Exercise 1 (dead line: 17 March)

1. Suppose we want to sample from the density

$$f(x) = x + 1/2, \quad 0 < x < 1.$$

(1) Using the inverse transform methods, simulate 1000 values from f;

(2) Using the acceptance-rejection method, simulate another 1000 values from f.

(3) Which algorithm is more efficient?

Exercise 1 (dead line: 17 March)

2. Suppose we want to simulate $|Z|$, where $Z \sim N(0,1)$.

The pdf of $|Z|$ is

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad 0 < x < +\infty.$$

Take $g(x) = e^{-x}, \quad 0 < x < +\infty.$

(1) **Determine the value of c such that** $c = \max \dfrac{f(x)}{g(x)}.$

(2) Using acceptance-rejection algorithm to simulate 1000 values of $|Z|$.

(3) How to recover Z from the simulated values of $|Z|$?

Exercise 1 (dead line: 26 March)

3. Suppose that 若干個同一個Ｘ的ＣＤＦ相乘

$$F(x) = \prod_{j=1}^{K} F_j(x),$$

where $F_j(\cdot)$ are CDFs from which we can sample easily (x is univariate).

Describe a way of sampling $X \sim F(x)$.

Exercise 1 (dead line: 17 March)

4. Suppose for some real numbers $a < b$, and some pdf $f(x)$ with associated CDF $F(x)$, $-\infty < x < \infty$, we want to generate random variates having the truncated pdf 截斷的

$$g(x) = \begin{cases} \dfrac{f(x)}{F(b) - F(a)}, & a \le x \le b, \\ 0, & \text{else.} \end{cases}$$

Assume the inverse CDF $F^{-1}(\cdot)$ can be computed. Explain how to generate variates from the above truncated pdf.

Exercise 1 (dead line: 17 March)

5. For the beta pdf

$$f(x) = 12x^2(1-x), 0 \le x \le 1$$

implement the acceptance-rejection approach, and for a sample of 100,000 beta variates, compute the average number of uniform variates required to output one beta variate.

Assignment 2 (Dead Line: April 9, 2019)

1. Prove the Brownian bridge theorem.

2. Prove that $(B(t_1),\ldots,B(t_n))^T \sim N(0,C)$, where $B(t_1),\ldots,B(t_n)$ are the values of BM at times t_1, …, t_n, and the entries of C are given by

$$c_{i,j} = \min(t_i, t_j).$$

3. **Prove that if $AA^T = C$, then $BB^T = C$, if and only if B can be written as $B = AU$ for some orthogonal matrix U.**

Assignment 2 (Dead Line: April 9, 2019)

4. Find the generating matrix A (analytically or numerically) corresponding to the Brownian bridge construction, i.e., the matrix A in

$$(\boldsymbol{B}(\boldsymbol{t}_1),\ldots,\boldsymbol{B}(\boldsymbol{t}_n))^T = \boldsymbol{AZ}, \textbf{ with } \boldsymbol{AA}^T = \boldsymbol{C} \textbf{ and } \boldsymbol{Z} \sim \boldsymbol{N}(0,\boldsymbol{I}),$$

with the values $B(t_i)$ generated by BB.

5. Write programs to generate BM for n being a power of 2, and being not a power of 2, respectively.

Assignment 2 (Dead Line: April 9, 2019)

6. Derive analytical formulas for the cumulative explained variability in the random walk construction, PCA (or even in BB construction).

7. Compare the cumulative explained variability in random walk construction, BB and PCA constructions in dimension 16, 64 and 256.

8. Suppose $\mathbf{C} = \mathbf{A}\mathbf{A}^{\mathrm{T}}, \Sigma = \mathbf{B}\mathbf{B}^{\mathrm{T}}$. Prove that

$$(\mathbf{C} \otimes \Sigma) = (\mathbf{A} \otimes \mathbf{B})(\mathbf{A} \otimes \mathbf{B})^{\mathrm{T}}.$$

Assignment 2 (Dead Line: April 9, 2019)

9. Let $X \sim \mathrm{BM}(\mu, \Sigma)$ and
$\boldsymbol{Y} := (\boldsymbol{X}_1(\boldsymbol{t}_1), \ldots, \boldsymbol{X}_1(\boldsymbol{t}_n), \boldsymbol{X}_2(\boldsymbol{t}_1), \ldots, \boldsymbol{X}_2(\boldsymbol{t}_n), \cdots, \boldsymbol{X}_d(\boldsymbol{t}_1), \ldots, \boldsymbol{X}_d(\boldsymbol{t}_n))^T$.
Prove that the covariance matrix of Y is $(\Sigma \otimes C)$.

10. How to obtain the eigenvalues and eigenvector of the matrix $(C \otimes \Sigma)$ from these of matrix C and matrix Σ.

11*. What is the difference in generating multi-dimensional BM between "one-step PCA" and "two-step-PCA" (i.e., use PCA for both matrix C and matrix Σ)?

Assignments 3 (dead line: April 23, 2019)

2. Show that in the CV method, var(f-bg) < var(f), if and only if b lies between 0 and b*, where b* is the optimal parameter which minimizes the variance of the CV estimate (note that b* may be negative).

3. (1) Show that for the function

$$f(x) = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d + b$$

the AV estimate has zero variance.

(2) What is the effect of using AV to function

$$f(x) = \sum_{j=1}^{d} \left(1 - 2x_j\right)^2 ?$$

Assignments 3 (dead line: April 23, 2019)

1. Prove the next **Lemma** for d>1:

   If X_1, …, X_d are independent, then for any increasing functions f_1 and f_2 of d variables,

$$\mathbf{E[f_1(X)f_2(X)] \geq E[f_1(X)]\ E[f_2(X)],}$$

$$\mathbf{or\ \ cov\big(f_1(X), f_2(X)\big) \geq 0,}$$

$$\mathbf{where\ \ X = (X_1, \ldots, X_d).}$$

Assignments 3:

4. Under the framework of Black-Scholes model, derive an analytical formula for the price of geometric Asian option with a payoff

$$f_G = \max\left(0,\ \Pi_{i=1}^{n} S(t_i)^{w_i} - K\right),$$
$$\textbf{where}\ w_i = 1/n.$$

5. Use geometric Asian option as an control variable (CV), write program to price arithmetic Asian option (and compute the variance reduction factor).

Assignments 3:

6. Show that stratified estimator with proportional allocation has a variance no larger than that of crude MC estimate. Show that optimal allocation gives smaller variance than proportional allocation.

7. Is the LHS estimate unbiased? Why?
   Is the WUS estimate unbiased? Why?
   (LHS --- Latin hypercube sampling;
   WUS --- Weighted Uniform Sampling)

Assignments 3:

8. Consider estimating $\theta = \int_0^1 4x^3 dx.$

(1) Using standard simulation method to estimate θ.

(2) Using antithetic variable technique to estimate θ.

(3) Construct a control variable estimate of θ.

(4) Using stratification, construct another estimate of θ.

(5) Can you combine the above methods to improve the results?

Assignments 3:

9. Consider the problem of estimating

$$\theta = P(Z > b),$$

where $Z \sim N(0,1)$ and b is a positive constant.

(1) Estimate θ via simulation without doing IS.

(2) Estimate θ by doing IS with a new random variable $Y \sim N(\mu, 1)$ with some appropriat e chice for μ (how to choose μ?)

Assignment 4 (Dead line: May 21)

1. Prove that for L2 star discrepancy, we have

$$\left(T_N^*(P)\right)^2 = \int_{[0,1]^d} \left| \frac{A(J(x);P)}{N} - m(J(x)) \right|^2 dx$$

$$= \frac{1}{3^d} - \frac{1}{2^{d-1}} \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{d} (1 - t_{i,k}^2) + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{k=1}^{d} [1 - \max(t_{i,k}, t_{j,k})],$$

where $t_i = (t_{i,1}, \ldots, t_{i,d}), P = \{t_i\}$.

Prove also that the square expected L2 star discrepancy of random points is

$$E\left(T_N^*(P)\right)^2 = \frac{1}{N}\left(2^{-d} - 3^{-d}\right).$$

Assignment 4 (Dead line: May 21)

2. Determine the L_1-discrepancy, L_2-discrepancy and the star-discrepancy of the following point sets

$$(1)\ P = \{n/N : n = 0,1,\dots,N-1\};$$

$$(2)\ P = \left\{\frac{2n-1}{2N} : n = 0,1,\dots,N-1\right\};$$

3. Construct ''by hand'' a (0, 2, 2)-net in base 3. (Refer to Faure points).

Assignment 4 (Dead line: May 21)

4. Write programs to generate Halton, Sobol, or Faure sequences in dimensions up to 50.

5. Empirically compare the L_2 discrepancies of Halton, Sobol, or Faure sequences, and compare with the expected L_2 discrepancy.

6. Compare the efficiency of Halton, Sobol, or Faure sequences for high-dimensional financial problems (say, option pricing) and compare their efficiency with MC.

Assignment 4:

7. **Using Halton, Sobol, or Faure sequences, in conjunction with random work construction, Brownian bridge construction and PCA construction of Brownian motion, for option pricing.**

8. **Using good lattice rules (say, Korobov lattice rules), in conjunction with BB or PCA, for option pricing.**

9*. **Using randomized QMC for error estimation and variance estimation when doing option pricing.**