American options: a cash flows approach

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Abstract

In the papers [2], [3], [4], [5] arbitrage and pricing of European options were studied in models that represent investment opportunities on the market by means of cash flows. This approach proved to be very powerful allowing for a large number of imperfections imposed on the market at the same time: proportional transaction costs, convex cone constraints on portfolio contents, defaultable bonds, etc. However, pricing of American options seemed to be impossible.

In this paper we construct a model that, preserving the flexibility of the above approach, enables us to price American options. We prove that under no transaction costs the price of an American option is equal to $\sup\{$price of the payoff $A(\tau)$ at $\tau : \tau$ is a stopping time$\}$, where $A(\tau)$ is a payoff resulting from execution of the option at τ. This is a well-known fact for complete market models. Here, however, we allow for all imperfections except transaction costs. Our considerations hold even under the presence of arbitrage. This points out praticular importance of transaction costs in the modelling of financial markets.

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1. Introduction

A new concept of modelling of financial markets was introduced in the paper [2]. Its roots came from an observation that investors practically, instead of being interested in their portfolios, are interested in the profit. Originally, all models were broken up into two parts: dynamics of asset prices and a set of admissible trading strategies. These, however, can be considered jointly. Each trading strategy defines a set of cash flows, either positive or negative, meaning, respectively, earning or paying money mostly as a result of buying or selling of assets. These cash flows are

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linked with random moments at which they occur. These moments, together with the cash flows, constitute an investment opportunity.

The models with cash flows were considered in several papers [2], [4], [3] and [5], where arbitrage and pricing of European options was studied. They proved to be very powerful since they allow for a large number of imperfections imposed on the market at the same time: convex cone constraints on portfolio contents, proportional transaction costs, lack of bank account, defaultable bonds, dividends, etc. However, there were no papers using cash flow approach to American options. The reason for it, as we point out later, is that it is impossible to price American or game type options directly. In the usual approach to financial modelling a trading strategy defines a portfolio process which says how many assets of each type we possess at each moment. It is done in two ways: by specifying the amount of money invested in each asset or the quantity of stocks or other quanta of the asset. Therefore, an American option with payoff $(A(t))_{t\in I}$, where $I \subseteq \mathbb{R}_+$ is a set of possible execution times, is hedged by a strategy Π if its portfolio process dominates in some sense $A(t)$ at all moments $t \in I$. This intuition seems to be difficult to adapt to models that represent investment strategies in terms of cash flows, since the information about contents of portfolios is lost.

In this paper we construct a model that, having all advantages of models with cash flows, allows for pricing of American contingent claims. We motivate this approach by an observation that models with cash flows enable us to cover all types of investments available on real markets: stocks, defaultable stocks and bonds, credits, currencies, derivatives together with all kinds of imperfections. Generality of the approach does not promise closed pricing formulas, but gives a lot of insight in mechanisms governing the market.

The intuition that stands behind our approach is very natural. We need to find a hedging price for an American claim with payoff $(A(t))_{t\in I}$, where I is a set of possible execution times. A hedging is a strategy that provides a payoff $A(t)$ if the holder executes the option at the moment $t \in I$. Hence, we propose to consider a family of investments $(\Pi(t))_{t\in I}$ with the following meaning:

i) we make exactly one choice of $t \in I$ during the investment period, i.e. we decide to comply to the strategy $\Pi(t)$,

ii) the choice can be made only at times $t \in I$,

iii) we can choose $\Pi(t)$ only at the moment t,

iv) two strategies $\Pi(t)$ and $\Pi(s)$ are identical on $[0, \min(t,s)[$, $t, s \in I$.

To illustrate this idea consider an American option with payoff $(A(t))_{t\in I}$. Let Π be a strategy denoting the investment in this option priced at C. Assume that $I \subseteq]0,\infty[$. For $t \in I$

$$\Pi(t) = \text{pay } C \text{ at } 0\text{, receive } A(t) \text{ at } t.$$

If we find a sequence of strategies converging to $\Pi = \big(\Pi(t)\big)_{t\in I}$, we call C a hedging price. At the moment $s \notin I$ we proceed either according to $\Pi(t^*)$ if we have already made a choice at $t^* \in I$, $t^* < s$, or otherwise according to any strategy $\Pi(t)$ for $t > s$ (they are identical, see iv)). At $s \in I$ we check if the holder of the option decides to execute it. If he does, we proceed according to $\Pi(s)$, otherwise we take up any strategy $\Pi(t)$ for $t > s$.

In Black-Scholes model it is shown that a hedging price of an American contingent claim $(A(t))_{t\in I}$ is equal to

$$\sup_{\tau\in\mathcal{S}} \big\{ \text{ hedging price of a European option with payoff } A(\tau) \text{ expiring at } \tau \ \big\},$$

where $\mathcal{S}$ is a set of stopping times with values in I. Therefore, the uncertainty about the moment of execution does not rise costs of hedging. We prove that this is also the case in the American model with cash flows provided that there are no transaction costs. Other imperfections are allowed. This generality of the result points out particular importance of transaction costs in the modelling of financial markets. Moreover, our result holds even in the presence of arbitrage.

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2. Model of the European and American market

Let $\mathcal{T} \subseteq \mathbb{R}_+$ be a countable set and $0 \in \mathcal{T}$. Consider a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$. We construct a space X that supports investment opportunities. We allow for such investments that have cash flows of the size which is appropriately measurable and integrable and occur only in moments from $\mathcal{T}$. Fix $p \geq 1$ and set $X = L^1\big(\mathcal{T}, 2^{\mathcal{T}}, \nu; L^p(\Omega, \mathcal{F})\big)$, where ν is a counting measure, i.e. for $\gamma \in X$

$$\|\gamma\|_X = \int_{\mathcal{T}} \|\gamma(t)\|_p \, \nu(dt) = \sum_{t\in\mathcal{T}} \big(\mathbb{E}\,|\gamma(t)|^p\big)^{1/p}.$$

We denote by

$$\sum_{i=1}^{N} \gamma_i \delta_{\tau_i}$$

an investment opportunity which has N random cash flows $\gamma_1, \ldots, \gamma_N$ that occur only at random times $\tau_1, \ldots, \tau_N$. Positive cash flow represents receiving money, negative – paying. We assume that $\tau_1, \ldots, \tau_N \in \mathcal{S}$, where $\mathcal{S}$ is a set of stopping times with values in $\mathcal{T}$ and $\gamma_i \in L^p(\Omega, \mathcal{F}_{\tau_i})$, $i = 1, \ldots, N$. Let Γ be a linear space of all such expressions. It can be considered as a linear subspace of X.

To make the notation clear, consider the following examples. A strategy consisting of buying of two stocks priced at $S(1)$ at 1 and selling them for $S(3)$ at 3 is represented by the investment opportunity

$$-2S(1)\delta_1 + 2S(3)\delta_3,$$

where δ_t is a notation for the moment when a cash flow occurs. Similarly, if $\big(S(t)\big)_{t\in\mathbb{R}_+}$ stands for a random dynamics of asset prices,

$$-S(\tau)\delta_\tau + S(\sigma)\delta_\sigma,$$

where $\tau \leq \sigma$ are stopping times, represents cash flows linked to the strategy of buying of one stock at τ and selling it at σ. Further examples are the following:

$$-2S(1)\delta_1 + S(2)\delta_2 + 1_A S(3)\delta_3 + 1_{A^c} S(4)\delta_4,$$

where A is a random event, or

$$\theta\big(- S(\tau)\delta_\tau + S(\sigma)\delta_\sigma\big),$$

where θ is a random variable. Observe that the information about the strategy that generates the investment opportunity is lost. For further motivation consider the following investment opportunities:

$$-S^1(0)\delta_0 + S^1(1)\delta_1, \qquad \frac{S^1(1)}{S^2(1)}\big(- S^2(1)\delta_1 + S^2(2)\delta_2\big),$$

where $S^1(t)$, $S^2(t)$ are prices of different assets. If we decide to use both opportunities, the resulting cash flows reduce to

$$-S^1(0)\delta_0 + \frac{S^1(1)S^2(2)}{S^2(1)}\delta_2.$$

Therefore, we lose the information about the investment strategy we executed.

We describe a market by a set of available investment opportunities, satisfying reasonable conditions of convexity (ability to follow two investment opportunities at the same time) and positive homogeneity (infinite divisibility of investments).

DEFINITION 2.1. A European market j is a positive convex cone in Γ, i.e.

i) $\gamma_1 + \gamma_2 \in j$ if $\gamma_1, \gamma_2 \in j$,
ii) $\alpha\gamma_1 \in j$ if $\gamma \in j$, $\alpha \geq 0$.

Arbitrage opportunities are described by the following set:

$$\Gamma_+ = \{\sum_{i=1}^N \gamma_i \delta_{\tau_i} \in \Gamma : \quad \gamma_i \geq 0\}.$$

A self-explanatory notion of fairness of the market J is called no arbitrage condition

(NA) $j \cap \Gamma_+ = \{0\}$.

Observe that $-\Gamma_+$ is a set of all consumption plans. Then $j - \Gamma_+ = \{\gamma - \tilde{\gamma} : \quad \gamma \in \Gamma, \quad \tilde{\gamma} \in \Gamma_+\}$ represents investments enriched with consumption. Easily, $j \cap \Gamma_+ = \{0\}$ if and only if $(j - \Gamma_+) \cap \Gamma_+ = \{0\}$.

Condition (NA) is too weak in most of the models to obtain any reasonable results (see [2], [3]). Therefore, we introduce a stronger condition of no free lunch

(NFL) $\overline{j - \Gamma_+} \cap \Gamma_+ = \{0\}$, where the closure is made with respect to the norm topology of X.

It assures that it is impossible to get infinitely close to an arbitrage opportunity. The distance is specified by the norm of X. For further discussion on arbitrage see [2], [3] and [5].

EXAMPLE 2.1 (An illuminating model). Take $\mathcal{T} = \{0, 1, 2, 3\}$ and some $p \geq 1$. Then

$$X = L^p(\Omega, \mathcal{F}_0) \times L^p(\Omega, \mathcal{F}_1) \times L^p(\Omega, \mathcal{F}_2) \times L^p(\Omega, \mathcal{F}_3),$$

where $(\mathcal{F}_t)_{t \in I}$ is a filtration. The norm on X is equal to l^1-norm, i.e. the norm of $(\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in X$ is equal to $\|\gamma_0\| + \|\gamma_1\| + \|\gamma_2\| + \|\gamma_3\|$. Certainly, $\Gamma = X$ by trivial embedding: each

$$\gamma_0\delta_0 + \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3 \in \Gamma$$

has a representation as $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ in X. $\square$

Let $I \subseteq \mathcal{T}$ and a probability measure μ on $(I, 2^I)$ such that $\mu(t) > 0$ for all $t \in I$. Define $\mathcal{N} = L^1\big(I \times \mathcal{T}, 2^I \otimes 2^{\mathcal{T}}, \mu \otimes \nu; L^p(\Omega, \mathcal{F})\big)$. One can check that $\mathcal{N}$ is isomorphic to $L^1\big(I, 2^I, \mu; X\big)$. Notice that for $\sigma \in \mathcal{N}$

$$\|\sigma\|_{\mathcal{N}} = \sum_{t \in I} a(t) \|\sigma(t)\|_X.$$

Therefore, we come up with a very important observation for the construction of the model: if $\gamma_n \to \gamma$ in X then $\gamma_n(t) \to \gamma(t)$ in $L^p(\Omega, \mathcal{F})$, $t \in \mathcal{T}$; if $\sigma_n \to \sigma$ in $\mathcal{N}$ then $\sigma_n(s)(t) \to \sigma(s)(t)$ in $L^p(\Omega, \mathcal{F})$, $s \in I, t \in \mathcal{T}$.

Let us introduce a notation. For $\gamma \in X$ and $t \in I$

$$\gamma\zeta(t)$$

denotes an element $\sigma \in \mathcal{N}$ such that $\sigma(s) \equiv 0$ for $s \in I \setminus \{t\}$ and $\sigma(t) = \gamma$. Now, we can define a model for the **American market**. Put

$$\Sigma = \Big\{\sigma \in \mathcal{N} : \quad \sigma(t) \in \Gamma, \quad \sigma(s)\Big|_{[0,s)} = \sigma(t)\Big|_{[0,s)} \quad \forall t, s \in I, \quad s < t\Big\}.$$

Therefore, $\sigma \in \Sigma$ if $\sigma(s)$ and $\sigma(t)$ stand for identical investment opportunities on interval $[0, s)$ for arbitrary $s, t \in I$, $s < t$. The set Σ plays the same role for the American market as Γ for the European one.

EXAMPLE 2.2 (The illuminating model, contd.). Take $I = \mathcal{T} = \{0, 1, 2, 3\}$. Space $\mathcal{N}$ is identical to X^I with the norm

$$\|\sigma\|_{\mathcal{N}} = \|\sigma(0)\|_X + \|\sigma(1)\|_X + \|\sigma(2)\|_X + \|\sigma(3)\|_X.$$

For the future considerations we pick up the following notation for elements $\sigma \in \mathcal{N}$:

$$\sigma(i) = \gamma_0^i\delta_0 + \gamma_1^i\delta_1 + \gamma_2^i\delta_2 + \gamma_3^i\delta_3, \quad i \in I.$$

Set Σ consists of those elements $\sigma \in \mathcal{N}$ that satisfy

- $\gamma_0^1 = \gamma_0^2 = \gamma_0^3$,
- $\gamma_1^2 = \gamma_1^3$,

so, for instance, the strategies $\sigma(2)$ and $\sigma(3)$ agree on $[0, 2)$, i.e. in points 0 and 1.

This simple example enables us to show the intuitions that explain the form of American investment opportunities. Consider a strategy $\sigma \in \Sigma$ hedging an American contingent claim. At the moment $t = 0$ an owner of the option can ask for its execution or decide to wait. If the agent decides to execute, we follow the strategy $\sigma(0)$. Otherwise, we start investing with the strategy $\sigma(1)$ or $\sigma(2)$ or $\sigma(3)$, since they are identical in $t = 0$. At $t = 1$ we check if the agent asks for the execution of the option. If yes, the strategy $\sigma(1)$ leads us to fulfilling the obligation and from this point we follow it. If no, we take up either $\sigma(2)$ or $\sigma(3)$, both identical at $t = 1$. We proceed at $t = 2, 3$ in the similar way.

However, we have to stress that the equality of cash flows of two investment opportunities on interval $[0, t)$ does not guarantee that both trading strategies are identical. Any two different strategies can be modified in such a way that all cash flows from the interval $[0, t)$ are displaced to some $s \geq t$ by lending/borrowing money (bank account, bonds). Hence, both investment opportunities are equal zero on $[0, t)$ even if they are basically different. Therefore, the form of Σ is only a necessary condition for the above explanation to fully true. In the sequel we shall elaborate more on this issue. $\square$

DEFINITION 2.2. An **American market** is a subset $J \subseteq \Sigma$ such that

- J is a positive convex cone,
- (COH1) for any $\sigma \in J$, $t \in I$ there exists $\tilde{\sigma} \in J$ such that $\tilde{\sigma}(s) = \sigma(t)$, $s \in I$.

Condition (COH1) guarantees that for any investment opportunity $\sigma \in J$ and arbitrary $t \in I$ there exists $\tilde{\sigma} \in J$ with all coordinates equal to $\sigma(t)$ i.e. a strategy independent on $s \in I$.

The strategy $\sigma \in \Sigma$ should be called an arbitrage if there exists at least one possibility of execution $t \in I$ that guaranties arbitrage in the European sense, i.e. $\sigma(t) \in \Gamma_+ \setminus \{0\}$. Therefore, a natural set of arbitrage opportunities has the form

$$\Sigma_A = \{\sigma \in \Sigma : \quad \exists\, t \in I \quad \sigma(t) \in \Gamma_+ \setminus \{0\}\}.$$

However, from technical reasons we restrict ourselves to the set

$$\Sigma_+ = \{\sigma \in \Sigma : \quad \forall\, t \in I \quad \sigma(t) \in \Gamma_+\}.$$

It is explained by the fact that the set $\Sigma_- = -\Sigma_+$ represents consumption. This relation will play a crucial role in the future considerations. However the choice of Σ_+ seems restrictive, it will be shown that there is no difference in the notion of no-arbitrage for Σ_A and Σ_+ (cf. lemma 2.3).

The notion of no free lunch is defined as

(NFL) $\overline{J - \Sigma_+} \cap \Sigma_+ = \{0\}$, where the closure is taken in $\mathcal{N}$.

The condition (COH1) allows to show that above condition is equivalent to $\overline{J - \Sigma_+} \cap \Sigma_A = \emptyset$, where Σ_A is the set of arbitrage opportunities. Notice that Σ_A is much bigger that Σ_+, omitting the element 0.

LEMMA 2.3. Let J be an American market, $C = \overline{J - \Sigma_+}$. Then

i) C satisfies (COH1),

ii) $C \cap \Sigma_+ \neq \{0\}$ if and only if $C \cap \Sigma_A \neq \emptyset$

Proof. First, notice that $J - \Sigma_+$ satisfies (COH1) trivially. Only its closure needs a proof. Consider $\sigma \in C$ and choose $t \in I$. There exists a sequence $\sigma_n \in J - \Sigma_+$ converging to σ. Hence, $\sigma_n(t)$ converges to $\sigma(t)$ in X. We also know that $\sum_{s \in I} \sigma_n(t)\zeta(s) \in J - \Sigma_+$ and $\sum_{s \in I} \sigma_n(t)\zeta(s) \to \sum_{s \in I} \sigma(t)\zeta(s)$ in $\mathcal{N}$, which finishes the proof of i).

For ii) assume that $C \cap \Sigma_+ = \{0\}$ and $\sigma \in C \cap \Sigma_A$. There exists $t \in I$ such that $\sigma(t) \in \Gamma_+ \setminus \{0\}$. Hence, by i) there exists $\tilde{\sigma} \in C$ with

$$\tilde{\sigma} = \sum_{s \in I} \sigma(t)\zeta(s),$$

which contradicts NFL assumption. ■

Notice that above considerations are valid for different models of the European market. The only requirement is that X is a Banach space (for example see [5]). Moreover, most of the results presented in what follows also hold in more general situation.

3. European vs. American market

The following sections are concerned with comparison of pricing rules of different contingent claims on American and European markets. A key issue arising here is how to make a link between these two concepts.

DEFINITION 3.1. A European market j is linked to J ($j \sim J$) if

$$j = \{\sigma(t) : \quad \sigma \in J, \quad t \in I\}.$$

The set j is indeed a European market in the spirit of definition 2.1. It consists of all investments available to an American market player. It can be seen as a subset of J since in the view of condition (COH1) j is isometric with $J \cap \Sigma_H$, where

$$\Sigma_H = \{\sigma \in \Sigma : \quad \sigma(t) = \sigma(s) \quad \forall t, s \in I\}$$

is a set of those American investment opportunities that are identical on all coordinates. In other words, for any $\sigma \in J$ a market player can choose any execution time $t \in I$ and realize the investment opportunity $\sigma(t)$.

Recall the definition of assumption (NFL) for a European market:

(NFL) $\overline{j - \Gamma_+} \cap \Gamma_+ = \{0\}$, where the closure is taken in X.

We can formulate a lemma relating assumption (NFL) for J and for a linked European market j, addressing an important question of coherence between European and American markets.

LEMMA 3.2. Assume $j \sim J$.

$$\overline{j - \Gamma_+} \cap \Gamma_+ = \{0\} \qquad \Longleftrightarrow \qquad \overline{J - \Sigma_+} \cap \Sigma_+ = \{0\}.$$

Moreover,

$$\overline{J - \Sigma_+} \cap \Sigma_+ = \{0\} \qquad \Longleftrightarrow \qquad \overline{J \cap \Sigma_H - \Sigma_+} \cap \Sigma_+ = \{0\}.$$

Proof. We showed, under condition (COH1), that j is isometric with a subset $J \cap \Sigma_H$ of J. Clearly, existence of NFL in j implies the same in J. To prove the opposite implication take a sequence σ_n converging to $\sigma \in \Sigma_+ \setminus \{0\}$. Fix $t \in I$ such that $\sigma(t) \in \Gamma_+ \setminus \{0\}$. Convergence in $\mathcal{N}$ implies point-wise convergence, because I is countable. Hence, $\sigma_n(t) \to \sigma(t)$ in X, which contradicts assumption (NFL) for j. Second part is obvious. ■

4. Pricing

In this section we give formal definitions of contingent claims in the European and American context. We introduce pricing functionals in both cases and show important connections between them. For these connections to hold we do not need condition (NFL). We only assume that $\mathcal{F}_0$ is a trivial σ-field.

Fix an American market J and the linked European counterpart j. Let us introduce a notation

$$\begin{aligned} \mathcal{S} &= \{\tau - \text{ stopping time}: \quad \tau \in I \text{ a.s.}\}, \\ \mathcal{S}^t &= \{\tau \in \mathcal{S}: \quad \tau \geq t \text{ a.s.}\}, \quad t \in I. \end{aligned}$$

A European contingent claim is a pair (A, η), where $A \in L^p(\Omega, \mathcal{F}_\eta)$ is the claim size and η – the execution time – is a stopping time with values in $\mathcal{T}$.

DEFINITION 4.1. A European pricing functional at time 0 is given by

$$\Pi_0^e(A, \tau) = \inf\{c \in \mathbb{R}: \quad -c\delta(0) + A\delta(\tau) \in \overline{j - \Gamma_+}\},$$

where the closure is taken in X.

One can see that $\Pi_0^e(A, \tau)$ is the minimal hedging price. More about pricing in a European market can be found in [4], [5].

DEFINITION 4.2. An American contingent claim is a family $\big(A(t)\big)_{t \in I}$, where $A(t) \in L^p(\Omega, \mathcal{F}_t)$ is a claim size at t and $A(t) \in L^p(\Omega, \mathcal{F}_\tau)$ for any stopping time $\tau \in \mathcal{S}$.

DEFINITION 4.3. An American pricing functional at time 0 is given by

$$\Pi_0^a(A) = \inf\{c \in \mathbb{R} : \sum_{t \in I} \big(-c\delta(0) + A(t)\delta(t)\big)\zeta(t) \in \overline{J - \Sigma_+}\},$$

where the closure is taken in $\mathcal{N}$.

Similarly as in the European case, $\Pi_0^a(A)$ is a hedging price, i.e.

$$\sum_{t \in I} \big(-\Pi_0^a(A) + A(t)\delta_t\big)\zeta(t) \in \overline{J - \Sigma_+}.$$

For obvious reasons the price of an American claim $\big(A(t)\big)$ cannot be smaller than the price of any of European claims $\big(A(\tau), \tau\big)$ for $\tau \in \mathcal{S}$. We shall prove this fact under a weak condition imposed on the market J. It guarantees mixing of strategies for different execution times.

(COH2) For any $t_1 < t_2 \in I$, $B \in \mathcal{F}_{t_1}$ and $\sigma \in J$

$$\sum_{t \neq t_1} \sigma(t)\zeta(t) + \big(1_B\sigma(t_1) + 1_{B^c}\sigma(t_2)\big)\zeta(t_1) \in J.$$

Observe that although it is formulated for two times t_1, t_2 it implies the result for any finite number of times.

LEMMA 4.4. Let $\big(A(t)\big)_{t \in I}$ be an American contingent claim and assume that J satisfies (COH2). Then

$$\sup_{\tau \in \mathcal{S}} \Pi_0^e\big(A(\tau), \tau\big) \le \Pi_0^a(A).$$

Proof. If $\Pi_0^a(A) = \infty$, the result is obvious. Otherwise, we show that $\Pi_0^e\big(A(\tau), \tau\big) \le \Pi_0^a(A)$ for any $\tau \in \mathcal{S}$. We start with stopping times with two values and generalize to finite and countable number of values. Recall that any $\tau \in \mathcal{S}$ has a countable number of values, since I is a countable set.

Denote $c = \Pi_0^a(A)$. Let $\sigma_n \in J - \Sigma_+$ converge to $\sum_{t \in I} \zeta(t)\big(-c\delta(0) + A(t)\delta(t)\big)$. Let $\tau \in \mathcal{S}$ have two values $t_1 \le t_2$. Put $B = \{\tau = t_1\}$ and $\gamma_n = 1_B\sigma_n(t_1) + 1_{B^c}\sigma_n(t_2)$. By (COH2) $\gamma_n \in j$. Moreover,

$$\gamma_n \to -c\delta(0) + 1_B A(t_1)\delta(t_1) + 1_{B^c} A(t_2)\delta(t_2),$$

so $c \ge \Pi_0^e\big(A(\tau), \tau\big)$. Similar consideration proves this result for τ with finite number of values.

Now take $\tau \in \mathcal{S}$ with infinite number of values $\{q_1, q_2, \dots\}$. Let $q_n^* = \max(q_1, \dots, q_n)$, $n \in \mathbb{N}$. We construct a sequence of stopping times $\tau_n \in \mathcal{S}$ that are restrictions of τ to the first n values. The probability mass of the remaining values is put on q_n^*:

$$\tau_n = \sum_{i=1}^{n} 1_{\tau = q_i} q_i + \sum_{i=n+1}^{\infty} 1_{\tau = q_i} q_n^*, \quad n \in \mathbb{N}.$$

Since τ_n has a finite number of values, previous considerations imply that $\Pi_0^e\big(A(\tau_n),\tau_n\big) \leq \Pi_0^a(A)$ for any $n \in \mathbb{N}$. Define $A_n = A(\tau_n)1_{\tau\neq q_1,\dots q_n}$ and observe that A_n is $\mathcal{F}_{\tau_n}$-measurable and $\Pi_0^e\big(A_n,\tau_n\big) \leq \Pi_0^e\big(A(\tau_n),\tau_n\big)$. Denote $a_n = \Pi_0^e\big(A_n,\tau_n\big)$. The sequence (a_n) is bounded from above and increasing, since $A_n = A_{n+1}$ on $\operatorname{supp} A_n$. Hence, there exists a limit a and

$$-a_n\delta(0) + A_n\delta(\tau_n) \to -a\delta(0) + A(\tau)\delta(\tau) \qquad \text{in } X.$$

This convergence is justified by the fact that (A_n) is dominated by $A(\tau)$ (it is a restriction of $A(\tau)$ to subsets of Ω). This implies that $-a\delta(0) + A(\tau)\delta(\tau) \in j$ and $a \geq \Pi_0^e\big(A(\tau),\tau\big)$. Obviously, $a \leq \Pi_0^a(A)$ since $a_n \leq \Pi_0^a(A)$. Therefore,

$$\Pi_0^e\big(A(\tau),\tau\big) \leq \Pi_0^a(A).$$

■

The following example shows that condition (COH2) cannot be skipped in the formulation of lemma 4.4. It guarantees appropriate mixing of strategies.

EXAMPLE 4.3. Consider a model with $\mathcal{T} = \{0,1,2\}$, $I = \{1,2\}$, $p \geq 1$. Let J be an American market generated by

$$\begin{gathered}\big(-S(0)\delta(0) + S(1)\delta(1)\big)\zeta(1) + \big(-S(0)\delta(0) + S(2)\delta(2)\big)\zeta(2),\\ \big(-S(0)\delta(0) + S(1)\delta(1)\big)\big(\zeta(1)+\zeta(2)\big),\\ \big(-S(0)\delta(0) + S(2)\delta(2)\big)\big(\zeta(1)+\zeta(2)\big),\end{gathered}$$

where $S(t)$ denotes price process of a stock with $S(t) \in L^p(\Omega,\mathcal{F}_t)$, $t \in \mathcal{T}$. Hence, the related European market j is generated by

$$-S(0)\delta(0) + S(1)\delta(1), \qquad -S(0)\delta(0) + S(2)\delta(2).$$

Take an American contingent claim with payoff $A(1) = S(1)$, $A(2) = S(2)$. Obviously, $\Pi_0^a(A) = S(0)$. However, $\Pi_0^e\big(A(\tau),\tau\big) = 2S(0)$ for any stoppping time τ such that $\mathbb{P}(\tau = 1) > 0$ and $\mathbb{P}(\tau = 2) > 0$. □

Our further efforts will be put on proving that

$$\sup_{\eta\in\mathcal{S}} \Pi_0^e\big(A(\eta),\eta\big) = \Pi_0^a(A)$$

if there are no transaction costs.

Extended market

The presented model of the market does not enable us to construct a set of investment strategies that start after some moment. The fact that the strategy is null before, say, t_1 does not assure that no transactions were held before t_1, e.g. we borrow some money from a bank and invest it in the stocks; at $t_2 > t_1$ we sell it and pay back the loan. Hence, we have only one cash flow at t_2, but the decision of this pursuit was made much before t_1.

For the simplicity of notation introduce

$$I^* = I \cup \{0\}.$$

DEFINITION 4.5. A family $\big(J^s\big)_{s\in I^*}$ is called an **extended market** if

i) J^s is an American market for $s \in I^*$,

ii) $J^s \subseteq J^t$ for $s, t \in I^*$, $s \le t$,

iii) $\bigcup_{s\in I^*} J^s = J^0 = J$,

iv) $\sigma(t)\big|_{[0,s)} = 0 \quad \forall \sigma \in J^s, \quad \forall t \in I^*$,

v) J^s satisfies COH2 for all $s \in I^*$,

vi) (COH3) $\forall s, \tilde{s}, t \in I^* \quad s \le t \quad \forall \sigma, \tilde{\sigma} \in J^t$

$$\sigma + \big(\tilde{\sigma}(\tilde{s}) - \sigma(s)\big)\zeta(s) \in J^t.$$

Condition (COH3) is only an extension of (COH1). Take $\sigma \in J^t$ and recall that the decision about choosing an investment strategy $\sigma(s)$, $s \le t$, is made before or at t. Hence, if trading starts at t, the coordinates $\sigma(s)$, $s \le t$, are independent (not constrained) of each other and of the remaining part of σ. So they can be replaced by any available investment opportunity.

The family $\big(J^s\big)_{s\in I^*}$ defines a related family of European markets $\big(j^s\big)_{s\in I^*}$. The sets of arbitrage opportunities or consumption plans for American and European markets are the following

$$\begin{aligned} \Sigma_+^s &= \{\sigma \in \Sigma_+ : \quad \sigma(t)\big|_{[0,s[} = 0 \quad \forall t \in I\}, \\ \Gamma_+^s &= \{\gamma \in \gamma_+ : \quad \gamma\big|_{[0,s[} = 0\}. \end{aligned}$$

Notice that $(J^s - \Sigma_+^s)_{s\in I^*}$ satisfies conditions i)-vi) of definition 4.5 and $(j^s - \Gamma_+^s)_{s\in I^*}$ is a related European market to $(J^s)_{s\in I^*}$.

DEFINITION 4.6. A **European pricing functional** at the time $s \in I^*$ is given by

$$\Pi_s^e(A, \tau) = \operatorname{ess\,inf}\big\{C \in L^p(\Omega, \mathcal{F}_s, \mathbb{P}) : \quad -C\delta(s) + A\delta(\tau) \in \overline{j^s - \Gamma_+^s}\big\}$$

for a European contingent claim (A, τ) with $\tau \in \mathcal{S}^s$ with convention $\operatorname{ess\,inf} \emptyset = \infty$.

DEFINITION 4.7. An **American pricing functional** at the time $s \in I^*$ is given by

$$\pi_s^a(A) = \operatorname{ess\,inf}\big\{C \in L^p(\Omega, \mathcal{F}_s) : \quad \sum_{t\in I\cap[s,\infty)} \big(- C\delta_s + A(t)\delta_t\big)\zeta(t) \in \overline{J^s - \Sigma_+^s}\big\}$$

for an American contingent claim $\big(A(t)\big)_{t\in I}$ with convention $\operatorname{ess\,inf} \emptyset = \infty$.

In what follows we shall assume that the following condition is satisfied:

(BND) the price $\Pi^e_s(A,\tau)$ is either infinite a.s. or is contained in $L^p(\Omega,\mathcal{F}_s)$ for any European claim (A,η) and $s\in I$, such that $s\le\eta$ a.s.

This condition is not restrictive. It assures that it is impossible to borrow infinite amount of money (in the sense of the space L^p) that can be paid up by a finite sum in the future.

Now we shall show that $\Pi^e_s(A,\eta)$ is a hedging price for the European option (A,τ). First, we prove a technical lemma.

LEMMA 4.8. For any $\gamma_1,\gamma_2\in\overline{j^s-\Gamma^s_+}$ and $B\in\mathcal{F}_s$

$$1_B\gamma_1+1_{B^c}\gamma_2\in\overline{j^s-\Gamma^s_+}.$$

Proof. From the definition of a related European market

$$\sigma_1=\sum_{t\in I}\gamma_1\zeta(t)\in\overline{J^s-\Sigma^s_+}\qquad\text{and}\qquad\sigma_2=\sum_{t\in I}\gamma_2\zeta(t)\in\overline{J^s-\Sigma^s_+}.$$

From (COH3)

$$\sigma_3=\sum_{t\neq s}\gamma_2\zeta(t)+\gamma_1\zeta(s)\in\overline{J^s-\Sigma^s_+}.$$

From (COH2)

$$\sigma=\sum t\neq s\gamma_2\zeta(t)+\big(1_B\gamma_1+1_{B^c}\gamma_2\big)\zeta(s)\in\overline{J^s-\Sigma^s_+}.$$

Hence, $\sigma(s)\in\overline{j^s-\Gamma^s_+}$. ■

LEMMA 4.9. Let $s\in I^*$ and (A,τ) be a European claim with $\tau\in\mathcal{S}^s$. The price $\Pi^e_s(A,\tau)$, if finite, is a hedging price, i.e.

$$-\Pi^e_s(A,\tau)\delta_s+A\delta_\tau\in\overline{j^s-\Gamma^s_+}.$$

Proof. We shall use the fact that the family

$$\mathcal{A}=\big\{C\in L^p(\Omega,\mathcal{F}_s):\quad -C\delta_s+A\delta_\tau\in\overline{j^s-\Gamma^s_+}\big\}$$

is directed downward, i.e. for any $C_1,C_2\in\mathcal{A}$ there exists $C\in\mathcal{A}$ such that $C\le\min(C_1,C_2)$. To prove it, take $C_1,C_2\in\mathcal{A}$ and put $C=\min(C_1,C_2)$. By lemma 4.8

$$1_{C_1\le C_2}\big(-C_1\delta_s+A\delta_\tau\big)+1_{C_2<C_1}\big(-C_2\delta_s+A\delta_\tau\big)\in\overline{j^s-\Gamma^s_+}.$$

Hence, $C\in\mathcal{A}$.

There exists a non-increasing sequence $(C_n)_{n\in\mathbb{N}}\subseteq\mathcal{A}$ point-wise convergent to $C=\operatorname{ess\,inf}\mathcal{A}$. From boundedness (from above by C_1, from below by C), this convergence holds in $L^p(\Omega,\mathcal{F}_s,\mathbb{P})$, too. Therefore $-C_n\delta_s+A\delta_\eta$ tends in X to $-C\delta_s+A\delta_\eta$, which proves that $C\in\mathcal{A}$. Hence, C is a hedging price. ■

It is also possible to prove that $\pi^a_s(A)$ is a hedging price of the American claim $\big(A(t)\big)$, but we do not require this fact.

No transaction costs

In the case of a complete market model the price of an American contingent claim A is equal to $\sup_{\tau\in\mathcal{S}} \mathbb{E}_{\mathcal{Q}} A(\tau)$, where $\mathcal{Q}$ is a unique martingale measure. Here we shall show an analogous relation

$$\sup_{\tau\in\mathcal{S}} \Pi_0^e\big(A(\tau),\tau\big) = \Pi_0^a(A)$$

under the condition that there are no transaction costs. Our reasoning will resemble the one for a complete discrete time model.

DEFINITION 4.10. An extended market $(J^s)_{s\in I^*}$ satisfies **NTC (no transaction costs)** if for all $t, s \in I^*$, $t \geq s$ and for any $\sigma \in \overline{J^s - \Sigma_+^s}$ there exists $\sigma_1 \in \overline{J^s - \Sigma_+^s}$ and $\sigma_2 \in \overline{J^t - \Sigma_+^t}$ such that

$$\sigma = \sigma_1 + \sigma_2 \qquad \text{and} \qquad \sigma_1(r)\big|_{]t,\infty[} = 0 \quad \forall r \in I^*,$$

where the closure is taken in $\mathcal{N}$.

This definition implies directly similar property of the related European market: for all $t, s \in I^*$, $t \geq s$ and for any $\gamma \in \overline{j^s - \Gamma_+^s}$ there exists $\gamma_1 \in \overline{j^s - \Gamma_+^s}$ and $\gamma_2 \in \overline{j^t - \Gamma_+^t}$ such that

$$\gamma = \gamma_1 + \gamma_2 \qquad \text{and} \qquad \gamma_1\big|_{]t,\infty[} = 0,$$

where the closure is taken in X.

THEOREM 4.11. If I is a finite set and $\big(J^s\big)_{s\in I^*}$ satisfies NTC, then for any American contingent claim $\big(A(t)\big)_{t\in I}$

$$\Pi_0^a(A) = \sup_{\tau\in\mathcal{S}} \Pi_0^e\big(A(\tau),\tau\big).$$

The proof of this theorem is divided into several lemmas. Notice that if not stated so, there are no assumptions concerning the set I. It must be finite only for the proof of the theorem.

LEMMA 4.12. Let $s \in I$ and $\tau \in \mathcal{S}^s$. Assume that a family of random variables $(\xi_k)_{k\in\theta} \subseteq L^p(\Omega, \mathcal{F}_\tau)$ is directed upward, i.e. for any $k, l \in \theta$ there exists $k^* \in \theta$ such that $\xi_{k^*} \geq \max(\xi_k, \xi_l)$. If $\operatorname*{ess\,sup}_{k\in\theta} \xi_k \in L^p(\Omega, \mathcal{F}_\tau)$ and the price $\Pi_s^e(\operatorname*{ess\,sup}_{k\in\theta} \xi_k, \tau)$ is finite, then

$$\Pi_s^e(\operatorname*{ess\,sup}_{k\in\theta} \xi_k, \tau) = \operatorname*{ess\,sup}_{k\in\theta} \Pi_s^e(\xi_k, \tau).$$

Proof. First notice that Π_s^e is a monotonic mapping i.e. $\xi, \xi' \in L^p(\Omega, \mathcal{F}_\tau)$, $\xi \leq \xi'$ yields $\Pi_s^e(\xi, \tau) \leq \Pi_s^e(\xi', \tau)$. Denote $\xi^* = \operatorname*{ess\,sup}_{k\in\theta} \xi_k$. Monotonicity of Π_s^e yields

$$\Pi_s^e(\xi_k, \tau) \leq \Pi_s^e(\xi^*, \tau) \text{ a.s.}$$

and therefore

$$\operatorname*{ess\,sup}_{k\in\theta} \Pi_s^e(\xi_k, \tau) \leq \Pi_s^e(\xi^*, \tau) \text{ a.s.}$$

Consequently, $\operatorname*{ess\,sup}_{k\in\theta} \Pi^e_s(\xi_k,\tau) \in L^p(\Omega,\mathcal{F}_s)$.

Now, we shall prove that

$$\Pi^e_s(\xi^*,\tau) \le \operatorname*{ess\,sup}_{k\in\theta} \Pi^e_s(\xi_k,\tau), \quad \text{a.s.}$$

Since $(\xi_k)_{k\in\theta}$ is directed upward there exists a sequence $(k_n)_{n\in\mathbb{N}} \subseteq \theta$ such that $\xi_{k_n} \nearrow \xi^*$ point-wise and in $L^p(\Omega,\mathcal{F}_\tau)$ from dominated convergence theorem. Hence, $C^n = \Pi^e_s(\xi_{k_n},\tau)$ is a non-decreasing sequence bounded by $\operatorname*{ess\,sup}_{k\in\theta} \Pi^e_s(\xi_k,\tau)$. Consequently, C^n is point-wise convergent and, from bounded convergence theorem, in L^p. Hence

$$-C^n\delta_s + \xi_{k_n}\delta_\tau \to -\big(\lim_{n\to\infty} C^n\big)\delta_s + \xi^*\delta_\tau \quad \text{in } X.$$

Thus

$$\Pi^e_s(\xi^*,\tau) \le \lim_{n\to\infty} C^n \le \operatorname*{ess\,sup}_{k\in\theta} \Pi^e_s(\xi_k,\tau).$$

■

LEMMA 4.13. Assume the market J satisfies NTC. For $s,t \in I^*$, $t \le s$, $\tau \in \mathcal{S}^t$ and $A \in L^p(\Omega,\mathcal{F}_\tau)$ such that $\Pi^e_t(A,\tau)$ is finite

$$\Pi^e_t\big(A,\tau\big) = \Pi^e_t\big(1_{\tau<s}A + 1_{\tau\ge s}\Pi^e_s(A,\tau)\big).$$

Proof. Let $C = \Pi^e_t\big(A,\tau\big)$. Since C is a hedging price $C \in L^p(\Omega,\mathcal{F}_t)$. By lemma 4.9 we have $-C\delta(t) + A\delta(\tau) \in \overline{j^s - \Gamma^s_+}$. From NTC there exists $W \in L^p(\Omega,\mathcal{F}^s)$ such that $-C\delta_t + A\delta_\tau = \gamma_1 + \gamma_2$, where

$$\begin{aligned} \gamma_1 &= -C\delta_t + 1_{\tau<s}A\delta_\tau + 1_{\tau\ge s}W\delta_s \quad &\in \overline{j^t - \Gamma^t_+}, \\ \gamma_2 &= 1_{\tau\ge s}\big(-W\delta_s + A\delta_\tau\big) \quad &\in \overline{j^s - \Gamma^s_+}. \end{aligned}$$

Hence, $W \ge \Pi^e_s\big(1_{\tau\ge s}A, \tau\vee s\big)$, where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. Obviously, we can take $W = \Pi^e_s\big(1_{\tau\ge s}A,\tau\vee s\big)$. By lemma 4.8 $\Pi^e_s\big(1_{\tau\ge s}A,\tau\vee s\big) = 1_{\tau\ge s}\Pi^e_s\big(1_{\tau\ge s}A,\tau\vee s\big)$. Considering γ_1 we obtain

$$C \ge \Pi^e_t\big(1_{\tau<s}A + 1_{\tau\ge s}W, \tau\wedge s\big).$$

Denote

$$C_1 = \Pi^e_t\big(1_{\tau<s}A + 1_{\tau\ge s}W, \tau\wedge s\big)$$

and observe that

$$\big(-C_1\delta_t + 1_{\tau<s}A\delta_\tau + 1_{\tau\ge s}W\delta_s\big) + 1_{\tau\ge s}\big(-W\delta_s) + A\delta_\tau\big) = -C_1\delta_t + A\delta_\tau,$$

which yields $C_1 = C$. ■

LEMMA 4.14. Assume that the market J satisfies NTC. For $s,t\in I^*$, $t\le s$ and an American contingent claim $\big(A(t)\big)_{t\in I}$ set

$$\hat A(r)\equiv Y(r),\quad r<s,$$
$$\hat A(s)=\operatorname*{ess\,sup}_{\tau\in\mathcal S^s}\Pi^e_s\big(A(\tau),\tau\big).$$

If $\hat A(s)\in L^p(\Omega,\mathcal F_s)$ and $\Pi^e_s\big(\hat A(\vartheta),\vartheta\big)<\infty$ for $\vartheta\in\mathcal S^t,\vartheta\le s$, then

$$\operatorname*{ess\,sup}_{\tau\in\mathcal S^t}\Pi^e_t\big(A(\tau),\eta\big)=\operatorname*{ess\,sup}_{\vartheta\in\mathcal S^t,\vartheta\le s}\Pi^e_s\big(\hat A(\vartheta),\vartheta\big).$$

Proof. Fix $\tau\in\mathcal S^t$ and put $C=\Pi^e_t\big(A(\tau),\tau\big)$. By lemma 4.13

$$C=\Pi^e_t\Big(1_{\tau<s}A(\tau)+\Pi^e_s\big(1_{\tau\ge s}A(\tau),\tau\vee s\big),\tau\wedge s\Big).$$

We know that $\Pi^e_s\big(A(\tau),\tau\vee s\big)\in L^p(\Omega,\mathcal F_s)$ since $\Pi^e_s\big(A(\tau),\tau\vee s\big)\le\hat A(s)$. By lemma 4.8

$$\Pi^e_s\big(1_{\tau\ge s}A(\tau),\tau\vee s\big)=1_{\tau\ge s}\Pi^e_s\big(A(\tau),\tau\vee s\big).$$

Thus $C\le\Pi^e_t\big(\hat A(\tau),\tau\big)$ and

$$\operatorname*{ess\,sup}_{\tau\in\mathcal S^t}\Pi^e_t\big(A(\tau),\tau\big)\le\operatorname*{ess\,sup}_{\vartheta\in\mathcal S^t,\vartheta\le s}\Pi^e_s\big(\hat A(\vartheta),\vartheta\big).$$

For the proof of the opposite inequality fix $\vartheta\in\mathcal S^t$, $\vartheta\le s$. Consider a family of stopping times $\theta=\big\{\tau\in\mathcal S^t:\quad\tau\equiv\vartheta\text{ on }\{\sigma<s\}\big\}$. For any $\tau\in\theta$ lemma 4.13 implies

$$\begin{aligned}\Pi^e_t\big(A(\tau),\tau\big)&=\Pi^e_t\Big(1_{\tau<s}A(\tau)+1_{\tau\ge s}\Pi^e_s\big(A(\tau),\tau\vee s\big),\tau\wedge s\Big)\\&=\Pi^e_t\Big(1_{\vartheta<s}A(\vartheta)+1_{\vartheta=s}\Pi^e_s\big(A(\tau),\tau\vee s\big),\tau\wedge s\Big).\end{aligned}$$

Similarly as above, by lemma 4.8

$$\Pi^e_s\big(1_{\vartheta=s}A(\tau),\tau\vee s\big)=1_{\vartheta=s}\Pi^e_s\big(A(\tau),\tau\vee s\big).$$

Observe also that $\tau\wedge s=\vartheta$. Hence, by lemma 4.12

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau\in\theta}\Pi^e_t\Big(1_{\vartheta<s}A(\vartheta)+1_{\vartheta=s}\Pi^e_s\big(A(\tau),\tau\vee s\big),\vartheta\Big)\\&\qquad=\Pi^e_t\Big(1_{\vartheta<s}A(\vartheta)+1_{\vartheta=s}\operatorname*{ess\,sup}_{\tau\in\theta}\Pi^e_s\big(A(\tau),\tau\vee s\big),\vartheta\Big)=\Pi^e_t\big(\hat A(\vartheta),\vartheta\big).\end{aligned}$$

Thus

$$\Pi^e_t\big(\hat A(\vartheta),\vartheta\big)=\operatorname*{ess\,sup}_{\tau\in\theta}\Pi^e_t\big(A(\tau),\tau\big)\le\operatorname*{ess\,sup}_{\tau\in\mathcal S^t}\Pi^e_t\big(A(\tau),\tau\big).$$

■

Proof of theorem 4.11. From lemma 4.4 we have

$$\Pi_0^a(A) \geq \sup_{\tau\in\mathcal{S}} \Pi_0^e\big(A(\tau),\tau\big).$$

If the right-hand side is infinite, the left-hand side is also infinite and the equality follows. Otherwise, $\Pi_0^e\big(A(\tau),\tau\big)$ are bounded for $\tau \in \mathcal{S}$. Moreover, $\Pi_s^e\big(A(\tau),\tau\big) \in L^p(\Omega,\mathcal{F}_s)$ for any $\tau \in \mathcal{S}^s$, $s \in I$, because of 4.13 for $t = 0$ i $A = A(\eta)$. Notice also that random variables

$$A = \operatorname*{ess\,sup}_{\eta\in\mathcal{S}^s} \Pi_s^e\big(A(\eta),\eta\big)$$

are in $L^p(\Omega,\mathcal{F}_s)$ since

$$A \leq \sum_{t\in I, t\geq s} \Pi_s^e\big(A(t),t\big)$$

and above sum consists of finite number of elements (the set I is finite), which are finite.

Without loss of generality we assume that $I = \{0, t_1, t_2\}$, $t_1 < t_2$. Let

$$C_1 = \Pi_{t_1}^e\big(A(t_2),t_2\big) \vee A(t_1) \qquad \text{and} \qquad C_0 = \Pi_0^e\big(C_1,t_1\big).$$

Lemma 4.14 implies that $C_0 = \sup_{\tau\in\mathcal{S}} \Pi_0^e\big(A(\tau),\tau\big)$. We prove the theorem once we construct a strategy $\sigma \in \overline{J - \Sigma_+}$ that hedges the American contingent claim $\big(A(t)\big)_{t\in I}$. Denote

$$\gamma^1 = -C_0\delta_0 + C_1\delta_{t_1} \in \overline{j^0 - \Gamma_+^0} \qquad \text{and} \qquad \gamma^2 = -C_1\delta_{t_1} + A(t_2)\delta_{t_2} \in \overline{j^{t_1} - \Gamma_+^{t_1}}.$$

Let $\gamma_n^1 \in j^0 - \Gamma_+^0$, $\gamma_n^2 \in j^{t_1} - \Gamma_+^{t_1}$ be sequences of European investment opportunities converging respectively to γ^1 and γ^2 in X. By (COH3)

$$\sigma_n = \gamma_n^1\zeta(t_1) + (\gamma_n^1 + \gamma_n^2)\zeta(t_2) \in J - \Sigma_+.$$

Obviously, $\sigma_n \to \sigma$ in $\mathcal{N}$

$$\sigma = \big(-C_0\delta_0 + C_1\delta_{t_1}\big)\zeta(t_1) + \big(-C_0\delta_0 + A(t_2)\delta_{t_2}\big)\zeta(t_2)$$

that hedges the claim $(A(t))$. ■

Observe that condition NTC was only used in the proof of lemma 4.13. Therefore, we can replace it in the assumptions of the theorem 4.11 by

(PD) For $s, t \in I^*$, $t \leq s$, $\tau \in \mathcal{S}^t$ and $A \in L^p(\Omega,\mathcal{F}_\tau)$

$$\Pi_t^e\big(A,\tau\big) = \Pi_t^e\big(1_{\tau<s}A + \Pi_s^e(1_{\tau\geq s}A, \tau\vee s), \tau\wedge s\big).$$

We shall also stress that in the proofs presented above we do not use the fact that the market satisfies (NFL). These results hold in the case of a market with arbitrage opportunities, too. And as it was pointed out these markets are not trivial in the face of the possibility that there is no bank account or any other similar instrument enabling to transfer money between different moments.

5. Example

Let $\mathcal{T}$ be any at most countable subset of $\mathbb{R}_+$, $I \subseteq \mathcal{T}$, $p \geq 1$. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$. It supports a family $(S^\alpha)_{\alpha \in \mathcal{A}}$ of adapted stochastic processes representing asset prices. We do not impose any restrictions with respect to cardinality of $\mathcal{A}$. We only require that $S^\alpha(\eta) \in L^p(\Omega, \mathcal{F}_\eta)$ for arbitrary $\alpha \in \mathcal{A}$, $\eta \in \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ is a set of stopping times with values in $\mathcal{T}$. By $\tilde{\mathcal{S}}^s$, $s \in I^*$, we denote a set of stopping times $\eta \in \tilde{\mathcal{S}}$ greater or equal to s. We impose constraints on portfolios by fixing a positive convex cone $C \subseteq \mathbb{R}^{\mathcal{A}}$ denoting admissible portfolio contents. We assume that one transaction can only affect a finite number of stocks.

We shall construct an American market with constraints on portfolio contents. The idea of construction is easy, but requires complicated notation. Let $\Theta(\eta)$, where $\eta \in \tilde{\mathcal{S}}$, denotes a family of functions $\theta : \mathcal{A} \to L^\infty(\Omega, \mathcal{F}_\eta)$ with the following properties:

i) $\theta(\alpha) = 0$ a.s. apart from a finite number of arguments $\alpha \in \mathcal{A}$,

ii) $\big(\theta(\alpha), \alpha \in \mathcal{A}\big) \in C$ a.s.

This is a set of available transactions at time η.

The construction of J^s is divided into two parts. By definition, if $\sigma \in J^s$, then $\sigma\big|_{[0,s)} \equiv 0$, and the coordinates $\sigma(t)$, $t \in I \cap [0, s]$, are independent, but the coordinates $\sigma(t)$, $t \in I \cap (s, \infty)$, are constrained by relations expressed in Σ. Set $\mathcal{Y}^s$ answers to first case and $\mathcal{X}^s$ – to the second one.

The set $\mathcal{X}^s$, $s \in I^*$, consists of functions $\Delta : I \cap (s, \infty) \to \tilde{\mathcal{S}}^s \times \tilde{\mathcal{S}}^s \times \Theta(\infty)$ with the following properties: taking up the notation of coordinates $\Delta(u) = (\eta_u, \vartheta_u, \theta_u)$, $u \in I \cap (s, \infty)$, for $u, v \in I \cap (s, \infty)$

$$\begin{aligned} \theta_u &\in \Theta(\eta_u), \\ \eta_u &< \vartheta_u, \\ \eta_u = \eta_v,\ \theta_u &= \theta_v \text{ on } \{\eta_u < u\} \cup \{\eta_v < u\}, \\ \vartheta_u &= \vartheta_v \text{ on } \{\vartheta_u < u\} \cup \{\vartheta_v < u\}. \end{aligned}$$

An element $\Delta \in \mathcal{X}^s$ defines an American investment opportunity σ having on coordinate $t > s$ the result of a trading strategy consisting of buying a portfolio θ_t at η_t and selling it at ϑ_t. Analogously we construct the set $\mathcal{Y}^s$, $s \in I^*$, that contains functions $\Delta : I \cap [0, s] \to \tilde{\mathcal{S}}^s \times \tilde{\mathcal{S}}^s \times \Theta(\infty)$ with the following properties: taking up the notation of coordinates $\Delta(u) = (\eta_u, \vartheta_u, \theta_u)$, $u \in I \cap [0, s]$,

$$\begin{aligned} \theta_u &\in \Theta(\eta_u), \\ \eta_u &< \vartheta_u. \end{aligned}$$

The extended American market J^s, $s \in I^*$, consists of smallest positive convex cones J^s

containing

$$\sum_{t\in I\cap(s,\infty)}\sum_{\alpha\in\mathcal{A}}\theta(t)(\alpha)\big(S^\alpha(\vartheta_t)\delta_{\vartheta_t}-S^\alpha(\eta_t)\delta_{\eta_t}\big)\zeta(t),\qquad(\eta_t,\vartheta_t,\theta_t)\in\mathcal{X}^s,$$
$$\sum_{t\in I\cap[0,s]}\sum_{\alpha\in\mathcal{A}}\theta(t)(\alpha)\big(S^\alpha(\vartheta_t)\delta_{\vartheta_t}-S^\alpha(\eta_t)\delta_{\eta_t}\big)\zeta(t),\qquad(\eta_t,\vartheta_t,\theta_t)\in\mathcal{Y}^s.$$

EXAMPLE 5.4 (The illuminating model, contd.). Recall that in this model $\mathcal{T}=I=\{0,1,2,3\}$ and every American investment opportunity σ can be expressed as

$$\sigma(i)=\gamma_0^i\delta_0+\gamma_1^i\delta_1+\gamma_2^i\delta_2+\gamma_3^i\delta_3,\quad i\in I.$$

For ease of presentation fix $\mathcal{A}=\{1,2\}$ and some positive convex cone $C\subseteq\mathbb{R}^2$. An extended American market consists of four elements: J^0, J^1, J^2 and J^3. The last one is trivial, having only one element: zero. We shall present a detailed construction of J^1. A counterpart to investments generated by $\mathcal{X}^1$ is a subset of Σ with the following properties:

$$\begin{aligned}&\gamma_j^0=\gamma_j^1=0,\quad j=0,1,2,3,\\&\gamma_1^2=\gamma_1^3=-\theta(1)S^1(1)-\theta(2)S^2(1),\\&\gamma_k^2=\gamma_k^3=\theta(1)S^1(k)+\theta(2)S^2(k)\end{aligned}$$

for $k\in\{2,3\}$, $\theta\in L^\infty(\Omega,\mathcal{F}_1;C)$ or

$$\begin{aligned}&\gamma_2^k=-\theta(1)S^1(2)-\theta(2)S^2(2),\\&\gamma_3^k=\theta(1)S^1(3)+\theta(2)S^2(3),\end{aligned}$$

for $k\in\{2,3\}$, $\theta\in L^\infty(\Omega,\mathcal{F}_2;C)$ with other coordinates equal to zero. The set $\mathcal{Y}^1$ generates investments in Σ with all coordinates equal to zero apart from

$$\begin{aligned}&\gamma_k^i=-\theta(1)S^1(k)-\theta(2)S^2(k),\\&\gamma_l^i=\theta(1)S^1(l)+\theta(2)S^2(l)\end{aligned}$$

for $k,l,i\in I$, $1\le k<l$, $\theta\in L^\infty(\Omega,\mathcal{F}_l;C)$. □

If I and $\mathcal{A}$ are finite sets, the market can be constructed in a much easier way. The market J^s is the smallest positive convex cone containing

$$\begin{aligned}&\bigcup_{t\in I\cap(t,\infty)}J^t,\\&\gamma\zeta(t),\quad\gamma\in\Xi(s),\quad t\in I\cap[0,s],\\&\sum_{u\in I\cap(t,\infty)}\gamma\zeta(u),\quad\gamma\in\Xi(s),\quad t\ge s,\end{aligned}$$

where

$$\Xi(s) = \Big\{ \sum_{\alpha\in\mathcal{A}} \theta(\alpha)\big(S^\alpha(\vartheta)\delta_\vartheta - S^\alpha(\eta)\delta_\eta : \quad \eta,\vartheta \in \tilde{\mathcal{S}}^s, \quad \eta < \vartheta, \quad \theta \in L^\infty(\Omega,\mathcal{F}_\eta, C)\Big\},$$

and C is treated as a subset of the euclidean space $\mathbb{R}^{\mathcal{A}}$.

Let us denote the constructed model by $\mathcal{J}\big((S^\alpha)_{\alpha\in\mathcal{A}}, C\big)$. This model is quite general. It covers financial instruments such as stocks, defaultable bonds, bank accounts, etc. We can also include investments in options. Notice that we can consider different interest rates for borrowing and lending.

LEMMA 5.1. The family $\mathcal{J}\big((S^\alpha)_{\alpha\in\mathcal{A}}, C\big)$ is an extended American market.

The proof of this fact is straightforward. Now, we shall prove condition (PD).

THEOREM 5.2. If asset prices are non-negative and $C \subseteq [0,\infty)^{\mathcal{A}}$, then condition (PD) is satisfied on the market $\mathcal{J}\big((S^\alpha)_{\alpha\in\mathcal{A}}, C\big)$:

$$\Pi^e_t\big(A,\eta\big) = \Pi^e_t\big(1_{\eta<s}A + \Pi^e_s(1_{\eta\geq s}A, \eta\vee s), \eta\wedge s\big), \quad s,t\in I, \quad s>t, \quad \eta\in\mathcal{S}^t$$

for pay-offs satisfying $\Pi^e_s(A,\eta) < \infty$.

Proof. If the left-hand side of (PD) is infinite, the equality is immediate. Assume that $\Pi^e_t\big(A,\eta\big)$ is finite. In the proof we will strongly use the fact that convergence of γ_n to γ is linked with convergence of cash flows $\gamma_n(t)$ at time $t\in\mathcal{T}$ to $\gamma(t)$ in $L^p(\Omega,\mathcal{F}_t)$.

By the proof of lemma 4.13 it is enough to prove NTC for investments of the form $-C\delta_t + A\delta_\eta$. Fix $t\in\mathcal{T}$ and $s\in I^*$, $s>t$, $s\leq\eta$ a.s. Let $\gamma = -C\delta_t + A\delta_\eta \in \overline{j^s - \Gamma^s_+}$ and $\gamma_n \in j^s - \Gamma^s_+$ be a sequence converging to γ. Each investment opportunity γ_n has a decomposition as a sum $\phi_n + \psi_n$ satisfying conditions of NTC, i.e. $\phi_n \in j^t - \Gamma^t_+$, $\psi_n \in j^s - \Gamma^s_+$ and $\phi|_{(s,\infty)} \equiv 0$. It is a result of the fact that there are no transaction costs. Let us denote by $w_n \in L^p(\Omega,\mathcal{F}_s)$ the cash flow at the moment s in the investment ϕ_n and by $v_n \in L^p(\Omega,\mathcal{F}_s)$ – the cash flow in the moment s in the investment ψ_n. Obviously, $w_n + v_n$ is equal to the cash flow of γ_n at time s, thus this sum tends to 0 in $L^p(\Omega,\mathcal{F}_s)$. Moreover, v_n is a non-negative random variable, since the market does no allow for credits.

Put $v = -\Pi^e_s(A,\eta)$ and $B = \{v_n < v\}$. The investment

$$\psi^*_n = 1_B(v\delta_s + A\delta_\eta) + 1_{B^c}\gamma_n$$

belongs to $\overline{j^s - \Gamma^s_+}$. The cash flow at s equals $v^*_n = 1_B v + 1_{B^c}v_n$. We shall modify the investments ϕ_n setting

$$\phi^*_n = \phi_n + (v_n - v^*_n)\delta_s.$$

Since $v_n - v^*_n \leq 0$ we have $\phi^*_n \in \overline{j^t - \Gamma^t_+}$.

By lemma A1.1 from [1] there exists a sequence $\tilde{v}^*_n \in \operatorname{conv}(v^*_n, v^*_{n+1}, \ldots)$ converging pointwise and in $L^p(\Omega,\mathcal{F}_s)$ to a random variable $\tilde{v}^*$. Let us take identical convex combinations of investments ψ^*_n and ϕ^*_n, and denote them, respectively, by $\tilde{\psi}^*_n$, $\tilde{\phi}^*_n$. Thus

$$\tilde{\psi}^*_n \to \tilde{v}^*\delta_s + A\delta_\eta \tag{1}$$

and

$$\tilde{\phi}_n^* \to -C\delta_t - v^*\delta_s,$$

because the cash flow of ϕ_N^* at s is equal to $w_n + v_n - v_n^*$, and $w_n + v_n$ tends to zero in $L^p(\Omega, \mathcal{F}_s)$. ■

Notice that $\tilde{v}^* = -\Pi_s^e(A, \eta)$. It results from the fact that

$$0 \geq v_n^* \geq -\Pi_s^e(A, \eta),$$

so $\tilde{v}^*$ satisfies the same inequalities. On the other hand, from (1) we have $\tilde{v}^* \leq -\Pi_s^e(A, \eta)$.

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