards

YES / NO ?

$$\hat{k}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{1}{1-0.5}}$$

- Example: assume country A has $a = 0.05$ and B has $a = 0.1$

$$\hat{k}^*_A = \left(\frac{0.1}{0.05 + 0 + 0.05}\right)^{\frac{1}{1-0.5}} = 1$$

$$\hat{k}^*_B = \left(\frac{0.1}{0.05 + 0 + 0.1}\right)^{\frac{1}{1-0.5}} = 0.44$$

2. The Solow Growth Model shows that only technological progress can explain persistently rising living standards

YES / NO ?

$$\hat{y}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{0.5}{1-0.5}} \quad \& \quad \frac{\Delta y^*}{y^*} = \frac{\Delta \hat{y}^*}{\hat{y}^*} + \frac{\Delta A^*}{A^*} = a$$

Cross-country differences in LEVELS of income per capita

- Savings rate (s)
- Depreciation rates (δ)
- Population growth (n)
- Technical progress (a) - only per effective worker
- Capital intensity (α)

Cross-country differences in GROWTH RATES of income per capita

- Only technical progress
- Population growth also affects aggregate income growth

3. Calculate steady-state for different values of savings rate with $y = k^{0.75}$.

Law of motion of capital per worker:

$$\Delta k = sk^{0.75} - (\delta + n)k$$

Steady State: $\Delta k = 0 \quad \longleftrightarrow \quad sk^{0.75} = (\delta + n)k \quad \longleftrightarrow \quad k^* = \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-0.75}}$

Substitute for different values of savings rate:

savings rate (s)	0	0.1	0.5	**0.8**	0.9	1
k^*	0	1	625	4096	6561	10000
y^*	0	1	125	512	729	1000
c^*	0	0.9	62.5	**102.4**	72.9	0
i^*	0	0.1	62.5	409.6	656.1	1000
MPk^*	-	0.75	0.15	0.094	0.083	0.075

How to find the k^* that gives us the MAX consumption per worker c^*?
We can see it NEXT CLASS!

Extras

Please refer if you find time.

Notes: Solow model with technical progress and population growth

- Define output with labour-augmenting technical progress:

$$Y = K^{0.5}(AL)^{0.5}$$

- Define output and capital per effective worker:

$$\hat{y} = \frac{Y}{AL} = \frac{K^{0.5}(AL)^{0.5}}{AL} = \left(\frac{K}{AL}\right)^{0.5} = \hat{k}^{0.5}$$

- Growth rate of capital per effective worker $\hat{k} = K/AL$:

$$\frac{\Delta \hat{k}}{\hat{k}} = \frac{\Delta K}{K} - \frac{\Delta A}{A} - \frac{\Delta L}{L} = \frac{\Delta K}{K} - a - n$$

- Capital per effective worker law of motion and steady-state $\hat{k}^*$:

$$\Delta \hat{k} = s\hat{k}^{0.5} - (\delta + n + a)\hat{k} \quad \longleftrightarrow \quad \hat{k}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{1}{1-0.5}}$$

2. The Solow Growth Model shows that only technological progress can explain persistently rising living standards

YES / NO ?

$$\hat{k}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{1}{1-0.5}} \quad \& \quad \hat{y}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{0.5}{1-0.5}}$$

- Recall growth rates formulas IN STEADY STATE:

$$\frac{\Delta K}{K} = ?$$

$$\frac{\Delta k}{k} = \frac{\Delta K}{K} - n$$

$$\frac{\Delta \hat{k}}{\hat{k}} = \frac{\Delta K}{K} - a - n$$

$$\frac{\Delta \hat{k}}{\hat{k}} = \frac{\Delta K}{K} - a - n = 0$$

$$\frac{\Delta k}{k} = \frac{\Delta K}{K} - n = a$$

$$\frac{\Delta K}{K} = a + n$$

2. The Solow Growth Model shows that only technological progress can explain persistently rising living standards

YES / NO ?

$$\hat{k}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{1}{1-0.5}} \quad \& \quad \hat{y}^* = \left(\frac{s}{\delta + n + a}\right)^{\frac{0.5}{1-0.5}}$$

- Recall growth rates formulas IN STEADY STATE:

$$\hat{y} = \frac{Y}{AL} = \hat{k}^{0.5}$$

$$y = \frac{Y}{L} = A\frac{Y}{AL}$$

$$Y = AL\frac{Y}{AL}$$

$$\frac{\Delta \hat{y}}{\hat{y}} = 0.5\frac{\Delta \hat{k}}{\hat{k}} = 0$$

$$\frac{\Delta y}{y} = \frac{\Delta A}{A} + \frac{\Delta \hat{y}}{\hat{y}} = a$$

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \frac{\Delta L}{L} + \frac{\Delta \hat{y}}{\hat{y}} = a + n$$