5

a. Random walk

For Standard Brownian Motion,
$B_{t+s} - B_s \sim N(0, (t+s) - s),$
$\forall B_0 = 0,\ 0 < t_1 < \ldots < t_n, B_{t_1} - B_0, \ldots, Bt_n - Bt_{n-1} are\ independent,$
$B_t\ is\ a\ continuous\ function\ of\ t\ on\ [0, T]$

$\Rightarrow d(B_{t_j}) = B_{t_j} - B_{t_{j-1}} = \sqrt{\Delta t} N(0, 1)$

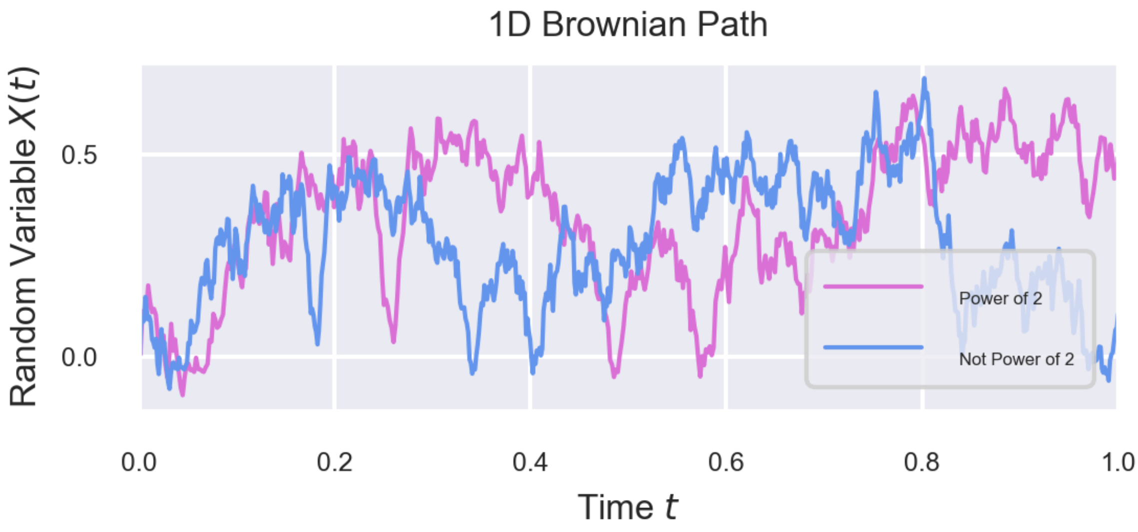
In [1]:
```
#We start the script by initializing parameters
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import warnings
warnings.filterwarnings('ignore')
import matplotlib as mpl
mpl.rcParams['figure.dpi'] = 300

T = 1.0
N = 501
dt = T/(N-1)

#create rv
B = [0] * N
dB = [0] * N
dB[0] = np.sqrt(dt) * np.random.randn()
B[0] = dB[0]
```

Because $d(B_{t_j}) = B_{t_j} - B_{t_{j-1}} = \sqrt{\Delta t} N(0, 1)$
$\Rightarrow B_{t_j} = d(B_{t_j}) + B_{t_{j-1}}$

In [2]:
```
for j in range(1, N):
    dB[j] = np.sqrt(dt) * np.random.randn()
    B[j] = B[j-1] + dB[j]
```

This time create power of 2 and not power of 2 respectively

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
#power of 2
np.random.seed(5)

T = 1
N = 2**9
dt = T/(N-1)
t = np.linspace(0,T,N)
dX = [0]*N
X = [0]*N
dX[0] = np.sqrt(dt)*np.random.randn() # Eq. (3)
X[0] = dX[0]
for i in range(1,N):
    dX[i] = np.sqrt(dt)*np.random.randn() # Eq. (3)
    X[i] = X[i-1] + dX[i] # Eq. (4)


#power of 3
np.random.seed(5)
T = 1
N = 3**6
dtt = T/(N-1)
tt = np.linspace(0,T,N)
dY = [0]*N
Y = [0]*N
dY[0] = np.sqrt(dtt)*np.random.randn() # Eq. (3)
Y[0] = dY[0]
for i in range(1,N):
    dY[i] = np.sqrt(dtt)*np.random.randn() # Eq. (3)
    Y[i] = Y[i-1] + dY[i] # Eq. (4)

#plot
sns.set_theme()
fig, ax = plt.subplots(figsize=(4,2))
sns.lineplot(t, X,color='orchid',linewidth=1,label='Power of 2')
sns.lineplot(tt, Y, color='cornflowerblue',linewidth=1,label='Not P
ower of 2')
plt.xlabel('Time $t$', fontsize=8)
plt.ylabel('Random Variable $X(t)$', fontsize=8)
plt.title('1D Brownian Path', fontsize=8)
axes = plt.gca()
axes.set_xlim([0,T])
plt.xticks(fontsize=6)
plt.yticks(fontsize=6)
plt.setp(ax.get_legend().get_texts(), fontsize='4') # for legend te
xt
plt.tight_layout()
plt.show()
```



b. Brownian Bridge

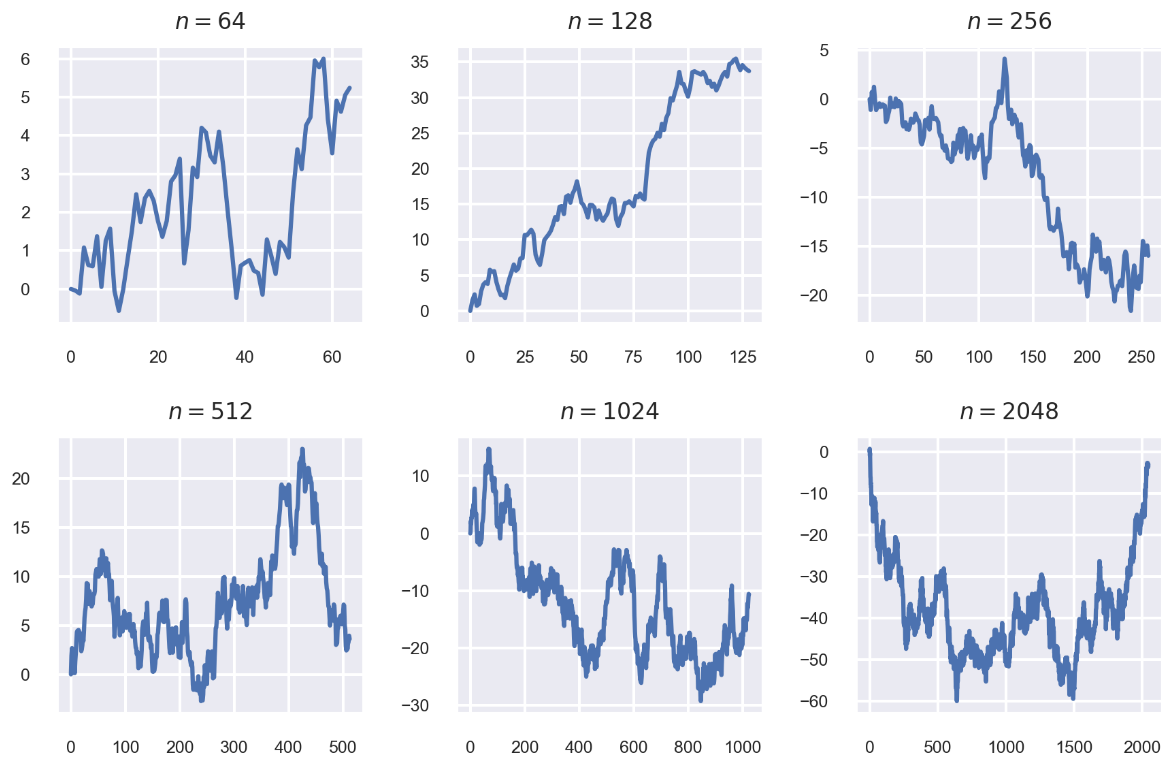
根据题4的结果来生成，令$n = 2^m, m \in \mathbb{Z}$，$n$表示数量，$b_0$ 表示出发点:

In [4]:
```
def Brown_brige(n, b0):
    B = np.zeros(n+1)
    #初始化
    B[0] = b0
    z = np.random.randn()
    B[n] = np.sqrt(n) * z
    M = int(np.log2(n))
    h=n
    #生成矩阵
    A = np.zeros((n+1, n+1))
    A[n, 1] = np.sqrt(n)
    cnt = 1

    for k in range(1, M+1):
            h = h // 2
            for j in range(1, 2**(k-1)+1):
                z = np.random.randn()
                B[(2*j-1)*h] = (B[2*(j-1)*h] + B[2*j*h]) / 2 + np.s
qrt(h/2) * z
                #更新生成矩阵
                cnt += 1
                A[(2*j-1)*h, :] = (A[2*(j-1)*h, :] + A[2*j*h, :]) /
2
                A[(2*j-1)*h, cnt] += np.sqrt(h/2)
    return B, A
```

```
In [5]: N = range(6, 12)
        l= []
        for n in N:
            m = 2**n
            l.append(m)
```

可以注意到X轴和Y轴的数字变化：

```
In [6]: fig, ax = plt.subplots(2, 3)
        ax[0,0].plot(Brown_brige(l[0], 0)[0])
        ax[0,0].set_title("$n={}$".format(l[0]), size=8)
        ax[0,0].tick_params(axis='both',labelsize = 6)
        ax[0,1].plot(Brown_brige(l[1], 0)[0])
        ax[0,1].set_title("$n={}$".format(l[1]), size=8)
        ax[0,1].tick_params(axis='both',labelsize = 6)
        ax[0,2].plot(Brown_brige(l[2], 0)[0])
        ax[0,2].set_title("$n={}$".format(l[2]), size=8)
        ax[0,2].tick_params(axis='both',labelsize = 6)
        ax[1,0].plot(Brown_brige(l[3], 0)[0])
        ax[1,0].set_title("$n={}$".format(l[3]), size=8)
        ax[1,0].tick_params(axis='both',labelsize = 6)
        ax[1,1].plot(Brown_brige(l[4], 0)[0])
        ax[1,1].set_title("$n={}$".format(l[4]), size=8)
        ax[1,1].tick_params(axis='both',labelsize = 6)
        ax[1,2].plot(Brown_brige(l[5], 0)[0])
        ax[1,2].set_title("$n={}$".format(l[5]), size=8)
        ax[1,2].tick_params(axis='both',labelsize = 6)
        fig.tight_layout()
        plt.show()
```



若$n \neq 2^m, m \in \mathbb{Z}$，$\Rightarrow n = \sum_{i=1}^{k} 2^{m_i}$ ，$m_i \geq m_j$, $i < j$:

得到如下算法:

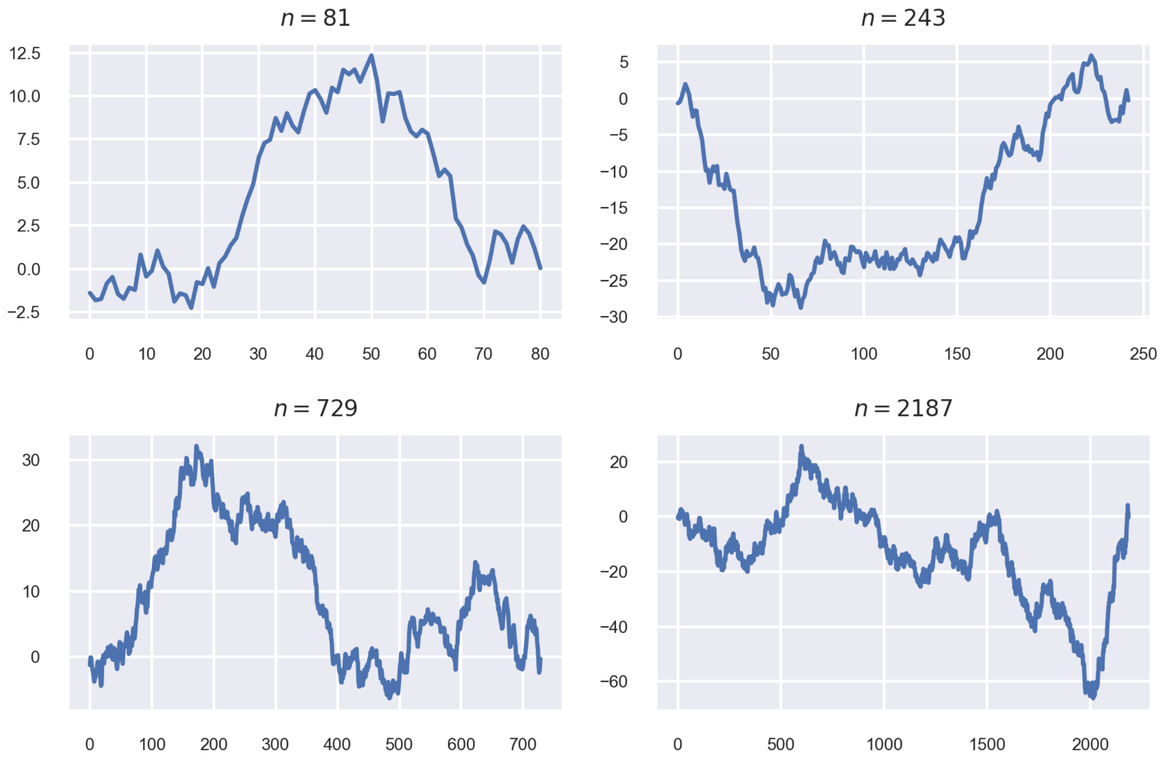
- B=[], b=0
- 对i=1,...,k:
  - 计算$B_i = Brown_bridge(2^{m_i}, b)$
  - 将B_i第二个元素开始的全部元素添加到B最后
  - 令$b = B_i$最后一个元素

In [7]:
```
def to_2(n):
    res = []
    i=0
    while n != 0:
        j=n%2
        if j:
            res.append(i)
        i += 1
        n = n // 2
    #反序输出
    return res[: : -1]
```

可以注意到X轴和Y轴的数字变化：

In [8]:

```python
N = [3**4, 3**5, 3**6, 3**7]

fig, ax = plt.subplots(2, 2)
k = to_2(N[0])
B = np.array([])
b0 = 0
for i in k:
    m = 2 ** i
    Bi = Brown_brige(m, b0)[0]
    B = np.append(B, Bi[1:])
    b0 = Bi[-1]
ax[0,0].plot(B)
ax[0,0].set_title("$n={}$".format(N[0]), size=8)
ax[0,0].tick_params(axis='both',labelsize = 6)

k = to_2(N[1])
B = np.array([])
b0 = 0
for i in k:
    m = 2 ** i
    Bi = Brown_brige(m, b0)[0]
    B = np.append(B, Bi[1:])
    b0 = Bi[-1]
ax[0,1].plot(B)
ax[0,1].set_title("$n={}$".format(N[1]), size=8)
ax[0,1].tick_params(axis='both',labelsize = 6)

k = to_2(N[2])
B = np.array([])
b0 = 0
for i in k:
    m = 2 ** i
    Bi = Brown_brige(m, b0)[0]
    B = np.append(B, Bi[1:])
    b0 = Bi[-1]
ax[1,0].plot(B)
ax[1,0].set_title("$n={}$".format(N[2]), size=8)
ax[1,0].tick_params(axis='both',labelsize = 6)

k = to_2(N[3])
B = np.array([])
b0 = 0
for i in k:
    m = 2 ** i
    Bi = Brown_brige(m, b0)[0]
    B = np.append(B, Bi[1:])
    b0 = Bi[-1]
ax[1,1].plot(B)
ax[1,1].set_title("$n={}$".format(N[3]), size=8)
ax[1,1].tick_params(axis='both',labelsize = 6)

fig.tight_layout()
plt.show()
```



7

已知$t_i = i\Delta t$

- random walk

$$\frac{\sum_{i=1}^{k} ||a_i^2||}{\sum_{i=1}^{n} ||a_i^2||} = \frac{\sum_{i=1}^{k} (n-i+1)(t_i - t_{i-1})}{\sum_{i=1}^{n} (n-i+1)(t_i - t_{i-1})} = \frac{(2n-k+1)k}{n(n+1)}$$

- PCA

$$\lambda_i = \frac{\Delta t}{4} sin^{-2}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$$

$$\frac{\sum_{i=1}^{k} ||a_i^2||}{\sum_{i=1}^{n} ||a_i^2||} = \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{n} \lambda_i} = \frac{\sum_{i=1}^{k} sin^{-2}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sum_{i=1}^{n} sin^{-2}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}$$

In [9]:
```python
def RW(n):
    r1 = np.arange(n, 0, -1)
    r2 = np.cumsum(r1) / np.sum(r1)
    return r2

def PCA(n):
    angle = ((2 * np.arange(1, n+1)) - 1) * np.pi / 2 / (2 * n + 1)
    r1 = 1 / (np.sin(angle) ** 2)
    r2 = np.cumsum(r1) / np.sum(r1)
    return r2

def BB_var(A):
    r1 = np.sum(A ** 2, axis=0)[1:]
    r2 = np.cumsum(r1) / np.sum(r1)
    return r2
```

In [10]:
```python
def show(n):
    r1 = RW(n)
    A = Brown_brige(256, 0)[1]
    r2 = BB_var(A)
    r3 = PCA(n)
    print("n = {}时".format(n))
    print("RW:", r1[:5])
    print("Brown_brige:", r2[:5])
    print("PCA:", r3[:5])
```

In [11]:
```python
show(16)
```

```
n = 16时
RW: [0.11764706 0.22794118 0.33088235 0.42647059 0.51470588]
Brown_brige: [0.66796875 0.83399197 0.87550158 0.91701119 0.927392
39]
PCA: [0.8119275  0.90268881 0.93576113 0.95294517 0.9635973 ]
```

In [12]:
```python
show(64)
```

```
n = 64时
RW: [0.03076923 0.06105769 0.09086538 0.12019231 0.14903846]
Brown_brige: [0.66796875 0.83399197 0.87550158 0.91701119 0.927392
39]
PCA: [0.81065825 0.90076701 0.93323183 0.94981518 0.95986297]
```

In [13]:
```python
show(256)
```

```
n = 256时
RW: [0.0077821  0.0155338  0.02325511 0.03094601 0.03860652]
Brown_brige: [0.66796875 0.83399197 0.87550158 0.91701119 0.927392
39]
PCA: [0.81057508 0.90064123 0.93306667 0.9496115  0.9596211 ]
```