EC502 Macroeconomics
Week 15 Seminar: Growth Theory II

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Calendar

- ~~Week 14: Growth Theory I~~
- Week 15: Growth Theory II
- Week 16: Borrowing, Lending and Intertemporal Budget Constraints
- Week 17: Consumption
- Week 18: Investment
- Week 19: READING WEEK
- Week 20: Real Business Cycles
- Week 21: Monetary Policy
- Week 22: Fiscal Policy
- Week 23: Limits to demand management
- Week 24: ICT TEST

1a. Derive the growth rate and the level of per capita income in SS:

- Step 1:
- Step 2:
- Step 3:

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- Step 1: Intensive form of production function

$$y = \frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} = AK^{\alpha}L^{-\alpha} = A\left(\frac{K}{L}\right)^{\alpha} = Ak^{\alpha}$$

- Step 2:

- Step 3:

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- Step 2: Law of motion of capital per worker

$$\Delta k = sy - (\delta + n)k = sAk^{\alpha} - (\delta + n)k$$

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- Step 2: Law of motion of capital per worker

$$\Delta k = sy - (\delta + n)k = sAk^{\alpha} - (\delta + n)k$$

- Step 3: SS - since $a = \Delta A/A = 0$, simply assume SS for capital per worker k

$$\Delta k = 0 \quad \longleftrightarrow \quad sAk^{\alpha} - (\delta + n)k = 0$$

$$sAk^{\alpha} = (\delta + n)k \quad \longleftrightarrow \quad \frac{sA}{\delta + n} = k^{1-\alpha}$$

$$k^{*} = \left(\frac{sA}{\delta + n}\right)^{1/(1-\alpha)}$$

1a. Derive the growth rate and the level of per capita income in SS:

- Capital per worker in SS:

$$k^* = \left(\frac{sA}{\delta + n}\right)^{1/(1-\alpha)}$$

- Step 4: replace k^* to get income per capita in SS

$$y^* = Ak^{*\alpha} = A\left[\left(\frac{sA}{\delta+n}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} = A\left(\frac{sA}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} =$$

$$= A \cdot A^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} = A^{1+\frac{\alpha}{1-\alpha}} \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} =$$

$$= A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

1b. Relative income per capita of DR Congo predictions

- Calculate predicted relative SS y^* between both countries:

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$$\frac{y^C}{y^{US}} = \frac{(A^C)^{\frac{1}{1-\alpha}}}{(A^{US})^{\frac{1}{1-\alpha}}} \frac{\left(\frac{s^C}{\delta^C+n^C}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{s^{US}}{\delta^{US}+n^{US}}\right)^{\frac{\alpha}{1-\alpha}}}$$

$$= \left(\frac{A^C}{A^{US}}\right)^{\frac{1}{1-\alpha}} \left[\frac{\frac{s^C}{\delta^C+n^C}}{\frac{s^{US}}{\delta^{US}+n^{US}}}\right]^{\frac{\alpha}{1-\alpha}}$$

- Since A, n and δ are equal for both countries we get that

$$\frac{y^C}{y^{US}} = \left(\frac{s^C}{s^{US}}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{0.154}{0.202}\right)^{\frac{0.3}{0.7}} = 0.89$$

1b. Relative income per capita of DR Congo predictions

- Calculate predicted relative SS y^* between both countries:

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$$= \left(\frac{A^C}{A^{US}}\right)^{\frac{1}{1-\alpha}} \left[\frac{\frac{s^C}{\delta^C+n^C}}{\frac{s^{US}}{\delta^{US}+n^{US}}}\right]^{\frac{\alpha}{1-\alpha}}$$

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- So Congo's income per capita is 89% that of US. But we know in 2008 it was actually 0.055 (5%). Is the Solow model bad?
- Such difference might come from a different source: α, A, n or δ?

1c. Alternative 1: differences in capital intensity α

- Suppose we want to calculate how much should α be to generate the correct value of income per capita differences:
- Step 1: assign the correct value

$$\frac{y^C}{y^{US}} = \left(\frac{0.154}{0.202}\right)^{\frac{\alpha}{1-\alpha}} = 0.055$$

- Step 2: apply logarithms

$$\log\left[\left(\frac{0.154}{0.202}\right)^{\frac{\alpha}{1-\alpha}}\right] = \log(0.055) \quad \longleftrightarrow \quad \frac{\alpha}{1-\alpha}\log\left(\frac{0.154}{0.202}\right) = \log(0.055)$$

- Solve for the ratios and α

$$\frac{\alpha}{1-\alpha}\cdot(-0.27) = -2.9 \quad \longleftrightarrow \quad \frac{\alpha}{1-\alpha} = 10.74$$

$$\alpha = 10.74(1-\alpha) \quad \longleftrightarrow \quad 11.74\alpha = 10.74$$

$$\alpha = \frac{10.74}{11.74} = 0.915 \quad \leftarrow \quad \text{very implausible value}$$

1c. Alternative 2: differences in technology A

- Step 1: assign the correct value (still assume same n and δ)

$$\frac{y^C}{y^{US}} = \left(\frac{A^C}{A^{US}}\right)^{\frac{1}{1-\alpha}} \left(\frac{s^C}{s^{US}}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{A^C}{A^{US}}\right)^{\frac{1}{0.7}} \left(\frac{0.154}{0.202}\right)^{\frac{0.3}{0.7}} = 0.055$$

- Step 2: solve for the A ratio:

$$\left(\frac{A^C}{A^{US}}\right)^{\frac{1}{0.7}} \cdot 0.89 = 0.055 \quad \longleftrightarrow \quad \frac{A^C}{A^{US}} = \left(\frac{0.055}{0.89}\right)^{0.7} = 0.1424$$

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- DR Congo has only 14.24% of the technology level of US
- Not only differences in savings rates, but also differences in technology cause Congo having only 5.5% of US income per capita
- Try out other candidates: δ^C/δ^{US} or n^C/n^{US}

2a. Find the steady state capital stock and discuss economic efficiency

Assumptions

$$Y = AK^{\alpha}L^{1-\alpha}, \quad A = 1, \quad a = 0, \quad \alpha = 0.3, \quad n = 0, \quad s = 0.5, \quad \delta = 0.1$$

- Step 1: intensive form of income per worker

$$y = \frac{Y}{L} = Ak^{\alpha}$$

- Step 2: law of motion of capital per worker

$$\Delta k = sAk^{\alpha} - (\delta + n)k$$

- Step 3: steady state

$$\Delta k = 0 \quad \longleftrightarrow \quad k^* = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.5 \cdot 1}{0.1 + 0}\right)^{\frac{1}{1-0.3}} = 9.97$$

2a. Find the steady state capital stock and discuss economic efficiency

- Step 4: find Golden Rule of capital - that level of k^* that maximizes consumption in SS
- Recall savings = investment. In SS, investment is $sf(k^*) = (\delta + n)k^*$
- Consumption = income - savings

$$c^* = (1-s)y^* \quad \longleftrightarrow \quad c^* = f(k^*) - sf(k^*) \quad \longleftrightarrow \quad c^* = f(k^*) - (\delta + n)k^*$$

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$$c^* = (1-s)y^* \quad \longleftrightarrow \quad c^* = f(k^*) - sf(k^*) \quad \longleftrightarrow \quad c^* = f(k^*) - (\delta + n)k^*$$

- Maximize consumption - derivate with respect to k^*

$$\max\{c^*\} = \max\{f(k^*) - (\delta + n)k^*\} \quad \longleftrightarrow \quad \frac{\partial c^*}{\partial k^*} = \frac{\partial f(k^*)}{\partial k^*} - (\delta + n) = 0$$

$$MPK \equiv \frac{\partial f(k^*)}{\partial k^*} = \alpha A k^{\alpha - 1} = (\delta + n)$$

2a. Find the steady state capital stock and discuss economic efficiency

- Step 5: Solve for k^{GR}

$$\frac{\alpha A}{\delta + n} = \frac{1}{k^{\alpha-1}} \quad \longleftrightarrow \quad \frac{\alpha A}{\delta + n} = k^{1-\alpha} \quad \longleftrightarrow$$

$$\longleftrightarrow \quad k^{GR} = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.3 \cdot 1}{0.1 + 0}\right)^{\frac{1}{1-0.3}} = 4.8$$

- Step 6 (most important): compare with steady-state k^*

$$k^{GR} = 4.8 < 9.97 = k^* = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

- Note only difference between k^* and k^{GR} is that one has s while the other has α in the formula

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- This economy is ***dynamically inefficient*** - saves too much, has capital stock overaccumulation
- We could increase consumption now by reducing savings, and end up with highest consumption in the SS - **Pareto improvement**

2a. Find the steady state capital stock and discuss economic efficiency

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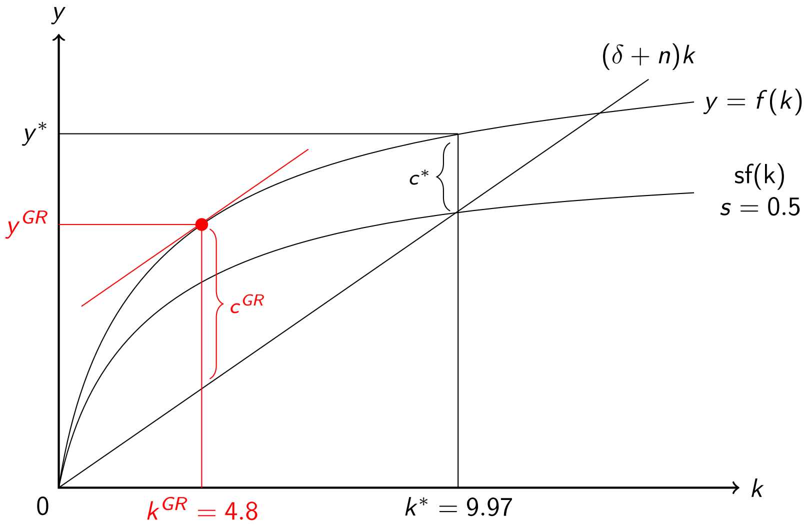
- Step 6 (most important): compare with steady-state k^*

$$k^{GR} = 4.8 < 9.97 = k^* = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

- Which savings rate would take us to k^{GR}?

$$k^{**} = 4.8 = \left(\frac{s \cdot 1}{0.1 + 0}\right)^{\frac{1}{0.7}} \quad \longleftrightarrow \quad s^{GR} = 0.3$$


y
$(\delta + n)k$
$y = f(k)$
y^*
c^*
sf(k)
$s = 0.5$
y^{GR}
c^{GR}
k
0
$k^{GR} = 4.8$
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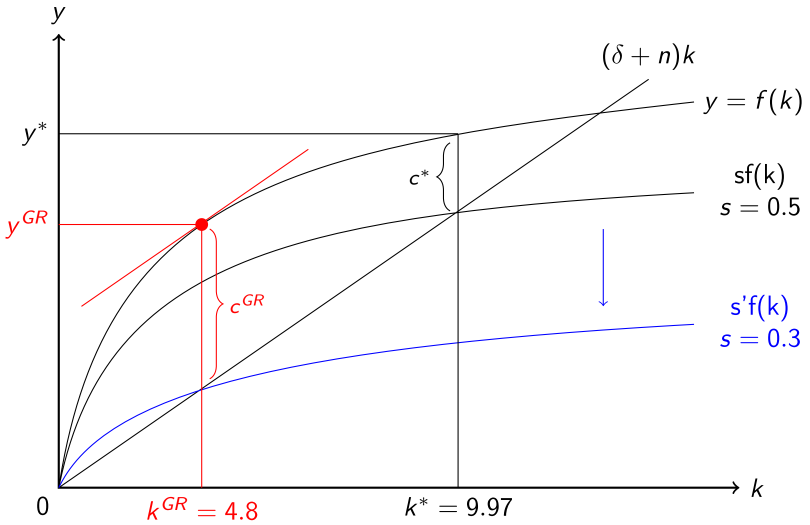
y
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k
0
$k^{GR} = 4.8$
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2b. Golden rule with population growth

- Just incorporate $n = 0.04$ to formulae

$$k^* = \left(\frac{sA}{\delta + n}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.5 \cdot 1}{0.1 + 0.04}\right)^{\frac{1}{1-0.3}} = 6.16$$

$$k^{GR} = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.3 \cdot 1}{0.1 + 0.04}\right)^{\frac{1}{1-0.3}} = 2.97$$

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- The economy is still saving too much: $k^* > k^{GR}$
- Without population growth: $k^{GR}/k^* = 4.8/9.97 = 0.48$
- With population growth: $k^{GR}/k^* = 2.97/6.16 = 0.48$
- ***Only the savings rate*** matters to close the distance between k^{GR} and k^*

3. Growth Accounting

- Suppose output is $Y = AF(K, L) = AK^{\alpha}L^{1-\alpha}$
- Apply differentiation with respect to time $dx \equiv \partial x / \partial t$

$$
\begin{aligned}
dY &= dA \cdot (K^{\alpha}L^{1-\alpha}) + A \cdot \frac{\partial(K^{\alpha}L^{1-\alpha})}{\partial K} dK + A \cdot \frac{\partial(K^{\alpha}L^{1-\alpha})}{\partial L} dL \\
&= dA \cdot (K^{\alpha}L^{1-\alpha}) + A\alpha K^{\alpha-1}L^{1-\alpha} dK + A \cdot (1-\alpha)K^{\alpha}L^{-\alpha} dL
\end{aligned}
$$

- Divide by $Y = AF(K, L)$

$$
\frac{dY}{Y} = \frac{dA \cdot (K^{\alpha}L^{1-\alpha})}{AK^{\alpha}L^{1-\alpha}} + \frac{A\alpha K^{\alpha-1}L^{1-\alpha}}{AK^{\alpha}L^{1-\alpha}} dK + \frac{A(1-\alpha)K^{\alpha}L^{-\alpha}}{AK^{\alpha}L^{1-\alpha}} dL
$$

$$
\longleftrightarrow \quad \frac{dY}{Y} = \frac{dA}{A} + \alpha \frac{dK}{K} + (1-\alpha)\frac{dL}{L}
$$

3. Growth Accounting

- Solve for the unknown growth rate of technology (TFP) and plug the index rate changes:

$$\frac{dA}{A} = \frac{dY}{Y} - \alpha\frac{dK}{K} - (1-\alpha)\frac{dL}{L}$$
$$= \frac{185-100}{100} - 0.3\cdot\frac{165-100}{100} - (1-0.3)\cdot\frac{150-100}{100}$$
$$= 0.85 - 0.3\cdot 0.65 - 0.7\cdot 0.5$$

$$\frac{dA}{A} = 0.305$$

- Then take all contributions over the growth rate $dY/Y = 0.85$

	Y	K	L	A
growth rates	0.85	0.65	0.5	0.305
% over Y	100%	64%		36%
		23%	41%	

Note

Per effective worker:

$$\hat{k} = \frac{K}{AL}$$

$$\Delta\hat{k} = s\hat{y} - (\delta + n + a)\hat{k}$$

$$\text{In SS: } \Delta\hat{k} = 0$$

$$\hat{y} = \frac{Y}{AL} = \hat{k}^{0.5}$$

$$\frac{\Delta\hat{y}}{\hat{y}} = 0.5 \cdot \frac{\Delta\hat{k}}{\hat{k}} = 0$$

Per worker:

$$k = \frac{K}{L}$$

$$\Delta k = sy - (\delta + n)k$$

$$y = f(k) = k^{0.5}$$

$$y = \frac{Y}{L} = A\frac{Y}{AL} = A \cdot \hat{y}$$

$$\frac{\Delta y}{y} = \frac{\Delta A}{A} + \frac{\Delta\hat{y}}{\hat{y}} = a$$

Aggregate:

$$K = AL\frac{K}{AL} = AL \cdot \hat{k}$$

$$\frac{\Delta K}{K} = \frac{\Delta A}{A} + \frac{\Delta L}{L} + \frac{\Delta\hat{k}}{\hat{k}}$$

$$\Delta K = sY - \delta K$$

$$Y = F(K, L) = K^{0.5}L^{0.5}$$

$$Y = AL\frac{Y}{AL} = AL \cdot \hat{y}$$

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \frac{\Delta L}{L} + \frac{\Delta\hat{y}}{\hat{y}} = a + n$$