計算金融学 2020270026 王姿文

1.

a. Prove $(T_N^*(P))^2 = \int_{[0,1]^d} \left| \frac{A(J(x);P)}{N} - m(J(x)) \right|^2 dx$

$$= \frac{1}{3^d} - \frac{1}{2^{d-1}} \cdot \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{d} (1 - t_{i,k}^2) + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{k=1}^{d} [1 - \max(t_{i,k}, t_{j,k})]$$

$\forall\, t_i = (t_{i1}, \cdots, t_{id})$, $P = \{t_i\}$

Proof:

$$(T_N^*(P))^2 = \int_{[0,1]^d} \left| \frac{A(J(x);P)}{N} - m(J(x)) \right|^2 dx$$

$$= \int_{[0,1]^d} \left| \frac{1}{N} \sum_{i=1}^{N} I_{[0,x)}(t_i) - x_1, \cdots, x_d \right|^2 dx$$

$$= \int_{[0,1]^d} x_1^2 \cdots x_d^2 \, dx - \frac{2}{N} \int_{[0,1]^d} x_1, \cdots, x_d \sum_{i=1}^{N} I_{[0,x)}(t_i)\, dx$$

$$+ \frac{1}{N^2} \int_{[0,1]^d} \left(\sum_{i=1}^{N} I_{[0,x)}(t_i) \right)^2 dx$$

$$= \frac{1}{3^d} - \frac{2}{N} \sum_{i=1}^{N} \int_{t_{i1}}^{1} \cdots \int_{t_{id}}^{1} x_1 \ldots x_d \, dx_1 \ldots dx_d + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{[0,1]^d} I_{[t_i \vee t_j, 1]}(x)\, dx$$

$$= \frac{1}{3^d} - \frac{2}{N} \sum_{i=1}^{N} \frac{1}{2^d} \prod_{k=1}^{d} (1 - t_{i,k}^2) + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{k=1}^{d} [1 - \max(t_{ik}, t_{jk})]$$

$$= \frac{1}{3^d} - \frac{1}{2^{d-1}} \cdot \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{d} (1 - t_{i,k}^2) + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{k=1}^{d} [1 - \max(t_{ik}, t_{jk})] \quad \#$$

b. Prove $E(T_N^*(P))^2 = \frac{1}{N}\left(\frac{1}{2^d} - \frac{1}{3^d}\right)$

Proof:

$$E(T_N^*(P))^2 = E \int_{[0,1]^d} \left| \frac{A(J(x);P)}{N} - m(J(x)) \right|^2 dx$$

$$= E \int_{[0,1]^d} \left| \frac{1}{N} \sum_{i=1}^{N} I_{[0,x)}(t_i) - x_1 \ldots x_d \right|^2 dx$$

$$= \int_{[0,1]^d} E \left| \frac{1}{N} \sum_{i=1}^{N} I_{[0,x)}(t_i) - x_1 \ldots x_d \right|^2 dx$$

$$= \int_{[0,1]^d} Var\left(\frac{1}{N} \sum_{i=1}^{N} I_{[0,x)}(t_i) \right) dx$$

$$= \int_{[0,1]^d} \frac{1}{N^2} N\, m(J(x))(1 - m(J(x)))\, dx$$

$$= \frac{1}{N} \int_{[0,1]^d} x_1 \cdots x_d - x_1^2 \cdots x_d^2 \, dx$$

$$= \frac{1}{N} \left[\frac{1}{2^d} - \frac{1}{3^d} \right] \quad \#$$

∵ $\sum_{i=1}^{N} I_{[0,x)}(t_i) \sim B(N, m(J(x)))$

ⓐ $\sum_{i=1}^{N} I_{[0,x)}(t_i)$ 有加成性故

可由 Bernoulli 加為 Binomial

ⓑ $P(I_{[0,x)}(t_i) = 1)$

$= P(t_i) = m(J(x))$ 故

$I_{[0,x)}(t_i) \sim Ber(m(J(x)))$

2.

(1) $P = \{ \frac{n}{N} ; n=0,1,\cdots,N-1 \}$

a. L_1 - Discrepancy.

$$T_{1N}(P) = \int_{[0,1]^{2d}} \left| \frac{A(J(x,y);P)}{N} - m(J(x,y)) \right| dx\,dy$$

$$= \int_0^1 \int_0^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{n}{N}) - |y-x| \right| dx\,dy$$

$$= \int_0^1 \int_0^y \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{n}{N}) - y + x \right| dx\,dy + \int_0^1 \int_y^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{n}{N}) - x + y \right| dx\,dy \quad \#$$

b. L_2 - Discrepancy

$$T_{2N}(P) = \left[\int_{[0,1]^{2d}} \left| \frac{A(J(x,y);P)}{N} - m(J(x,y)) \right|^2 dx\,dy \right]^{1/2}$$

$$= \left[\int_0^1 \int_0^y \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{n}{N}) - y + x \right|^2 dx\,dy + \int_0^1 \int_y^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{n}{N}) - x + y \right|^2 dx\,dy \right]^{1/2}$$

$$= \left[2 \int_0^1 \left(\frac{1}{N} \sum_{n=0}^{N-1} I_{[0,x)}(\tfrac{n}{N}) - x \right)^2 dx - 2 \left[\int_0^1 \left(\frac{1}{N} \sum_{n=0}^{N-1} I_{[0,x)}(\tfrac{n}{N}) - x \right) dx \right]^2 \right]^{1/2}$$

$$= \left[\frac{2}{3} \cdot \frac{1}{N^2} - \frac{1}{N^2} \cdot \frac{1}{4} \cdot 2 \right]^{1/2} = \sqrt{\frac{1}{6} \cdot \frac{1}{N^2}} = \frac{\sqrt{6}}{6N} \quad \#$$

c. Star - Discrepancy

$$D_N^*(P) = \sup_{E \in J^*} \left| \frac{A(E;P)}{N} - m(E) \right|$$

$$\forall\, 0 \le x_1 \le \cdots \le \underset{\le x_N}{x_{N-1}} \le 1 \Rightarrow D_N^*(P) = \frac{1}{2N} + \max_{1 \le n \le N} \left| \frac{n-1}{N} - \frac{2n-1}{2N} \right|$$

$$= \frac{1}{2N} + \left| \frac{-1}{2N} \right| = \frac{2}{2N} = \frac{1}{N} \quad \#$$

(2) $P = \{ \frac{2n-1}{2N} ; n=0,1,\cdots,N-1 \}$

a. L_1 - Discrepancy

$$T_{1N}(P) = \int_0^1 \int_0^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{2n-1}{2N}) - |y-x| \right| dx\,dy$$

$$= \int_0^1 \int_0^y \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{2n-1}{2N}) - y + x \right| dx\,dy + \int_0^1 \int_y^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{2n-1}{2N}) - x + y \right| dx\,dy \quad \#$$

b. L_2 - Discrepancy

$$T_{2N}(P) = \left[\int_0^1 \int_0^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{2n-1}{2N}) - |y-x| \right|^2 dx\,dy \right]^{1/2}$$

$$= \left[\int_0^1 \int_0^y \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[x,y)}(\tfrac{2n-1}{2N}) - y + x \right|^2 dx\,dy + \int_0^1 \int_y^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} I_{[y,x)}(\tfrac{2n-1}{2N}) - x + y \right|^2 dx\,dy \right]^{1/2}$$

$$= \left[2 \int_0^1 \left(\frac{1}{N} \sum_{n=0}^{N-1} I_{[0,x)}(\tfrac{2n-1}{2N}) - x \right)^2 dx - 2 \left(\int_0^1 \left(\frac{1}{N} \sum_{n=0}^{N-1} I_{[0,x)}(\tfrac{2n-1}{2N}) - x \right) dx \right)^2 \right]^{1/2}$$

$$= \left[\frac{2}{3} \cdot \frac{13}{4N^2} - \frac{1}{N^2} \cdot 2 \right]^{1/2} = \sqrt{\frac{1}{6} \cdot \frac{1}{N^2}} = \frac{\sqrt{6}}{6N} \quad \#$$

c. Star - Discrepancy

$$\forall\, 0 \le x_1 \le \cdots \le x_N \le 1 \Rightarrow D_N^*(P) = \frac{1}{2N} + \max_{1 \le n \le N} \left| \frac{2n-3}{2N} - \frac{2n-1}{2N} \right| = \frac{1}{2N} + \left| \frac{-2}{2N} \right| = \frac{3}{2N} \quad \#$$

3. Construct "By hand" a (0, 2, 2) net in base 3

$\underset{t}{0}, \underset{m}{2}, \underset{d}{2}$

∵ Faure points. ∴ 用下述方法來建立:

↳ 意思是只有一个 point 在 $\frac{1}{3^2}$ 中且維度是 $[0,1)^2$.

$C^{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

当 $K = 0, 1, \cdots, 8$ 則 $a^1(k)$: $\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$

故 $a^2(k) = C^{(1)} a^1(k) \pmod{\underset{3}{b}}$: $\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$

↳ 計算过程是 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \Rightarrow \bmod 3 = \begin{pmatrix}0\\2\end{pmatrix}$

接著左乘 $(\frac{1}{3}, \frac{1}{9})$

得到: $\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1/3\\1/3\end{pmatrix}\begin{pmatrix}2/3\\2/3\end{pmatrix}\begin{pmatrix}1/9\\4/9\end{pmatrix}\begin{pmatrix}4/9\\7/9\end{pmatrix}\begin{pmatrix}7/9\\1/9\end{pmatrix}\begin{pmatrix}2/9\\8/9\end{pmatrix}\begin{pmatrix}5/9\\2/9\end{pmatrix}\begin{pmatrix}8/9\\5/9\end{pmatrix}$ ※

↳ 計算过程是 $\left(\frac{1}{3}\ \frac{1}{9}\right)\overset{a^1(4)}{\begin{pmatrix}1\\1\end{pmatrix}} = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

$\left(\frac{1}{3}\ \frac{1}{9}\right)\begin{pmatrix}2\\1\end{pmatrix} = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

$\Rightarrow \begin{pmatrix}4/9\\7/9\end{pmatrix}$ ↳ $a^2(4)$

4, ~ 8, Coding 附在後面

CH7.

1. 用 PW 和 LR 來估計於 Delta, Vega, Gamma

(1) EU Put Option

首先來看定义: $V(\theta) = E[f(S(T))]$

在 BS Model 下 $Y = e^{-rT}[K - S(T)]^+ = f(S(T))$

$S(T) = S(0)\exp[(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T} Z]$, $Z \sim N(0,1)$

$\Rightarrow V(\theta) = \int f(S(T)) P_Z(z)\, dz = E[f(S(T))] = E[Y]$

a. PW

Goal: $\frac{d}{d\theta} E[Y(\theta)] = E\left[\frac{d}{d\theta} Y(\theta)\right] \approx \frac{1}{N}\sum_{i=1}^{N} \frac{d}{d\theta} Y^{(i)}(\theta)$

$\Rightarrow \frac{d}{d\theta} V = E\left[\frac{d}{d\theta} f(S(T))\right] = E\left[\frac{\partial f(S(T))}{\partial S(T)} \frac{\partial S(T)}{\partial \theta}\right]$

$\approx \frac{1}{N}\sum_{i=1}^{N} \frac{d}{d\theta} Y^{(i)}(\theta) = \frac{1}{N}\sum_{i=1}^{N} \frac{\partial f(S^i(T))}{\partial S^i(T)} \frac{\partial S^i(T)}{\partial \theta}$

a.1 Delta

By def, $\theta = S(0)$

$\Rightarrow \frac{d}{d\theta} E[Y] = E\left[\frac{d}{d\theta} Y\right] = E\left[\frac{d}{dS(0)} Y\right]$

$$\frac{dY}{dS(0)} = \frac{dY}{dS(T)} \cdot \frac{dS(T)}{dS(0)}$$

$$\frac{dY}{dS(T)} = \frac{d}{dS(T)} e^{-rT} [K-S(T)]^{+} = e^{-rT} \frac{d}{dS(T)} \text{MAX}(0, K-S(T))$$

$$= e^{-rT} \cdot -I\{K>S(T)\} \quad \forall\ I = \begin{cases} 0, & K<S(T) \\ 1, & K>S(T) \\ \text{不存在}, & K=S(T) \end{cases}$$

$$\frac{dS(T)}{dS(0)} = \frac{S(T)}{S(0)}$$

$$\Rightarrow \frac{dY}{dS(0)} = -e^{-rT} I\{K>S(T)\} \frac{S(T)}{S(0)}$$

$$\Rightarrow E\left[\frac{dY}{dS(0)}\right] = E\left[-e^{-rT} I\{K>S(T)\}\right] \frac{S(T)}{S(0)} \approx \frac{1}{N} \sum_{i=1}^{N} I\{K>S^{(i)}(T)\} \frac{S^{(i)}(T)}{S(0)} \ _{*}$$

a.2 Vega

By def, $\theta = \sigma$

$$\Rightarrow \frac{d}{d\theta} E[Y] = E\left[\frac{d}{d\sigma} Y\right]$$

$$\frac{d}{d\sigma} Y = \frac{dY}{dS(T)} \frac{dS(T)}{d\sigma}$$

$$\frac{dS(T)}{d\sigma} = S(T)\left[\sqrt{T} Z - \sigma T\right]$$

$$\Rightarrow E\left[\frac{d}{d\sigma} Y\right] = E\left[-e^{-rT} I\{K>S(T)\} S(T) (\sqrt{T} Z - \sigma T)\right]$$

$$\because S(T) = S(0) \exp\left[(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T} Z\right]$$

$$\Rightarrow \ln \frac{S(T)}{S(0)} = (r-\frac{\sigma^2}{2})T + \sigma\sqrt{T} Z$$

$$\Rightarrow \sqrt{T} Z = \frac{\ln \frac{S(T)}{S(0)} - (r-\frac{\sigma^2}{2})T}{\sigma}$$

$$\Rightarrow \sqrt{T} Z - \sigma T = \frac{\ln \frac{S(T)}{S(0)} - rT + \frac{\sigma^2}{2} T - \sigma^2 T}{\sigma} = \frac{\ln \frac{S(T)}{S(0)} - (r+\frac{\sigma^2}{2})T}{\sigma}$$

$$\Rightarrow E\left[\frac{d}{d\sigma} Y\right] = E\left[-e^{-rT} I\{K>S(T)\} S(T) \frac{\ln \frac{S(T)}{S(0)} - (r+\frac{\sigma^2}{2})T}{\sigma}\right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} -e^{-rT} I\{K>S^{(i)}(T)\} S^{(i)}(T) \left[\frac{\ln \frac{S^{(i)}(T)}{S(0)} - (r+\frac{\sigma^2}{2})T}{\sigma}\right]_{*}$$

a.3 Gamma.

By def, $\theta = S(0)$

$$\Rightarrow \frac{d^2}{dS(0)^2} E[Y] = E\left[\frac{d^2}{dS(0)^2} Y\right] = E\left[\frac{d^2 Y}{dS(T)^2} \cdot \left(\frac{dS(T)}{dS(0)}\right)^2 + \frac{dY}{dS(T)} \cdot \frac{d^2 S(T)}{dS_0^2}\right]$$

$$\frac{d^2 Y}{dS(T)^2} = \frac{d}{dS(T)} -e^{-rT} I\{K>S(T)\} = 0$$

$$\frac{dY}{dS(T)} = -e^{-rT} I\{K > S(T)\}$$

$$\frac{d^2 S(T)}{dS_0^2} = \frac{d}{dS(0)} \frac{S(T)}{S(0)} = \frac{d}{dS(0)} S(T)\, S(0)^{-1} = S(T)\,(-1)\,S(0)^{-2} = \frac{-S(T)}{S(0)^2}$$

$$\Rightarrow E\left[\frac{d^2}{dS(0)^2} Y\right] = E\left[e^{-rT} I\{K > S(T)\} \frac{S(T)}{S(0)^2}\right]$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT} I\{K > S^{(i)}(T)\} \cdot \frac{S^{(i)}(T)}{S(0)^2} \quad *$$

b. LR.

Goal: $\frac{d}{d\theta} E[Y] = \frac{d}{d\theta} \int_{R^N} f(x)\, g_\theta(x)\, dx$

$$= \int_{R^N} \frac{d}{d\theta} f(x)\, g_\theta(x)\, dx$$

$$= \int_{R^N} f(x) \frac{d g_\theta(x)/d\theta}{g_\theta(x)} g_\theta(x)\, dx$$

$$= E\left[f(x) \cdot \frac{d g_\theta(x)/d\theta}{g_\theta(x)}\right], \ \forall\ f(x) = Y = f(x_1, \dots, x_N)$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} f(x^i) \frac{d g_\theta(x^i)/d\theta}{g_\theta(x^i)}$$

b.1 Delta

By def, $\theta = S(0)$

$$\Rightarrow \frac{d}{dS(0)} E[Y] = E\left[f(x) \frac{d g_{S(0)}(x)/dS(0)}{g_{S(0)}(x)}\right]$$

$\forall\ x = S(T)$,

$$g_{S(0)}(x) = \frac{1}{x\sigma\sqrt{T}} \phi(\xi(x)),$$

$\phi(\cdot)$ is pdf of $N(0,1)$,

$$\xi(x) = \frac{\ln\left(\frac{x}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\Rightarrow \frac{d g_{S(0)}(x)/dS(0)}{g_{S(0)}(x)} = \frac{\ln\frac{x}{S(0)} - \left(r - \frac{\sigma^2}{2}\right)T}{S(0)\,\sigma^2 T}$$

$$\because\ z = \frac{\ln\frac{x}{S(0)} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\Rightarrow \frac{d g_{S(0)}(x)/dS(0)}{g_{S(0)}(x)} = \frac{z}{S(0)\,\sigma\sqrt{T}}$$

$$\Rightarrow \frac{d}{dS(0)} E[Y] = E\left[e^{-rT}(K - S(T))^+ \frac{z}{S(0)\sigma\sqrt{T}}\right] \approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}(K - S^{(i)}(T))^+ \cdot \frac{z^i}{S(0)\sigma\sqrt{T}} \quad *$$

b.2 Vega

By def, $\theta = \sigma$

$$\frac{d}{d\sigma} E[Y] = E\left[e^{-rT} (K-x)^+ \frac{d g_\sigma(x)/d\sigma}{g_\sigma(x)} \right]$$

$$\frac{d g_\sigma(x)/d\sigma}{g_\sigma(x)} = \frac{g(x)}{\sigma} \frac{d g(x)}{d\sigma} = \frac{1}{\sigma} \frac{\ln\left(\frac{x}{S(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \cdot \frac{\ln\left(\frac{S(0)}{x}\right) + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma^2 T}$$

$$\because S(T) = x = S(0) \exp\left[\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T} Z \right]$$

$$\Rightarrow \frac{S(0)}{x} = \exp\left[-\left(r-\frac{\sigma^2}{2}\right)T - \sigma\sqrt{T} Z \right]$$

$$\Rightarrow \ln\frac{S(0)}{x} = -\left(r-\frac{\sigma^2}{2}\right)T - \sigma\sqrt{T} Z$$

$$\Rightarrow \ln\frac{S(0)}{x} + \left(r+\frac{\sigma^2}{2}\right)T = -\cancel{rT} + \frac{\sigma^2}{2}T - \sigma\sqrt{T}Z + \cancel{rT} + \frac{\sigma^2}{2}T = \sigma^2 T - \sigma\sqrt{T} Z$$

$$\Rightarrow \frac{\ln\frac{S(0)}{x} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma^2 T} = 1 - \frac{Z}{\sigma\sqrt{T}}$$

$$\Rightarrow \frac{d g_\sigma(x)/d\sigma}{g_\sigma(x)} = \frac{Z}{\sigma} \cdot \left(1 - \frac{Z}{\sigma\sqrt{T}}\right) = \frac{Z^2-1}{\sigma} - Z\sqrt{T}$$

$$\Rightarrow \frac{d}{d\sigma} E[Y] = E\left[e^{-rT}(K-S(T))^+ \left[\frac{Z^2-1}{\sigma} - Z\sqrt{T}\right] \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (K - S^{(i)}(T))^+ \left(\frac{Z_i^2-1}{\sigma} - Z_i\sqrt{T}\right) *$$

b.3 Gamma.

By def, $\theta = S(0)$

$$\frac{d^2}{dS(0)^2} E[Y] = \int f(x) \frac{d^2 g_\theta(x)/d\theta^2}{g_\theta(x)} g_\theta(x) dx = E\left[f(x) \frac{d^2 g_\theta(x)/d\theta^2}{g_\theta(x)} \right]$$

$$\Rightarrow \frac{d^2}{dS(0)^2} E[Y] = E\left[e^{-rT} (K-x)^+ \frac{d^2 g_{S(0)}(x)/dS(0)^2}{g_{S(0)}(x)} \right]$$

$$\frac{d^2 g_{S(0)}(x)/dS(0)^2}{g_{S(0)}(x)} = \frac{g^2(x) - g(x)\sigma\sqrt{T} - 1}{S^2(0)\sigma^2 T} = \frac{Z^2 - Z\sigma\sqrt{T} - 1}{S^2(0)\sigma^2 T}$$

$$\Rightarrow \frac{d^2}{dS(0)^2} E[Y] = E\left[e^{-rT}(K-S(T))^+ \frac{Z^2 - Z\sigma\sqrt{T} - 1}{S^2(0)\sigma^2 T} \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (K - S^{(i)}(T))^+ \left[\frac{Z_i^2 - Z_i\sigma\sqrt{T} - 1}{S^2(0)\sigma^2 T} \right] *$$

2. ASIAN Put Option

a. PW

首先来看 Asian Put Option 定义：$Y = e^{-rT}[K - S(T)]^+$

a.1 Delta

$$\bar{S} = \frac{1}{m} \sum_{i=1}^{m} S(t_i)$$

$$S(t_i) = S(t_{i-1}) \exp\left[\left(r-\frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}\, Z_i \right]$$

$Z_i \sim N(0,1)$,

可以根据上題修改（未提到的定义与上題一样），

重複的計算不再寫下来.

By def, $\theta = S(0)$

$$E\left[\frac{dY}{dS(0)}\right] = E\left[\frac{dY}{d\bar{S}}\,\frac{d\bar{S}}{d(S(0))}\right]$$

$$\frac{d\bar{S}}{dS(0)} = \frac{d}{dS(0)}\,\frac{1}{m}\sum_{i=1}^{m} S(t_i) = \frac{1}{m}\sum_{i=1}^{m}\frac{S(t_i)}{S(0)} = \frac{\bar{S}}{S(0)}$$

$$\Rightarrow E\left[\frac{dY}{dS(0)}\right] = E\left[-e^{-rT} I\{K>\bar{S}\}\,\frac{\bar{S}}{S(0)}\right] \approx \frac{1}{N}\sum_{i=1}^{N} -e^{-rT} I\{K>\bar{S}^i\}\,\frac{\bar{S}^i}{S(0)}$$

a.2 Vega

By def, $\theta = \sigma$

$$E\left[\frac{dY}{d\sigma}\right] = E\left[\frac{dY}{d\bar{S}}\,\frac{d\bar{S}}{d\sigma}\right]$$

$$= E\left[-e^{-rT} I\{K>\bar{S}\}\,\frac{1}{\sigma}\,\frac{1}{m}\sum_{i=1}^{m} S(t_i)\left(\ln\frac{S(t_i)}{S(0)} - \left(r+\frac{\sigma^2}{2}\right)t_i\right)\right]$$

$$\approx \frac{1}{N}\sum_{j=1}^{N} -e^{-rT} I\{K>\bar{S}^j\}\,\frac{1}{m\sigma}\sum_{i=1}^{m} S^j(t_i)\left[\ln\frac{S^j(t_i)}{S(0)} - \left(r+\frac{\sigma^2}{2}\right)t_i\right]$$

a.3 Gamma

By def, $\theta = S(0)$

$$\Rightarrow E\left[\frac{d^2}{dS(0)^2}Y\right] = E\left[\frac{d^2Y}{d\bar{S}^2}\cdot\left(\frac{d\bar{S}}{dS(0)}\right)^2 + \frac{dY}{d\bar{S}}\cdot\frac{d^2\bar{S}}{dS(0)^2}\right]$$

$$= E\left[e^{-rT} I\{K>\bar{S}\}\cdot\frac{\bar{S}}{S(0)^2}\right]$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT} I\{K>\bar{S}^i\}\cdot\frac{\bar{S}^i}{S(0)^2}$$

b. LR

b.1 Delta

By def, $\theta = S(0)$

$$\Rightarrow \frac{d}{dS(0)}E[Y] = E\left[f(x)\,\frac{dg_{S(0)}(\tilde{x})/dS(0)}{g_{S_0}(\tilde{x})}\right]$$

$$\frac{dg_{S(0)}(x_1,\cdots,x_m)/dS(0)}{g_{S_0}(x_1,\cdots,x_m)} = \frac{d\,g_1(x_1|S(0))\,g_2(x_2|x_1)\cdots g_m(x_m|x_{m-1})/dS(0)}{g_{S_0}(x_1,\cdots,x_m)}$$

$$= \frac{dg_1(x_1|S(0))/dS(0)}{g_1(x_1|S(0))} = \frac{\ln\frac{S(t_1)}{S(0)} - \left(r-\frac{\sigma^2}{2}\right)t_1}{S_0\sigma^2 t_1} = \frac{Z_1}{S_0\sigma\sqrt{t_1}}$$

$$\Rightarrow \frac{d}{dS(0)}E[Y] = E\left[e^{-rT}(K-\bar{S})^+\,\frac{Z_1}{S_0\sigma\sqrt{t_1}}\right] \approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}(K-\bar{S}^i)^+\,\frac{Z_1^i}{S(0)\sigma\sqrt{t_1}}$$

b.2 Vega.

By def, $\theta = \sigma$

$$\frac{d}{d\sigma} E[Y] = E\left[e^{-rT}(K-x)^{+} \frac{dg_\sigma(x_1,\cdots,x_m)/d\sigma}{g_\sigma(x_1,\cdots,x_m)}\right]$$

$$\frac{dg_\sigma(x_1,\cdots,x_m)/d\sigma}{g_\sigma(x_1,\cdots,x_m)} = \frac{d\ln g_\sigma(x_1,\cdots,x_m)}{d\sigma} = \sum_{i=1}^{m} \frac{d\ln g_i}{d\sigma}$$

$$= \sum_{i=1}^{m} \frac{dg_i(x_i|x_{i-1})/d\sigma}{g_i(x_i|x_{i-1})} = \sum_{i=1}^{m}\left(\frac{z_i^2-1}{\sigma} - z_i\sqrt{t_i-t_{i-1}}\right)$$

$$\Rightarrow \frac{d}{d\sigma} E[Y] = E\left[e^{-rT}(K-\bar{S})^{+} \sum_{i=1}^{m}\left(\frac{z_i^2-1}{\sigma} - z_i\sqrt{t_i-t_{i-1}}\right)\right]$$

$$\approx \frac{1}{N}\sum_{j=1}^{m} e^{-rT}(K-\bar{S}^{j})^{+}\left[\sum_{i=1}^{m}\left(\frac{z_i^2-1}{\sigma} - z_i\sqrt{t_i-t_{i-1}}\right)\right]$$

b.3 Gamma.

By def, $\theta = S(0)$

$$\frac{d^2}{dS(0)^2} E[Y] = E\left[e^{-rT}(K-x)^{+} \frac{d^2 g_{S(0)}(\tilde{x})/dS(0)^2}{g_{S(0)}(\tilde{x})}\right]$$

$$\frac{d^2 g_{S(0)}(x_1,\cdots,x_m)/dS(0)^2}{g_{S(0)}(x_1,\cdots,x_m)} = \frac{dg_1^2(x_1|S(0))/dS^2(0)}{g_1(x_1|S(0))} = \frac{z_1^2 - z_1\sigma\sqrt{t_1} - 1}{S^2(0)\sigma^2 t_1}$$

$$\Rightarrow \frac{d^2}{dS(0)^2} E[Y] = E\left[e^{-rT}(K-x)^{+} \frac{z_1^2 - z_1\sigma\sqrt{t_1} - 1}{S^2(0)\sigma^2 t_1}\right]$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}(K-\bar{S}^{i})^{+} \frac{z_1^{i\,2} - z_1^{i}\sigma\sqrt{t_1} - 1}{S^2(0)\sigma^2 t_1}$$