ECON5020 – Macroeconomics

Week 25 - Growth Theory

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Growth Theory

Two Countries with production function: $Y_t = F(K_t, L_t) = K_t^{0.5} L_t^{0.5}$

- Constant Returns to Scale
  - **Interpretation:** If inputs are increased proportionately, output grows in the same proportion.
  - **Condition:** Production function is homogenous of degree 1.

$$F(K_t, L_t) = K_t^{0.5} L_t^{0.5} \implies F(\lambda K_t, \lambda L_t) = (\lambda K_t)^{0.5} (\lambda L_t)^{0.5} = \lambda K_t^{0.5} L_t^{0.5} = \lambda F(K_t, L_t)$$

- Per worker production function
  - λ above can be any value, including L, so

$$\frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = \left(\frac{K_t}{L_t}\right)^{0.5} \left(\frac{L_t}{L_t}\right)^{0.5} \implies \frac{Y_t}{L_t} \equiv y_t = f(k_t) = k_t^{0.5}$$

- Steady state levels of capital per worker, consumption and income per worker
  - Evolution of capital:

$$K_{t+1} = (1-\delta)K_t + sY_t \implies \frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\frac{Y_t}{L} \implies k_{t+1} = (1-\delta)k_t + sf(k_t)$$

  - In the steady state $k_{t+1} = k_t = k_{ss}$. Replacing in the equation above and solving for k: $k_{ss} = \left(\frac{s}{\delta}\right)^2$. Income and consumption are given by $y_{ss} = k_{ss}^{0.5}$ and $c_{ss} = (1-s)y_{ss}$

Growth Theory

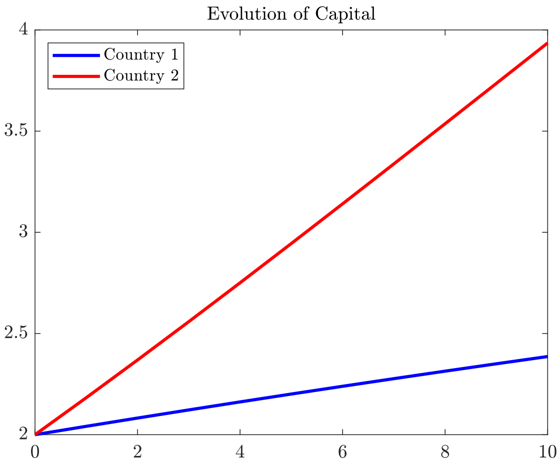
Two Countries with production function: $y_t = f(k_t) = k_t^{0.5}$

	Country 1			**Country 2**		
Year	k	$y = k^{0.5}$	$c = (1 - s_1)y$	k	$y = k^{0.5}$	$c = (1 - s_2)y$
0	2.000	1.414	1.273	2.000	1.414	1.131
1	2.041	1.429	1.286	2.183	1.477	1.182
2	2.082	1.443	1.299	2.369	1.539	1.231
3	2.122	1.457	1.311	2.559	1.600	1.280
4	2.162	1.470	1.323	2.751	1.658	1.327
5	2.201	1.484	1.335	2.945	1.716	1.373
6	2.239	1.496	1.347	3.141	1.772	1.418
7	2.277	1.509	1.358	3.338	1.827	1.462
8	2.314	1.521	1.369	3.537	1.881	1.504
9	2.350	1.533	1.380	3.736	1.933	1.546
10	2.386	1.545	1.390	3.936	1.984	1.587
SS	**4.000**	**2.000**	**1.800**	**16.000**	**4.000**	**3.200**

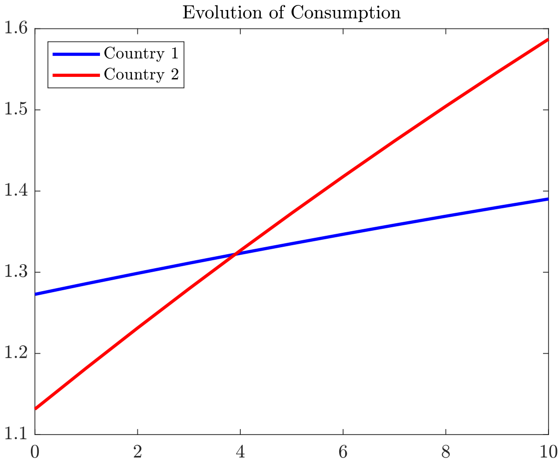
Growth Theory

Two Countries with production function: $y_t = f(k_t) = k_t^{0.5}$ and $s_1 = 10\%$, $s_2 = 20\%$.

- Evolution of capital stock per worker



(a) Capital



(b) Consumption

Figure: Overall caption for the figures

Technological progress and persistently rising living standards

- Say now that

$$Y_t = K_t^{0.5}\left(A_t L_t\right)^{0.5} \text{ and we can define } \hat{y}_t \equiv \frac{Y_t}{A_t L_t} = \hat{k}_t^{0.5}$$

- Which will imply that Derivation here

$$\Delta \hat{k}_t = s\hat{k}_t^{0.5} - (\delta + a + n)\hat{k}_t$$

- And in the steady state

$$\hat{k}_{ss} = \left(\frac{s}{\delta + a + n}\right)^{\frac{1}{1-.5}} \text{ and } \hat{y}_{ss} = \left(\frac{s}{\delta + a + n}\right)^{\frac{.5}{1-.5}}$$

- If we measure living standards as out-per-capita, in steady state:

$$y_{ss} \equiv \frac{Y_{ss}}{L_{ss}} = A_{ss}\underbrace{\frac{Y_{ss}}{A_{ss}L_{ss}}}_{\hat{y}_{ss}} \implies \Delta^{\%} y_{ss} = \Delta^{\%} A_{ss} + \underbrace{\Delta^{\%}\hat{y}_{ss}}_{=0} \implies \Delta^{\%} y_{ss} = a$$

Savings and Consumption

We know that $k_{ss} = \left(\frac{s}{\delta}\right)^{\frac{1}{.25}}$, $y_{ss} = k_{ss}^{.75}$, $c = (1-s)y_{ss}$, $i = sy_{ss}$ **and** $\text{MPK} = .75\left(\frac{1}{k_{ss}}\right)^{.25}$

Table: Add caption

s	$k_{ss} = \left(\frac{s}{\delta}\right)^{\frac{1}{.25}}$	$y_{ss} = k_{ss}^{.75}$	$c = (1-s)y_{ss}$	$i = sy_{ss}$	$\text{MPK} = .75\left(\frac{1}{k_{ss}}\right)^{.25}$
0	0.00	0	0	0	-
0.5	625	125	63	63	0.150
0.8	4096	512	102	410	0.094
1	10000	1000	0	1000	0.075

- What would happened if $y_t = k_t^{.25}$ instead?

Here we show how to get $\Delta\widehat{\mathbf{k}}_\mathbf{t} = \mathbf{s}\widehat{\mathbf{k}}_\mathbf{t}^{\mathbf{0.5}} - (\delta + \mathbf{a} + \mathbf{n})\widehat{\mathbf{k}}_\mathbf{t}$:

From the standard version of the model, we know:

$$\Delta K_t = sY_t - \delta K_t$$

Which if we define "hat" variables as the original variable divided by $A_t L_t$ we get

$$\frac{\Delta K_t}{A_t L_t} = s\hat{y}_t - \delta \hat{k}_t \text{ where } \hat{y}_t = \frac{Y_t}{A_t L_t} \text{ and } \hat{k}_t = \frac{K_t}{A_t L_t}$$

In order to get the proper $\Delta\hat{k}_t$ we use total differentiation. Notice that if $\hat{k}_t = \frac{K_t}{A_t L_t}$ we can write

$$\Delta\hat{k}_t = \frac{\partial\hat{k}_t}{\partial K_t}\Delta K_t + \frac{\partial\hat{k}_t}{\partial A_t}\Delta A_t + \frac{\partial\hat{k}_t}{\partial L_t}\Delta L_t$$

Calculating the partial derivatives we get

$$\frac{\partial\hat{k}_t}{\partial K_t} = \frac{1}{A_t L_t} \qquad \frac{\partial\hat{k}_t}{\partial A_t} = -\frac{K_t L_t}{(A_t L_t)^2} \qquad \frac{\partial\hat{k}_t}{\partial L_t} = -\frac{K_t A_t}{(A_t L_t)^2}$$

Now that we know

$$\Delta \hat{k}_t = \frac{\partial \hat{k}_t}{\partial K_t}\Delta K_t + \frac{\partial \hat{k}_t}{\partial A_t}\Delta A_t + \frac{\partial \hat{k}_t}{\partial L_t}\Delta L_t$$

and

$$\frac{\partial \hat{k}_t}{\partial K_t} = \frac{1}{A_t L_t} \qquad \frac{\partial \hat{k}_t}{\partial A_t} = -\frac{K_t L_t}{(A_t L_t)^2} \qquad \frac{\partial \hat{k}_t}{\partial L_t} = -\frac{K_t A_t}{(A_t L_t)^2}$$

we can write

$$\begin{aligned}
\Delta \hat{k}_t &= \frac{\Delta K_t}{A_t L_t} - \frac{K_t L_t}{(A_t L_t)^2}\Delta A_t - \frac{K_t A_t}{(A_t L_t)^2}\Delta L_t \\
&= \frac{\Delta K_t}{A_t L_t} - \frac{K_t L_t}{A_t (L_t)^2}\frac{\Delta A_t}{A_t} - \frac{K_t A_t}{(A_t)^2 L_t}\frac{\Delta L_t}{L_t} \\
&= \frac{\Delta K_t}{A_t L_t} - \underbrace{\frac{K_t}{A_t L_t}}_{\hat{k}_t}\frac{\Delta A_t}{A_t} - \underbrace{\frac{K_t}{A_t L_t}}_{\hat{k}_t}\frac{\Delta L_t}{L_t}
\end{aligned}$$

Remember now that $\frac{\Delta A_t}{A_t} = a_t$ and $\frac{\Delta L_t}{L_t} = n_t$ so we can write

$$\Delta \hat{k}_t = \frac{\Delta K_t}{A_t L_t} - (a_t + n_t)\,\hat{k}_t \implies \frac{\Delta K_t}{A_t L_t} = \Delta \hat{k}_t + (a_t + n_t)\,\hat{k}_t$$

continue in next slide...

Plugging

$$\frac{\Delta K_t}{A_t L_t} = \Delta \hat{k}_t + (a_t + n_t)\,\hat{k}_t$$

into

$$\frac{\Delta K_t}{A_t L_t} = \hat{y}_t - \delta \hat{k}_t$$

we get

$$\Delta \hat{k}_t = s\hat{y}_t - (\delta + a_t + n_t)\,\hat{k}_t$$

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