Log-Gaussian Cox Process for London crime data

Jan Povala

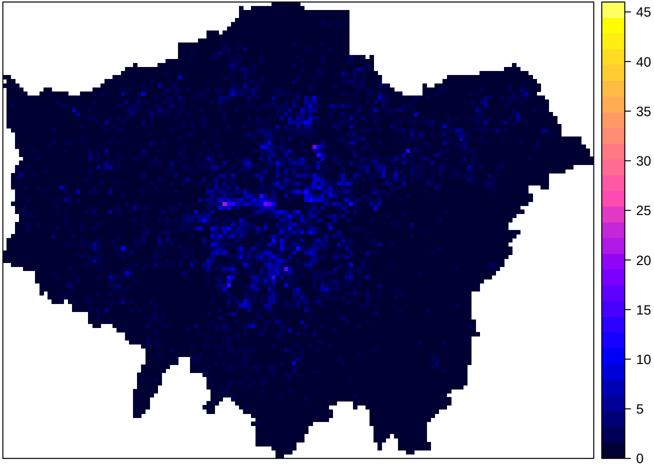
May 3, 2018

Outline

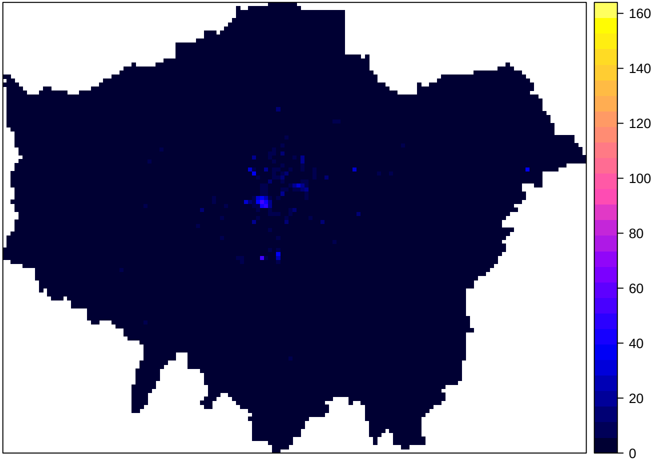
Aims and Objectives

- Modelling of crime and short-term forecasting.
- Two stages involved:
  1. *inference* - what is the underlying process that generated the observations?
  2. *prediction* - use the inferred process's properties to forecast future values.

Burglary



Theft from the person



Outline

Cox Process

Cox process is a natural choice for an environmentally driven point process (Diggle et al., 2013).

Definition

Cox process $\mathsf{Y}(\boldsymbol{x})$ is defined by two postulates:

1. $\Lambda(\boldsymbol{x})$ is a nonnegative-valued stochastic process;
2. conditional on the realisation $\lambda(\boldsymbol{x})$ of the process $\Lambda(\boldsymbol{x})$, the point process $\mathsf{Y}(\boldsymbol{x})$ is an inhomogeneous Poisson process with intensity $\lambda(\boldsymbol{x})$.

Log-Gaussian Cox Process

- Cox process with intensity driven by a Gaussian Process $f(\boldsymbol{x})$:

$$\Lambda(\boldsymbol{x}) = \exp\left(f(\boldsymbol{x})\right).$$

- The latent surface f is modelled by placing a GP prior:

$$f(\boldsymbol{x}) \sim \mathcal{GP}(0, k_\theta(\cdot,\cdot)).$$

- Discretised version of the model over a regular grid on the observation window is:

$$\mathsf{y}_i | f(\boldsymbol{x}_i) \sim \mathrm{Poisson}(\exp\left[f(\boldsymbol{x}_i)\right]).$$

Field inference

Given the observations $\mathbf{y}$ on the grid X, our goal is to find the distribution of the latent field $\mathbf{f}$:

$$p(\mathbf{f}|\mathbf{y}, X, \boldsymbol{\theta}) = \frac{p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})p(\mathbf{f}|X, \boldsymbol{\theta})}{p(\mathbf{y}|X, \boldsymbol{\theta})},$$

where

$$p(\mathbf{y}|X, \boldsymbol{\theta}) = \int p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})p(\mathbf{f}|X, \boldsymbol{\theta})d\mathbf{f}$$

which is intractable.

Laplace Approximation

Flaxman et al. (2015)

- One approach to overcome intractability is *Laplace approximation*.
- Approximate the posterior distribution of the latent surface by:

$$p(\mathbf{f}|\mathbf{y}, X, \boldsymbol{\theta}) \approx \mathcal{N}\left(\hat{\mathbf{f}}, -\left(\nabla\nabla\Psi(\mathbf{f})|_{\hat{\mathbf{f}}}\right)^{-1}\right),$$

  where $\Psi(\mathbf{f}) := \log p(\mathbf{f}|\mathbf{y}, X, \boldsymbol{\theta}) \overset{\text{const}}{=} \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta}) + \log p(\mathbf{f}|X, \boldsymbol{\theta})$ is unnormalised log posterior, and $\hat{\mathbf{f}}$ is the mode of the distribution.
- Newton's method to find $\hat{\mathbf{f}}$.

Hyperparameters - Marginal Likelihood

Flaxman et al. (2015)

- Accurate inferences/predictions require knowing $\boldsymbol{\theta}$.
- Marginal log-likelihood:

$$\begin{aligned}\log p(\mathbf{y}|X,\boldsymbol{\theta}) &= \log \int \exp\left[\Psi(\mathbf{f})\right] d\mathbf{f} \\ &\approx \log p(\mathbf{y}|\hat{\mathbf{f}}) - \frac{1}{2}\hat{\mathbf{f}}^\top \boldsymbol{K}^{-1}\hat{\mathbf{f}} - \frac{1}{2}\log\left|\boldsymbol{I} + \boldsymbol{K}\boldsymbol{W}\right|,\end{aligned}$$

where $K_{ij} = k_{\boldsymbol{\theta}}(\boldsymbol{x}_i, \boldsymbol{x}_j)$ describes covariance between pairwise locations, and $\boldsymbol{W} \coloneqq -\nabla\nabla \log p(\mathbf{y}|\hat{\mathbf{f}}, X, \boldsymbol{\theta})$.

Computation I

Flaxman et al. (2015)

- The calculations above require $\mathcal{O}(n^3)$ operations and $\mathcal{O}(n^2)$ space.
- Cheaper linear algebra available if separable kernel functions are assumed, *e.g.* in $D = 2$ dimensions:

$$k((x_1, x_2), (x_1', x_2')) = k_1(x_1, x_1')k_2(x_2, x_2')$$

  implies that $\boldsymbol{K} = \boldsymbol{K}_1 \otimes \boldsymbol{K}_2$.
- Determinant approximation due to Fiedler (1971):

$$\begin{aligned}\log|\boldsymbol{I} + \boldsymbol{K}\boldsymbol{W}| &= \log\left(|\boldsymbol{K} + \boldsymbol{W}^{-1}||\boldsymbol{W}|\right) \\ &\leq \log\left\{\prod_i \left(e_i + W_{ii}^{-1}\right)\prod_i W_{ii}\right\} \\ &= \sum_i \log\left(1 + e_i W_{ii}\right),\end{aligned}$$

  where $e_1, \ldots, e_n$ are sorted eigenvalues of $\boldsymbol{K}$.

Computation II

Flaxman et al. (2015)

Applying the above properties, the inference and predictions can be computed using $\mathcal{O}\left(Dn^{\frac{D+1}{D}}\right)$ operations and $\mathcal{O}\left(Dn^{\frac{2}{D}}\right)$ space thanks to:

- Conjugate gradient for solving $\boldsymbol{K}^{-1}\boldsymbol{b} = \boldsymbol{x}$, where matrix-vector multiplication is efficient due to Kronecker structure.
- Eigendecomposition utilising Kronecker structure.

Outline

Motivation

Methodology

Results

Current work, Next steps

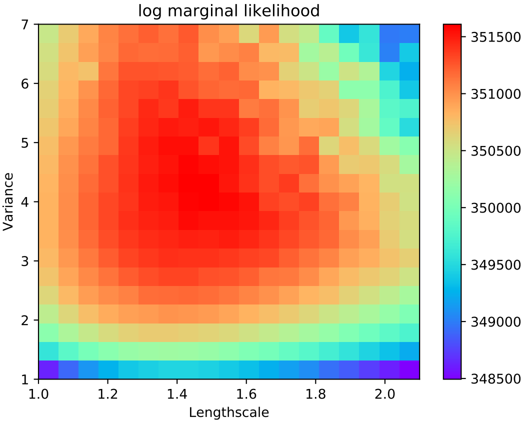
Experiment

Spatial model with isotropic Matérn covariance function:

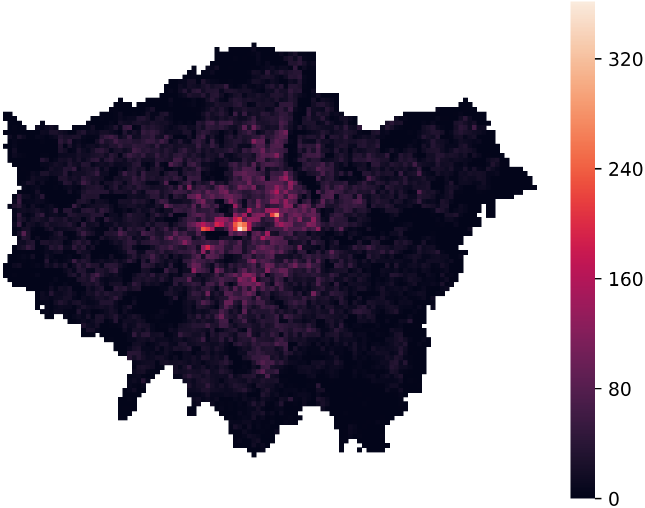
- Dataset used: 2016 data
- Crime types: Burglary, Theft from the person
- Grid: 117x91, one cell is an area of 500m by 500m.
- Missing locations were treated as imaginary with a special noise model.
- Two hyperparameters inferred: lengthscale(ℓ), marginal variance (σ^2)

Burglary - inferred hyperparameters

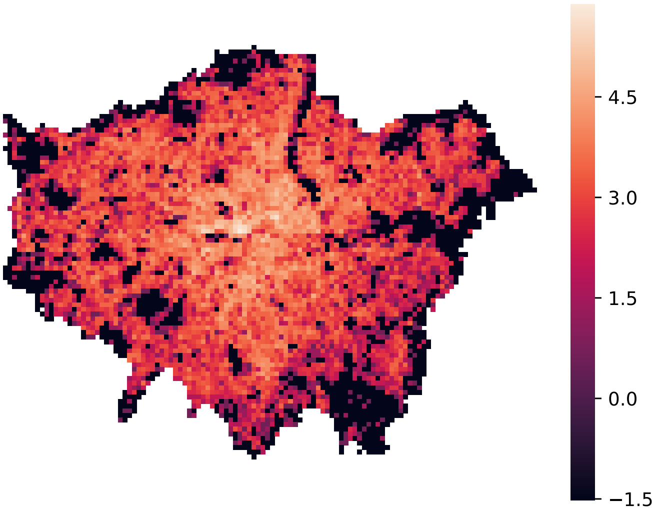
Inferred hyperparameters: $\ell = 1.41$, and $\sigma^2 = 4.16$



Burglary - counts

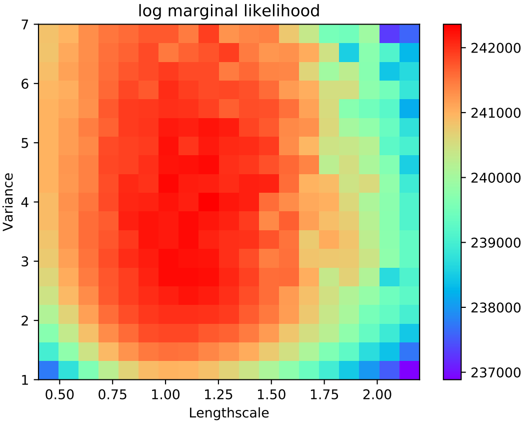


Burglary - latent field



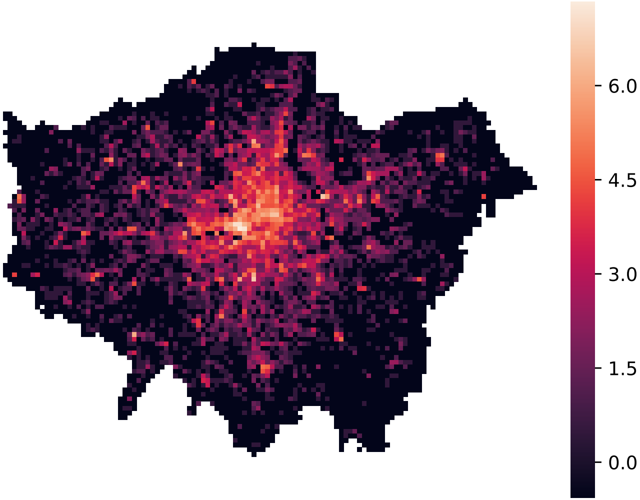
Theft from the person - inferred hyperparameters

Inferred hyperparameters: $\ell = 1.16$, and $\sigma^2 = 3.84$



Theft from the person - counts

Theft from the person - latent field



Comments

- The inference confirmed that number of occurrences in a cell influences neighbouring locations.
- The process driving Burglary is ‘smoother’ than the process driving Theft from the person.

Outline

Motivation

Methodology

Results

Current work, Next steps

Forecasting

The domain will now be $X_1 \times X_2 \times T$ and the kernel will be of the form

$$k((x_1, x_2, t), (x'_1, y'_2, t')) = k_1(x_1, x'_1) k_2(x_2, x'_2) k_t(|t - t'|)$$

with $k_1(\cdot, \cdot)$, $k_2(\cdot, \cdot)$ as before and $k_t(\cdot)$ as one of the below:

- A kernel with period of 12 months for seasonal variation (Flaxman, 2014):

$$k_t(\tau) = \exp\left(-\frac{2\sin^2\left(\frac{\tau\pi}{12}\right)}{\ell^2}\right)$$

- Spectral mixture kernel with Q components (Flaxman et al., 2015):

$$k_t(\tau) = \sum_{q=1}^{Q} w_q \exp\left(-2\pi^2\tau^2 v_q\right) \cos\left(2\pi\tau\mu_q\right)$$

Stochastic PDEs

Another computationally tractable, and more mechanistic, approach is describing the crime activity using stochastic PDEs:

- Finite Element Method to solve SPDEs as described in Lindgren, Rue, and Lindström (2011).
- Sigrist, Künsch, and Stahel (2015) solve transport-diffusion SPDE using spectral methods on a grid.

Bibliography I

Diggle, Peter J. et al. (2013). "Spatial and Spatio-Temporal Log-Gaussian Cox Processes: Extending the Geostatistical Paradigm". en. In: *Statistical Science* 28.4, pp. 542–563. ISSN: 0883-4237. DOI: 10.1214/13-STS441. URL: http://projecteuclid.org/euclid.ss/1386078878.

Fiedler, Miroslav (1971). "Bounds for the Determinant of the Sum of Hermitian Matrices". In: *Proceedings of the American Mathematical Society* 30.1, p. 27. ISSN: 00029939. DOI: 10.2307/2038212. URL: http://www.jstor.org/stable/2038212?origin=crossref.

Flaxman, Seth et al. (2015). "Fast Kronecker Inference in Gaussian Processes with non-Gaussian Likelihoods". In: *Proceedings of the 32nd International Conference on International Conference on Machine Learning*. Vol. 37. ICML'15. Lille, France: JMLR.org, pp. 607–616.

Bibliography II

Flaxman, Seth R. (2014). *A General Approach to Prediction and Forecasting Crime Rates with Gaussian Processes*. Tech. rep. Heinz College Technical Report, 2014. URL https://www. ml. cmu. edu/research/dap-papers/dap_flaxman. pdf.

Lindgren, Finn, Håvard Rue, and Johan Lindström (2011). "An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach". en. In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 73.4, pp. 423–498. ISSN: 1467-9868. DOI: 10.1111/j.1467-9868.2011.00777.x. URL: http://onlinelibrary.wiley.com/doi/10.1111/j.1467-9868.2011.00777.x/abstract.

Saatçi, Yunus (2012). "Scalable inference for structured Gaussian process models". PhD Thesis. Citeseer.

Bibliography III

Sigrist, Fabio, Hans R. Künsch, and Werner A. Stahel (2015). "Stochastic partial differential equation based modelling of large space-time data sets". en. In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 77.1, pp. 3–33. ISSN: 13697412. DOI: 10.1111/rssb.12061. URL: http://doi.wiley.com/10.1111/rssb.12061.

Wilson, Andrew Gordon et al. (2014). "Fast Kernel Learning for Multidimensional Pattern Extrapolation". In: *Proceedings of the 27th International Conference on Neural Information Processing Systems - Volume 2*. NIPS'14. Cambridge, MA, USA: MIT Press, pp. 3626–3634. URL: http://dl.acm.org/citation.cfm?id=2969033.2969231.

Matérn Covariance Function

$$k(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} r}{\ell} \right)^{\nu} K_{\nu} \left(\frac{\sqrt{2\nu} r}{\ell} \right)$$

We fix $\nu = 2.5$ as it is difficult to jointly estimate ℓ and ν due to identifiability issues.

Kronecker Algebra

Saatçi (2012)

- Matrix-vector multiplication $(\otimes_d \mathbf{A}_d)\, \boldsymbol{b}$ in $\mathcal{O}(n)$ time and space.
- Matrix inverse: $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$
- Let $\boldsymbol{K}_d = \boldsymbol{Q}_d \boldsymbol{\Lambda}_d \boldsymbol{Q}_d^\top$ be the eigendecomposition of $\boldsymbol{K}_d$. Then, the eigendecomposition of $\boldsymbol{K} = \otimes_d \boldsymbol{K}_d$ is given by $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^\top$, where $\boldsymbol{Q} = \otimes_d \boldsymbol{Q}_d$, and $\boldsymbol{\Lambda} = \otimes_d \boldsymbol{\Lambda}_d$. The number of steps required is $\mathcal{O}\left(Dn^{\frac{3}{D}}\right)$.

Field inference - Newton Optimisation

Flaxman et al. (2015)

- The Newton optimisation step:

$$\mathbf{f}^{\text{new}} \leftarrow \mathbf{f}^{\text{old}} - \left(\nabla\nabla\Psi\right)^{-1} \nabla\Psi.$$

- $\nabla\nabla\Psi$ and $\nabla\Psi$ require inverting the covariance matrix of the GP:

$$\begin{aligned} \nabla\Psi(\mathbf{f}) =& \nabla \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta}) - K^{-1}\mathbf{f} \\ \nabla\nabla\Psi(\mathbf{f}) =& -\boldsymbol{W} - \boldsymbol{K}^{-1}, \end{aligned}$$

  where $\boldsymbol{W} := -\nabla\nabla \log p(\mathbf{y}|\mathbf{f}, X, \boldsymbol{\theta})$.

Incomplete grids

Wilson et al. (2014)

We have that $\mathsf{y}_i \sim \mathrm{Poisson}(\exp(\mathsf{f}_i))$. For the points of the grid that are not in the domain, we let $\mathsf{y}_i \sim \mathcal{N}(\mathsf{f}_i, \epsilon^{-1})$ and $\epsilon \to 0$. Hence,

$$p(\mathbf{y}|\mathbf{f}) = \prod_{i \in \mathcal{D}} \frac{\left(e^{\mathsf{f}_i}\right)^{\mathsf{y}_i} e^{-e^{\mathbf{f}_i}}}{\mathsf{y}_i!} \prod_{i \notin \mathcal{D}} \frac{1}{\sqrt{2\pi\epsilon^{-1}}} e^{\frac{-\epsilon(\mathbf{y}_i - \mathbf{f}_i)^2}{2}}$$

The log-likelihood is thus:

$$\sum_{i \in \mathcal{D}} [\mathsf{y}_i \mathsf{f}_i - \exp(f_i) + \mathsf{const}] - \frac{1}{2} \sum_{i \notin \mathcal{D}} \epsilon(\mathsf{y}_i - \mathsf{f}_i)^2$$

We now take the gradient of the log-likelihood as

$$\nabla \log p(\mathbf{y}|\mathbf{f})_i = \begin{cases} \mathsf{y}_i - \exp(\mathsf{f}_i), & \text{if } i \in \mathcal{D} \\ \epsilon(\mathsf{y}_i - \mathsf{f}_i), & \text{if } i \notin \mathcal{D} \end{cases}$$

and the hessian of the log-likelihood as

$$\nabla\nabla \log p(\mathbf{y}|\mathbf{f})_{ii} = \begin{cases} -\exp(\mathsf{f}_i), & \text{if } i \in \mathcal{D} \\ -\epsilon & \text{if } i \notin \mathcal{D} \end{cases}.$$

Fiedler's bound

For Hermitian positive semidefinite matrices mU and mV:

$$\prod_i (u_i + v_i) \leq |\boldsymbol{U} + \boldsymbol{V}| \leq \prod_i (u_i + v_{n-i+1}),$$

where u_i and v_j are sorted eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively.