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1,

Brownian Bridge THM: $B_{t_j} \mid B_{t_i}=b_i,\ B_{t_k}=b_k \sim N\left(\frac{t_k-t_j}{t_k-t_i}b_i+\frac{t_j-t_i}{t_k-t_i}b_k,\ \frac{(t_k-t_j)(t_j-t_i)}{t_k-t_i}\right)$

$\forall\ i<j<k$

$t_i \quad t_j \quad t_k$

下証:

$$\begin{pmatrix} B_{t_i} \\ B_{t_j} \\ B_{t_k} \end{pmatrix} \sim N\left(0,\ \begin{pmatrix} t_i & t_i & t_i \\ t_i & t_j & t_j \\ t_i & t_j & t_k \end{pmatrix}\right) \underset{\Rightarrow 的元素}{\text{交換向量}} \begin{pmatrix} B_{t_j} \\ B_{t_i} \\ B_{t_k} \end{pmatrix} \sim N\left(0,\ \begin{pmatrix} t_j & t_i & t_j \\ t_i & t_i & t_i \\ t_j & t_i & t_k \end{pmatrix}\right)$$

given $B_{t_i}=b_i,\ B_{t_k}=b_k \Rightarrow E\left[B_{t_j} \mid B_{t_i}=b_i,\ B_{t_k}=b_k\right] = 0-(t_i\ t_j)\begin{pmatrix} t_i & t_i \\ t_i & t_k \end{pmatrix}\begin{pmatrix} b_i \\ b_k \end{pmatrix}$

$= \frac{(t_k-t_j)b_i+(t_j-t_i)b_k}{t_k-t_i}$

$Var\left[B_{t_j} \mid B_{t_i}=b_i,\ B_{t_k}=b_k\right] = t_j - (t_i\ t_j)\begin{pmatrix} t_i & t_i \\ t_i & t_k \end{pmatrix}\begin{pmatrix} t_i \\ t_j \end{pmatrix}$

$= \frac{(t_k-t_j)(t_j-t_i)}{t_k-t_i}$

$\because B_{t_j} \mid B_{t_i}=b_i,\ B_{t_k}=b_k \sim$ Normal Distribution

$\therefore B_{t_j} \mid B_{t_i}=b_i,\ B_{t_k}=b_k \sim N\left(\frac{(t_k-t_j)b_i+(t_j-t_i)b_k}{t_k-t_i},\ \frac{(t_k-t_j)(t_j-t_i)}{t_k-t_i}\right)$ #

2. $(B_{t_1},\cdots,B_{t_n})^T \sim N(0,C)$, $C_{i,j}=\min(t_i,t_j)$

下証:

$\forall\ 0<t_i<t_j\le t_n$, $Cov(B_{t_i},B_{t_j}) = Cov(B_{t_i},(B_{t_j}-B_{t_i})+B_{t_i})$

$= Cov(B_{t_i},B_{t_j}-B_{t_i})+Cov(B_{t_i},B_{t_i}) = 0+t_i=t_i \quad \Rightarrow C_{i,j}=\min(t_i,t_j)$ #

3, if $AA^T=C \Rightarrow BB^T=C$ iff $B=AU$ for some orthogonal matrix U

下証:

if $B=AU$, U is orthogonal matrix $\Rightarrow BB^T = AUU^TA^T = AA^T = C \Rightarrow BB^T=C$

Let $U=A^{-1}B \Rightarrow B=AU$

$\because AA^T=C=BB^T=AUU^TA^T \Rightarrow UU^T=I \Rightarrow U$ is orthogonal matrix #

4, $(B_{t_1}\ \cdots\ B_{t_n})^T = AZ\quad \forall\ AA^T=C$, $Z\sim N(0,I)$, B_{t_i} generated from BB

$\Rightarrow$ 令 A_i 為 A 的 ith row , 在 $n=2^M$ 時.

根據課件上的公式知，$B_{(2j-1)h} = \frac{1}{2}\left(B_{2(j-1)h} + B_{2jh}\right) + \sqrt{\frac{h}{2}}\, z_i$，$\forall\, j = 1, \dots, 2^{K-1}$

$K = 1, \dots, M$

$z_i \sim N(0,1)$

$h = n, \frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \dots$

$B_0 = 0$，$B_n = \sqrt{n}\, z_i$，

$A_{n,i+} = 1$

$$\Rightarrow A_{(2j-1)h} = \frac{1}{2}\left(A_{2(j-1)h} + A_{2jh}\right)$$

$$\forall\, A_{(2j-1)h,\, i+} = \sqrt{\frac{h}{2}}$$

5. 後面言附上 coding and results

6. 下求 Random walk 和 PCA 的累計解釋變異

a. Random walk:

令 a_i 為 A 的 ith column，$t_0 = 0$ $\Rightarrow a_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{t_i - t_{i-1}} \\ \vdots \\ \sqrt{t_i - t_{i-1}} \end{pmatrix}$ $\because A = \begin{pmatrix} \sqrt{t_1} & & & \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & & \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \sqrt{t_3 - t_2} & \\ \sqrt{t_1} & \cdots & & \sqrt{t_n - t_{n-1}} \end{pmatrix}$

$$\Rightarrow \|a_i\|^2 = (n - i + 1)(t_i - t_{i-1})$$

$$\Rightarrow \text{Cumulative explained variability} = \frac{\sum_{i=1}^{k} \|a_i\|^2}{\sum_{i=1}^{n} \|a_i\|^2} = \frac{\sum_{i=1}^{k} (n-i+1)(t_i - t_{i-1})}{\sum_{i=1}^{n} (n-i+1)(t_i - t_{i-1})}$$

b. PCA:

和 Random walk 不同的是 $\|a_i\|^2$ 的計算方法.

$B = AZ$，且 $B_i = v_i^T B$ $\Rightarrow A = VD^{\frac{1}{2}} = (\sqrt{\lambda_1}\, v_1, \dots, \sqrt{\lambda_n}\, v_n)$ $v_i^T v_j = \delta_{ij}$

$\because Cv_i = \lambda v_i \Leftrightarrow CV = VD \Rightarrow C = VDV^T = (VD^{1/2})(VD^{1/2})^T = AA^T$

故 $\|a_i\|^2 = \lambda_i \|v_i\|^2 = \lambda_i$

$$\Rightarrow \text{Cumulative explained variability} = \frac{\sum_{i=1}^{k} \|a_i\|^2}{\sum_{i=1}^{n} \|a_i\|^2} = \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{n} \lambda_i}$$ #

7. 後面言附上 coding and results.

8. $C = AA^T$，$\Sigma = BB^T$，欲證 $(C \otimes \Sigma) = (A \otimes B)(A \otimes B)^T$

$(C \otimes \Sigma) = ((AA^T) \otimes (BB^T))$

By Kronecker Product 的性質：$(M \otimes N)^T = M^T \otimes N^T$、$(A \otimes B)(C \otimes D) = AC \otimes BD$

$\because \begin{pmatrix} m_{11}N & m_{12}N & \\ \vdots & & \\ & & m_{nn}N \end{pmatrix}^T = \begin{pmatrix} m_{11}N^T & & \\ m_{12}N^T & & \\ \cdots & & m_{nn}N^T \end{pmatrix} = M^T \otimes N^T$

$\because \begin{pmatrix} a_{11}B & & \\ & & a_{nn}B \end{pmatrix}\begin{pmatrix} c_{11}D & & \\ & & c_{nn}D \end{pmatrix} = \begin{pmatrix} a_{11}c_{11}BD & & \\ a_{nn}c_{n1}BD & & \\ \vdots & & a_{nn}c_{nn}BD \end{pmatrix} = AC \otimes BD$

故 $(A \otimes B)(A \otimes B)^T$

$= (A \otimes B)(A^T \otimes B^T) = (AA^T) \otimes (BB^T) = C \otimes \Sigma$ #

9. $Y = (X_1(t_1), \ldots, X_1(t_n), X_2(t_1), \ldots, X_2(t_n), \ldots, X_d(t_1), \ldots, X_d(t_n))^T$，$\forall X \sim BM(\mu, \Sigma)$，

欲證 $Cov(Y) = (\Sigma \otimes C)$ $\forall C = \begin{bmatrix} t_1 & \cdots & & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 \cdots & t_3 \\ \vdots \\ t_1 & \cdots & \cdots & t_n \end{bmatrix} = \min(t_i, t_j)$

令 $X_1(t) = (X_1(t_1), \cdots, X_1(t_n))^T$，已知 $X \sim BM(\mu, \Sigma)$，

若 $a<b$，則 $Cov(X_1(a), X_1(b)) = Cov(X_1(a), X_1(a) + X_1(b) - X_1(a)) = Cov(X_1(a), X_1(a)) + Cov(X_1(a), X_1(b) - X_1(a)) = Cov(X_1(a), X_1(a)) = \Sigma_{11} a = \Sigma_{11} \min(a, b)$

$\Rightarrow Cov(X_1(t), X_1(t)) = \Sigma_{11} \min(t_i, t_j)$

以此類推，$Cov(X_1(t), X_2(t)) = \Sigma_{12} \min(t_i, t_j)$

故 $Cov(Y) = \Sigma \otimes C$ #

10. 若有一 $A \otimes B$，A 的 eigenvalue 為 λ，eigenvector 為 x $(Ax = \lambda x)$

B 的 eigenvalue 為 μ，eigenvector 為 y $(By = \mu y)$

則 $A \otimes B$ 的 eigenvalue 為 $\lambda\mu$，eigenvector 為 $x \otimes y$

$\because (A \otimes B)(x \otimes y) = (Ax) \otimes (By) = (\lambda x) \otimes (\mu y) = \lambda(x) \otimes \mu(y) = \lambda\mu(x \otimes y)$

故 $C \otimes \Sigma$ 的 eigenvalue 為 $\lambda_i \eta_j$，eigenvector 為 $v_i \otimes w_j$

$\forall\ Cv = \lambda v$，$\Sigma w = \eta w$ #

11. one-step PCA：只在其中一步選 PCA

two-step PCA：兩步都選 PCA

$\Leftrightarrow$ Step 1：建構 C，可用 Standard Method / Random walk / Brownian Bridge / PCA

Step 2：建構 Σ，可用 Cholesky / PCA

兩個步驟使用不同方法會有不同結果，例如 PCA、BB 較不受維度災難影響，而且計算的時間複雜度也不同。