MATH 217 Theorems

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Lemma 1. Let $\mathbb{O}_1$ be an ordered field. Then for every $x \in \mathbb{O}_1$ we have that $x^2 = x \cdot x > 0$.

Corollary 1. The field $\mathbb{C}$ of complex numbers is NOT orderable. (hint for proof:)

Lemma 2. Let $\mathbb{O}_1$ be an ordered field, and consider the following subset of it:

$$\mathcal{P}_{\mathbb{O}_1} = \{x \in \mathbb{O}_1 : 0 < x\}$$

then, $\forall y, z \in \mathcal{P}_{\mathbb{O}_1}$ we have that $y + z \in \mathcal{P}_{\mathbb{O}_1}$ and $y \cdot z \in \mathcal{P}_{\mathbb{O}_1}$

Theorem 1. Let $\mathbb{F} = (\mathbb{F}_1, +_{\mathbb{F}}, \cdot_{\mathbb{F}})$ be a field, and suppose that there is a subset $\mathcal{P}_1$ of $\mathbb{F}$ for which properties (O1'), (O2'), and (O3') hold true. Then $\mathbb{F}$ is orderable. More specifically, the order relation '$\leq$' that is given by the strict order relation '$<$' which is defined by:

$$x < y \text{ if } y - x \in \mathcal{P}_1$$

is a total order on $\mathbb{F}$ which is compatible with the addition and multiplication on $\mathbb{F}$.

Proposition 1. let $\mathbb{O}_1$ be an ordered field. Then, for any prime $p \in \mathbb{N}$ we have that no subfield of $\mathbb{O}_1$ is isomorphic to $\mathbb{Z}_p$. More generally, we have that NO subfield of $\mathbb{O}_1$ is finite.