Log-Gaussian Cox Process for London crime data

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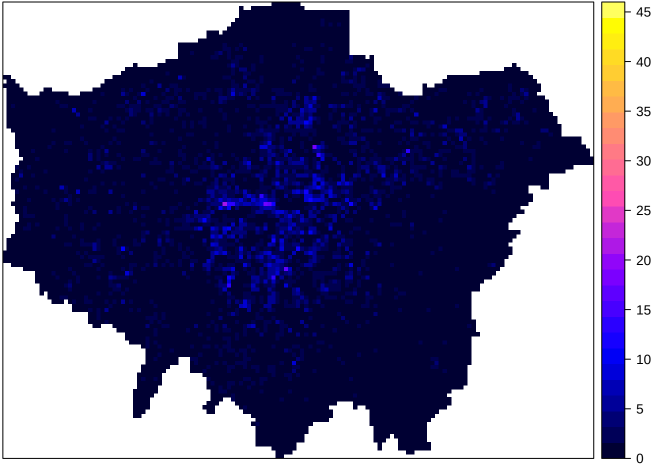
Outline

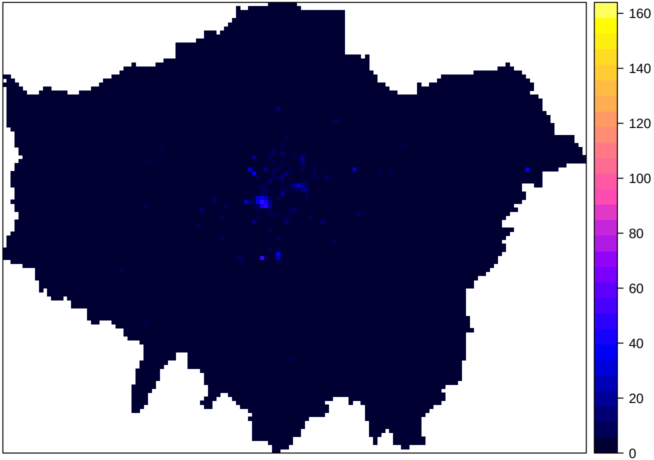
Aims and Objectives

- Modelling of crime and short-term forecasting.
- Two stages involved:
  1. *inference* - what is the underlying process that generated the observations?
  2. *prediction* - use the inferred process's properties to forecast future values.

Burglary



Theft from the person



Outline

Cox Process

Cox process is a natural choice for an environmentally driven point process (Diggle et al., 2013).

Definition

Cox process $\mathsf{Y}(\boldsymbol{x})$ is defined by two postulates:

1. $\Lambda(\boldsymbol{x})$ is a nonnegative-valued stochastic process;
2. conditional on the realisation $\lambda(\boldsymbol{x})$ of the process $\Lambda(\boldsymbol{x})$, the point process $\mathsf{Y}(\boldsymbol{x})$ is an inhomogeneous Poisson process with intensity $\lambda(\boldsymbol{x})$.

Log-Gaussian Cox Process

- Cox process with intensity driven by a Gaussian Process $f(\boldsymbol{x})$:

$$\Lambda(\boldsymbol{x}) = \exp\left(f(\boldsymbol{x})\right).$$

- The latent surface f is modelled by placing a GP prior:

$$f(\boldsymbol{x}) \sim \mathcal{GP}(0, k_\theta(\cdot,\cdot)).$$

- Discretised version of the model over a regular grid on the observation window is:

$$\mathrm{y}_i | f(\boldsymbol{x}_i) \sim \mathrm{Poisson}(\exp\left[f(\boldsymbol{x}_i)\right]).$$

Field inference

Given the observations $\mathbf{y}$ on the grid X, our goal is to find the distribution of the latent field $\mathbf{f}$:

$$p(\mathbf{f}|\mathbf{y}, X, \boldsymbol{\theta}) = \frac{p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})p(\mathbf{f}|X, \boldsymbol{\theta})}{p(\mathbf{y}|X, \boldsymbol{\theta})},$$

where

$$p(\mathbf{y}|X, \boldsymbol{\theta}) = \int p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})p(\mathbf{f}|X, \boldsymbol{\theta})d\mathbf{f}$$

which is intractable.

Laplace Approximation

Flaxman et al. (2015)

- One approach to overcome intractability is *Laplace approximation*.
- Approximate the posterior distribution of the Gaussian Process by:

$$p(\mathbf{f}|\mathbf{y}, X, \boldsymbol{\theta}) \approx \mathcal{N}\left(\hat{\mathbf{f}}, -\left(\nabla\nabla\Psi(\mathbf{f})|_{\hat{\mathbf{f}}}\right)^{-1}\right),$$

  where $\Psi(\mathbf{f}) := \log p(\mathbf{f}|\mathbf{y}, X, \boldsymbol{\theta}) \overset{\text{const}}{=} \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta}) + \log p(\mathbf{f}|X, \boldsymbol{\theta})$ is unormalised log posterior, and $\hat{\mathbf{f}}$ is the mode of the distribution.
- Newton's method to find $\hat{\mathbf{f}}$.

Hyperparameters - Marginal Likelihood

Flaxman et al. (2015)

- Accurate inferences/predictions require knowing $\boldsymbol{\theta}$.
- Marginal log-likelihood:

$$\log p(\mathbf{y}|X, \boldsymbol{\theta}) = \log \int \exp\left[\Psi(\mathbf{f})\right] d\mathbf{f}$$
$$\approx \log p(\mathbf{y}|\hat{\mathbf{f}}) - \frac{1}{2}\mathbf{f}^\top \boldsymbol{K}^{-1}\mathbf{f} - \frac{1}{2}\log\left|\boldsymbol{I} + \boldsymbol{K}\boldsymbol{W}\right|,$$

where $K_{ij} = k_{\boldsymbol{\theta}}(\boldsymbol{x}_i, \boldsymbol{x}_j)$ describes covariance between pairwise locations, and $\boldsymbol{W} := -\nabla\nabla \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})$.

Computation I

Flaxman et al. (2015)

- The calculations above require $\mathcal{O}(n^3)$ operations and $\mathcal{O}(n^2)$ space.
- Cheaper linear algebra available if separable kernel functions are assumed, *e.g.* in $D = 2$ dimensions:

$$k((x_1, x_2), (x'_1, y'_2)) = k_1(x_1, x'_1)k_2(x_2, x'_2)$$

  implies that $\boldsymbol{K} = \boldsymbol{K}_1 \otimes \boldsymbol{K}_2$.
- Determinant approximation due to Fiedler (1971):

$$\begin{aligned}\log|\boldsymbol{I} + \boldsymbol{K}\boldsymbol{W}| &= \log\left(|\boldsymbol{K} + \boldsymbol{W}^{-1}||\boldsymbol{W}|\right) \\ &\leq \log\left\{\prod_i \left(e_i + W_{ii}^{-1}\right)\prod_i W_{ii}\right\} \\ &= \sum_i \log\left(1 + e_i W_{ii}\right),\end{aligned}$$

  where $e_1, \ldots, e_n$ are sorted eigenvalues of $\boldsymbol{K}$.

Computation II

Flaxman et al. (2015)

Applying the above properties, the inference and predictions can be computed using $\mathcal{O}\left(Dn^{\frac{D+1}{D}}\right)$ operations and $\mathcal{O}\left(Dn^{\frac{2}{D}}\right)$ space thanks to:

- Conjugate gradient for solving $\boldsymbol{K}^{-1}\boldsymbol{b} = \boldsymbol{x}$, where matrix-vector multiplication is effificent due to Kronecker structure.
- Eigendecomposition utilising Kronecker structure.

Outline

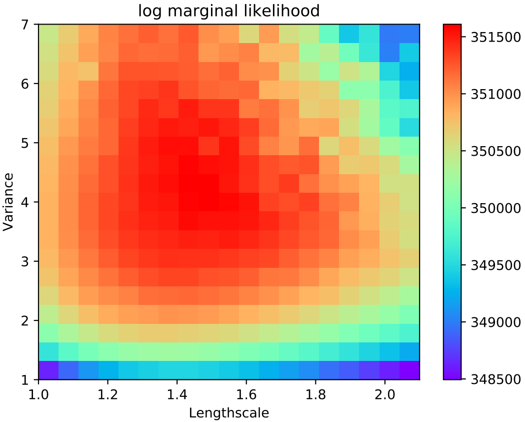
Experiment

Spatial model with isotropic Matérn covariance function:

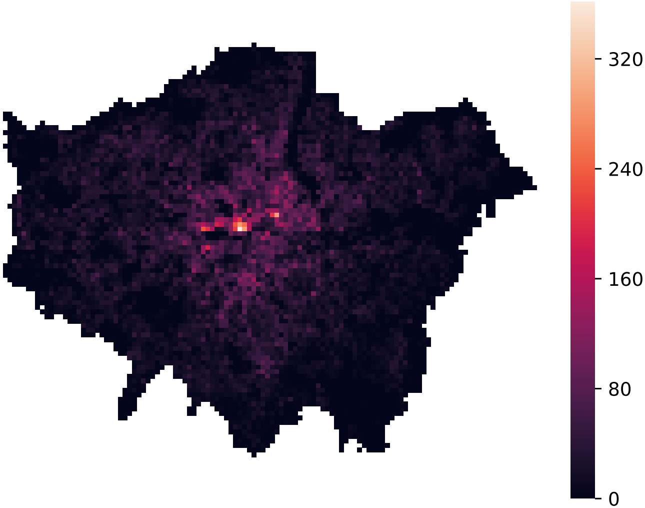
- Dataset used: 2016 data
- Crime types: Burglary, Theft from the person
- Grid: 117x91, one cell is an area of 500m by 500m.
- Missing locations were treated as imaginary with a special noise model.
- Two hyperparameters inferred: lengthscale(ℓ), marginal variance (σ^2)

Burglary - inferred hyperparameters

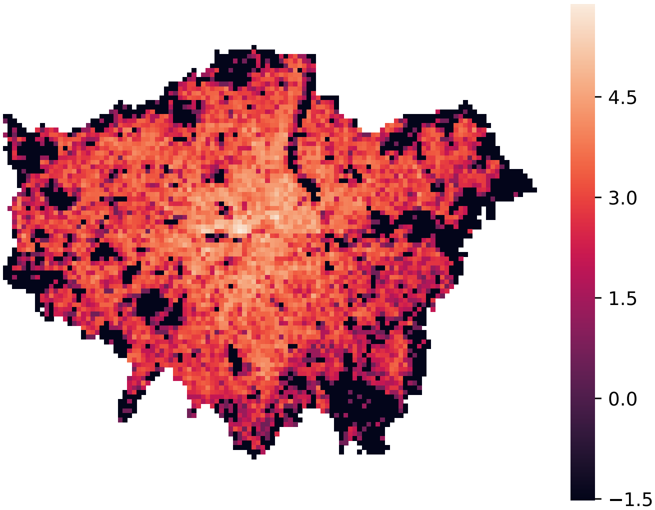
Inferred parameters: $\ell = 1.41$, and $\sigma^2 = 4.16$



Burglary - counts

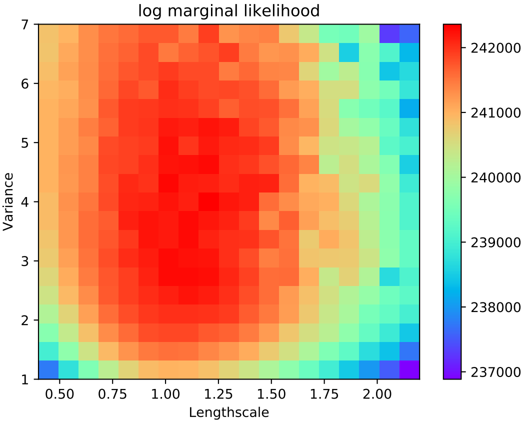


Burglary - latent field



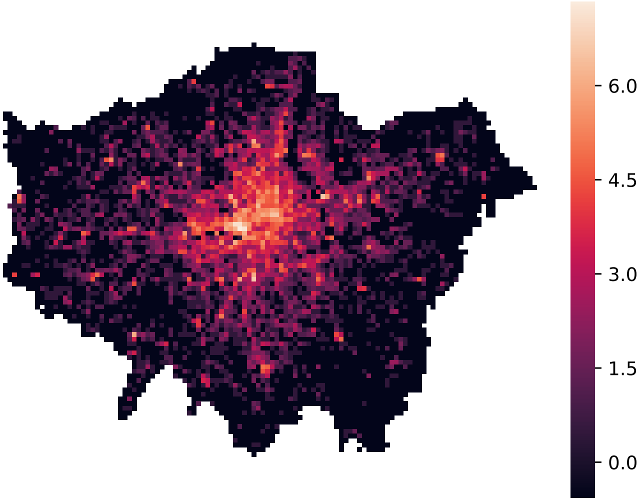
Theft from the person - inferred hyperparameters

Inferred parameters: $\ell = 1.16$, and $\sigma^2 = 3.84$



Theft from the person - counts

Theft from the person - latent field



Comments

- The inference confirmed that number of occurences in a cell influences neighbouring locations.
- The process driving Burglary is ‘smoother’ than the process driving Theft from the person.

Outline

Forecasting

We are considering a few options for the temporal covariance function:

- A kernel with period of 12 months for seasonal variation (Flaxman, 2014):

$$k(\tau) = \exp\left(-\frac{2\sin^2\left(\frac{\tau\pi}{12}\right)}{\ell^2}\right)$$

- Spectral mixture kernel with Q components (Flaxman et al., 2015):

$$k(\tau) = \sum_{q=1}^{Q} w_q \exp\left(-2\pi^2\tau^2 v_q\right)\cos\left(2\pi\tau\mu_q\right)$$

Stochastic PDEs

Another computationally tractable, and more mechanistic, approach is describing the latent Gaussian field using stochastic PDEs:

- Finite Element Method to solve SPDEs as described in Lindgren, Rue, and Lindström (2011).
- Sigrist, Künsch, and Stahel (2015) solve transport-diffusion SPDE using spectral methods on a grid.

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Matérn Covariance Function

$$k(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} r}{\ell} \right)^{\nu} K_{\nu} \left(\frac{\sqrt{2\nu} r}{\ell} \right)$$

We fix $\nu = 2.5$ as it is difficult to jointly estimate ℓ and ν due to identifiability issues.

Kronecker Algebra

Saatçi (2012)

- Matrix-vector multiplication $(\otimes_d \mathbf{A}_d)\,\boldsymbol{b}$ in $\mathcal{O}(n)$ time and space.
- Matrix inverse: $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$
- Let $\boldsymbol{K}_d = \boldsymbol{Q}_d \boldsymbol{\Lambda}_d \boldsymbol{Q}_d^\top$ be the eigendecomposition of $\boldsymbol{K}_d$. Then, the eigendecomposition of $\boldsymbol{K} = \otimes_d \boldsymbol{K}_d$ is given by $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^\top$, where $\boldsymbol{Q} = \otimes_d \boldsymbol{Q}_d$, and $\boldsymbol{\Lambda} = \otimes_d \boldsymbol{\Lambda}_d$. The number of steps required is $\mathcal{O}\left(Dn^{\frac{3}{D}}\right)$.

Field inference - Newton Optimisation

Flaxman et al. (2015)

- The Newton optimisation step:

$$\mathbf{f}^{\text{new}} \leftarrow \mathbf{f}^{\text{old}} - (\nabla\nabla\Psi)^{-1}\,\nabla\Psi.$$

- $\nabla\nabla\Psi$ and $\nabla\Psi$ require inverting the covariance matrix of the GP:

$$\begin{aligned}\nabla\Psi(\mathbf{f}) &= \nabla \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta}) - K^{-1}\mathbf{f} \\ \nabla\nabla\Psi(\mathbf{f}) &= -\boldsymbol{W} - \boldsymbol{K}^{-1},\end{aligned}$$

where $\boldsymbol{W} := -\nabla\nabla \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})$.

Incomplete grids

Wilson et al. (2014)

We have that $\mathsf{y}_i \sim \mathrm{Poisson}(\exp(\mathsf{f}_i))$. For the points of the grid that are not in the domain, we let $\mathsf{y}_i \sim \mathcal{N}(\mathsf{f}_i, \epsilon^{-1})$ and $\epsilon \to 0$. Hence,

$$p(\mathbf{y}|\mathbf{f}) = \prod_{i \in \mathcal{D}} \frac{\left(e^{\mathsf{f}_i}\right)^{\mathsf{y}_i} e^{-e^{\mathbf{f}_i}}}{\mathsf{y}_i!} \prod_{i \notin \mathcal{D}} \frac{1}{\sqrt{2\pi\epsilon^{-1}}} e^{\frac{-\epsilon(\mathbf{y}_i - \mathbf{f}_i)^2}{2}}$$

The log-likelihood is thus:

$$\sum_{i \in \mathcal{D}} [\mathsf{y}_i \mathsf{f}_i - \exp(f_i) + \mathsf{const}] - \frac{1}{2} \sum_{i \notin \mathcal{D}} \epsilon(\mathsf{y}_i - \mathsf{f}_i)^2$$

We now take the gradient of the log-likelihood as

$$\nabla \log p(\mathbf{y}|\mathbf{f})_i = \begin{cases} \mathsf{y}_i - \exp(\mathsf{f}_i), & \text{if } i \in \mathcal{D} \\ \epsilon(\mathsf{y}_i - \mathsf{f}_i), & \text{if } i \notin \mathcal{D} \end{cases}$$

and the hessian of the log-likelihood as

$$\nabla\nabla \log p(\mathbf{y}|\mathbf{f})_{ii} = \begin{cases} -\exp(\mathsf{f}_i), & \text{if } i \in \mathcal{D} \\ -\epsilon & \text{if } i \notin \mathcal{D} \end{cases}.$$