Computational Finance

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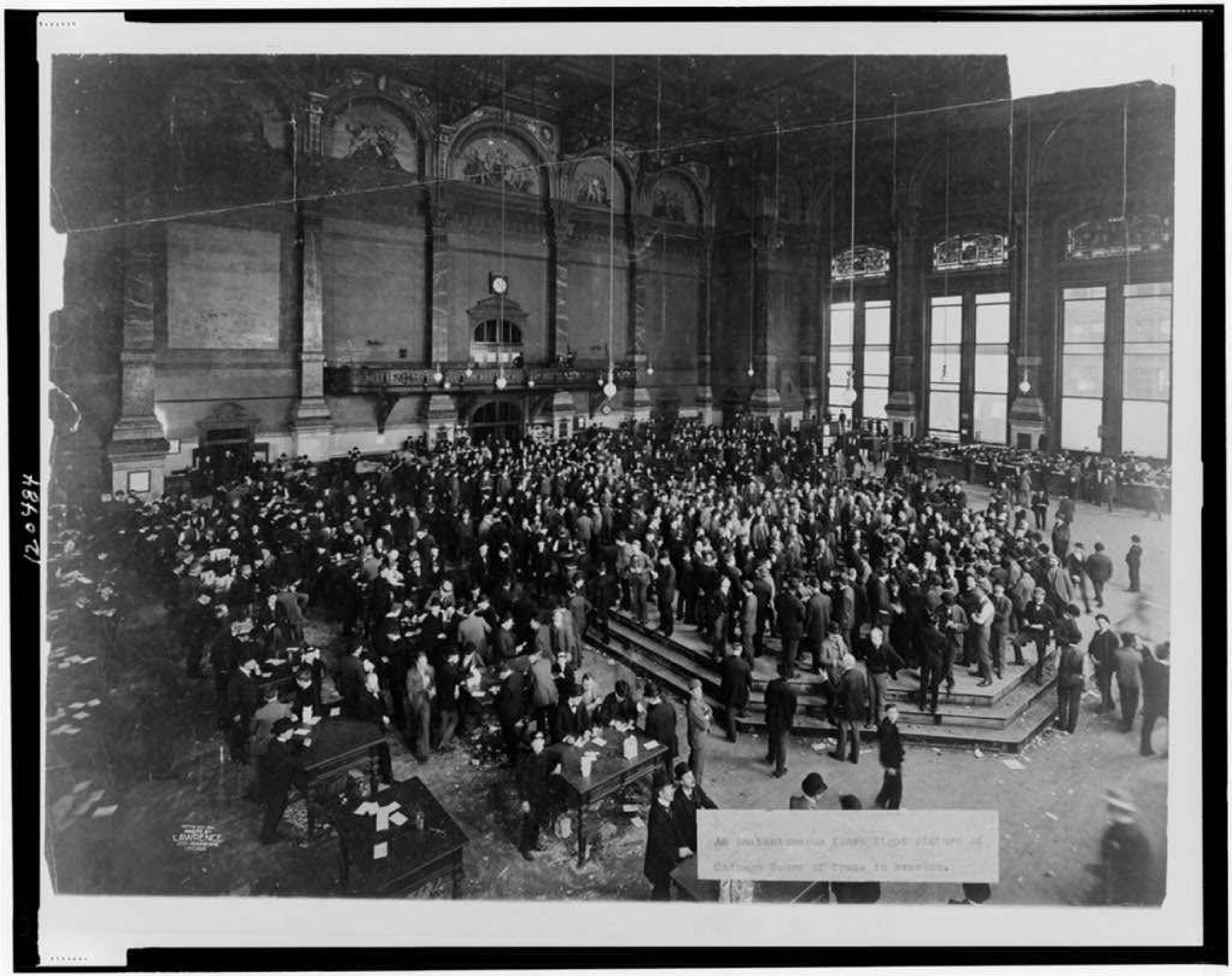
Finance without computers

Chicago Board of Trade, year c. 1900



The Library of Congress (pan 6a20124)

Finance pre-pandemic

New York Stock Exchange, year 2013

(Photo by Richard Drew / Associated Press)

Why computers?

- Even closed-form fomulas for prices, risk, etc., require evaluation of named but computationally laborious functions (e.g. Black-Scholes formula)

Contingent claims - recap

- European path-independent options
  - payoff given by $h(S_T)$
  - European call and put options
  - European binary call and put options
- European path-dependent options
  - Asian options (calls and puts on the average of the stock price)
  - Barrier options
  - Lookback options
- American options
  - they can be exercised anytime before or at maturity
  - the payoff at the exercise moment t is $h(S_t)$

One-period binomial model 1/2

$$\begin{tikzcd}[row sep=small]
 & \boxed{S_1 = S_0 u} \\
\boxed{S_0} \arrow[ur] \arrow[dr] & \\
 & \boxed{S_1 = S_0 d}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{B_0 = 1} \arrow[r] & \boxed{B_1 = e^{r\Delta t}}
\end{tikzcd}$$

Parameters: u, d, r, Δt, S_0.

Risk-neutral measure (martingale measure):

$$p^* = \mathbb{P}^*(S_1 = S_0 u) = \frac{e^{r\Delta t} - d}{u - d} = \frac{B_1/B_0 - d}{u - d}$$

Price of option with payoff $h(s)$:

$$e^{-r\Delta t}\Big(p^*\, h(S_0 u) + (1 - p^*) h(S_0 d)\Big)$$

One-period binomial model 2/2

Why risk-neutral measure?

$$\mathbb{E}_{\mathbb{P}^*}[S_1] = p^* S_0 u + (1-p^*) S_0 d = e^{r\Delta t} S_0.$$

Arbitrage: The model is arbitrate free iff $0 < \frac{e^{r\Delta t} - d}{u-d} < 1$.

Important for understanding of the rest of the lecture

How to think about this price of payoff $h(s)$:

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-r\Delta t} h(S_1)\right] = e^{-r\Delta t}\Big(p^*\, h(S_0 u) + (1-p^*) h(S_0 d)\Big)$$

- This is the smallest amount of money required to fully hedge the payoff.
- It is the only amount of money that allows to construct a portfolio that replicates $h(S_1)$.

A multiperiod binomial model

$$
\begin{tikzcd}[row sep=small]
 & & \boxed{S_2 = S_0 u^2} & \omega_1 \\
 & \boxed{S_1 = S_0 u} \arrow[ur] \arrow[dr] & & \\
 & & \boxed{S_2 = S_0 u d} & \omega_2 \\
\boxed{S_0} \arrow[uur] \arrow[ddr] & & & \\
 & & \boxed{S_2 = S_0 d u} & \omega_3 \\
 & \boxed{S_1 = S_0 d} \arrow[ur] \arrow[dr] & & \\
 & & \boxed{S_2 = S_0 d^2} & \omega_4
\end{tikzcd}
$$

$$
\boxed{B_0 = 1} \longrightarrow \boxed{B_1 = e^{r\Delta t}} \longrightarrow \boxed{B_2 = e^{2r\Delta t}}
$$

Parameters: u, d, r, S_0, T, N

$$\Delta t = T/N.$$

Pricing of European options - first draft

Input: u, d, r, S_0, T, N (number of periods).

Compute one-period risk neutral probability p^*.

Construct a tree.

Compute values in the leaves of the tree (payoffs).

For i from $N-1$ to 0 do

compute values in each node in the period i using one-period binomial formula (using p^*) and using values computed in the previous step

---

Output: the value in the root of the tree is the price of the option

In the above sketch of an algorithm the period length $\Delta t = \frac{T}{N}$.

Computational complexity

Computational complexity plays a major role in the construction of numerical algorithms. It says how the amount of resources required by the algorithm (program) grows with the size of the problem (e.g. the number of periods in the Binomial Model). There are two main types of resources:

- memory (RAM)
- time

For our pricing algorithm

- memory requirement is proportional to 2^{N+1}
- time is proportional to 2^{N+1}

Practice of computational complexity

Why is the computational complexity so important?

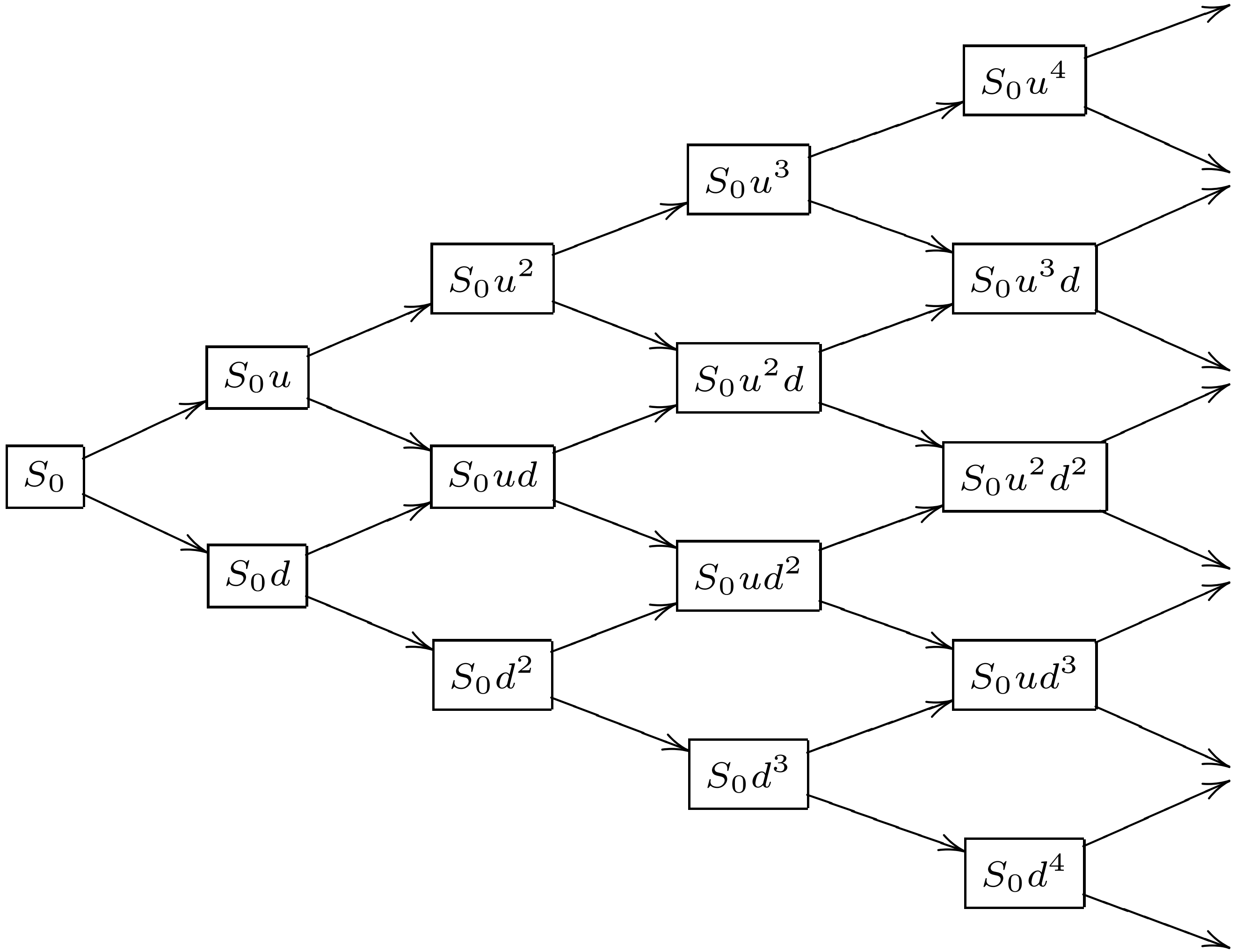
If it were sufficient to have one byte for each node, to simulate $N = 100$ periods we would need

$$2^{101} \text{ Bytes} = 2^{91} \text{ kBytes} = 2^{81} \text{ MBytes} = 2^{61} \text{ TBytes (!)}$$

$$= 2,305,843,009,213,693,952 \text{ TBytes}$$

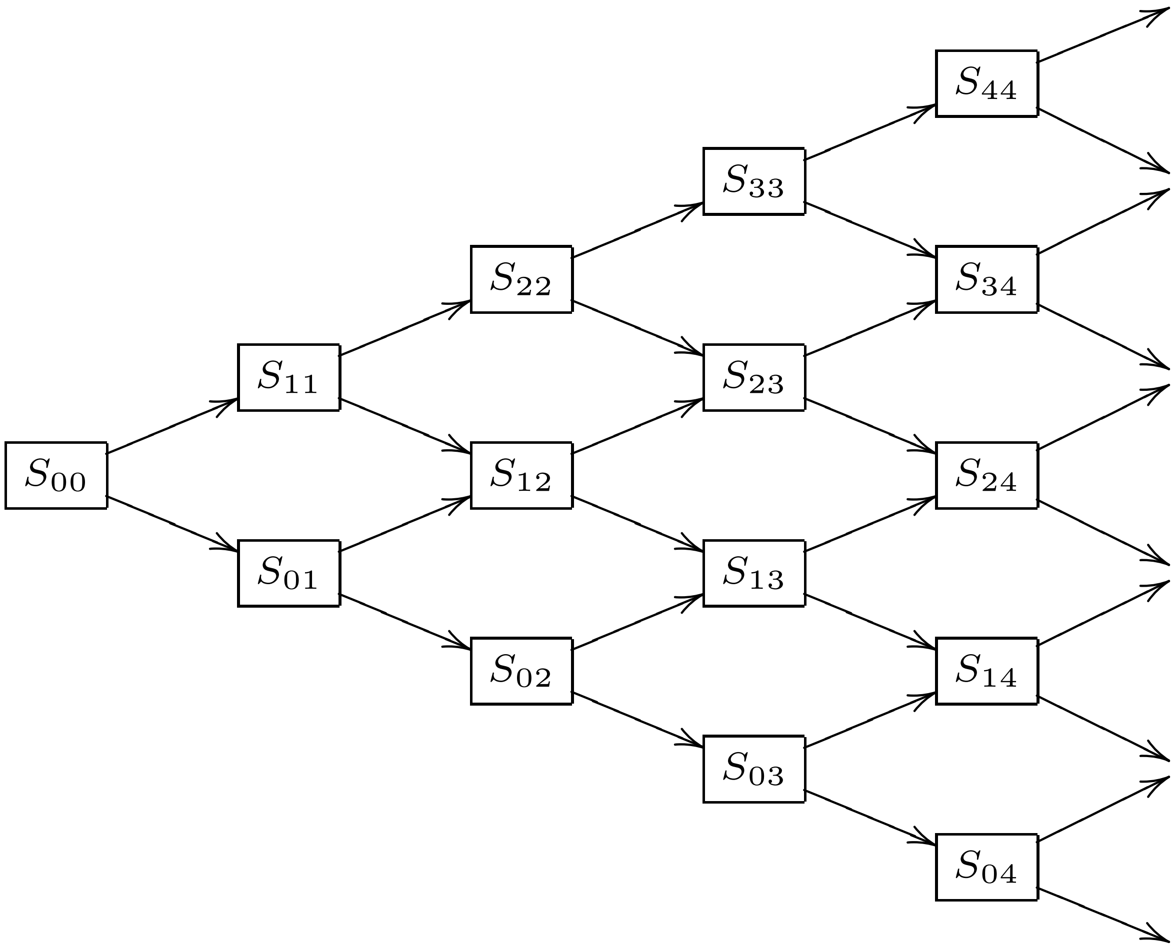
Path-indepenent options: the values in many nodes of the tree are identical (for a simple example draw a tree for $N = 3$). We can glue identical nodes into one node -> recombining binomial tree.

Recombination of the binomial tree



Recombining tree can be used to price path independent European and American options and some path-dependent options (using tricks). General path-dependent options cannot be priced!!!

Recombination of the binomial tree



Pricing of path-indep. options

Denote by S_{ji} the price after i periods with j being the number of up moves:

$$S_{ji} = S_0 u^j d^{i-j}.$$

Let h be the payoff function and V_{ji} denote the value process of the replicating strategy in the node S_{ji}. Then

$$V_{jN} = h(S_{jN}), \qquad j = 0, 1, \ldots, N.$$

European options:

$$V_{ji} = e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*) V_{j,i+1}\Big)$$

American options:

$$V_{ji} = \max\Big(h(S_{ji}), e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*) V_{j,i+1}\Big)\Big)$$

European (path-indep.) options

Input: u, d, r, S_0, T, N (number of periods).

$$\Delta t = T/M, \qquad p^* = \frac{e^{r\Delta t} - d}{u - d}$$

If $p^* \leq 0$ or $p^* \geq 1$ then Error: there is arbitrage on the market.

For j from 0 to N

$\quad S_{jN} = S_0 u^j d^{N-j}$

$\quad V_{jN} = h(S_{jN})$

For i from $N-1$ to 0

$\quad$ For j from 0 to i

$\quad\quad V_{ji} = e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)$

Output: V_{00}

American (path-indep.) options

Input: u, d, r, S_0, T, N (number of periods).

$$\Delta t = T/N, \qquad p^* = \frac{e^{r\Delta t} - d}{u - d}$$

If $p^* \le 0$ or $p^* \ge 1$ then Error: there is arbitrage on the market.

For i from 0 to N

For j from 0 to i do $S_{ji} = S_0 u^j d^{i-j}$

For j from 0 to N do $V_{jN} = h(S_{jN})$

For i from $N-1$ to 0

For j from 0 to i

$$V_{ji} = \max\Big(h(S_{ji}), e^{-r\Delta t}\Big(p^* V_{j+1,i+1} + (1-p^*)V_{j,i+1}\Big)\Big)$$

Output: V_{00}

Computational complexity

For the algorithm from the previous slide

- memory is proportional to $\frac{(N+1)(N+2)}{2}$, which means it grows like N^2
- time is proportional to $\frac{(N+1)(N+2)}{2}$, which means it grows like N^2

To represent $N = 100$ periods we need only

$$101 * 102/2 = 5151 \text{ nodes.}$$

Is there any hope for path-dependent options? Yes, Monte Carlo methods... and some can be priced on recombining trees.

Binomial model as a numerical scheme for Black-Scholes model

Goal: Calibrate the Binomial Model in such a way that its pricing approximates the pricing of the Black-Scholes model.

Black-Scholes model under risk-neutral measure $\mathbb{Q}$:

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

where r is a riskfree interest rate, σ is the volatility and (W_t) is a Brownian motion (Wiener process).

$$\boxed{\text{price in Black-Scholes model} \approx \text{price in binomial model for small } \Delta t}$$

Calibration of the binomial model

$$
\begin{tikzcd}
 & & \boxed{S_2 = S_0u^2} & \omega_1 \\
 & \boxed{S_1 = S_0u} \arrow[ur] \arrow[dr] & & \\
 & & \boxed{S_2 = S_0ud} & \omega_2 \\
\boxed{S_0} \arrow[uur] \arrow[ddr] & & & \\
 & & \boxed{S_2 = S_0du} & \omega_3 \\
 & \boxed{S_1 = S_0d} \arrow[ur] \arrow[dr] & & \\
 & & \boxed{S_2 = S_0d^2} & \omega_4
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{B_0 = 1} \arrow[r] & \boxed{B_1 = e^{r\Delta t}} \arrow[r] & \boxed{B_2 = e^{2r\Delta t}}
\end{tikzcd}
$$

Parameters:

unknown: u, d known: r, S_0, Δt

Moment matching

Find u and d such that under the risk-neutral measure p^* the one period expectation and variance of the stock price change in the Binomial Model agree with the stock price change in the Black-Scholes model under its risk-neutral measure.

$\mathbb{E}$ and Var in Black-Scholes

Let Δt be the length of one period in the Binomial Model.

Stock price in Black-Scholes model under the risk-neutral measure satisfies:

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t},$$

from which we obtain

$$\log\left(\frac{S_{t+\Delta t}}{S_t}\right) \sim N\big((r-\frac{1}{2}\sigma^2)\Delta t, \sigma^2 \Delta t\big).$$

The expectation and variance of $S_{t+\Delta t}/S_t$ are equal to

$$\mathbb{E}\left(\frac{S_{t+\Delta t}}{S_t}\right) = e^{r\Delta t}, \qquad Var\Big(\frac{S_{t+\Delta t}}{S_t}\Big) = e^{2r\Delta t}\Big(e^{\sigma^2\Delta t} - 1\Big).$$

$\mathbb{E}$ and Var in Binomial Model

Remember that Δt is the length of one period in the Binomial Model.

The expectation of S_{i+1}/S_i is

$$\mathbb{E}\left(\frac{S_{i+1}}{S_i}\right) = p^* u + (1 - p^*) d.$$

The variance of S_{i+1}/S_i is

$$\begin{aligned} Var\Big(\frac{S_{i+1}}{S_i}\Big) &= \mathbb{E}\left(\frac{S_{i+1}}{S_i}\right)^2 - \left(\mathbb{E}\left(\frac{S_{i+1}}{S_i}\right)\right)^2 \\ &= p^* u^2 + (1 - p^*) d^2 - \big(p^* u + (1 - p^*) d\big)^2. \end{aligned}$$

Matching $\mathbb{E}$ and Var

Expectation

$$p^* u + (1 - p^*)d = e^{r\Delta t}$$

Variance

$$p^* u^2 + (1 - p^*)d^2 - \big(p^* u + (1 - p^*)d\big)^2 = e^{2r\Delta t}\Big(e^{\sigma^2 \Delta t} - 1\Big)$$

After simplification:

$$\boxed{\begin{aligned} p^* u + (1 - p^*)d &= e^{r\Delta t} \\ p^* u^2 + (1 - p^*)d^2 &= e^{2r\Delta t + \sigma^2 \Delta t} \end{aligned}}$$

The story is not finished yet

$$p^* u + (1 - p^*) d = e^{r\Delta t}$$
$$p^* u^2 + (1 - p^*) d^2 = e^{2r\Delta t + \sigma^2 \Delta t}$$

There are two equations with three unknowns, so we may expect more than one solution.

Popular choices:

- $u = d^{-1}$ leads to the Cox-Ross-Rubinstein model – industry standard
- $p^* = 1 - p^* = 1/2$ – mostly of theoretical importance

Cox-Ross-Rubinstein (CRR)

Assume that $u = d^{-1}$. We obtain the following set of parameters

$$
\begin{aligned}
u &= \beta + \sqrt{\beta^2 - 1}, \\
d &= 1/u = \beta - \sqrt{\beta^2 - 1}, \\
p^* &= \frac{e^{r\Delta t} - d}{u - d},
\end{aligned}
$$

where

$$
\beta = \frac{1}{2}\Big(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\Big).
$$

In practice, there is a tendency to use a simplified version of the CRR model (approximate CRR). It can be shown that up to the terms of higher order, we can set

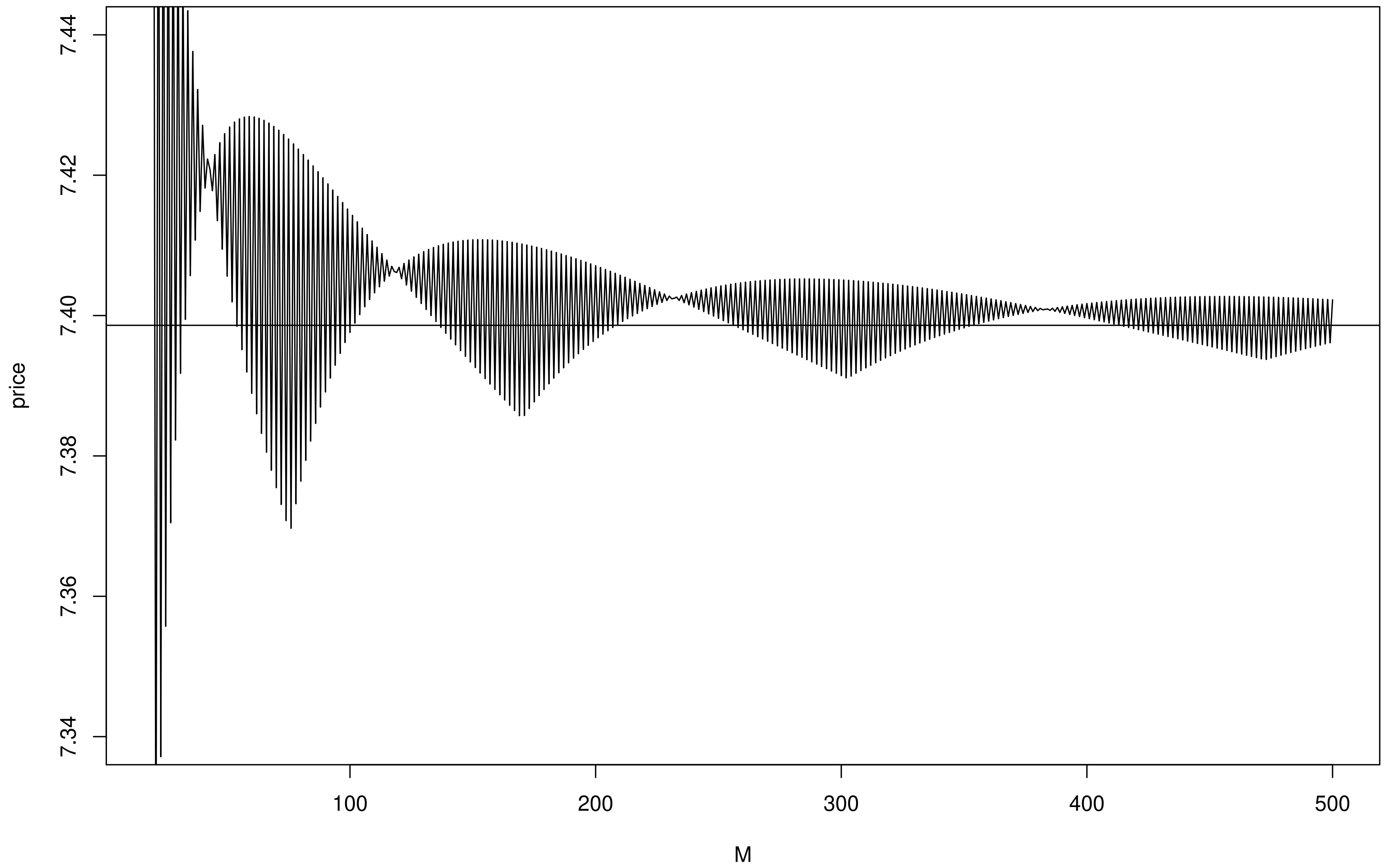
$$
u = e^{\sigma\sqrt{\Delta t}}, \qquad d = 1/u = e^{-\sigma\sqrt{\Delta t}},
$$

and p^* computed as the risk neutral measure for these u and d.

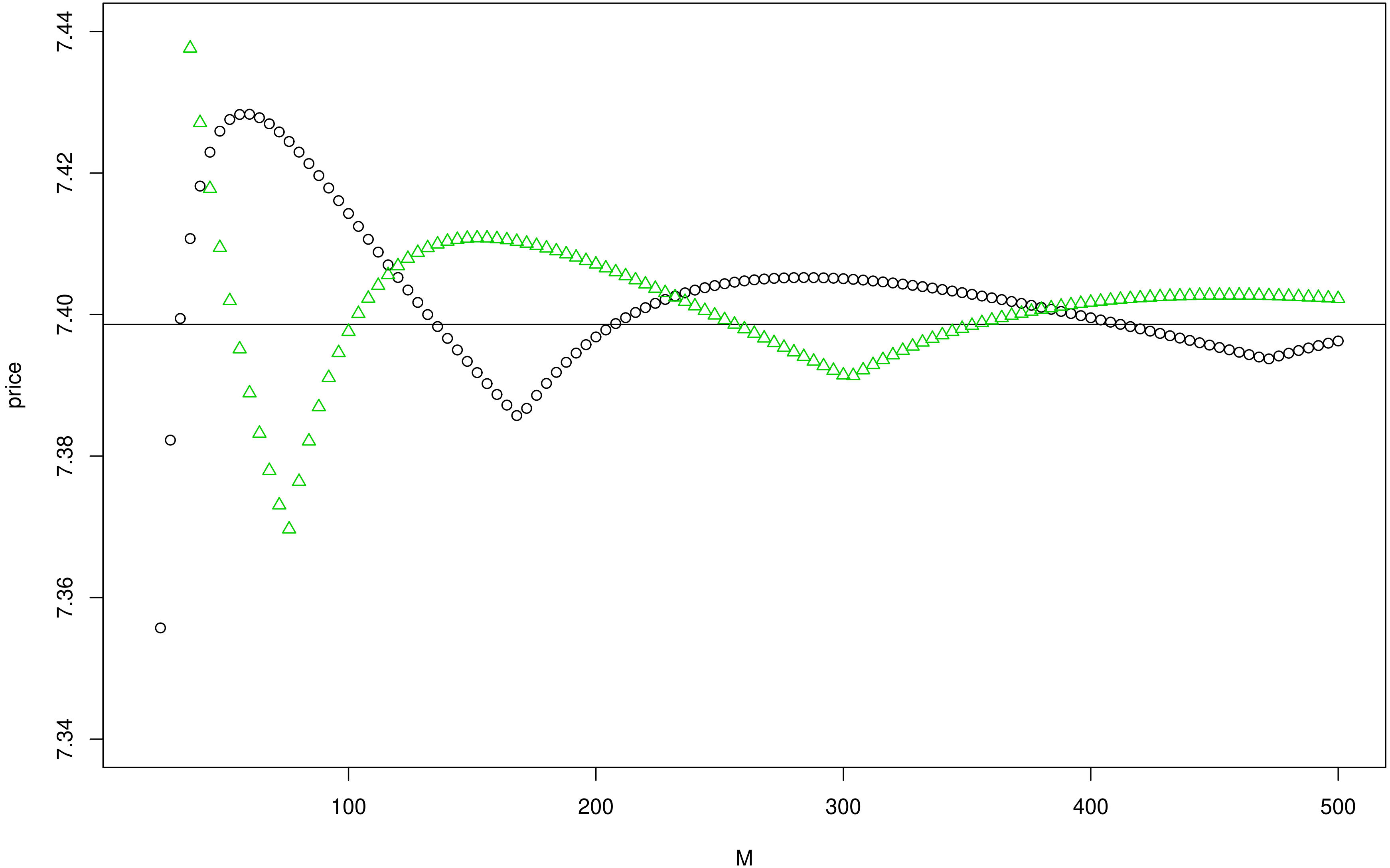
Convergence

Theorem. When the number of time periods N goes to infinity then the option price computed by a calibrated binomial model converges to the option price in the Black-Scholes model.

Oscillatory behaviour of prices



Prices for even and odd N



Task

Construct a recombining tree for the Brownian motion:

$$x \to x \pm \sqrt{\Delta t}$$

with probabilities $1/2$ each and use the closed form formula for the stock price for the calculation of payoffs

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}.$$

In this model:

1. Implement (in Python) the pricing of a European call payoff with maturity T and strike K.
2. Study how the result depends on the choice of the number of periods N. Do you see the oscillatory behaviour again?
3. Use a closed-form pricing formula (Black-Scholes formula) to obtain the benchmark and see how the accuracy depends on the number of periods taking into account that the convergence is not monotone - you need to think carefully how to measure it so that your results convey the message.
4. Have a couple of slides (in pdf) ready for the next week to discuss your results - this should be done in groups of 4 students.