Math 216 Theorems

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I. Logic, Proofs, Sets, Functions

Theorem 1. For every integer k, the following statements are equivalent:

(i) 2 divides k

(ii) 2 divides k^2

Theorem 2. Let a, b be positive real nubmers. Then

$$AM(\frac{1}{a}, \frac{1}{b}) \geq \frac{1}{AM(a,b)} \tag{$\star$}$$

Furthermore, $(*)$ is strict unless $a = b$

Theorem 3. Let k, l be integers with $l \neq 0$. Then $\frac{k^2}{l^2} \neq 2$.

Proposition 1. Let $A, B \neq \varnothing$ be non-empty sets

(i) Either $\#A \leq \#B$ or $\#B \leq \#A$

(ii) $\#A \leq \#B \iff \#B \geq \#A$

(iii) $\#A = \#B \iff \#A \leq \#B \ \& \ \#B \leq \#A$

Lemma 1. $\forall \ n \in \mathbb{N} \setminus \{1\}$, the set

$$\{j \in \mathbb{N} \setminus \{1\} : j|n\} := A_n \tag{1}$$

is non-empty and its smallest element, denoted $p(n)$, is prime.

Theorem 4. For all $n \in \mathbb{N}$ there exists a prime p with $p > n$.

III. Sequences in $\mathbb{R}$

Lemma. Let (a_n) be a sequence.

(i) (a_n) can have at most one limit.

(ii) If (a_n) is convergent, then (a_n) is bounded.

Theorem. Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that $(a_n) \to a$ and $(b_n) \to b$. Then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} (a_n) \pm \lim_{n\to\infty} (b_n)$$

$$\lim_{n\to\infty} (a_n) \cdot (b_n) = \lim_{n\to\infty} (a_n) \cdot \lim_{n\to\infty} (b_n)$$

$$\lim_{n\to\infty} \frac{(a_n)}{b_n} = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$$

$$\lim_{n\to\infty} |(a_n)| = |\lim_{n\to\infty} (a_n)|$$

Theorem. Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that $(a_n) \to a$ and $(b_n) \to b$. Assume now that $\forall n \in \mathbb{N}, a_n \leq b_n$. Then

$$\lim_{n\to\infty}(a_n) \leq \lim_{n\to\infty}(b_n)$$

Theorem. Let $(a_n), (b_n), (c_n)$ be sequences in $\mathbb{R}$. If $(a_n) \to a, (c_n) \to a,$ and $\forall n \in \mathbb{N}, a_n \leq b_n \leq c_n$, then $(b_n) \to a$.

Theorem. For every monotone sequence (a_n) in $\mathbb{R}$, the following are equivalent:

(i) (a_n) is convergent

(ii) (a_n) is bounded

Theorem. Let $I_1 \supset I_2 \supset I_3 \supset \dots$ be non-empty closed intervals. Then the set

$$\bigcap_{n\in\mathbb{N}} I_n := \{c \in \mathbb{R} : c \in I_n\}$$

is non-empty.

Lemma. Assume that A_j is a countable set $\forall j \in \mathbb{N}$. Then

$$\bigcup_{j\in\mathbb{N}} A_j := \{a : a \in A_j \text{ for some } j \in \mathbb{N}\}$$

is also countable.

Theorem. $\mathbb{Q}$ is countable.

Theorem. $\mathbb{R}$ is uncountable.

Lemma. Let (a_n) be a sequence in $\mathbb{R}$. If (a_n) converges to $a \in \mathbb{R}$ then any subsequence (a_{n_j}) of (a_n) also converges to a. i.e. $\mathbb{S}[a_n] = \{a\}$

Lemma. Every sequence (a_n) in $\mathbb{R}$ has a monotone subsequence.

Theorem. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Lemma. Let (a_n) be a sequence in $\mathbb{R}$. Then

(i) $\mathbb{S}[a_n] \neq \varnothing$

(ii) $\mathbb{S}[a_n] \subset \mathbb{R} \iff (a_n)$ is bounded.

(iii) $\mathbb{S}[a_n] = \{a\} \iff \lim_{n\to\infty}(a_n) = a$