202027026 应統碩 王翌文 HW3

1. 先令 $d=1$, $[f_1(x)-f_1(y)][f_2(x)-f_2(y)]\ge 0$, $\forall x,y$

$\Rightarrow E\{[f_1(x)-f_1(y)][f_2(x)-f_2(y)]\}\ge 0$

$\Rightarrow E[f_1(X)f_2(X)-f_1(X)f_2(Y)-f_1(Y)f_2(X)+f_1(Y)f_2(Y)]\ge 0$

$\Rightarrow E[f_1(X)f_2(X)]+E[f_1(Y)f_2(Y)]\ge E[f_1(X)f_2(Y)]+E[f_1(Y)f_2(X)]$

再令 X, Y i.i.d

$\Rightarrow 2E[f_1(X)f_2(X)]\ge 2E[f_1(X)]E[f_2(X)]$

$\Rightarrow E[f_1(X)f_2(X)]\ge E[f_1(X)]E[f_2(X)]$ ($\Leftarrow Cov(f_1(X),f_2(X))\ge 0$)

假設 $d=k$ 時成立，即 $\mathbb{X}=(X_1,\dots,X_k)$, $E[f_1(\mathbb{X})f_2(\mathbb{X})]\ge E[f_1(\mathbb{X})]E[f_2(\mathbb{X})]$

則 $d=k+1$ 時，$\mathbb{X}'=(X_1,\dots,X_{k+1})$，令有一 $\mathbb{Y}' \overset{iid}{\sim} \mathbb{X}$

$\Rightarrow [f_1(\mathbb{X}')-f_1(\mathbb{Y}')][f_2(\mathbb{X}')-f_2(\mathbb{Y}')]\ge 0$ ($\because$ 已知 $[f_1(\mathbb{X})-f_1(\mathbb{Y})][f_2(\mathbb{X})-f_2(\mathbb{Y})]\ge 0$, $\forall \mathbb{Y}=(Y_1,\dots,Y_k)\overset{iid}{\sim}\mathbb{X}$，且 $[f_1(X_{k+1})-f_1(Y_{k+1})][f_2(X_{k+1})-f_2(Y_{k+1})]\ge 0$)

$\Rightarrow E[f_1(\mathbb{X}')f_2(\mathbb{X}')]\ge E[f_1(\mathbb{X}')]E[f_2(\mathbb{X}')]$

根据歸納法得証#

2. 欲証 $Var(f-bg)<Var(f)$ iff $b\in[0,b^*]$

先求 b^*：欲 $\min Var(f-bg)\Rightarrow Var(f-bg)=Var(f)-2b\,Cov(f,g)+b^2Var(g)$

$$\Rightarrow \frac{d}{db}Var(f-bg)=2b\,Var(g)-2Cov(f,g)=0$$

$$\Rightarrow b^*=\frac{Cov(f,g)}{Var(g)}=\frac{Cov(f,g)}{\sqrt{Var(g)}\sqrt{Var(f)}}\frac{\sqrt{Var(f)}}{\sqrt{Var(g)}}=\rho\frac{\sqrt{Var(f)}}{\sqrt{Var(g)}}$$

$\because \rho\in[-1,1]$ $\therefore b^*$ 可能為正可能為負.

① $b^*\ge 0$

欲求 $Var(f-bg)=Var(f)-2b\,Cov(f,g)+b^2Var(g)<Var(f)$

$\Rightarrow b^2Var(g)-2b\,Cov(f,g)<0$

$\Rightarrow b\,Var(g)<2Cov(f,g)$

$$\Rightarrow b<2\frac{Cov(f,g)}{Var(g)}$$

$\Rightarrow b<2b^*$

② $b^*<0$

$b^2Var(g)-2b\,Cov(f,g)<0$，且知 $b^*<0\Rightarrow \dfrac{Cov(f,g)}{Var(g)}<0\Rightarrow Cov(f,g)<0$

故為滿足 $b^2Var(g)-2b\,Cov(f,g)<0$，則 $b<0$

$$\Rightarrow b^2-2b\frac{Cov(f,g)}{Var(g)}<0\Rightarrow b^2-2bb^*<0\Rightarrow b-2b^*>0\Rightarrow b>2b^*$$

線上 $Var(f-bg) < Var(f)$ iff $b \in [0, b^*]$ 得証

3. (1) 先建構 AV estimate $g = \frac{1}{2}[f(x)+f(1-x)]$

欲求 $Var(g) = 0$；

$g = \frac{1}{2}[f(x)+f(1-x)]$

$= \frac{1}{2}[(a_1x_1 + a_2x_2 + \cdots + a_dx_d + b) + (a_1(1-x_1) + a_2(1-x_2) + \cdots + a_d(1-x_d) + b)]$

$= \frac{1}{2}[(a_1x_1 + a_2x_2 + \cdots + a_dx_d + b) + \sum_{i=1}^{d} a_i - (a_1x_1 + a_2x_2 + \cdots a_dx_d) + b]$

$= \frac{1}{2}[\sum_{i=1}^{d} a_i + 2b] = \frac{1}{2}\sum_{i=1}^{d} a_i + b$

$Var(g) = Var(\frac{1}{2}\sum_{i=1}^{d} a_i + b) = 0$

(2) 在 AV 下，已知 $Var(g) = \frac{1}{2}[Var(f(x)) + Cov(f(x), f(1-x))] \le Var(f(x))$

且 $Cov(f(x), f(1-x)) \le 0$

因此只要 var 下降程度 > 2倍 就得到正面效益 ($\because g = \frac{1}{2}[f(x)+f(1-x)]$

$\therefore$ 此時 Benefit > Cost)

$f(x) = \sum_{j=1}^{d}(1-2x_j)^2 \Rightarrow f(1-x) = \sum_{j=1}^{d}(1-2(1-x_j))^2 = \sum_{j=1}^{d}(1-2+2x_j)^2 = \sum_{j=1}^{d}(2x_j-1)^2 = f(x)$

$g = \frac{1}{2}[f(x)+f(1-x)] = \frac{1}{2}[f(x)+f(x)] = f(x)$

$\Rightarrow Var(g) = Var(f(x)) \Rightarrow$ 只有耗到成本卻无效益

4. 已知 $f(x)$ 二階連續可微，且要估 $E[f(x)]$

$\Rightarrow$ standard MC : $Q_N[f] = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$ $\Rightarrow$ $Var[Q_N(f)] = \frac{1}{N} Var(f)$

AV : $Q_N^{AV}[f] = \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)+f(1-x_i)}{2}$ $\quad Var[Q_N^{AV}(f)] = \frac{1}{N} Var(\frac{f(x)+f(1-x)}{2})$

欲求 $Var[Q_N(f)] = O(\frac{c^2}{N})$， $Var[Q_N^{AV}(f)] = O(\frac{c^4}{N})$

則表示欲求 $Var(f(x)) = O(c^2)$， $Var(\frac{f(x)+f(1-x)}{2}) = O(c^4)$；

$\forall x \sim N(0, c)$, $c \ll 1$

x 对 $x = \underset{(0)}{\mu}$ 泰勒展開， $f(x) = f(\mu) + f'(\mu)(x-\mu) + \frac{f''(a)}{2}(x-\mu)^2$, $\forall a \in (\mu, x)$

$\Rightarrow Var(f(x)) = f'^2(\mu) Var(x) + \frac{f''^2(a)}{2^2} Var(x^2)$

$= f'^2(0)c + \frac{f''^2(a)}{2^2}(3 \cdot c^2 - c^2) = O(c^2)$

$f(1-x)$ 对 $x = \underset{(0)}{\mu}$ 做泰勒展開， $f(1-x) = f(\mu) + f'(\mu)(1-x-\mu) + \frac{f''(a)}{2}(1-x-\mu)^2$, $\forall a \in (\mu, x)$

$= f(0) + f'(0)(1-x) + \frac{f''(a)}{2}(1-x)^2$

$Var(f(x)+f(1-x)) = Var(2f(0) + f'(0)x + f'(0) - f'(0)x + \frac{f''(a)}{2}x^2 + \frac{f''(a)}{2} + \frac{f''(a)}{2}x^2 - f''(a)x) = Var(2f(0) + f'(0) + \frac{f''(a)}{2} + f''(a)x^2 - f''(a)x)$

$= f''(a)^2 \, Var(X^2) + f''(a)^2 \, Var(X) - Cov(X^2, X) \cdot 2 f''(a)^2$

$= f''(a)^2 \cdot 3c^2 - Cov(X^2, X) \cdot 2f''(a)^2 = O(c^4)$

$\Rightarrow$ 故 $Var[Q_N(f)] = O(\frac{c^2}{N})$, $Var[Q_N^{AV}(f)] = O(\frac{c^4}{N})$ ∎

5. $Q_{cv} = f - b_1(g_1 - \mu_{g_1}) - b_2(g_2 - \mu_{g_2})$

令 $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $\mu_G = \begin{pmatrix} \mu_{g_1} \\ \mu_{g_2} \end{pmatrix}$ $\Rightarrow$ $Q_{cv} = f - b^T(G - \mu_G)$

$= (f - b^T G) + b^T \mu_G$

$Var(Q_{cv}) = Var(f - b^T G) = Var(f) + b^T \Sigma_{GG} b - 2 b^T \Sigma_{Gf}$, $\forall \Sigma_{GG} = \begin{bmatrix} \sigma_{g_1}^2 & \sigma_{g_1, g_2} \\ \sigma_{g_1, g_2} & \sigma_{g_2}^2 \end{bmatrix}$

$\Sigma_{Gf} = \begin{bmatrix} \sigma_{g_1, y} \\ \sigma_{g_2, y} \end{bmatrix}$

$\Rightarrow \frac{\partial}{\partial b} Var(Q_{cv}) = 2\Sigma_{GG} b - 2\Sigma_{Gf} = 0$

$\Rightarrow \Sigma_{GG} b - \Sigma_{Gf} = 0 \Rightarrow b^* = \Sigma_{GG}^{-1} \Sigma_{Gf}$

$$\Rightarrow \begin{bmatrix} b_1^* \\ b_2^* \end{bmatrix} = \frac{1}{\sigma_{g_1}^2 \sigma_{g_2}^2 - (\sigma_{g_1, g_2})^2} \begin{bmatrix} \sigma_{g_2}^2 & -\sigma_{g_1, g_2} \\ -\sigma_{g_1, g_2} & \sigma_{g_1}^2 \end{bmatrix} \begin{bmatrix} \sigma_{g_1 y} \\ \sigma_{g_2 y} \end{bmatrix}$$

$$= \frac{1}{\sigma_{g_1}^2 \sigma_{g_2}^2 - (\sigma_{g_1}, \sigma_{g_2})^2} \begin{bmatrix} \sigma_{g_2}^2 \sigma_{g_1 y} - \sigma_{g_1 g_2} \cdot \sigma_{g_2 y} \\ \sigma_{g_1}^2 \sigma_{g_2 y} - \sigma_{g_1 g_2} \sigma_{g_1 y} \end{bmatrix}$$ ∎

6.

(1) $f_G = \max(0, \prod_{i=1}^n S(t_i)^{w_i} - K)$ $\forall w_i = \frac{1}{n}$ By 証①

$S_G = \prod_{i=1}^n S(t_i)^{w_i} \Rightarrow \ln S_G \sim N(m, s^2)$ ($\because \ln S_G = \sum_{i=1}^n w_i \ln S(t_i)$).

証① $S(t_i) = S(0) \exp[(r - \frac{\sigma^2}{2}) t_i + \sigma B(t_i)]$, $\forall B(t_i)$ is standard BM at time t_i

$\Rightarrow S(t_i)^{w_i} = S(0)^{w_i} \exp[(r - \frac{\sigma^2}{2}) t_i w_i + \sigma w_i B t_i]$

$\Rightarrow \prod_{i=1}^n S(t_i)^{w_i} = S(0) \exp[(r - \frac{\sigma^2}{2}) \sum_{i=1}^n t_i w_i + \sigma \sum_{i=1}^n w_i B(t_i)]$ $= \Delta t \frac{n+1}{2}$

$= \exp(m + \sigma \underset{\sim}{W}^T \underset{\sim}{B})$, $\forall$ $m = \ln S_0 + (r - \frac{\sigma^2}{2}) \underbrace{\sum_{i=1}^n t_i w_i}$

$\underset{\sim}{B} = (B(t_1) \cdots B(t_n))^T \sim N(0, \underset{\sim}{C})$

$\underset{\sim}{C} = (c_{ij})_{n \times n}$, $c_{ij} = \min(t_i, t_j)$

$\underset{\sim}{W} = (w_1, \cdots, w_n)^T$

$\underset{\sim}{W}^T \underset{\sim}{B} \sim N(0, \underset{\sim}{W}^T \underset{\sim}{C} \underset{\sim}{W})$.

$\Rightarrow \ln S_G = m + \sigma \underset{\sim}{W}^T \underset{\sim}{B} \sim N(m, s^2)$, $\forall s = \sigma \sqrt{\underset{\sim}{W}^T \underset{\sim}{C} \underset{\sim}{W}} = \frac{\sigma \sqrt{\Delta t}}{n} \sqrt{\frac{n(n+1)(2n+1)}{6}}$ ∎

已知有 Lemma : $\ln(V) \sim N(m, s^2) \Rightarrow E(\max(V - K, 0)) = e^{m + s^2/2} N(d_1) - KN(d_2)$

$\forall d_1 = \frac{m - \ln K + s^2}{s}$, $d_2 = \frac{m - \ln K}{s} = d_1 - s$, $N(x)$ 是 $N(0,1)$ 的 cdf

$\Rightarrow$ call option price of Asian option $C_G = e^{-rT} [e^{m + s^2/2} N(d_1) - KN(d_2)]$ ∎

(2) $\frac{1}{n}\sum_{i=1}^{n} Y_i$ 為 Q_{CV}，$Y_i = A_{w_i} - b^*(G_{w_i} - P_G)$ ∀ $w_i = 1/n$

$$A_{w_i} = e^{-rT}\left(\sum_{i=1}^{n} w_i S(t_i)^{w_i} - K\right)^+$$

$$G_{w_i} = e^{-rT}\left(\prod_{i=1}^{n} (S(t_i))^{w_i} - K\right)^+$$

$$b^* = \frac{Cov(A, G)}{Var(G)}$$

$$P_G = e^{-rT}\left[e^{m+\frac{s^2}{2}}N(d_1) - KN(d_2)\right]$$

$$\Rightarrow Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}Var(Y_i) = \frac{1}{n}Var(A_{w_i} - b^*(G_{w_i} - P_G))$$

$$= \frac{1}{n}Var(A_{w_i})(1-\rho^2),\ \forall \rho = \rho(A_{w_i}, G_{w_i}),\ -1 \le \rho \le 1$$

而 crude MC: $Q = \frac{1}{n}\sum_{i=1}^{n} A_{w_i} \Rightarrow Var\left(\frac{1}{n}\sum_{i=1}^{n} A_{w_i}\right) = \frac{1}{n}Var(A_{w_i})$

故 variace reduction factor = $(1-\rho^2)$ #

7.(a) SS with proportional allocation.

$\bar{Y}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} f(x_{ij})$, $\bar{Y}_{SS} = \sum_{j=1}^{M} \bar{Y}_j p_j$, ∀ M = strata, p_j = width in each strata, n_j = samples from each strata.

$$Var(\bar{Y}_{SS}) = \sum_{j=1}^{M} p_j^2 Var(\bar{Y}_j) = \sum_{j=1}^{M}\frac{p_j^2}{n_j}\sigma_j^2 = \frac{1}{N}\sum_{j=1}^{M}\frac{p_j^2}{q_j}\sigma_j^2,\ \forall q_j = \frac{n_j}{N}$$

proportional allocation: $n_j = p_j N$ (a.k.a $p_j = q_j$)

$$\Rightarrow Var(\bar{Y}_{PS}) = \frac{1}{N}\sum_{j=1}^{M} p_j\sigma_j^2 = \frac{1}{N}E[Var(f(X)|X)] \le \frac{1}{N}Var(f(X))$$ = var of Crude MC #

∵ $Var(f(X)) = E(Var(f(X)|X)) + Var(E(f(X)|X))$

(b) SS with optimal allocation

$$\min var(\bar{Y}_{SS}) = \frac{1}{N}\sum_{j=1}^{M}\frac{p_j^2}{q_j}\sigma_j^2 \quad s.t.\ \sum_{j=1}^{M} q_j = 1,\ q_j \ge 0$$

$$\Rightarrow \text{optimal allocation: } q_j^* = \frac{p_j\sigma_j}{\sum_{i=1}^{M} p_i\sigma_i},\ j = 1, \dots, M$$

$$\Rightarrow Var(\bar{Y}_{OPT}) = \frac{1}{N}\sum_{j=1}^{M}\frac{p_j^2}{q_j^*}\sigma_j^2$$

$$= \frac{1}{N}\sum_{j=1}^{M} p_j^{\cancel{2}}\frac{\sum_{i=1}^{M} p_i\sigma_i}{\cancel{p_j}\cancel{\sigma_j}}\sigma_j^{\cancel{2}}$$

$$= \frac{1}{N}\left(\sum_{j=1}^{M} p_j\sigma_j\right)^2 < \frac{1}{N}\sum_{j=1}^{M} p_j\sigma_j^2 = Var(\bar{Y}_{PS})$$ #

$$\because Var(\bar{Y}_{MC}) = \frac{1}{N}Var(f(X)) = \frac{1}{N}\sigma^2 = \frac{1}{N}\sum_j p_j(\mu_j^2 + \sigma_j^2) - \mu^2) = \frac{1}{N}\sum_j p_j((\mu_j - \mu)^2 + \sigma_j^2)$$

$$= \frac{1}{N}\left[\sum_j p_j(\mu_j - \mu)^2 + \sum_j p_j(\sigma_j - \bar{\sigma})^2 + \bar{\sigma}^2\right]$$ (∵ $\sum_j p_j(\sigma_j - \bar{\sigma})^2 = \sum_j p_j\sigma_j^2 - \bar{\sigma}^2$)

$$Var(\overline{Y}_{PS}) = \frac{1}{N}\sum_j P_j \sigma_j^2 = \frac{1}{N}\left(\sum_j P_j(\sigma_j - \overline{\sigma})^2 + \overline{\sigma}^2\right)$$

$$Var(\overline{Y}_{OPT}) = \frac{1}{N}\left(\sum_j P_j \sigma_j\right)^2 = \frac{1}{N}\overline{\sigma}^2$$ #

8. LHS estimate is U.E :

$$\overline{Y}_{LHS} = \frac{1}{N}\sum_{j=1}^{N} f(V^j),\ \forall V^j = (V_1^j, \cdots, V_d^j),\quad V_i^j = \frac{\pi_i(j) - 1 + U_i^j}{N},\ i = 1, \cdots, d,\ j = 1, \cdots, N$$

$\because$ $f(V^j)$ 是在 hypercube 內均勻分布的点

$\therefore E\left[\frac{1}{N}\sum_{i=1}^{N} f(V^j)\right] = \int_{[0,1)^d} f(x)\,dx \Rightarrow$ U.E #

WUS estimate is biased (slightly).

$$\overline{Y}_N^{WUS} = \frac{\sum_{i=1}^{N} f(x_i)p(x_i)}{\sum_{i=1}^{N} p(x_i)},\quad x_i \sim U[0,1]^d$$

$$E\left[\frac{\sum_{i=1}^{N} f(x_i)p(x_i)}{\sum_{i=1}^{N} p(x_i)}\right] \not= \int f(x)p(x)\,dx$$

$$\because \int \frac{\sum_{i=1}^{N} f(x_i)p(x_i)}{\sum_{i=1}^{N} p(x_i)}\,dx_1\,dx_2\cdots dx_N \neq \int f(x)p(x)\,dx$$ #

⊙ 詳細的 LHS U.E 过程.

$$V_j^i \sim U(A_i^j)\quad \forall A_i^j \in \left\{[0,\tfrac{1}{N}), [\tfrac{1}{N},\tfrac{2}{N}), \cdots, [\tfrac{N-1}{N}, 1)\right\},\ A_i^1 \cup A_i^2 \cdots \cup A_i^N = [0,1)$$

$$\begin{aligned}
E[\overline{Y}_{LHS}] &= E\left[\frac{1}{N}\sum_{j=1}^{N} f(V_1^j, V_2^j, \cdots, V_d^j)\right] \\
&= E\left[E\left[\frac{1}{N}\sum_{j=1}^{N} f(V_1^j, V_2^j, \cdots, V_d^j)\right] \Big|\ V_i^j \in A_i^j,\ i = 1, \cdots, d,\ j = 1, \cdots, N\right] \\
&= E\left[\frac{1}{N}\sum_{j=1}^{N} \frac{\int_{A_1^j \times A_2^j \times \cdots \times A_d^j} f(x)}{\int_{A_1^j \times A_2^j \times \cdots \times A_d^j} dx}\ \Big|\ A_i^j,\ i = 1, \cdots, d,\ j = 1, \cdots, N\right] \\
&= E\left[\frac{1}{N} N^d \sum_{j=1}^{N} \int_{A_1^j \times A_2^j \times \cdots A_d^j} f(x)\,dx\ \Big|\ \bigcup_{j=1}^{N} A_i^j = [0,1),\ A_i^j \in \left\{[0,\tfrac{1}{N}), \cdots [\tfrac{N-1}{N}, 1)\right\}\right] \\
&= \frac{1}{N} \cdot N^d \cdot \sum_{j=1}^{N} \frac{1}{N^d} \int_{[0,1)^d} f(x)\,dx \\
&= \int_{[0,1)^d} f(x)\,dx
\end{aligned}$$ #

9. 附在後面 with coding.

10. (1) Use crude MC:

$$P(Z>b) = E[I\{Z>b\}]$$

$$\Rightarrow \hat{\theta}_{MC} = \frac{1}{N}\sum_{i=1}^{N} I_{\{Z_i>b\}} \#$$

(2)

$$Q_N^{IS} = \frac{1}{N}\sum_{i=1}^{N} f(x_i)w(x_i),\ \forall x_i \sim g(x),\ w(x_i) = \frac{p(x_i)}{g(x_i)}$$

$$w(z_i) = \frac{p(z_i)}{g(z_i)} = \frac{\frac{1}{\sqrt{2\pi}}e^{-z_i^2/2}}{\frac{1}{\sqrt{2\pi}}e^{-(z_i-\mu)^2/2}} = \text{Likelihood ration}$$

$$\Rightarrow E[I\{z>b\}] = \int_{-\infty}^{\infty} I_{\{z_i>b\}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{\infty} I_{\{z_i>b\}} \frac{\frac{1}{\sqrt{2\pi}}e^{-z^2/2}}{\frac{1}{\sqrt{2\pi}}e^{-(z-\mu)^2/2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-(z-\mu)^2/2} dz$$

$$= \int_{-\infty}^{\infty} I_{\{z_i>b\}} e^{\frac{\mu^2}{2}-\mu z} \cdot \frac{1}{\sqrt{2\pi}} e^{-(z-\mu)^2/2} dz$$

Let $Y \sim N(\mu, 1)$

$$\Rightarrow E[I_{\{z>b\}}] = E_\mu\{I\{Y>b\} e^{\frac{\mu^2}{2}-\mu Y}\}$$

$$\Rightarrow \hat{\theta}_{IS} = \frac{1}{N}\sum_{i=1}^{N} I_{\{Y_i>b\}} e^{\frac{\mu^2}{2}-\mu y_i} \#$$