STAT 310: Homework #4

Due on October 17, 2017 at 1:00 PM

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Problem 1

RT 2.6.3. The cdf of a discrete random variable Y is given by the following: $F(-1) = 0.1, F(0) = 0.15, F(2) = 0.4, F(5) = 0.8, F(6) = 1$.

(a) Find $\mathbb{E}[Y], \mathbb{E}[Y^2], \mathbb{E}[Y^3]$, and $\text{Var}(Y)$

(b) Find the mgf of Y.

Solution

(a) f(-1) = 0.1, f(0) = 0.05, f(2) = 0.25, f(5) = 0.4, f(6) = 0.2. Thus, $\mu = \mathbb{E}[X] = (-1)(0.1) + 2(0.25) + 5(0.4) + 6(0.2) = 3.6$, $\mathbb{E}[X^2] = (-1)^2(0.1) + 2^2(0.25) + 5^2(0.4) + 6^2(0.2) = 18.3$, $\mathbb{E}[X^3] = (-1)^3(0.1) + 2^3(0.25) + 5^3(0.4) + 6^3(0.2) = 95.1$, and $\text{Var}(x) = \mathbb{E}[X^2] - \mu^2 = 18.3 - 3.6^2 = 5.34$.

(b) $M_x(t) = \sum e^{tx}p(x) = 0.1e^{-t} + 0.05e^0 + 0.25e^{2t} + 0.4e^{5t} + 0.2e^{6t} = 0.05 + 0.1e^{-t} + 0.25e^{2t} + 0.4e^{5t} + 0.2e^{6t}$. Then, $M_x(t)^{(1)} = \frac{6e^{6t}}{5} + \frac{2e^{5t}}{5} + \frac{e^{2t}}{2} - \frac{e^{-t}}{10}$ and $M_x(0)^{(1)} = 3.6$. $M_x(t)^{(2)} = \frac{72e^{6t}}{10} + 10e^{5t} + e^{2t} + \frac{e^{-t}}{10}$ and $M_x(0)^{(2)} = 18.3$. $M_x(t)^{(3)} = \frac{432e^{6t}}{10} + 50e^{5t} + 2e^{2t} - \frac{e^{-t}}{10}$ and $M_x(0)^{(3)} = 95.1$. Thus, $\text{Var}(x) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = M_x(t)^{(2)} - (M_x(t)^{(1)})^2 = 18.3 - 3.6^2 = 5.34$

Problem 2

RT 2.6.15. Let X be a random variable with geometric pdf

$$f(x) = p(1-p)^{x-1}, x = 1, 2, 3, ...$$

(a) Find $\mathbb{E}[X]$ and $\text{Var}(X)$

(b) Show that $M_x(t) = \frac{pe^t}{1-(1-p)e^t}, t < -\ln(1-p)$

Solution

(a)

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} xf(x) = \sum_{x=1}^{\infty} xp(1-p)^{x-1} = p\sum_{x=1}^{\infty} x(1-p)^{x-1}.$$

Since

$$\sum_{x=1}^{\infty} x(1-p)^{x-1} = \sum_{x=1}^{\infty} -\frac{d}{dp}(1-p)^x = -\frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^x = -\frac{d}{dp}\frac{1-p}{p} = \frac{d}{dp}(1-\frac{1}{p}) = \frac{1}{p^2}$$

via chain rule and the definition of a power series, we have

$$\mathbb{E}[X] = p\frac{1}{p^2} = \frac{1}{p}$$

Then we use the expected value of X squared to find variance,

$$\mathbb{E}[X^2] = \sum_{x=1}^{\infty} x^2 f(x) = \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1} = p\sum_{x=1}^{\infty} x^2(1-p)^{x-1} = p\sum_{x=1}^{\infty} -\frac{d}{dp}k(1-p)^k$$

$$= -p\frac{d}{dp}\frac{1-p}{p}\sum_{x=1}^{\infty} k(1-p)^{k-1}p = -p\frac{d}{dp}\frac{1-p}{p}\mathbb{E}[X] = -p\frac{d}{dp}\frac{1-p}{p^2} = -p(\frac{-2}{p^3} + \frac{1}{p^2}) = \frac{2}{p^2} - \frac{1}{p} = \frac{2-p}{p^2}$$

Thus,

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Problem 2 continued on next page...

(b) We use the definition of a power series to get

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} = pe^t \sum_{x=1}^{\infty} e^{t(x-1)}(1-p)^{x-1} = pe^t \sum_{x=1}^{\infty} (e^t(1-p))^{x-1} = \frac{pe^t}{1-e^t(1-p)}$$

Problem 3

RT 2.6.16. Find $\mathbb{E}[X]$ and $\text{Var}(X)$ for a random variable X with pdf $f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$. Also find the mgf of X.

Solution

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|} dx = \frac{1}{2} \int_0^{\infty} x e^{-x} dx + \frac{1}{2} \int_0^{\infty} -x e^{-x} dx = 0.$$

By symmetry, we get $\mathbb{E}[X] = 0$. We can calculate $\mathbb{E}[X^2]$ using L'Hospital's Rule to yield:

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} \frac{1}{2} x^2 e^{-|x|} dx = \int_0^{\infty} x^2 e^{-x} dx = \left[-x^2 e^{-x}\right]_0^{\infty} - 2\int_0^{\infty} x e^{-x} dx = \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 2.$$

Thus, $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2 - 0^2 = 2$. Then, the mgf of X is

$$M_x(t) = \int_{-\infty}^{\infty} \frac{1}{2} e^{tx} e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^{0} e^{(t+1)x} dx + \frac{1}{2} \int_0^{\infty} e^{(t-1)x} dx = \frac{1}{2} \int_0^{\infty} e^{-x(t+1)} dx + e^{x(t-1)} dx$$

$$= \frac{1}{2} \left[\frac{\mathrm{e}^{(t-1)x}}{t-1} - \frac{\mathrm{e}^{(-t-1)x}}{t+1} \right]_0^{\infty} = \frac{1}{2} \left[\frac{\mathrm{e}^{(t-1)x}}{t-1} \right]_0^{\infty} - \frac{1}{2} \left[\frac{\mathrm{e}^{(-t-1)x}}{t+1} \right]_0^{\infty} = \frac{1}{2-2t} + \frac{1}{2+2t} = \frac{1}{1-t^2}$$

Problem 4

Problem 4.15. A large parabolic antenna is designed against wind load. During a wind storm, the maximum wind-induced pressure on the antenna, P, is computed as

$$P = \frac{1}{2} CRV^2$$

where C = drag coefficient; R = air mass density in slugs/ft^3; V = maximum wind speed in ft/sec; and P = pressure in lb/ft^2. C, R, and V are statistically independent lognormal variates with the following respective means and c.o.v.'s:

$$\mu_C = 1.80, \qquad \delta_C = 0.20$$
$$\mu_R = 2.3 * 10^{-3}, \quad \delta_R = 0.10$$
$$\mu_V = 120, \qquad \delta_V = 0.45$$

(a) Determine the probability distribution of the maximum wind pressure P and evaluate its parameters.

(b) What is the probability that the maximum wind pressure will exceed 30 lb/ft^2?

(c) The actual wind resistance capacity of the antenna is also a lognormal random variable with a mean of 90 lb/ft^2 and a c.o.v. of 0.15. Failure in the antenna will occur whenever the maximum applied wind pressure exceeds its wind resistance capacity. During a wind storm, what is the probability of failure of the antenna?

Problem 4 continued on next page...

(d) If the occurances of wind storms in (c) constitute a Poisson process with a mean occurance rate of once every 5 years, what is the probability of failure of the antenna in 25 years?

(e) Suppose five antennas were built and installed in a given region. What is the probability that at least two of the five antennas will not fail in 25 years? Assume that failures between antennas are statistically independent.

Solution

(a) First, we calculate the mean pressure given by:

(b)