DEPARTMENT OF HUMANITIES & SCIENCE

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Unit – 1 ⇝ Probability Theory - I

Introduction

→ Probability theory is the branch of mathematics that is concerned with random (or chance) phenomena. It has attracted people to its study both because of its intrinsic interest and its successful applications to many areas within the physical, biological, social sciences, in engineering and in the business world.

→ The words PROBABLE and POSSIBLE CHANCES are quite familiar to us. We use these words when we are sure of the result of certain events. These words convey the sense of uncertainty of occurrence of events.

→ Probability is the word we use to calculate the degree of the certainty of events.

→ There are two types of approaches in the theory of Probability:

- Classical Approach – By Blaise Pascal
- Axiomatic Approach – By A. Kolmogorov

Method – 1 ⇝ Counting

Factorial Notation

→ The notation n! represents the product of first n natural numbers, i.e.,

$$\mathbf{n! = 1 \times 2 \times 3 \times \dots \dots \times (n-1) \times n}$$

→ For example:

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

→ **Some important results:**

(1) $n! = n \times (n-1)!$

(2) $0! = 1$

Permutation (Arrangement)

→ A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

→ Suppose that we are given ‘n’ distinct objects and wish to arrange ‘r’ of these objects in a line without repeating.

→ Then number of such different arrangements is given as below. It is denoted by ${}^{n}P_{r}$.

$$\mathbf{{}^{n}P_{r} = \frac{n!}{(n-r)!}} \; ; \; 0 < r \leq n.$$

→ For example:

A number of 3 lettered words which can be formed by without repeating letters of the word “NUMBER” is,

$${}^{6}P_{3} = \frac{6\,!}{3\,!} = 120.$$

→ **Some important results:**

(1) ${}^{n}P_{n} = n!$

(2) ${}^{n}P_{0} = 1$

(3) When all the objects are distinct, the number of permutations with repeating the object is given by $\mathbf{n^{r}}$.

- For example:

  A number of 3 lettered words which can be formed by repeating letters of the word "NUMBER" is

  $6^3 = 216$.

(4) When all the objects are **not distinct**, the number of permutations of n object in which n_1 are of one kind, n_2 are of second kind, …, and n_k are of k^{th} kind is given by,

$$\frac{\mathbf{n!}}{\mathbf{n_1!\ n_2!\ \dots\ n_k!}}.$$

Note that, $n = n_1 + n_2 + \cdots + n_k$.

- For example:

  A number of different permutations of letters of the word MISSISSIPPI is

  Here, total number of letters are 11 among which M repeats 1 time, I & S repeat 4 times each and P repeats 2 times.

  $$\frac{11!}{1!\ 4!\ 4!\ 2!} = 34650$$

Combination (Selection)

→ A combination is selection of a number of objects from given set of objects.

→ We denote the number of unique r selections or combinations out of a group of n objects by ^nC_r and defined as below.

$$\mathbf{{}^nC_r = C(n,\ r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}}\ ;\ 0 < r \le n.$$

→ For example:

- The number of ways in which 3 card can be chosen from 8 cards is

  $$\binom{8}{3} = \frac{8!}{3!\,(8-3)!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

- A club has 10 male and 8 female members. A committee composed of 3 men and 4 women is formed. In how many ways this can be done?

  $$\binom{10}{3}\binom{8}{4} = 120 \times 70 = 8400$$

- Out of 6 boys and 4 girls, in how many ways a committee of five members can be formed in which there are at most 2 girls are included?

$$\binom{4}{2}\binom{6}{3}+\binom{4}{1}\binom{6}{4}+\binom{4}{0}\binom{6}{5}=120+60+6=186$$

→ **Some important results:**

(1) $^{n}C_{0} = {}^{n}C_{n} = 1$

(2) $^{n}C_{1} = n$

(3) $^{n}C_{k} = {}^{n}C_{n-k}$

Example of Method-1: Counting

H	1	How many four digit numbers are there with no digit repeated? **Answer: 4536**
T	2	How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated? What if repetition is allowed? **Answer: 504 (without repeating), 729 (with repeating)**
T	3	How many words, with or without meaning, can be formed using all letters of the word EQUATION, using each letter exactly once? **Answer: 40320**
C	4	How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if **(1)** 4 – letters are used at a time, **(2)** All letters are used at a time, **(3)** All letters are used but the first letter is vowel? **Answer: (1) 360, (2) 720, (3) 240**
H	5	Find the number of permutations of the letters of the word ALLAHABAD. **Answer: 7560**
H	6	How many ways to select 2 students from 5 students? **Answer: 10**

H	7	In how many different ways can 4 of 15 laboratory assistants be chosen to assist with an experiment? **Answer: 1365**
C	8	How many different poker hands consists of 5 cards being either 2 or 7? **Answer: 56**
C	9	A bag contains 5 black ball and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. **Answer: 200**
T	10	Out of 6 boys and 4 girls in how many ways a committee of five members can be formed in which there are at most 2 girls are included? **Answer: 186**
C	11	What is the number of ways of choosing 4 cards of below choices from pack of 52 cards? **(1)** In how many of these four cards are of the same suit. **(2)** Four cards belong to four different suits. **(3)** Two are red card and two are black card. **Answer: (1) 2860, (2) 28561, (3) 105625**

Method – 2 ⇝ Basic Terminology and Definition of Probability

2.1 Sample Space and Event

Random Experiment

→ An experiment conducted under identical conditions is known as random experiment if it satisfies the following conditions:

(1) It has more than one possible outcome.

(2) It is not possible to predict the outcome in advance.

→ For example:

- Tossing a coin.
- Throwing/rolling a die.
- Selecting a card from a pack of playing cards.

→ The result of a random experiment is known as **outcome**.

Sample Space

→ The set of all possible outcome is known as sample space of an experiment.

→ Sample space is denoted by the symbol **S**.

→ Elements of a sample space is known as **sample points**.

→ The sample space of an experiment may consist of a finite or an infinite number of possible outcomes.

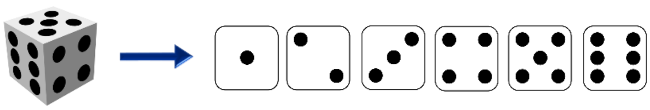
→ For example:

- Consider an experiment of tossing a coin. The outcomes of this experiment are head (H) or tail (T).

  Therefore, sample space of this experiment is written as follow:

  $S = \{H,\ T\}$

- Consider an experiment of rolling a die. The outcomes of this experiment are $1,\ 2,\ 3,\ 4,\ 5$ or 6.

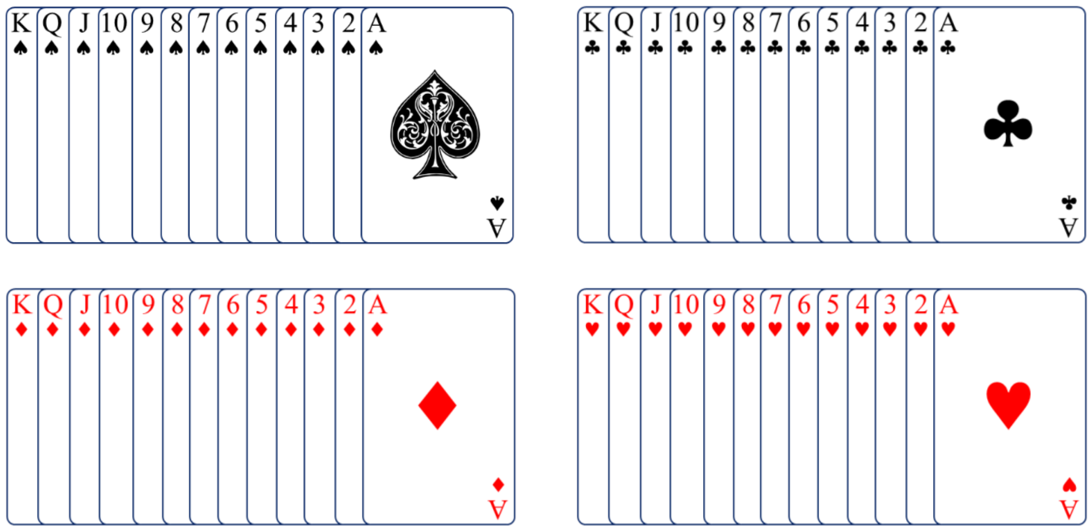


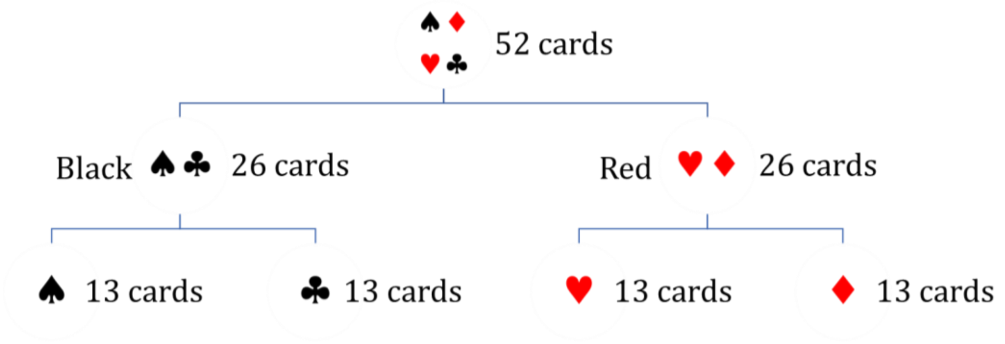
  Therefore, sample space of this experiment is written as follow:

  $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

→ Deck of 52 playing cards which has four suits as follow:

SPADES ♠, CLUB ♣, DIAMOND ♦, HEART ♥





Ace → A Picture cards = 12

Experiment **with** Replacement

→ You select something from sample, **put it back** and then select again.

→ Each selection is independent and the size of the sample remains constant throughout the experiment.

Experiment **without** Replacement

→ You select something from sample, **don't put it back** and then select again.

→ Each selection affects the subsequent selections, they are dependent on previous ones and size of a sample space is reduced after each selection.

Event

→ Any subset of a sample space is known as an event.

→ For example:

- In experiment of Rolling a six-sided die and observing the number (or dots) that appear on top.

  A sample space of an experiment is

  $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$.

  - Let A be an event that **even** number appears on top.

    $\therefore\ A = \{2,\ 4,\ 6\}$

  - Let B be an event that number **4** appears on top.

    $\therefore\ B = \{4\}$

  - Let C be an event that number **7** appears on top.

    $\therefore\ C = \phi$

  - Let D be an event that number **less than 7** appears on top.

    $\therefore\ D = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} = S$

Impossible Event

→ The event E is known as impossible event if $\mathbf{E = \phi}$.

→ In above example, event $C = \phi$ therefore, it is impossible event.

Sure Event

→ The event E is known as sure event if $\mathbf{E = S}$.

→ In above example, event $D = S$ therefore, it is sure event.

Simple **or** Elementary Event

→ If an event E has **only one** sample point of a sample space, then it is known as a simple (or elementary) event.

→ In above example, event B has only one sample point therefore, it is simple event.

Compound Event

→ If an event E has **more than one** sample point of a sample space, then it is known as a compound event.

→ In above example, event A and D are compound events.

Complementary Event

→ For every event E, there corresponds another event $\mathbf{E'}$ (or $\overline{\mathbf{E}}$) known as the complementary event to E.

→ Which is defined as $\mathbf{E' = S - E}$.

→ It is also known as the event "**not E**".

→ For example:

- For experiment of tossing a coin thrice, a sample space is

  $S = \{ HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT \}$

  Consider the following events:

  - Event A : **Exactly one head** appeared

    $\therefore\ A = \{ HTT,\ THT,\ TTH \}$

  The complementary event corresponds to event A is as follow:

    $\therefore\ A' = \{ HHH,\ HHT,\ HTH,\ THH,\ TTT \}$

Mutually Exclusive Event

→ Events A and B are known as mutually exclusive even if $\mathbf{A \cap B = \phi}$.

→ For example:

- For experiment of throwing a die, a sample space is

  $S = \{ 1,\ 2,\ 3,\ 4,\ 5,\ 6 \}$

  Consider the following events:

  - Event A : an **even** number appeared

    $\therefore\ A = \{ 2,\ 4,\ 6 \}$

  - Event B : an **odd** number appeared

    $\therefore\ B = \{ 1,\ 3,\ 5 \}$

  - Event C : a **prime** number appeared

    $\therefore\ C = \{ 2,\ 3,\ 5 \}$

  For events A and B

  $A \cap B = \phi$

  Hence, Events A and B are mutually exclusive events.

For events B and C

$B \cap C = \{3,\ 5\} \neq \phi$

Hence, Events B and C are not mutually exclusive events.

Exhaustive Events

→ Events A and B of a sample space S are known as exhaustive events, if $\mathbf{A \cup B = S}$.

→ For example:

- In experiment of rolling a die, $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$.

  Let us define the following events:

  - Event A : A number less than 4 appears

    $\therefore\ A = \{1,\ 2,\ 3\}$

  - Event B : A number greater than 2 but less than 5 appears

    $\therefore\ B = \{3,\ 4\}$

  - Event C : A number greater than 4 appears

    $\therefore\ C = \{5,\ 6\}$

  Now,

  $A \cup B \cup C = \{1,\ 2,\ 3\} \cup \{3,\ 4\} \cup \{5,\ 6\}$

  $= \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

  $= S$

  Hence, events A, B and C are exhaustive events.

→ Furthermore, if

$\mathbf{A \cap B = \phi}$ and $\mathbf{A \cup B = S}$,

then events A and B are known as mutually exclusive and exhaustive events.

Example of Method-2.1: Definitions of Probability

C	1	A coin is tossed twice, and their up faces are recorded. What is the sample space for this experiment? **Answer**: $\mathbf{S = \{HH,\ HT,\ TH,\ TT\}}$

H	2	Describe the sample space for the indicated random experiments. **(1)** A coin is tossed 3 times. **(2)** A coin and die are tossed together. **Answer:** (1) $S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}$ (2) $S = \{ H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6 \}$
H	3	Find the sample space associated with the experiment of rolling a pair of dice once. Also, find the number of elements of this sample space. **Answer:** $S = \left\{ \begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix} \right\}$ **Number of elements in sample space** $= 6 \times 6 = 6^2 = 36$
C	4	A coin is tossed. If it shows head, we draw a ball from a bag which contains 3 blue and 4 white balls; if it shows tail, we throw a die. Describe the sample space of this experiment. **Answer:** $S = \left\{ \begin{matrix} HB_1, & HB_2, & HB_3, & HW_1, & HW_2, & HW_3, & HW_4, \\ & T1, & T2, & T3, & T4, & T5, & T6 \end{matrix} \right\}$

H	5	One die of red color, one of white color and one of blue color are placed in a bag. One die is selected at random and rolled, its color and the number on its uppermost face is noted. Describe the sample space. **Answer**: $S=\left\{\begin{matrix} \mathbf{R1}, & \mathbf{R2}, & \mathbf{R3}, & \mathbf{R4}, & \mathbf{R5}, & \mathbf{R6} \\ \mathbf{W1}, & \mathbf{W2}, & \mathbf{W3}, & \mathbf{W4}, & \mathbf{W5}, & \mathbf{W6} \\ \mathbf{B1}, & \mathbf{B2}, & \mathbf{B3}, & \mathbf{B4}, & \mathbf{B5}, & \mathbf{B6} \end{matrix}\right\}$
C	6	The letters A, B, C and D are written separately on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other. Describe the sample space for the experiment in following cases: **(1)** With replacement **(2)** Without replacement **Answer**: $(1)\ S=\left\{\begin{matrix} \mathbf{AA}, & \mathbf{AB}, & \mathbf{AC}, & \mathbf{AD}, & \mathbf{BA}, & \mathbf{BB}, & \mathbf{BC}, & \mathbf{BD}, \\ \mathbf{CA}, & \mathbf{CB}, & \mathbf{CC}, & \mathbf{CD}, & \mathbf{DA}, & \mathbf{DB}, & \mathbf{DC}, & \mathbf{DD}, \end{matrix}\right\}$ $(2)\ S=\left\{\begin{matrix} \mathbf{AB}, & \mathbf{AC}, & \mathbf{AD}, & \mathbf{BA}, & \mathbf{BC}, & \mathbf{BD}, \\ \mathbf{CA}, & \mathbf{CB}, & \mathbf{CD}, & \mathbf{DA}, & \mathbf{DB}, & \mathbf{DC} \end{matrix}\right\}$
H	7	A balanced coin is tossed thrice. If three tails are obtained, a balance die is rolled. Otherwise, the experiment is terminated. Write down elements of the sample space. **Answer**: $S=\left\{\begin{matrix} \mathbf{HHH}, & \mathbf{HHT}, & \mathbf{HTH}, & \mathbf{HTT}, & \mathbf{THH}, & \mathbf{THT}, & \mathbf{TTH}, \\ \mathbf{TTT1}, & \mathbf{TTT2}, & \mathbf{TTT3}, & \mathbf{TTT4}, & \mathbf{TTT5}, & \mathbf{TTT6} \end{matrix}\right\}$

C	8	A coin is tossed 3 times. Give the elements of the following events: Event A: Getting at least two heads Event B: Getting exactly two tails Event C: Getting at most one tail Event D: Getting at least one tail **Answer**: $\mathbf{S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}}$ $\mathbf{Event\ A = \{ HHH, HHT, HTH, THH \}}$ $\mathbf{Event\ B = \{ HTT, THT, TTH \}}$ $\mathbf{Event\ C = \{ HHH, HHT, HTH, THH \}}$ $\mathbf{Event\ D = \{ HHT, HTH, HTT, THH, THT, TTH, TTT \}}$
C	9	Two unbiased dice are thrown. Write down the following events: Event A: Both the dice show the same number. Event B: The total of the numbers on the dice is 8. Event C: The total of the numbers on the dice is 13. Event D: The total of the number on the dice is any number from 2 to 12. **Answer**: $\mathbf{A = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \}}$ $\mathbf{B = \{ (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) \}}$ $\mathbf{C = \emptyset}$ $\mathbf{D = \{ (1, 1), \ldots, (1, 6), \ldots, (6, 1), \ldots, (6, 6) \}}$

2.2 Probability of an Event

Probability of an Event

→ The probability of an event E is denoted as **P(E)** and read as “probability of event E” and defined as follow:

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in S}} = \frac{n(E)}{n(S)}$$

or

$$P(E) = \frac{\text{number of outcomes favorable to E}}{\text{total number of all possible outcomes of the experiment}}$$

→ For example:

- One card is drawn from a well-shuffled deck of 52 cards.

  The probability of getting a queen $= \frac{1}{52}$

- A box contains 3 blue, 2 white, and 4 red marbles. A marble is drawn at random from the box, what is the probability that it will be white?

  Solution:

  $S = \{ B_1, B_2, B_3, W_1, W_2, R_1, R_2, R_3, R_4 \}$

  Number of possible outcomes $= 3 + 2 + 4 = 9$

  i.e., $n(S) = 9$

  Let W be the event that marble is white.

  $W = \{ W_1, W_2 \}$

  i.e., $n(W) = 2$

  $$\text{Therefore, } P(W) = \frac{n(W)}{n(S)}$$

  $$= \frac{2}{9}$$

→ **Some Important Results:**

(1) For every event A, $0 \le P(A) \le 1$.

(2) $\mathbf{P(A) = 0}$ if and only if event A is **impossible** event.

(3) $\mathbf{P(A) = 1}$ if and only if event A is **certain** event.

(4) $P(A') = 1 - P(A)$ **or** $P(A) = 1 - P(A')$

(5) If A and B are mutually exclusive events, , $P(A \cap B) = 0$.

(6) If A and B are mutually exhaustive events, $P(A \cup B) = 1$

(7) The probability that **at least one** out of the events A and B will occur is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(8) The probability that **only event A** out of the events A and B will occur is,

$$P(A \cap B') = P(A) - P(A \cap B)$$

(9) The probability that **only event B** out of the events A and B will occur is,

$$P(A' \cap B) = P(B) - P(A \cap B)$$

(10) The probability that **none** of the event A and B occur is,

$$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B) \quad \text{(De Morgan's Rule)}$$

(11) The probability that events A and B **not occur together** is,

$$P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B) \quad \text{(De Morgan's Rule)}$$

(12) The probability that **at least one** of the events A, B and C will occur is,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

(13) The probability that **at least two** of the three events occur is,

$$P[(A \cap B) \cup (B \cap C) \cup (C \cap A)] = P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C)$$

(14) The probability that **exactly two** of the three events occur is,

$$P\,[(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)]$$

$$= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$$

(15) The probability that **exactly one** of the three events occur is,

$$P[(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)]$$

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$$

(16) The probability that **none** of the event A, B and C occur is,

$$P(A' \cap B' \cap C') = P(A \cup B \cup C)' = 1 - P(A \cup B \cup C) \quad \text{(De Morgan's Rule)}$$

(17) The probability that events A, B and C **not occur together** is,

$$P(A' \cup B' \cup C') = P(A \cap B \cap C)' = 1 - P(A \cap B \cap C) \quad \text{(De Morgan's Rule)}$$

Example of Method-2.2: Probability of an Event

H	1	If probability of event A is $\frac{9}{10}$, what is the probability of the event "not A"? **Answer:** $\mathbf{0.1}$
C	2	If A and B are two mutually exclusive events with $P(A) = 0.30, P(B) = 0.45$. Find the probability of A', $A \cap B$, $A \cup B$, $A' \cap B'$. **Answer:** $\mathbf{P(A') = 0.7}$, $\mathbf{P(A \cap B) = 0}$, $\mathbf{P(A \cup B) = 0.75}$, $\mathbf{P(A' \cap B') = 0.25}$
H	3	A single die is tossed once. Find the probability of a 2 or 5 turning up. **Answer:** $\mathbf{\frac{1}{3}}$
H	4	Two unbiased dice are thrown. Find the probability that: **(1)** Both the dice show the same number. **(2)** The first die shows 6. **(3)** The total of the numbers on the dice is 8. **(4)** The total of the numbers on the dice is divisible by 2 or 3. **Answer:** $\mathbf{(1)\ \frac{1}{6}}$, $\mathbf{(2)\ \frac{1}{6}}$, $\mathbf{(3)\ \frac{5}{36}}$, $\mathbf{(4)\ \frac{2}{3}}$
C	5	Two unbiased dice are thrown. Find the probability that: **(1)** The total of the numbers on the dice is greater than 8. **(2)** The total of the numbers on the dice is 13. **(3)** Total of numbers on the dice is any number from 2 to 12, both inclusive. **Answer:** $\mathbf{(1)\ \frac{5}{18}}$, $\mathbf{(2)\ 0}$, $\mathbf{(3)\ 1}$
H	6	Three coins are tossed. Find the probability of **(1)** Getting at least 2 heads. **(2)** Getting exactly 2 heads. **Answer:** $\mathbf{(1)\ 0.5}$, $\mathbf{(2)\ 0.375}$,

T	7	A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card. **Answer**: $\frac{7}{13}$
T	8	One card is drawn at random from a well shuffled pack of 52 cards. Find probability that the card will be an ace, a card of black color, a diamond, and not an ace. **Answer**: $0.0769, \ 0.5, \ 0.25, \ 0.9231$
H	9	Four cards are drawn from the pack of cards. Find the probability that (1) All are diamonds (2) There is one card of each suit (3) There are two spades and two hearts (4) All are red or all are picture(face) cards. **Answer**: $(1)\ \frac{11}{4165}, \quad (2)\ \frac{2197}{20825}, \quad (3)\ \frac{468}{20825}, \quad (4)\ \frac{3086}{54145}$
T	10	4 cards are drawn at random from a pack of 52 cards. Find probability that (1) They are a king, a queen, a jack and an ace. (2) Two are kings and two are queens. (3) Two are black and two are red. (4) Two cards of hearts and two cards of diamonds. **Answer**: $(1)\ \frac{256}{270725}, \quad (2)\ \frac{36}{270725},$ $(3)\ \frac{325}{833}, \quad (4)\ \frac{486}{20825}$
T	11	Consider a poker hand of five cards. Find the probability of getting four of a kind (i.e., four cards of the same face value) assuming the five cards are chosen at random. **Answer**: $\frac{1}{4165}$

C	12	If 5 cards are drawn from a pack of 52 well-shuffled cards, find the probability of **(1)** 4 ace **(2)** 4 aces and 1 is a king **(3)** 3 are tens and 2 are jacks **(4)** a nine, ten, jack, queen, king is obtained in any order **(5)** 3 are of any one suit and 2 are of another **(6)** at least one ace is obtained **Answer**: $(1)\ \frac{1}{54145}$, $(2)\ \frac{1}{649740}$, $(3)\ \frac{1}{108290}$ $(4)\ \frac{64}{162435}$, $(5)\ \frac{429}{4165}$, $(6)\ \frac{18472}{54145}$
H	13	A box contains 5 red, 6 white and 2 black balls. The balls are identical in all aspects other than color. **(1)** One ball is drawn at random from the box. Find the probability that the selected ball is black. **(2)** Two balls are drawn at random from the box. Find the probability that one ball is white and one is red. **Answer**: $(1)\ \frac{2}{13}$, $(2)\ \frac{5}{13}$
T	14	If 3 balls are “randomly drawn” from a bowl containing 6 white and 5 black balls. What is the probability that one of the balls is white and the other two black? **Answer**: 0.3636
T	15	There are 5 yellow, 2 red, and 3 white balls in the box. Three balls are randomly selected from the box. Find the probability of the following events. **(1)** All balls are of different color **(2)** 2 yellow and 1 red color ball **(3)** all balls are of same color. **Answer**: $(1)\ 0.25$, $(2)\ 0.1667$, $(3)\ 0.0917$

C	16	An urn contains 6 green, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that at least one is green. **Answer:** $\frac{683}{969}$
T	17	A box contains 6 red balls, 4 white balls, 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each color. **Answer:** 0.5275
T	18	A machine produces a total of 12000 bolts a day, which are on the average 3% defective. Find the probability that out 600 bolts chosen at random, 12 will be defective. **Answer:** $\frac{\binom{360}{12}\binom{11640}{588}}{\binom{12000}{600}}$
H	19	If 5 of 20 tyres in storage are defective and 5 of them are randomly chosen for inspection (that is, each tire has the same chance of being selected), what is the probability that the two of the defective tires will be included? **Answer:** 0.2935
C	20	An integer is chosen at random from the first 200 positive integers. What is the probability that the integer is divisible by 6 or 8? **Answer:** 0.25
T	21	In a group of 1000 persons, there are 650 who can speak Hindi, 400 can speak English, and 150 can speak both Hindi and English. If a person selected at random, what is the probability that a person speaks **(1)** Hindi only, **(2)** English only, **(3)** Only one of two languages, **(4)** At least one of the two languages. **Answer:** (1) 0.5, (2) 0.25, (3) 0.75, (4) 0.9

H	22	A person applies for a job in two firms A and B, the probability of his being selected in firm A is 0.7 and being rejected in firm B is 0.5. The probability of at least one of the applications being rejected is 0.6. What is the probability that person will be selected in one of the two firms? **Answer**: $\mathbf{0.8}$
T	23	A basket contains 20 apples and 10 oranges of which 5 apples and 3 oranges are bad. If a person takes 2 at random, what is the probability that either both are apples or both are good? **Answer**: $\mathbf{\frac{316}{435}}$
T	24	A problem in statistics is given to three students A, B and C, whose chances of solving it independently are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that **(1)** the problem is solved **(2)** at least two of them are able to solve the problem **(3)** exactly two of them are able to solve the problem **(4)** exactly one of them is able to solve the problem **Answer**: $\mathbf{(1)\ \frac{3}{4},\quad (2)\ \frac{7}{24},\quad (3)\ \frac{1}{4},\quad (4)\ \frac{11}{24}}$
H	25	Three newspapers $A,\ B,\ C$ are published in a certain city. It is estimated from a survey that of the adult population: 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read B and C, 2% read all three. Find what percentage read at least one of the papers? **Answer**: $\mathbf{35\%}$

H	26	Do as directed: **(1)** Find the probability that there will be 5 Sundays in the month of July. **(2)** Find the probability that there will be 5 Sundays in the month of June. **(3)** What is the probability that a non-leap year contains 53 Sundays? **(4)** What is the probability that a leap year contains 53 Sundays? **Answer:** (1) $\frac{3}{7}$, (2) $\frac{2}{7}$, (3) $\frac{1}{7}$, (4) $\frac{2}{7}$
T	27	A room has three lamp sockets. From a collection of 10 light bulbs of which only 6 are good. A person selects 3 at random and puts them in the socket. What is the probability that the room will have light? **Answer:** $\frac{29}{30}$
T	28	A class has 10 boys and 5 girls. Three students are selected at random one after the other. Find the probability that **(1)** First two are boys and third is girl. **(2)** First and third of same gender and second is of opposite gender. **Answer:** (1) $\frac{15}{91}$, (2) $\frac{5}{21}$
T	29	Four letters of the word THURSDAY are arranged in all possible ways. Find the probability that the word formed is HURT. **Answer:** $\frac{1}{1680}$

Method – 3 ⇝ Conditional Probability and Independent Events

Conditional Probability

→ The probability of an event occurring given that another event has already occurred is known as conditional probability.

→ Let A and B be any two events in same sample space S.

→ The probability of the occurrence of event A when it is given that B has already occurred is known as conditional probability.

→ It is denoted as **P(A | B)** and read as "conditional probability of A given B" and defined as follow:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

→ Similarly, "conditional probability of B given A" is defined as follow:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

→ For example:

Consider the experiment of tossing three fair coins. To find probability of "getting at least two heads given that first coin shows tail" is conditional probability.

The sample space of the experiment is

$S = \{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}.$

Let E and F denote the following events :

E : at least two head appear

$E = \{HHH,\ HHT,\ HTH,\ THH\}$

F : first coin shows tail

$F = \{THH,\ THT,\ TTH,\ TTT\}$

Here, event F has occurred before event E.

Now, to find $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$

$E \cap F = \{THH\}$

Thus, $P(F) = \dfrac{4}{8} = \dfrac{1}{2}$ and $P(E \cap F) = \dfrac{1}{8}$

Therefore, $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\left(\frac{1}{8}\right)}{\left(\frac{1}{2}\right)} = \dfrac{1}{4}$

→ **Properties of Conditional Probability**

- Let A_1, A_2 and B be any three events of a sample space S, then

  $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B);\ P(B) > 0.$

- Let A and B be any two events of a sample space S, then

  $P(A' \mid B) = 1 - P(A \mid B);\ P(B) > 0.$

Multiplicative Law of Probability

→ **Statement:**

Let A and B be any two events in the sample space S, then

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B \mid A);\ P(A) \neq 0 \\ &= P(B) \cdot P(A \mid B);\ P(B) \neq 0 \end{aligned}$$

Let A, B and C be any three events in the sample space S, then

$$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot (C \mid A \cap B)$$

Independent Events

→ Two events are known as independent events if the probability of occurrence of one of them is not affected by occurrence of the other.

→ Let A and B be two events associated with the same random experiment, then A and B are known as independent if and only if

$\mathbf{P(A \cap B) = P(A) \cdot P(B)}$

→ Furthermore, if events A and B are independent events, then

(1) $P(A \mid B) = P(A)$

(2) $P(B \mid A) = P(B)$

Mutually Independent Events

→ Three events A, B and C are known as mutually independent if

$P(A \cap B) = P(A) \cdot P(B)$

$P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap C) = P(A) \cdot P(C)$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

→ If the last condition is not satisfied, the events are said to be pairwise independent.

Example of Method-3: Conditional Probability and Independent Events

H	1	If $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{11}{12}$. Find $P(A \mid B)$. **Answer**: $\mathbf{\frac{2}{9}}$
C	2	$P(A) = \frac{1}{3}$, $P(B') = \frac{1}{4}$, $P(A \cap B) = \frac{1}{6}$, then find $P(A \cup B)$, $P(A' \cap B')$ and $P(A' \mid B')$. **Answer**: $\mathbf{\frac{11}{12}}$, $\mathbf{\frac{1}{12}}$, $\mathbf{\frac{1}{3}}$
C	3	If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$, then find $P(B \mid A)$, $P(A \mid B')$. **Answer**: $\mathbf{\frac{1}{4}}$, $\mathbf{\frac{1}{3}}$
H	4	For two independent events A & B if $P(A) = 0.3$ and $P(A \cup B) = 0.6$. Find $P(B)$. **Answer**: $\mathbf{0.4286}$
T	5	If A, B are independent events and $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{3}$. Find $P(A \cup B)$. **Answer**: $\mathbf{0.75}$

C	6	If A and B are independent events, with $P(A) = \frac{3}{8}$, $P(B) = \frac{7}{8}$. Find $P(A \cup B)$, $P(A \mid B)$ and $P(B \mid A)$. **Answer:** $\frac{59}{64}$, $\frac{3}{8}$, $\frac{7}{8}$
H	7	Check weather events A and B are independent or not if $P(A) = 0.20$, $P(B) = 0.40$ and $P(A \cup B) = 0.50$. **Answer: Not Independent**
T	8	If A and B are independent events with $P(A) = 0.26$, $P(B) = 0.45$, find $P(A \cap B)$, $P(A \cap B')$, $P(A' \cap B')$. **Answer:** 0.117, 0.143, 0.407
H	9	In producing screws, let A mean "screw is too slim" and B "screw is too short". Let $P(A) = 0.1$ and $P(B \cap A) = 0.02$. A screw, selected randomly, is of type A, what is probability that a screw is of type B? **Answer:** 0.2
C	10	A person is known to hit the target in 3 out of 4 shots, whereas another person is known to hit the target in 2 out of 3 shots. What is probability that target will be hit? **Answer:** $\frac{11}{12}$
H	11	A problem in statistics is given to three students A, B, C whose chances of solving it are 0.5, 0.75 and 0.25 respectively. What is the probability that the problem will be solved if all of them try independently? **Answer:** 0.90625

C	12	A market survey was conducted in four cities to find out the preference for brand X soap. The responses are shown below: | | Delhi | Kolkata | Chennai | Mumbai | | Yes | 45 | 55 | 60 | 50 | | No | 35 | 45 | 35 | 45 | | No opinion | 5 | 5 | 5 | 5 | **(1)** What is the probability that a consumer preferred brand X, given that he was from Chennai? **(2)** Given that a consumer preferred brand X, what is the probability that he was from Mumbai? **Answer**: $(1)\ 0.6,\quad (2)\ 0.23$
C	13	In a group of 200 students 40 are taking English, 50 are taking Mathematics, 12 are taking both. **(1)** If a student is selected at random, what is the probability that the student is taking English? **(2)** A student is selected at random from those taking Mathematics. What is the probability that the student is taking English? **(3)** A student is selected at random from those taking English, what is the probability that the student is taking Mathematics? **Answer**: $(1)\ 0.20,\quad (2)\ 0.24,\quad (3)\ 0.3$
T	14	In a box, 100 bulbs are supplied out of which 10 bulbs have defects of type A, 5 bulbs have defects of type B and 2 bulbs have defects of both the type. Find the probability that, **(1)** A bulb to be drawn at random has a B – type defect under the condition that it has an A – type defect. **(2)** A bulb to be drawn at random has no B – type defect under the condition that it has no A – type defect. **Answer**: $(1)\ 0.2,\quad (2)\ 0.9667$

H	15	In a certain college 25% of the students failed in probability and 15% of the student failed in statistics and 10% of the students failed in both. A student is selected at random, if he failed in probability, what is probability that he failed in statistics? **Answer**: $\mathbf{0.4}$
T	16	Two integers are selected at random from 1 to 11. If the sum is even, find the probability that both the integers are odd. **Answer**: $\mathbf{0.6}$
H	17	A card is drawn from a well-shuffled deck of 52 cards and then second card is drawn, find the probability that one card is a spade and then second card is club if the first card is not replaced. **Answer**: $\mathbf{\frac{13}{204}}$
T	18	From a bag containing 4 white and 6 black balls, two balls are drawn at random. If the balls are drawn one after the other without replacements, find the probability that one is white and one is black. **Answer**: $\mathbf{\frac{8}{15}}$
C	19	A bag contains 6 white, 9 black balls. 4 balls are drawn at a time. Find the probability for first draw to give 4 white & second draw to give 4 black balls in each of following cases: **(1)** The balls are replaced before the second draw. **(2)** The balls are not replaced before the second draw. **Answer**: $\mathbf{(1)\ \frac{6}{5915}},\qquad \mathbf{(2)\ \frac{3}{715}}$

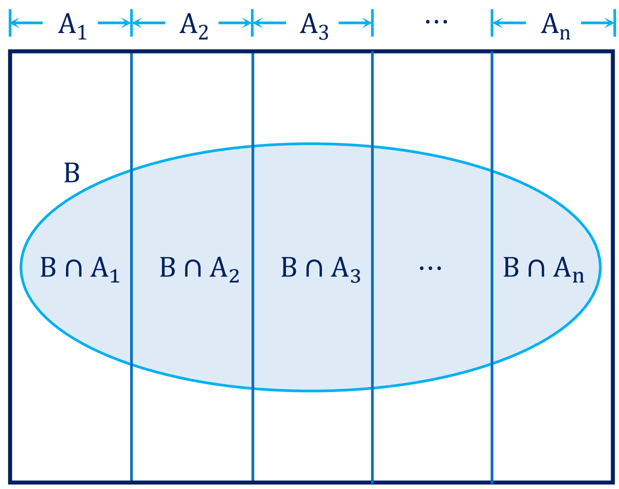
Method – 4 ⇝ Total Probability and Bayes' Theorem

Total Probability

→ Let $A_1, A_2, \dots, A_n$ be mutually exclusive and exhaustive events of the sample space S with $P(A_i) \neq 0$, for $i = 1, 2, \dots, n$. Let B be any event associated with S. Then, the probability of B is

$$\mathbf{P(B) = P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_n) \cdot P(B \mid A_n)}$$

Explanation:



S

Using multiplicative law of probability, we get

$P(B \cap A_i) = P(A_i) \cdot P(B \mid A_i), \ \forall\, i = 1, 2, \dots, n$ ('∀' means **for every**)

Thus,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_i)$$

$$P(B) = P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_n) \cdot P(B \mid A_n)$$

→ For example:

Factory A and Factory B, that produce electronic components. Factory A produces 60% of the components and among those, 10% are defective. Factory B produces the remaining 40% of the components and among those, 5% are defective.

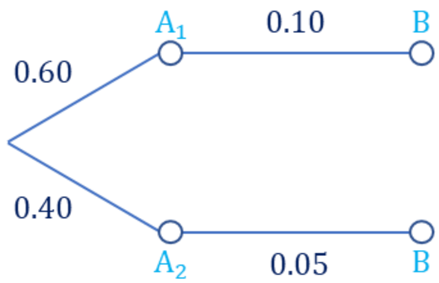
Find the probability that a randomly selected electronic component is defective.

Solution:

Let A_1 : The component is from Factory A.

A_2 : The component is from Factory B.

B : The component is defective.



Therefore,

$P(A_1) = 60\% = 0.60$ $\qquad$ $P(B \mid A_1) = 10\% = 0.10$

$P(A_2) = 40\% = 0.40$ $\qquad$ $P(B \mid A_2) = 5\% = 0.05$

Thus,

$$
\begin{aligned}
P(B) &= P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) \\
&= (0.60) \cdot (0.10) + (0.40) \cdot (0.05) \\
&= 0.06 + 0.02 \\
&= 0.08
\end{aligned}
$$

Bayes' Theorem

→ **Statement:**

Let $A_1, A_2, A_3, \dots, A_n$ be mutually exclusive and exhaustive events of the sample space S with $P(A_i) \neq 0$, for $i = 1, 2, 3, \dots, n$. Let B be any event associated with S with $P(B) \neq 0$. The probability of an event A_i when the event B has occurred is

$$
\mathbf{P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_n) \cdot P(B \mid A_n)}}
$$

Example of Method-4: Total Probability and Bayes' Theorem

C	1	In a certain assembly plant, three machines, B_1, B_2 and B_3, make 30%, 45% and 25%, respectively, of the products. It is known form the past experience that 2%, 3% and 2% of the products made by each machine respectively are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective? **Answer: 0.0245**

H	2	There are three boxes, Box - I contains 10 light bulbs of which 4 are defective, Box - II contains 6 light bulbs of which 1 is defective and Box - III contains 8 light bulbs of which 3 are defective. A box is chosen and a bulb is drawn. Find the probability that the bulb is defective. **Answer**: $\mathbf{0.3139}$
C	3	An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the later. What is the probability that it is a white ball? **Answer**: $\mathbf{\frac{59}{130}}$
H	4	Suppose that the population of a certain city is 40% male & 60% female. Suppose also that 50% of male & 30% of female smokes. Find the probability that a smoker is male. **Answer**: $\mathbf{\frac{10}{19}}$
T	5	A card from a pack of 52 cards is lost. From the remaining cards of pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart. **Answer**: $\mathbf{\frac{11}{50}}$
C	6	Consider two boxes, first with 5 green and 2 pink and second with 4 green and 3 pink balls. Two balls are selected from random box. If both balls are pink, find the probability that they are from second box. **Answer**: $\mathbf{\frac{3}{4}}$

C	7	A company has two plants to manufacture hydraulic machine. Plant I manufacture 70% of the hydraulic machines and plant II manufactures 30%. At plant I, 80% of hydraulic machines are rated standard quality and at plant II, 90% of hydraulic machine are rated standard quality. A machine is picked up at random and is found to be of standard quality. What is the chance that it has come from plant I? **Answer: $\mathbf{0.6747}$**
H	8	A microchip company has two machines that produce the chips. Machine-I produces 65% of the chips, but 5% of its chips are defective. Machine-II produces 35% of the chips, but 15% of its chips are defective. A chip is selected at random and found to be defective. What is the probability that it came from Machine-I? **Answer: $\mathbf{0.3824}$**
T	9	There are two boxes A and B containing 4 white, 3 red and 3 white, 7 red balls respectively. A box is chosen at random and a ball is drawn from it, if the ball is white, find the probability that it is from box A. **Answer:** $\mathbf{\frac{40}{61}}$
T	10	In a computer engineering class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of student are boys. If a student is selected random and found to have IQ more than 150, find the probability that the student is a boy. **Answer:** $\mathbf{\frac{3}{7}}$
C	11	If proposed medical screening on a population, the probability that the test correctly identifies someone with illness as positive is 0.99 and the probability that test correctly identifies someone without illness as negative is 0.95. The incident of illness in general population is 0.0001. You take the test the result is positive then what is the probability that you have illness? **Answer: $\mathbf{0.002}$**

H	12	Three hospitals contain 10%, 20% and 30% of diabetes patients. A Patient is selected at random who is diabetes patient. Determine the probability that this patient comes from first hospital. **Answer**: $\mathbf{0.1667}$
T	13	Suppose there are three chests each having two drawers. The first chest has a gold coin in each drawer, the second chest has a gold coin in one drawer and a silver coin in the other drawer and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer opened. If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin? **Answer**: $\mathbf{\frac{2}{3}}$
T	14	State Bayes' theorem. In a bolt factory, three machines A, B and C manufacture 25%, 35% and 40% of the total product respectively. Out of these outputs 5%, 4% and 2% respectively, are defective bolts. A bolt is picked up at random and found to be defective. What are the probabilities that it was manufactured by machine A, B and C? **Answer**: $\mathbf{0.3623}$, $\mathbf{0.4058}$, $\mathbf{0.2319}$
H	15	A factory has three machines X, Y, Z producing 1000, 2000, 3000 bolts per day respectively. Machine X produces 1% defective bolts, Y produces 1.5%, Z produces 2% defective bolts. At end of the day, a bolt is drawn at random and it is found to be defective. What is the probability that this defective bolt has been produced by the machine X? **Answer**: $\mathbf{0.1}$
H	16	Urn A contain 1 white, 2 black, 3 red balls; Urn B contain 2 white, 1 black, 1 red balls; Urn C contain 4 white, 5 black, 3 red balls. One urn is chosen at random & two balls are drawn. These happen to be one white & one red. What is probability that they come from urn A? **Answer**: $\mathbf{0.2797}$

T	17	An insurance company insured 2000 bike drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a bike driver, a car driver and a truck driver is 0.10, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a bike driver? **Answer**: $\mathbf{0.1639}$

Method – 5 ⇝ Random Variable and Probability Function

Random Variable

→ A variable whose values can be obtained from the results of a random experiment is known as random variable.

→ A random variable is a function associated with a sample space of a random experiment.

→ Random variable is often denoted by X, Y.

→ Random variable can be classified as bellow:

(1) Discrete random variable

(2) Continuous random variable

Discrete Random variable:

→ A random variable can take only finite values or countable infinite values is known as discrete random variable.

→ Discrete random variables can be measured exactly.

→ For example:

If two coins are tossed simultaneously the following sample space is generated:

$S = \{ HH,\ HT,\ TH,\ TT \}$

Now if X denotes the number of heads, then

$X(HH) = 2,\ X(HT) = 1,\ X(TH) = 1,\ X(TT) = 0$

i.e., random variable X can take values 0, 1, 2.

→ Some other examples of discrete random variable.

- Number of children in a family.
- Numbers of stars in the sky.
- Profit made by investor in a day.

Continuous random variable:

→ If a random variable can take all values within an interval, is known as continuous random variable.

→ Continuous random variable cannot be measured exactly.

→ For example:

- The age of a person
- Weight and height of a person
- Life of an electric bulb

Probability Distribution of random variable

→ Probability distribution of random variable is the set of its possible values together with their respective probabilities.

i.e.,

$\mathbf{X}$	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	...	...	$\mathbf{x_n}$
$\mathbf{P(X = x)}$	$\mathbf{p(x_1)}$	$\mathbf{p(x_2)}$	$\mathbf{p(x_3)}$	...	...	$\mathbf{p(x_n)}$

Where, $p(x_i) \geq 0$ and $\sum_{i=1}^{n} p(x_i) = 1$.

→ For example:

Two balanced coins are tossed, then $S = \{HH, HT, TH, TT\}$

We find the probability distribution of head

$$P(X = 0) = P(\text{no head}) = \frac{1}{4} = 0.25$$

$$P(X = 1) = P(\text{one head}) = \frac{2}{4} = \frac{1}{2} = 0.5$$

$$P(X = 2) = P(\text{two heads}) = \frac{1}{4} = 0.25$$

Probability distribution is as follow:

$\mathbf{X}$	0	1	2
$\mathbf{P(X = x)}$	0.25	0.5	0.25

Probability Function

→ If for random variable X, the real valued function $f(x)$ is such that $P(X = x) = f(x)$, then $f(x)$ is called probability function of random variable X.

→ Probability function $f(x)$ gives the measures of probability for different values of X say $x_1, x_2, \ldots, x_n$.

→ Probability functions can be classified as

(1) Probability Mass Function (P. M. F.) (for discrete random variable)

(2) Probability Density Function (P. D. F.) (for continuous random variable)

Probability Mass Function

→ If X is a **discrete** random variable, then its probability function $f(x)$ or $P(X = x_i) = p(x_i)$ is called probability mass function, if it satisfies below conditions:

(1) $p(x_i) \geq 0,$ for all i

(2) $\sum_{i=1}^{n} p(x_i) = 1$

→ Note:

If $a < x_1 < x_2 < \cdots < x_k < \cdots < x_{n-1} < x_n < b$, then

$\mathbf{P(x < b)} = P(X = a) + P(X = x_1) + \cdots + P(X = x_{n-1}) + P(X = x_n)$

$\mathbf{P(x_1 \leq x \leq x_k)} = P(X = x_1) + P(X = x_2) + \cdots + P(X = x_k)$

$\mathbf{P(x > a)} = P(X = x_1) + P(X = x_2) + \cdots + P(X = b)$

Probability Density Function

→ If X is a **continuous** random variable, then its probability function $f(x)$ is called continuous probability function, if it satisfies below conditions:

(1) $f(x_i) \geq 0,$ for all i

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

→ Note:

if $a < x_1 < x_2 < \cdots < x_k < \cdots < x_{n-1} < x_n < b$, then

$$\mathbf{P(x \leq b)} = \int_{-\infty}^{b} f(x)\, dx$$

$$\mathbf{P(a < x < b)} = \int_{a}^{b} f(x)\, dx$$

$$\mathbf{P(a < x)} = \int_{a}^{\infty} f(x)\, dx$$

Example of Method-5: Random Variable and probability Function

C	1	Is $P(X=x)=\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{1-x}$; $x=0,\ 1$ a probability function? **Answer: Yes**
H	2	Is $P(X=x)=\left(-\frac{1}{2}\right)^{x}$; $x=0,\ 1,\ 2$ a probability function? **Answer: No**
T	3	If $P(x)=\frac{2x+1}{48}$, $x=1,\ 2,\ 3,\ 4,\ 5,\ 6.$ Verify whether $P(x)$ is probability function or not. **Answer: Yes**
C	4	A random variable X has the following probability function. Find the value of k and then evaluate $P(x<6)$, $P(x\geq 6)$ and $P(0<x<5)$. <table><tr><td>X</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr><tr><td>P(X = x)</td><td>0</td><td>k</td><td>2k</td><td>2k</td><td>3k</td><td>k^2</td><td>$2k^2$</td><td>$7k^2+k$</td></tr></table> **Answer:** $\mathbf{k=0.1}$, $\mathbf{P(x<6)=0.81}$, $\mathbf{P(x\geq 6)=0.19}$, $\mathbf{P(0<x<5)=0.8}$
H	5	Probability distribution of a random variable X is given below. Find $P(2\leq x\leq 4)$, $P(x>2)$, $P(x \text{ is odd})$ and $P(x \text{ is even})$ <table><tr><td>X</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>P(X = x)</td><td>0.1</td><td>0.2</td><td>0.5</td><td>0.2</td></tr></table> **Answer:** $\mathbf{P(2\leq x\leq 4)=0.9}$, $\mathbf{P(x>2)=0.7}$, $\mathbf{P(x \text{ is odd})=0.6}$, $\mathbf{P(x \text{ is even})=0.4}$
T	6	Find k for the probability distribution $p(x)=k\binom{4}{x}$, $x=0,\ 1,\ 2,\ 3,\ 4.$ **Answer:** $\mathbf{k=\frac{1}{16}}$

T	7	If $P(X=x)=\frac{x}{15},\ x=1 \text{ to } 5.$ Find $P(1 \text{ or } 2)$ & $P(0.5<X<2.5\,\vert\,X>1)$. **Answer**: $\mathbf{P(1 \text{ or } 2)=\frac{1}{5}},\quad \mathbf{P(0.5<X<2.5\,\vert\,X>1)=\frac{1}{7}}$
C	8	Verify that the following function is P.D.F or not? $f(x)=\begin{cases}\frac{2x}{9}\left(2-\frac{x}{2}\right) & ;\ 0\le x\le 3\\ 0 & ;\text{elsewhere}\end{cases}.$ **Answer**: **Yes**
T	9	Verify that the following function is P.D.F or not? $f(x)=\begin{cases}\frac{x}{8} & ;\ 0\le x<2\\ \frac{1}{4} & ;\ 2\le x<4\\ \frac{6-x}{8} & ;\ 4\le x<6\end{cases}$ **Answer**: **Yes**
T	10	Is the function f(x) defined as below is a probability function? If so, find the probability that the variate having this density falls in the interval $(1,2)$. $f(x)=\begin{cases}e^{-x} & ;\ x\ge 0\\ 0 & ;x<0\end{cases}$ **Answer**: **Yes**, $\mathbf{P(1\le X<2)=0.2325}$
H	11	Check whether the following function $f(x)=\begin{cases}\frac{3+2x}{18} & ;\ 2\le x\le 4\\ 0 & ;\ \text{otherwise}\end{cases}$ is a probability density function? If yes, then find $P(3\le X\le 4)$. **Answer**: **Yes**, $\mathbf{\frac{5}{9}}$

T	12	Find the constant c such that the function $f(x) = \begin{cases} cx^2 & ; 0 < x < 3 \\ 0 & ; \text{elsewhere} \end{cases}$ is a probability density function and Compute $P(1 < X < 2),\ P(X \le 2),\ P(X \ge 2)$. **Answer:** $\mathbf{c = \frac{1}{9}}$, $\mathbf{P(1 < X < 2) = \frac{7}{27}}$, $\mathbf{P(X \le 2) = \frac{8}{27}}$, $\mathbf{P(X \ge 2) \frac{19}{27}}$

Method – 6 ⇝ Various Measures of Statistics

Mathematical Expectation

→ If X is a **discrete** random variable having various possible values $x_1, x_2, \dots, x_n$ and $P(X = x)$ is the probability mass function, the mathematical expectation of X is defined & denoted by

$$\mathbf{E(X) = \sum_{i=1}^{n} x_i \cdot p(x_i).}$$

→ If X is a **continuous** random variable which has probability density function $f(x)$, the mathematical expectation of X is defined as

$$\mathbf{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.}$$

→ $E(X)$ is also known as the mean value of the probability distribution of x.

Properties of Mathematical Expectation

(1) Expected value of constant term is constant. i.e., $E(c) = c$.

(2) If a, b and c are constants, then

$$E\left(\frac{aX \pm b}{c}\right) = \frac{1}{c}[a \cdot E(X) \pm b]$$

(3) For Probability mass function,

$$E(X^2) = \sum_{i=1}^{n} x_i^2 \cdot p(x_i)$$

(4) For Probability density function,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

(5) If X and Y are two random variables, then $E(X + Y) = E(X) + E(Y)$.

(6) If X and Y are two **independent** random variables, then $E(X \cdot Y) = E(X) \cdot E(Y)$.

Variance of Random Variable:

→ Variance is a characteristic of random variable X and it is used to measure dispersion (or variation) of X.

→ If X is a random variable with probability mass function P(X), then expected value of $[X - E(X)]^2$ is known as the variance of X and it is denoted by $\mathbf{V(X)}$.

$$\mathbf{V(X) = E(X^2) - [E(X)]^2}$$

Properties of Variance

(1) $V(c) = 0$, where, c is a constant.

(2) If a and b are constants, then $V(aX + b) = a^2 \cdot V(X)$.

(3) If X and Y are the **independent** random variables, then $V(X + Y) = V(X) + V(Y)$.

Standard Deviation of Random Variable

→ The positive square root of V(X) (Variance of X) is called standard deviation of random variable X and is denoted by $\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = \sqrt{\mathbf{V(X)}}$$

Example of Method-6: Various Measures of Statistics

<table>
<tr><td>C</td><td>1</td><td>Probability distribution of a random variable X is given below. Find $E(X)$, $V(X)$, $\sigma(X)$, $E(3X + 2)$, $V\,(3X + 2)$.

X	1	2	3	4
$P(X = x)$	0.1	0.2	0.5	0.2

Answer: $\mathbf{E(X) = 2.8}$, $\mathbf{V(X) = 0.76}$, $\boldsymbol{\sigma}\mathbf{(X) = 0.8718}$,
$\mathbf{E(3X + 2) = 10.4}$, $\mathbf{V\,(3X + 2) = 6.84}$</td></tr>
</table>

H 2

The probability distribution of a random variable X is given below. Find a, $E(X)$, $E(2X+3)$, $E(X^2+2)$, $V(X)$, $V(3X+2)$.

X	-2	-1	0	1	2
P(X)	$\frac{1}{12}$	$\frac{1}{3}$	a	$\frac{1}{4}$	$\frac{1}{6}$

Answer: $\mathbf{a=\frac{1}{6}}$, $\mathbf{E(X)=\frac{1}{12}}$, $\mathbf{E(2X+3)=\frac{19}{6}}$,

$\mathbf{E(X^2+2)=\frac{43}{12}}$, $\mathbf{V(X)=\frac{227}{144}}$, $\mathbf{V(3X+2)=\frac{227}{16}}$

T 3

The probability distribution of a random variable X is given below.

X	-1	0	1	2	3
P(X)	$\frac{3}{10}$	$\frac{1}{10}$	k	$\frac{3}{10}$	$\frac{1}{10}$

Find k, $E(X)$, $E\,(4X+3)$, $E(X^2)$, $V(X)$, $V(2X+3)$.

Answer: $\mathbf{k=\frac{1}{5}}$, $\mathbf{E(X)=\frac{4}{5}}$, $\mathbf{E\,(4X+3)=\frac{31}{5}}$,

$\mathbf{E(X^2)=\frac{13}{5}}$, $\mathbf{V(X)=\frac{49}{25}}$, $\mathbf{V(2X+3)=\frac{196}{25}}$

C 4

Let mean and standard deviation of a random variable X be 5 & 5 respectively, find $E(X^2)$ and $E(2X+5)^2$.

Answer: $\mathbf{E(X^2)=50}$, $\mathbf{E(2X+5)^2=325}$

C 5

The following table gives the probabilities that a certain computer will malfunction 0, 1, 2, 3, 4, 5 or 6 times on any one day.

Number of malfunctions x	0	1	2	3	4	5	6
Probability p(x)	0.17	0.29	0.27	0.16	0.07	0.03	0.01

Find the mean and variance of this probability distribution.

Answer: $\mathbf{Mean=1.8}$, $\mathbf{Variance=1.8}$

C	6	4 raw mangoes are mixed accidentally with the 16 ripe mangoes. Find the probability distribution of the raw mangoes in a draw of 2 mangoes. **Answer**: $\mathbf{P(X=0)=\frac{12}{19}}$, $\mathbf{P(X=1)=\frac{32}{95}}$, $\mathbf{P(X=2)=\frac{3}{95}}$
H	7	Three balanced coins are tossed, find the mathematical expectation of tails. **Answer**: $\mathbf{\frac{3}{2}}$
T	8	**(1)** A contestant tosses a coin and receives \$5 if head appears and \$1 if tail appears. What is the expected value of a trial? **(2)** A contestant receives \$4.00 if a coin turns up heads and pays \$3.00 if it turns tails. What is the expected value of a trail? **Answer**: $\mathbf{\$\ 3.00}$, $\mathbf{\$\ 0.50}$
H	9	A machine produces on average of 500 items during first week of the month & average of 400 items during the last week of the month. The probability for these being 0.68 and 0.32. Determine the expected value of the production. **Answer**: $\mathbf{468}$
T	10	In a business, the probability that a trader can get profit of Rs. 5000 is 0.4 and probability for loss of Rs. 2000 is 0.6. Find his expected gain or loss. **Answer**: $\mathbf{800}$
C	11	There are 3 red and 2 white balls in a box and 2 balls are taken at random from it. A person gets Rs. 20 for each red ball and Rs. 10 for each white ball. Find his expected gain. **Answer**: $\mathbf{32}$
H	12	There are 8 apples in a box, of which 2 are rotten. A person selects 3 Apples at random from it. Find the expected value of the rotten apples. **Answer**: $\mathbf{0.75}$

H	13	There are 10 bulbs in a box, out of which 4 are defectives. If 3 bulbs are taken at random, find the expected number of defective bulbs. **Answer: 1.2**
C	14	A random variable X has P.D.F $f(x) = \begin{cases} \dfrac{3+2x}{18} & ;\ 2 \le x \le 4 \\ 0 & ;\text{otherwise} \end{cases}$. Find the standard deviation of the distribution. **Answer: 0.5726**
T	15	A random variable X has pdf $f(x) = kx^2(4-x)\ ;\ 0 < x < 4$. Find the value of k and hence find its mean and standard deviation. **Answer:** $\mathbf{k = \frac{3}{64}}$, **Mean** $\mathbf{= 2.4}$, **SD** $\mathbf{= 0.8}$
H	16	A random variable X has pdf $f(x) = kx^2(1-x^3)\ ;\ 0 < x < 1$. Find the value of k and hence find its mean and variance. **Answer:** $\mathbf{k = 6}$, **Mean** $\mathbf{= \frac{9}{14}}$, **SD** $\mathbf{= \frac{9}{245}}$

Method – 7 ⇝ Cumulative Distribution Function

Cumulative Distribution Function

(1) **Discrete Distribution Function**

- The distribution function **F(x)** of the discrete random variable X is defined by

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$

where, X be a discrete random variable which takes the values $x_1, x_2, x_3 \dots$.

such that $x_1 < x_2 < \cdots$ with probabilities $p(x_1),\ p(x_2),\ p(x_3) \dots$ &

$p(x_i) \ge 0$ for all values of i and

$$\sum_{i=1}^{x} p(x_i) = 1$$

where, x is any integer.

- The set of pairs $\{x_i, F(x)\}, i = 1, 2, \dots$ is known as the cumulative probability distribution.

$\mathbf{x}$	$\mathbf{x_1}$	$\mathbf{x_2}$	…
$\mathbf{F(x)}$	$\mathbf{p(x_1)}$	$\mathbf{p(x_1) + p(x_2)}$	…

(2) **Continuous Distribution Function**

- If X is a continuous random variable having the probability density function f(x) then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx \ ; \ -\infty < x < \infty$$

is known as the continuous distribution function.

Properties of Continuous Distribution Function

(1) For $-\infty < x < \infty,\ 0 \le F(x) \le 1$

(2) $F(-\infty) = 0, \quad F(\infty) = 1$

(3) $P(a < X < b) = F(b) - F(a)$

(4) $P(X > x) = 1 - P(X \le x) = 1 - F(x)$

(5) $F'(x) = f(x);\ \ f(x) \ge 0$

Example of Method-7: Cumulative Distribution Function

T	1	Two dies are rolled. Let X denotes the random variable which counts the total number of points on the upturned faces, construct a table giving the non – zero values of the probability mass function. Also find the distribution of X.

Answer:

X	1	2	3	4	5	6	7	8	9	10	11	12
P(X)	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
F(x)	0	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

C	2	A random variable X takes the values $-3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3$ such that $P(X=0)=P(X>0)=P(X<0)$ $P(X=-3)=P(X=-2)=P(X=-1) = P(X=1)=P(X=2)=P(X=3)$. Obtain the probability distribution and the distribution function of X.

Answer:

X	-3	-2	-1	0	1	2	3
P(X)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
F(x)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1

C	3	A discrete random variable X has the following distribution function: $$F(x) = \begin{cases} 0 \ ; & x < 1 \\ \frac{1}{3}; & 1 \le x < 4 \\ \frac{1}{2}; & 4 \le x < 6 \\ \frac{5}{6}; & 6 \le x < 10 \\ 1 \ ; & x \ge 10 \end{cases}$$ Find $P(2 < X \le 6), P(X = 5), P(X = 4), P(X \le 6), P(X = 6)$. **Answer**: $\mathbf{P(2 < X \le 6) = \frac{1}{2}}$, $\mathbf{P(X = 5) = 0}$, $\mathbf{P(X = 4) = \frac{1}{6}}$, $\mathbf{P(X \le 6) = \frac{5}{6}}$, $\mathbf{P(X = 6) = \frac{1}{3}}$
H	4	The following is the distribution function $F(x)$ of a discrete random variable X. Find probability distribution of X, $P(-2 \le X \le 1)$ and $P(X \ge 1)$. <table><tr><td>X</td><td>−3</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>F(x)</td><td>0.08</td><td>0.2</td><td>0.4</td><td>0.65</td><td>0.8</td><td>0.9</td><td>1</td></tr></table> **Answer**: <table><tr><td>X</td><td>−3</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>P(X)</td><td>0.08</td><td>0.12</td><td>0.2</td><td>0.25</td><td>0.15</td><td>0.1</td><td>0.1</td></tr></table> $\mathbf{P(-2 \le X \le 1) = 0.72}$, $\mathbf{P(X \ge 1) = 0.35}$

C	5	The probability density function of a continuous random variable X is given by $$f(x)=\begin{cases} ax & ;\ 0\le x<1 \\ a & ;\ 1\le x<2 \\ 3a-ax & ;\ 2\le x<3 \\ 0 & ;\ \text{otherwise} \end{cases}.$$ Find the value of a and also find C.D.F of X. **Answer:** $\mathbf{a=\frac{1}{2}}$, $$\mathbf{F(x)=\begin{cases} \frac{ax^2}{2} & ;\ 0\le x<1 \\ ax-\frac{a}{2} & ;\ 1\le x<2 \\ 3ax-\frac{ax^2}{2}-\frac{5a}{2} & ;\ 2\le x<3 \\ 0 & ;\ \text{otherwise} \end{cases}}.$$
T	6	Find the value of k and the distribution function F(x) given the probability density function of a random variable X as, $f(x)=\frac{k}{1+x^2}$; $-\infty<x<\infty$. **Answer:** $\mathbf{k=\frac{1}{\pi}}$, $\mathbf{F(x)=\frac{1}{\pi}\left[\tan^{-1}x+\frac{\pi}{2}\right]}$
T	7	The life in hours of a certain kind of radio tube has the probability density $$f(x)=\begin{cases} \frac{100}{x^2} & ;\quad x\ge 100 \\ 0 & ;\quad \text{elsewhere} \end{cases}.$$ **(1)** Find the distribution function and use it to determine the probability that the life of tube is more than 150 hrs. **(2)** What is the probability that a tube will last less than 200 hrs. if it is known that the tube is still functioning after 150 hrs. of service? **Answer:** (1) $$\mathbf{F(x)=\begin{cases} 1-\frac{100}{x} & ;\quad x\ge 100 \\ 0 & ;\quad \text{elsewhere} \end{cases}},\quad \mathbf{P(x>150)=\frac{2}{3}}$$ (2) $\mathbf{P(X<200\,\lvert X>150)=0.25}$

*** * * * * End of the Unit * * * * ***