Math 216 Theorems

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III. Sequences in $\mathbb{R}$

Lemma 1. Let (a_n) be a sequence.

(i) (a_n) can have at most one limit.

(ii) If (a_n) is convergent, then (a_n) is bounded.

Theorem 2. Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that $(a_n) \to a$ and $(b_n) \to b$. Then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} (a_n) \pm \lim_{n\to\infty} (b_n)$$

$$\lim_{n\to\infty} (a_n) \cdot (b_n) = \lim_{n\to\infty} (a_n) \cdot \lim_{n\to\infty} (b_n)$$

$$\lim_{n\to\infty} \frac{(a_n)}{b_n} = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$$

$$\lim_{n\to\infty} |(a_n)| = |\lim_{n\to\infty} (a_n)|$$

Theorem 3. Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that $(a_n) \to a$ and $(b_n) \to b$. Assume now that $\forall n \in \mathbb{N}, a_n \leq b_n$. Then

$$\lim_{n\to\infty} (a_n) \leq \lim_{n\to\infty} (b_n)$$

Theorem 4. Let $(a_n), (b_n), (c_n)$ be sequences in $\mathbb{R}$. If $(a_n) \to a, (c_n) \to a,$ and $\forall n \in \mathbb{N}, a_n \leq b_n \leq c_n$, then $(b_n) \to a$.

Theorem 5. For every monotone sequence (a_n) in $\mathbb{R}$, the following are equivalent:

(i) (a_n) is convergent

(ii) (a_n) is bounded

Theorem 6. Let $I_1 \supset I_2 \supset I_3 \supset \dots$ be non-empty closed intervals. Then the set

$$\bigcap_{n\in\mathbb{N}} I_n := \{c \in \mathbb{R} : c \in I_n\}$$

is non-empty.

Lemma 7. Assume that A_j is a countable set $\forall j \in \mathbb{N}$. Then

$$\bigcup_{j\in\mathbb{N}} A_j := \{a : a \in A_j \text{ for some } j \in \mathbb{N}\}$$

is also countable.

Theorem 8. $\mathbb{Q}$ is countable.

Theorem 9. $\mathbb{R}$ is uncountable.

Lemma 10. Let (a_n) be a sequence in $\mathbb{R}$. If (a_n) converges to $a \in \mathbb{R}$ then any subsequence (a_{n_j}) of (a_n) also converges to a. i.e. $\mathbb{S}[a_n] = \{a\}$

Lemma 11. Every sequence (a_n) in $\mathbb{R}$ has a monotone subsequence.

Theorem 12. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Lemma 13. Let (a_n) be a sequence in $\mathbb{R}$. Then

(i) $\mathbb{S}[a_n] \neq \varnothing$

(ii) $\mathbb{S}[a_n] \subset \mathbb{R} \iff (a_n)$ is bounded.

(iii) $\mathbb{S}[a_n] = \{a\} \iff \lim_{n\to\infty}(a_n) = a$

Lemma 14. Let (a_n) be a sequence in $\mathbb{R}$.

(i) If (a_n) is convergent then (a_n) is Cauchy

(ii) If (a_n) is Cauchy then (a_n) is bounded

Theorem 15. For every sequence (a_n) in $\mathbb{R}$ the following are equivalent:

(i) (a_n) is convergent

(ii) (a_n) is Cauchy

Lemma 16. If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty}(a_n) = 0$.

Theorem 17 (Alternating Series Test)**.** Assume that (b_n) is monotone. Then the following are equivalent:

(i) $\sum_{n=1}^{\infty}(-1)^n b_n$ is convergent

(ii) $(b_n) \to 0$

Lemma 18. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

Lemma 19. Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ with $|a_n| \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

Theorem 20. Let (a_n) be a sequence in $\mathbb{R}$ and $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. (so $r \in \{0\} \cup \mathbb{R}^+ \cup \{\infty\}$)

(i) if $0 \leq r < 1$ then $\sum_{n=1}^{\infty}$ is absolutely convergent

(ii) if $r > 1$ then $\sum_{n=1}^{\infty}$ is divergent.

Corollary 21. Let (a_n) be a sequence in $\mathbb{R} \setminus \{0\}$ and assume that $r := \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists

(i) If $0 \leq r < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $r > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 22. Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then their Cauchy product $\sum_{n=1}^{\infty} c_n$ is also absolutely convergent and $\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n$.

Theorem 23. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then for every bijection $f : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{f(n)}$ is also absolutely convergent, and

$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_n \tag{1}$$