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用不同方法估计 $\theta = \int_0^1 4x^3 dx$，先直接计算出$\theta = 1$以比较下面几种方法：
在[0,1]上生成均匀随机数$x_i,\ \forall i = 1, 2, \ldots, n$，令$n = 1000$，以下几种方法使用同样的随机数

1. MC

在[0,1]上生成均匀随机数$x_i,\ \forall i = 1, 2, \ldots, n$
$\hat{\theta}_{MC} = \frac{1}{n} \sum_{i=1}^n 4x_i^3$

$\hat{\theta}_{MC} =$

```
0.9873149760335518
```

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2. AV

在[0,1]上生成均匀随机数$x_i,\ \forall i = 1, 2, \ldots, n$
$\hat{\theta}_{AV} = \frac{1}{n} \sum_{i=1}^n \frac{4x_i^3 + 4(1-x_i)^3}{2}$，令$n = 1000$

$\hat{\theta}_{AV} =$

```
0.9865257974290198
```

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3. CV

在[0,1]上生成均匀随机数$x_i,\ \forall i = 1, 2, \ldots, n$
$\hat{\theta}_{CV} = \frac{1}{n} \sum_{i=1}^n f(x_i) - b^*(g(x_i) - I(g(x_i)),\ \forall f(x_i) = 4x_i^3$

因为$f(x) \approx g(x)$，所以令$g(x) = e^x - 1 \ \Rightarrow I(g(x)) = e^1 - 2$

$b^* = \frac{\sum_{i=1}^n (f(x_i) - \bar{f(x)})(g(x_i) - \bar{g(x)})}{\sum_{i=1}^n (g(x_i) - \bar{g(x)})^2}$

$\hat{\theta}_{CV} =$

```
0.9941301787463542
```

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4. stratification

在[0,1]上生成均匀随机数$x_i, \ \forall i = 1, 2, ..., n$
$\hat{\theta}_{CV} = \frac{1}{M} \sum_{j=1}^{M} \bar{Y}_j, \ \forall \bar{Y}_j = \frac{1}{L} \sum_{i=1}^{L} f(x_{ij}), \ f(x_{ij}) = 4x_{ij}^3$

令层级$M = 5$，每层内的样本数$L = 200$

$\hat{\theta}_{SS} =$

```
0.9873149760335517
```

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5. Improvement

在[0,1]上生成均匀随机数$x_i, \ \forall i = 1, 2, ..., n$
综合上述AV、CV、SS方法来估计$\hat{\theta}$，但结果和CV差不多。

融合方法：
将CV用SS来分层，并且其使用的$f(x)$以AV来替换

$\hat{\theta} =$

```
0.9933410001418223
```