

Arbitrage and topology in modelling of financial markets by cash flows.

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Abstract

We consider a problem of the definition of no free lunch (NFL) condition in the family of models describing the market by means of cash flows, see [3], [4], [7]. Each investment strategy is represented by cashflows that it generates. The market is given as a positive convex cone of such investment strategies. We use a generalization of Kreps-Yan theorem given in [4] to obtain an equivalent characterization of NFL: existence of a discount for the market such that the expected discounted payoff of all investments is non-positive. We give an example of a pair of normed spaces in natural duality that satisfy assumptions of Kreps-Yan theorem – the norm topology is easier to work with (eg. topological closure and sequential closure are identical).

The topology chosen for the space of all investment strategies plays a crucial role in the definition of NFL, i.e. decides of the quality of the model. We characterize minimal requirements for "good models" and provide examples that satisfy these requirements and assumptions of Kreps-Yan theorem.

1. Introduction

Consider a model where any investment strategy is described by the cash flows it generates. The expression

$$\Phi = \sum_{i=1}^N \gamma_i \delta_{\tau_i} \in \Gamma$$

represents an investment opportunity in which random cash flows $\gamma_1, \dots, \gamma_N$ occur only in random times τ_1, \dots, τ_N . Positive cash flow represents receiving money, negative – paying.

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Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \in I}$ satisfying usual conditions, where either $I = [0, T]$, $T < \infty$ or $I = \mathbb{R}_+$. Let Γ be a linear subspace of

$$\tilde{\Gamma} = \left\{ \sum_{i=1}^N \gamma_i \delta_{\tau_i} : \text{for some } N \in \mathbb{N}, \gamma_i \in L^1(\Omega, \mathcal{F}_{\tau_i}, \mathbb{R}), \tau_i \in I - \text{stopping time} \right\}. \quad (1)$$

Γ is a set of all investment opportunities that we can model. The set J of investment strategies available on the market is a subset of Γ .

DEFINITION 1.1. A **market** is a positive convex cone in Γ .

Arbitrage opportunities are described by the following set:

$$\Gamma_+ = \left\{ \sum_{i=1}^N \gamma_i \delta_{\tau_i} \in \Gamma : \gamma_i \geq 0 \right\}.$$

We introduce a locally convex topology τ on Γ . It will be inevitable to the definition of fairness of the market. A self-explanatory notion of fairness of the market J is called no arbitrage condition

$$(NA) \quad J \cap \Gamma_+ = \{0\}.$$

However, it is too weak in most of the models to obtain any reasonable results. Therefore we introduce a stronger condition of no free lunch

$$(NFL) \quad \overline{J - \Gamma_+} \cap \Gamma_+ = \{0\}, \text{ where the closure is made with respect to the topology } \tau.$$

It assures that it is impossible to get infinitely close to an arbitrage opportunity. The distance is specified by the topology τ defined on Γ . This topology plays a key role in the model construction. It can be linked to the set of all continuous linear functionals Γ^* defined on Γ . We will see that instead of dealing with topologies we can concentrate on the set of linear functionals they generate. The characterization of the NFL condition can be done in terms of positive linear functionals i.e. NFL holds if and only if there exists a continuous linear functional y on Γ such that $y|_J \leq 0$ and $y|_{\Gamma_+ \setminus \{0\}} > 0$.

DEFINITION 1.2. A **market model** is a pair (Γ, τ) , where Γ is a linear subspace of $\tilde{\Gamma}$, τ is a locally convex topology on Γ .

First version of the above model of the financial market appeared in [3] and was generalized and researched in [6], [4] and [7].

The definition of (NFL) heavily depends on the choice of topology τ . In some topologies the market can satisfy (NFL), in others – not. Therefore, the choice of the "right" topology is a very important question. In the first part of the paper we compare several topologies and try to find conditions that would grant good properties of the model from a practical point of view. In the second part we propose models that satisfy those properties. One of them is a generalization of the results from [7]. We remove a technical condition following the approach in [4]. As a result

we obtain a pair of normed spaces in natural duality that satisfies assumptions of the Kreps-Yan theorem presented in [4], which we consider a new result. Moreover, we give a representation of positive continuous linear functionals in terms of adapted stochastic processes. We show that any RCLL process with bounded and positive trajectories satisfying a "conditional Lipschitz property" defines a positive functional.

Section 2 presents a slight generalization of the Kreps-Yan separation theorem and its application to constructing an equivalent formulation to NFL assumption. Section 3 presents our survey of properties of topologies on Γ . Section 4 is devoted to the generalization of the model from [7]. In sections 5 and 6 we propose new models of the market.

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2. Kreps-Yan separation theorem and NFL

The Kreps-Yan theorem is a main building block of the present paper. Although it is used much later, it is presented here for better understanding of the ideas in section 3. We slightly generalize the approach presented in [4]. Let (X, τ) be a locally convex linear topological space. By Y we denote a dual to (X, τ) . We assume that Y separates points of X . Let Γ be a linear subspace of X with topology τ_Γ induced by τ . In Γ we distinguish the set of non-negative elements Γ_+ .

We define two subsets of Y . The set of linear functionals non-negative on Γ_+ is denoted by $Y_+^{\Gamma_+}$, i.e. $y \in Y_+^{\Gamma_+} \equiv \forall x \in \Gamma_+ \langle x, y \rangle \geq 0$. The set of linear functionals positive on Γ_+ is denoted by $Y_{++}^{\Gamma_+}$, i.e. $y \in Y_{++}^{\Gamma_+} \equiv \forall x \in \Gamma_+ \setminus \{0\} \langle x, y \rangle > 0$.

We impose two key assumptions concerning the spaces X, Y, Γ_+ . We refer the reader to [4] for the discussion of these assumptions.

Assumption C. For every sequence $(y_n)_{n=1}^\infty \in Y$ there exist a sequence of strictly positive numbers $(\alpha_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \alpha_n y_n$ converges in Y with respect to $\sigma(Y, X)$ topology.

Assumption L. For any family $(y_\alpha)_{\alpha \in I} \subset Y_+^{\Gamma_+}$ there exist a countable subset $(y_{\alpha_n})_{n=1}^\infty$ such that if $x \in \Gamma_+$ and $\langle x, y_\alpha \rangle > 0$ for some $\alpha \in I$ then one can find n such that $\langle x, y_{\alpha_n} \rangle > 0$.

THEOREM 2.1. Assume that the spaces X, Y, Γ satisfy C and L. For any positive, convex cone $C \subset \Gamma$, which is closed in topology τ_Γ and such that $-\Gamma_+ \subset C$ the following statements are equivalent:

- i) $C \cap \Gamma_+ = \{0\}$,
- ii) $\exists y \in Y_{++}^{\Gamma_+} \ y|_C \leq 0$.

Proof. We start with i) \implies ii). Once we have assumptions C and L the proof is straightforward. First note that i) implies that $\bar{C} \cap \Gamma_+ = \{0\}$, where the closure is taken in the space (X, τ) . Therefore for any $x \in \Gamma_+ \setminus \{0\}$ we can find a continuous linear functional y_x on (X, τ) such that $\langle x, y_x \rangle > 1$ and $y_x|_C \leq 1$ (see [1] V.2.12). In fact, $y_x|_C \leq 0$, since C is a positive cone. Every

linear functional on (X, τ) is an element of Y , so we can treat y_x as such. Moreover, $y_x|_{-\Gamma_+} \leq 0$, which yields $y_x|_{\Gamma_+} \geq 0$ and finally $y_x \in Y_+^{\Gamma_+}$.

Consider a family $(y_x)_{x \in \Gamma_+ \setminus \{0\}}$. By assumption L we can find a countable family $(y_{x_n})_{n=1}^\infty$ such that for any $x \in \Gamma_+ \setminus \{0\}$ there exists y_{x_n} such that $\langle x, y_{x_n} \rangle > 0$. Assumption C gives us a sequence of strictly positive numbers α_n such that $\sum_{n=1}^\infty \alpha_n y_{x_n} \rightarrow y$ in $\sigma(Y, X)$ topology. Clearly $y \in Y_{++}^{\Gamma_+}$ and $y|_C \leq 0$, that is exactly ii).

For the reverse implication take any $x \in \Gamma_+ \cap C$. From the fact that $x \in C$ we obtain $\langle x, y \rangle \leq 0$. From the fact that $x \in \Gamma_+$ we get $\langle x, y \rangle \geq 0$. Therefore $\langle x, y \rangle = 0$ and $x = 0$ by the definition of $Y_{++}^{\Gamma_+}$. ■

Now, let us give a sketch of the application of the above theorem. X is a linear space that supports the set of all investments opportunities Γ . We take $C = \overline{J - \Gamma_+}$, the closure in the topology on Γ , where J is the cone of investments available on the market. Hence, i) is exactly the NFL condition. By theorem 2.1, it is equivalent to the existence of $y \in Y_{++}^{\Gamma_+}$ such that $y|_C \leq 0$. Moreover, it is easy to see that for $y \in Y_{++}^{\Gamma_+}$ the conditions $y|_J \leq 0$ and $y|_C \leq 0$ are equivalent. We call any $y \in Y_{++}^{\Gamma_+}$ such that $y|_J \leq 0$ a **discount**. To sum it up, NFL for J is equivalent to the existence of a discount.

3. Discussion of various topologies on Γ

The property NFL strongly depends on the topology used to make the closure. Hence, as was shown above, it depends on the set of positive continuous linear functionals $Y_{++}^{\Gamma_+}$ on X , the supporting space of Γ . We restrict ourselves to the topologies τ on Γ for which the set of continuous linear functionals Y can be represented as a subspace of measurable (i.e. measurable as a function of two variables) stochastic process with the duality given as follows: for $(y(t))_{t \in I} \in Y$ and $x = \sum_{i=1}^N \gamma_i \delta_{\tau_i} \in \Gamma$

$$\langle x, y \rangle = \mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i)$$

with the right-hand side well-defined. It captures all the models proposed in [4] and [7].

In the beggining we show that not all measurable processes are allowed to be in Y . We make a non restrictive

Assumption S1. $\{\delta_\tau : \tau \text{ is a bounded stopping time}\} \subseteq \Gamma$

to prove that the optional projection of processes in Y is well-defined. Let τ be a bounded stopping time, $y \in Y$. For $x = \delta_\tau \in \Gamma$ $\langle x, y \rangle = \mathbb{E} y(\tau)$ implies that $y(\tau)$ is integrable, from the definition of duality. Therefore, there exists an optional projection ${}^o y$ of y (see theorem 5.1 in

[2]). Let $x = \sum_{i=1}^N \gamma_i \delta_{\tau_i} \in \Gamma$:

$$\begin{aligned} \langle x, y \rangle &= \mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) = \mathbb{E} \sum_{i=1}^N \mathbb{E} [\gamma_i y(\tau_i) | \mathcal{F}_{\tau_i}] \\ &= \mathbb{E} \sum_{i=1}^N \gamma_i \mathbb{E} [y(\tau_i) | \mathcal{F}_{\tau_i}] = \mathbb{E} \sum_{i=1}^N \gamma_i {}^o y(\tau_i). \end{aligned} \tag{2}$$

Hence, only optional projections of elements of Y play a role in the duality.

LEMMA 3.1. Assume that $\Gamma = \tilde{\Gamma}$ (see (1)). Let τ be a stopping time :

- i) ${}^o y(\tau) \in L^\infty(\Omega, \mathcal{F}_\tau, \mathbb{R})$ for $y \in Y$,
- ii) ${}^o y(\tau) > 0$ a.s. for $y \in Y_{++}^{\Gamma+}$.

Proof. i) Obviously, $\{\gamma \delta_\tau : \gamma \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{R})\} \subseteq \Gamma$. Hence, ${}^o y(\tau) \gamma \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{R})$ for all $\gamma \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{R})$. It implies that ${}^o y(\tau)$ is bounded. For the completeness of reasoning we present a short proof of this fact. Assume on the contrary that ${}^o y(\tau)$ is unbounded. Without loss of generality a positive part of ${}^o y(\tau)$ is unbounded. Let $A_n = \{n \leq {}^o y(\tau) < n+1\}$, $p_n = \mathbb{P}(A_n)$. Put $a_n = \frac{1}{n^2 p_n}$ and define $\gamma^* = \sum_{n=1}^\infty 1_{A_n} a_n$. By $\sum_{n=1}^\infty a_n p_n = \sum_{n=1}^\infty n^{-2} < \infty$ we obtain $\gamma^* \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{R})$, but $\mathbb{E} \gamma^* {}^o y(\tau) \geq \sum_{n=1}^\infty n a_n p_n = \sum_{n=1}^\infty n^{-1} = \infty$, a contradiction.

ii) Take $x = 1_A \delta_\tau$, where $A = \{{}^o y(\tau) \leq 0\}$ and assume that $\mathbb{P}(A) > 0$. Since $x \in \Gamma_+ \setminus \{0\}$, we obtain $\mathbb{E} {}^o y(\tau) 1_A > 0$. However, $\mathbb{E} {}^o y(\tau) 1_A \leq 0$ from the choice of A . Hence, $x = 0$, which leads to a contradiction. ■

Consider a simple example of the market. Let $(S(t))_{t \in \mathbb{R}_+}$ be an adapted process representing asset prices. We impose a technical requirement that $S(\tau) \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{R})$ for any bounded stopping time τ . We define a market $J(S)$ as a smallest positive convex cone containing basic investment opportunities of the following form

$$\Phi_t = \theta (-\delta_\tau S(\tau) + \delta_\sigma S(\sigma)),$$

where τ, σ are bounded stopping times, $\tau < \sigma$ and $\theta \in L^\infty(\Omega, \mathcal{F}_\tau, \mathbb{R})$. Let us call $J(S)$ a perfect market. It can be easily proved (see [3]) that $J(S)$ satisfies NFL if and only if there exists $y \in Y_{++}^{\Gamma+}$ such that ${}^o y(t) S(t)$ is a martingale.

In the sequel, we list all important, in our opinion, properties of discount processes that appeared in the literature. We look at them from a financial point of view and argue which of them are "reasonable". We end the list with an alternative approach.

Left-continuity

We give an example of the price process S for which the market $J(S)$ is intuitively fair but any adapted discount process y such that $S(t)y(t)$ is a martingale cannot be left-continuous. Let us take S to be a purely discontinuous Levy process with jumps bigger than $\epsilon > 0$ and filtration

generated by this model. We denote by τ a moment of the first jump. The filtration up to time $\tau-$ is trivial. An adapted process must be deterministic on $[0, \tau[$. Any adapted discount process y has to satisfy the following relation: $\mathbb{E} S(\tau)y(\tau) = S(\tau-)y(\tau-)$. If y is left-continuous $y(\tau) = y(\tau-)$. It is possible only if $\mathbb{E} S(\tau) = S(\tau-) = S(0)$, that is S is itself a martingale on $[0, \tau]$ and this is a rare event in financial models. Hence, the family of processes with left-continuous optional projections is too small to characterize all reasonable NFL markets.

Right-continuity

We argue that the family of processes with RCLL optional projections is too big. It defines a topology on Γ that is too weak to grab intuitive meaning of the fairness of the market. Consider a deterministic process $S(t) = 1 + 1_{t \geq 2}$. A process $y(t) = 1 - \frac{1}{2}1_{t \geq 2}$ is a discount process for $J(S)$. However,

$$\phi_n = -S\left(1 - \frac{1}{n}\right)\delta_{1-\frac{1}{n}} + S(1)\delta_1 \in J(S)$$

and ϕ_n converges intuitively to an arbitrage opportunity δ_1 . It is even more striking because the market $J(S')$, where $S'(t) = 1 + 1_{t > 2}$ (S' is left-continuous in 1), lacks NFL. The sequence

$$\Psi_n = -S(1)\delta_1 + S\left(1 + \frac{1}{n}\right)\delta_{1+\frac{1}{n}}$$

converges to δ_1 , which complies with the intuition.

Continuity

Assume that Y consists of continuous measurable processes. Their optional projections are RCLL but they form a strict subset of RCLL processes satisfying integrability requirements for Y . It is easy to see that they induce such a topology on Γ that both Φ_n and Ψ_n converge to δ_1 . Therefore, this topology lacks asymmetry and points out unfairness. Moreover, it does not point out NFL for the case of purely discontinuous Levy process with jumps bounded away from 0. However, we do not have any hint that this is the right class of processes. We only showed that it lacks a few evident drawbacks.

Alternative approach

We have seen that it is easy to point out drawbacks of different kinds of linear functionals on Γ . However, we did not manage to find any satisfactory positive result. In this subsection, instead of beginning our considerations from the set Y of linear functionals we start with the properties of topology τ on Γ .

Assumption CV. For any monotone (increasing or decreasing) sequence of stopping times $\sigma_n \rightarrow \sigma$ a.s. and any sequence of random variables Z_n converging to Z in $L^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$, such that $(Z_n\delta_{\sigma_n}) \subseteq \Gamma$, $Z\delta_\sigma \in \Gamma$,

$$Z_n\delta_{\sigma_n} \rightarrow Z\delta_\sigma$$

in topology τ , which is equivalent to

$$\mathbb{E} Z_n y(\sigma_n) \rightarrow \mathbb{E} Z y(\sigma)$$

for all $y \in Y$.

The meaning of assumption CV is completely clear once we pick up a simple example. Assume that a sequence of investment opportunities

$$\Phi_n = -Z_n \delta_{\sigma_n} + \tilde{Z} \delta_\sigma$$

is contained in Γ , Z_n converges to Z in $L^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$, σ_n converges to σ and $Z_n \delta_{\sigma_n} \rightarrow Z \delta_\sigma$ in Γ . Intuitively, Φ_n tends to $(\tilde{Z} - Z) \delta_\sigma$. Assumption CV grasps exactly this intuition.

We also need a technical assumption:

Assumption R. The set Y of linear functionals can be represented as a family of adapted RCLL processes.

Above properties enable us to improve the result of lemma 3.1.

LEMMA 3.2. Assume CV, R and $\Gamma = \tilde{\Gamma}$. Any $y \in Y$ is bounded on $[0, \eta]$ for any stopping time η .

Proof. Assume by contradiction that y is unbounded on some $[0, \eta]$. Since y is RCLL, random moments $\sigma_n = \inf\{t \leq \eta : y(t) \geq n\} \wedge \eta$, with $\inf \emptyset = \infty$, are stopping times that form an increasing sequence. Moreover, $\sigma = \lim_{n \rightarrow \infty} \sigma_n$ is a stopping time. Let $a_n = \mathbb{P}(y_{\sigma_n} \geq n)$. We can easily check that y is unbounded if and only if $a_n > 0$ for all $n \in \mathbb{N}$. We distinguish two cases:

1) $a_n \rightarrow 0$

Consider $Z_n = 1_{y_{\sigma_n} \geq n} (a_n \sqrt{n})^{-1}$. Clearly, $\mathbb{E} Z_n = n^{-1/2}$, so $Z_n \in L^1(\Omega, \mathcal{F}_{\sigma_n}, \mathbb{P})$ and $Z_n \rightarrow 0$ in $L^1(\Omega, \mathcal{F}_\sigma, \mathbb{P})$. However, $\langle y, Z_n \delta_{\sigma_n} \rangle = \mathbb{E} Z_n y(\sigma_n) \geq \sqrt{n} \rightarrow \infty$, which contradicts property CV.

2) $a_n > \epsilon > 0$

In this simpler case, we take $Z_n = 1$. We calculate $\langle y, Z_n \delta_{\sigma_n} \rangle = \mathbb{E} y(\sigma_n) \geq \epsilon n \rightarrow \infty$ and $\langle y, \delta_\sigma \rangle = \mathbb{E} y(\sigma) < \infty$. This contradicts CV, too.

■

Although we leave the examples to the following sections, we wish to present here two more results. They should shed some light on the assumption CV and R.

LEMMA 3.3. The topology on $\Gamma = \tilde{\Gamma}$ with a space of linear functionals Y , that can be represented as a set of bounded continuous measurable functions, satisfies R and CV.

Proof. The space Y can be replaced by the space Y^* of optional projections of processes in Y . Moreover, optional projection of a continuous process is RCLL. Hence, we have R. To prove CV consider a sequence of Z_n, σ_n as in the definition. By continuity and uniform boundedness $y(\sigma_n)$ converges uniformly to a bounded random variable $y(\sigma)$. Therefore,

$$\mathbb{E} Z_n y(\sigma_n) \rightarrow \mathbb{E} Z y(\sigma).$$

■

LEMMA 3.4. Assume that $\Gamma = \tilde{\Gamma}$ and Y is a set of all bounded RCLL adapted processes. Then Y does not satisfy CV.

Proof. We point out a sequence Z_n, σ_n and $y \in Y$ such that the requirement of CV is not fulfilled. Consider $y(t) = 1_{t \geq \eta}$ for any stopping time η . Take $Z_n = Z = 1$ and σ_n any increasing sequence of stopping times converging to η . Hence, $\mathbb{E} Z_n y(\sigma_n) = 0$, but $\mathbb{E} Z y(\eta) = 1$. ■

COROLLARY 3.5. The models with topologies given by sets of all right-continuous processes fails to satisfy CV.

Survey of models

Jouini, Napp and Schachermayer explore a bundle of models in [4]. However, their approach leads to two distinct sets $Y_{++}^{\Gamma+}$ consisting of left- or right-continuous adapted processes. Therefore, as was shown above, they lack desirable financial properties. In the following sections, we wish to propose two models that satisfy the requirement of continuity and one that satisfies solely CV and R, and therefore in view of above discussion, seems to be the correct one.

4. Generalization of the model from [7]

This model is based on the results obtained in [7]. We have put the necessary information about spaces \mathcal{M} and $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$ in the appendix A. We use the notation from the section 2. We set $X = L_{\mathbb{P}}^1(\Omega, \mathcal{M})$ equipped with the norm topology. Y is a Banach space $L_*^\infty(\Omega, \mathcal{M}')$. It is dual to X . The topology $\sigma(Y, X)$ coincides with the norm topology of Y . We take $\Gamma = \tilde{\Gamma}$. We consider a model (Γ, τ) , where τ is a topology on Γ induced by a norm topology of X . This is an example of a pair of normed spaces with normed topologies that satisfy assumptions of Kreps-Yan theorem.

Take $\mu \in \Gamma$ with representation $\mu = \sum_{i=1}^N \gamma_i \delta_{\tau_i}$ and $y \in L_*^\infty(\Omega, \mathcal{M}')$. Theorem 7.11 suggests that y can be regarded as a bounded measurable stochastic process. Hence, from (2)

$$\langle \mu, y \rangle_{(L_{\mathbb{P}}^1(\Omega, \mathcal{M}), L_*^\infty(\Omega, \mathcal{M}'))} = \mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) = \mathbb{E} \sum_{i=1}^N \gamma_i^o y(\tau_i).$$

THEOREM 4.1. Let J be a positive convex cone of available investment opportunities in the model (Γ, τ) . J satisfies NFL if and only if there exists a positive linear functional $y \in Y_{++}^{\Gamma+}$ such that $y|_J \leq 0$.

Proof. We will use theorem 2.1. We have to check the following conditions:

- 1) Y separates points of X ,
- 2) X is a locally convex linear topological space,
- 3) assumption C,
- 4) assumption L.

Statements 1) and 2) result from the fact that X and Y are Banach spaces in duality. Assumption C is a straightforward implication of the fact that the topology $\sigma(Y, X)$ is equal to the topology under which Y is a Banach space: for instance, if we take $\alpha_n = \|y_n\|_{L_*^\infty(\Omega, \mathcal{M}')}^{-1} 2^{-n}$, the sequence

$$\left(\sum_{i=1}^n \alpha_n y_n \right)_{n>0}$$

forms a Cauchy sequence in $L_*^\infty(\Omega, \mathcal{M}')$ and is therefore convergent. For the proof of assumption L we will use the approach from [4].

Let $(y_\alpha)_{\alpha \in I} \subset Y_+^{\Gamma+}$ be a given family of functionals. We will construct a sequence $(\alpha_n) \subset I$ and then show that it satisfies desired properties. In the proof we will widely use the optional projection of y that we will denote by ${}^o y$. We recall that ${}^o y$ is an RCLL (cádlág) process since y is continuous. We also point out that $y \in Y_+^{\Gamma+}$ if and only if ${}^o y(t) \geq 0$ for all $t \in \mathbb{R}_+$. Let $(U_i)_{i=1}^\infty$ be an enumeration of all open intervals in \mathbb{R}_+ with rational endpoints. We put

$$A_{i,\alpha} = \{\omega : {}^o y_\alpha(t)(\omega) > 0 \ \forall t \in U_i\}, \quad i \in \mathbb{N}, \alpha \in I.$$

Fix $i \in \mathbb{N}$. We can find a sequence (not necessarily unique) $(\alpha_n(i))_{n=1}^\infty \subset I$ such that $A_{i,\alpha} \subseteq A_i$ a.s. for any $\alpha \in I$, where $A_i = \bigcup_{n=1}^\infty A_{i,\alpha_n(i)}$. One way to obtain the sequence $\alpha_n(i)$ is to build it iteratively; once we have defined it for $n < N$ we put $\alpha_N(i) \in \{\alpha : \mathbb{P}(A_{i,\alpha} \setminus B_N(i)) \geq \sup_{\alpha' \in I} \mathbb{P}(A_{i,\alpha'} \setminus B_N(i)) - 2^{-N}\}$, where $B_N(i) = \bigcup_{i=1}^{N-1} A_{i,\alpha_n(i)}$. We define

$$\begin{aligned} S_\alpha &= \{(t, \omega) : {}^o y_\alpha(t)(\omega) > 0\}, \quad \alpha \in I, \\ S_\alpha(\omega) &= \{t : {}^o y_\alpha(t)(\omega) > 0\}, \quad \alpha \in I, \\ L_\alpha(\omega) &= \{t : \exists \epsilon > 0 \ {}^o y_\alpha(s)(\omega) > 0 \text{ for } s \in (t, t + \epsilon) \text{ and } {}^o y_\alpha(t)(\omega) = 0\}, \quad \alpha \in I \end{aligned}$$

and

$$\begin{aligned} S &= \{(t, \omega) : \exists n, i \ {}^o y_{\alpha_n(i)}(t)(\omega) > 0\}, \\ S(\omega) &= \{t : \exists n, i \ {}^o y_{\alpha_n(i)}(t)(\omega) > 0\}, \\ L(\omega) &= \bigcup_{n, i \in \mathbb{N}} L_{\alpha_n(i)}(\omega). \end{aligned}$$

By construction of $(\alpha_n(i))_{n=1}^\infty$ we obtain that $\mathbb{P}(\omega : S_\alpha(\omega) \supseteq U_i, S(\omega) \not\supseteq U_i) = 0$ for any $\alpha \in N, i \in \mathbb{N}$. Therefore $S_\alpha(\omega) \setminus S(\omega) \subseteq L(\omega)$ a.s. We have to find a countable subset of functionals that would exhaust L . First note that $L = \bigcup_{\omega \in \Omega} \{\omega\} \times L(\omega)$ is optional. Obviously, $L = \bigcup_{n,i \in \mathbb{N}} L_{\alpha_n(i)}$, where $L_\alpha = \bigcup_{\omega \in \Omega} \{\omega\} \times L_\alpha(\omega)$. Hence, we need to prove that L_α is an optional set. Random moments of successive jumps of the process ${}^o y_\alpha$ from 0 to some positive level are well-defined (right-continuity implies that there is a positive distance between consecutive jumps) and form a sequence of stopping times. Graphs of stopping times are optional sets, therefore L_α as a countable sum of optional sets is itself optional. Similarly, L as a union of countable subfamily of $(L_\alpha)_{\alpha \in I}$ is optional. Moreover, $L(\omega)$ is a countable subset of \mathbb{R}_+ . If it is evanescent then we have nothing to do. Otherwise, by Optional Section Theorem (see VI.5.1 in [8]) there exists a sequence of stopping times T_k such that

$$L = \bigcup_{k=1}^{\infty} \{(\omega, T_k(\omega)) : T_k < \infty\}$$

up to an evanescent set (all statements are made up to evanescence). We proceed in the same way as in the construction of $A_{i,\alpha}$ and A_i to obtain

$$B_{k,\alpha} = \{\omega : T_k(\omega) < \infty \text{ and } {}^o y_\alpha(T_k)(\omega) > 0\}$$

and a sequence $(\beta_n(k))_{n \in \mathbb{N}}$ such that $B_{k,\alpha} \subseteq B_k$ a.s. for any $\alpha \in I$, where $B_k = \bigcup_{n \in \mathbb{N}} B_{k,\beta_n(k)}$.

We easily verify that the family $(y_{\alpha_n(i)})_{i,n \in \mathbb{N}} \cup (y_{\beta_n(k)})_{k,n \in \mathbb{N}}$ satisfies requirements of assumption L. We take any $x \in \Gamma_+$ and $\alpha \in I$ such that $\langle x, y_\alpha \rangle > 0$. We want to show that $\langle x, y_{\alpha_n(i)} \rangle > 0$ for some n, i . We can write x in the form $x = \sum_{i=1}^N \gamma_i \delta_{\tau_i}$, for some $N \in \mathbb{N}, \gamma_i \in L^1(\Omega, \mathcal{F}_{\tau_i}, \mathbb{R}_+)$ and stopping times τ_i . So, straight from the definition

$$\langle x, y_\alpha \rangle = \mathbb{E} \langle x, y_\alpha \rangle_{\langle \mathcal{M}, \mathcal{M}' \rangle} = \mathbb{E} \sum_{i=1}^N \gamma_i y_\alpha(\tau_i).$$

Hence, it suffices to prove that if $\langle \gamma \delta_\tau, y_\alpha \rangle > 0$ for $\gamma \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{R}_+)$ a.s. and some α then $\langle \gamma \delta_\tau, y_{\alpha_n(i)} \rangle > 0$ for some n, i . Put $D = \{(t, \omega) : \gamma(\omega) > 0, \tau(\omega) = t\}$. Condition $\langle \gamma \delta_\tau, y_\alpha \rangle = \mathbb{E} \gamma y_\alpha(\tau) > 0$ yields that $S_\alpha \cap D$ is not evanescent. Hence, $(S \cup L) \cap D$ is not evanescent. There exist n, i such that $(S_{\alpha_n(i)} \cup L_{\alpha_n(i)}) \cap D$ is not evanescent and $\mathbb{E} \gamma y_{\alpha_n(i)}(\tau) > 0$. This finishes the proof of the assumption L. \blacksquare

Theorem 4.1 can be rewritten in the language of stochastic processes.

THEOREM 4.2. J satisfies NFL condition if and only if there exists a process $y(t)(\omega)$ and a constant M such that

- i) $y(t)$ is a measurable process,
- ii) $\mathbb{P}(|y(t)| \leq M \forall t \in I) = 1$,
- iii) $\mathbb{P}(|y(t) - y(s)| \leq M|t - s|) = 1$ (Lipschitz continuity),
- iv) $\mathbb{P}({}^o y(t) > 0 \forall t \in I) = 1$,

v) $\mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) \leq 0$ for any element $\sum_{i=1}^N \gamma_i \delta_{\tau_i} \in J$.

Proof. It suffices to show that any element of $Y_{++}^{\Gamma+}$ satisfies conditions i) – iv) and vice versa. Take $y \in Y_{++}^{\Gamma+}$. By theorem 7.11 it is a measurable, bounded Lipschitz continuous process. Therefore we only have to show that almost all trajectories of ${}^o y$ are positive. Define $\sigma = \inf\{t \geq 0 : {}^o y(t) \leq 0\}$, with convention that $\inf \emptyset = \infty$. From right-continuity of ${}^o y$ we conclude $1_{\sigma < \infty} {}^o y(\sigma) \leq 0$ and $\mathbb{E} 1_{\sigma < \infty} {}^o y(\sigma) \leq 0$. On the other hand, $x = 1_{\sigma < \infty} \delta_\sigma \in \Gamma_+$ and $\mathbb{E} 1_{\sigma < \infty} y(\sigma) \geq 0$. This yields $\langle x, y \rangle = 0$. However, $y \in Y_{++}^{\Gamma+}$, so $x = 0$ and $\sigma = \infty$ a.s. Therefore iv) is proved.

To prove the opposite implication, we take a process y satisfying conditions i) – iv). By theorem 7.11 y defines a linear functional on $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$. Hence, we have to show that $y \in Y_{++}^{\Gamma+}$, which is clear from positivity of trajectories of the optional projection of y . ■

Now, we shall investigate the set $Y_{++}^{\Gamma+}$. The space Γ does not separate points in $L_*^\infty(\Omega, \mathcal{M}')$, since it looks at the functionals in $L_*^\infty(\Omega, \mathcal{M}')$ from the perspective of their optional projections, i.e. two functionals act on Γ in the same way if their optional projections are indistinguishable. Thus, we shall look closer on the properties of optional projections of bounded, Lipschitz continuous measurable processes.

DEFINITION 4.3. An adapted, RCLL process $(X_t)_{t \in I}$ is called **conditionally Lipschitz** with constant K if there exists a right-continuous version of the process $(\mathbb{E}[X_s | \mathcal{F}_t])_{s \geq t}$ that is Lipschitz with constant K for any $t \in I$.

Note that all right-continuous versions of the process $(\mathbb{E}[X_s | \mathcal{F}_t])_{s \geq t}$ considered in the above definition are indistinguishable. The process is called Lipschitz if almost all its trajectories are Lipschitz functions.

We recall that all processes are defined for $t \in I$, where $I = [0, T]$ or $I = [0, \infty)$.

LEMMA 4.4. If (X_t) is a measurable process with Lipschitz trajectories with constant K , then its optional projection $({}^o X_t)$ is conditionally Lipschitz with constant K .

Proof. Take $t \in I$ and notice that

$$\begin{aligned} |\mathbb{E}[{}^o X_{s_2} | \mathcal{F}_t] - \mathbb{E}[{}^o X_{s_1} | \mathcal{F}_t]| &= |\mathbb{E}[X_{s_2} | \mathcal{F}_t] - \mathbb{E}[X_{s_1} | \mathcal{F}_t]| = |\mathbb{E}[X_{s_2} - X_{s_1} | \mathcal{F}_t]| \\ &\leq \mathbb{E}[|X_{s_2} - X_{s_1}| | \mathcal{F}_t] \leq K|s_2 - s_1|. \end{aligned}$$

■

THEOREM 4.5. Let $(Z_t)_{t \in [0, T]}$ be a RCLL adapted process bounded by K and conditionally Lipschitz with constant K . Then there exists a measurable process $(X_t)_{t \in [0, T]}$ such that

- i) (X_t) is bounded by KT ,
- ii) (X_t) is Lipschitz continuous with constant K ,
- iii) (Z_t) is indistinguishable from ${}^o X_t$.

Proof. We introduce

$$D^n = \left\{ \frac{i}{2^n} T : i = 0, \dots, 2^n \right\},$$

$$D = \bigcup_{n \in \mathbb{N}} D^n.$$

The sketch of the proof goes as follows. For $n \in \mathbb{N}$ we define a bounded, Lipschitz continuous process X_t^n on D and such that $\mathbb{E}[X_t^n | \mathcal{F}_t] = Z_t$ a.s. for $t \in D^n$. We find a subsequence of X^n converging almost everywhere on $D \times \Omega$ to X and such that $\mathbb{E}[X_t | \mathcal{F}_t] = Z_t$ a.s. for $t \in D$ and X is bounded and Lipschitz. We define X on the whole of $[0, T]$ by continuity and show that $Z_t = {}^o X_t$.

LEMMA 4.6. Let $\mathcal{G}_1 \subseteq \mathcal{G}_2$ be two σ -fields. For any $Z_1 \in \mathcal{G}_1$, $Z_2 \in \mathcal{G}_2$ such that $|\mathbb{E}[Z_2 | \mathcal{G}_1] - Z_1| \leq K$ we can find $H \in \mathcal{G}_2$ with $|H - Z_2| \leq K$ and $Z_1 = \mathbb{E}[H | \mathcal{G}_1]$.

Proof of the lemma. Take $H = Z_2 - (\mathbb{E}[Z_2 | \mathcal{G}_1] - Z_1)$. ■

Let $(t_i)_{i=0, \dots, 2^n} \subseteq D^n$ be the enumeration of points in D^n in ascending order. We put $X_T^n = Z_T$. By the above lemma we calculate inductively $X_{t_{2^n-1}}^n, X_{t_{2^n-2}}^n, \dots, X_0^n$ and extend X^n to the whole of D by linear interpolation. It is clear that the resulting process is measurable, bounded by KT and Lipschitz continuous with constant K .

We shall find a subsequence of X^n converging almost everywhere on $D \times \Omega$ to some X and satisfying $\mathbb{E}[X_t | \mathcal{F}_t] = Z_t$ a.s. for $t \in D$. We will show two different approaches. We take any measure μ on D putting positive weight to any subset of D and satisfying $\mu(D) = 1$. We consider a space $L^2(D \times \Omega, \mu \otimes \mathbb{P})$. All processes X^n are elements of this space, since they are bounded. Therefore there exists a sequence $H^n \in \text{conv}(X^n, X^{n+1}, \dots)$ converging to X a.s. It is easy to see that all elements of $\text{conv}(X^n, X^{n+1}, \dots)$ are uniformly bounded by KT and Lipschitz continuous with constant K . So is the pointwise limit. Moreover, by the dominated convergence theorem

$$\mathbb{E}[X_t | \mathcal{F}_t] = \lim_{n \rightarrow \infty} \mathbb{E}[X_t^n | \mathcal{F}_t] = Z_t, \quad t \in D.$$

We extend X to the whole of $[0, T]$ by continuity. Then $({}^o X_t) = (Z_t)$ up to indistinguishability from the right continuity of (Z_t) and the equality $\mathbb{E}[X_t | \mathcal{F}_t] = Z_t$ on a dense subset of $[0, T]$. ■

Unfortunately the results concerning conditional Lipschitz continuity are valid only for a finite time interval. Hence, we obtain the following result

THEOREM 4.7. Assume that $I = [0, T]$, $T < \infty$. J satisfies NFL condition if and only if there exists a process $(y(t))_{t \in [0, T]}$ and a constant M such that

- i) $y(t)$ is an adapted RCLL process,
- ii) $\mathbb{P}(0 < y(t) \leq M \forall t \in \mathbb{R}_+) = 1$,
- iii) $y(t)$ is conditionally Lipschitz,
- iv) $\mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) \leq 0$ for any element $\sum_{i=1}^N \gamma_i \delta_{\tau_i} \in J$.

Proof. The proof is a straightforward consequence of theorems 4.2, 4.5 and lemma 4.4. ■

5. A model with continuous discount processes

We will present a new model of the market. We set $\Gamma = \tilde{\Gamma}$. We construct a topology on Γ by specifying the set Y of continuous linear functionals. Y consists of all bounded measurable processes with continuous trajectories with duality with Γ given by

$$\langle y, x \rangle_{\langle Y, \Gamma \rangle} = \mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i),$$

for any element $x = \sum_{i=1}^N \gamma_i \delta_{\tau_i} \in \Gamma$. We equip Γ with the topology $\sigma(\Gamma, Y)$.

THEOREM 5.1. Let J be a positive convex cone of investment opportunities in the market model $(\Gamma, \sigma(\Gamma, Y))$. J satisfies NFL condition if and only if there exists a process $y(t)(\omega)$ and a constant M such that

- i) $y(t)$ is a measurable process,
- ii) $\mathbb{P}(|y(t)| \leq M \forall t \in \mathbb{R}_+) = 1$,
- iii) almost all trajectories of $y(t)$ are continuous,
- iv) $\mathbb{P}({}^o y(t) > 0 \forall t \in \mathbb{R}_+) = 1$,
- v) $\mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) \leq 0$ for any element $\sum_{i=1}^N \gamma_i \delta_{\tau_i} \in J$.

Proof. We shall use the theorem 2.1 in the proof of this result. We set $X = \Gamma$ and verify all the assumptions. First we reason that Y separates points of X . Elements of Y act on $x \in \Gamma$ in the same way as elements of $L_*^\infty(\Omega, \mathcal{M}')$, so Y contains $L_*^\infty(\Omega, \mathcal{M}')$ (Lipschitz processes are continuous). We proved that $L_*^\infty(\Omega, \mathcal{M}')$ separates points of Γ , so does Y . Moreover, the topology $\tau = \sigma(X, Y)$ is a locally convex topology (see [1] lemma V.3.3). The assumption L can be proved in the similar way as in theorem 4.1 (we take optional projections of elements in Y). To prove assumption C we view Y as a subspace of $L^\infty(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), \mathbb{P} \otimes |\cdot|)$. We can easily prove that the topology of L^∞ on Y is stronger than $\sigma(Y, X)$: if $\|y_n - y\|_{L^\infty} \rightarrow 0$ then $\mathbb{E} \gamma y_n(\sigma) \rightarrow \mathbb{E} \gamma y(\sigma)$ for any stopping time σ and $\gamma \in L^1(\Omega, \mathcal{F}_\sigma, \mathbb{R})$ from bounded convergence theorem (bound is γM for some $M \in \mathbb{R}_+$). Therefore, the assumption C is satisfied. Hence, by theorem 2.1 NFL in J is equivalent to the existence of positive functional y in Y_{++}^{Γ} such that $y|_J \leq 0$. Conditions i)-iii), v) are equivalent to $y \in Y$. The condition iv) is equivalent to $y \in Y_{++}^{\Gamma}$ (see proof of theorem 4.2). ■

If we assume quasi-left-continuity of filtration, i.e. $\mathcal{F}_\sigma = \mathcal{F}_{\sigma-}$ for any previsible stopping time σ , we can strengthen the above result observing that the optional and previsible projections are indistinguishable.

LEMMA 5.2. Let J be a positive convex cone of investment opportunities in the market model $(\Gamma, \sigma(\Gamma, Y))$ and assume the filtration is quasi-left continuous. J satisfies NFL condition if and only if there exists a process $y(t)(\omega)$ and a constant M such that

- i) y is an adapted continuous process,
- ii) $\mathbb{P}(0 < y(t) \leq M \ \forall t \in \mathbb{R}_+) = 1$,
- iii) $\mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) \leq 0$ for any element $\sum_{i=1}^N \gamma_i \delta_{\tau_i} \in J$.

Proof. Left implication is straightforward since any process y satisfying i)-iii) complies with i)-v) of the theorem 5.1. For the opposite implication there exists a process g satisfying i)-v) of the theorem 5.1. Denote by $y = {}^o g$ its optional projection and by $z = {}^p g$ its previsible projection. Then y is RCLL and z is LCRL (see [8] section VI.7). Observe that for any previsible stopping time σ

$$y_\sigma 1_{\sigma < \infty} = \mathbb{E}[g_\sigma 1_{\sigma < \infty} | \mathcal{F}_\sigma] = \mathbb{E}[g_\sigma 1_{\sigma < \infty} | \mathcal{F}_{\sigma-}].$$

Thus y is also a previsible projection which is defined up to indistinguishability. Hence, y is indistinguishable from z and as a result y is continuous. \blacksquare

6. An alternative approach model

In this section we present a representation theorem for the model satisfying R and CV. We start from a reasonable property of convergence in Γ and consider the biggest topology τ on Γ in which this property holds. Therefore, this is a well motivated example of "a good model." We take Y – the set of continuous linear functionals for the topology τ . It consists of all RCLL processes satisfying CV. Unfortunately, we cannot prove a representation property in general situation. We have to restrict ourselves to investment opportunities with transactions in bounded stopping times

$$\Gamma = \{\Phi \in \tilde{\Gamma} : \Phi \text{ has representation } \sum_{i=1}^N \gamma_i \delta_{\tau_i} \text{ with } \tau_i \text{-- bounded}\}.$$

THEOREM 6.1. Let J be a positive convex cone of investment opportunities in the market model $(\Gamma, \sigma(\Gamma, Y))$. J satisfies NFL condition if and only if there exists a process $y(t)(\omega)$ and a constant M such that

- i) y is adapted and RCLL,
- ii) y satisfies CV,
- iii) $\mathbb{P}(y(t) > 0 \ \forall t \in \mathbb{R}_+) = 1$,
- iv) $\mathbb{E} \sum_{i=1}^N \gamma_i y(\tau_i) \leq 0$ for any element $\sum_{i=1}^N \gamma_i \delta_{\tau_i} \in J$.

Proof. The proof is in parts identical to the one of theorem 5.1. We use the theorem 2.1. We set $X = \Gamma$ and verify all the assumptions. First we reason that Y separates points of X . Elements of Y act on $x \in \Gamma$ in the same way as elements of $L_*^\infty(\Omega, \mathcal{M}')$, so Y contains $L_*^\infty(\Omega, \mathcal{M}')$ (bounded and Lipschitz processes are continuous, so they satisfy CV). We proved that $L_*^\infty(\Omega, \mathcal{M}')$ separates points of Γ , so does Y . Therefore, the topology $\tau = \sigma(X, Y)$ is a locally convex topology (see [1] lemma V.3.3). The assumption L can be proved in the similar way as in theorem 4.1 (we only omit optional projections appearing in the proof since Y consists of optional processes).

Assumption C requires a kind of localization technique. Let $(y_n)_{n \in \mathbb{N}}$ be a family of processes in Y . We will construct a sequence of positive numbers α_n such that $\sum_{i=1}^N \alpha_i y_i$ is convergent in Γ . For each $T > 0$, by lemma 3.2, any process $y \in Y$ is uniformly bounded by a constant. Let $y_n^T(t) = y_n(t)1_{t \leq T}$ for $n \in \mathbb{N}$. Therefore, (y_n^T) is a subset of $L^\infty(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \mathbb{P} \otimes |\cdot|)$. By completeness of this space we construct a sequence $(\alpha_n^T)_{n \in \mathbb{N}}$ of positive numbers such that $\sum_{i=1}^N \alpha_i^T y_i^T$ converges to y^T .

To construct a global sequence (α_n) we use diagonal extraction method. First observe that if $T > S$ then $\sum_{i=1}^N \alpha_i^T y_i^S$ is convergent in $L^\infty(\Omega \times [0, S], \mathcal{F} \otimes \mathcal{B}([0, S]), \mathbb{P} \otimes |\cdot|)$. This hints that we should set $\alpha_n = \alpha_n^T$ for $n \in \mathbb{N}$. Denote by z_n partial sums: $z_n = \sum_{i=1}^n \alpha_i y_i$. In order to find a limit of z_n we turn to the above observation. The sequence $z_n(t)1_{t \leq T}$ converges in $L^\infty(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \mathbb{P} \otimes |\cdot|)$. We denote by z^T its limit that is defined up to indistinguishability. Evidently, $(z^T(t)1_{t \leq S})$ is indistinguishable from $(z^S(t))$ for $0 \leq S \leq T$. Hence, there is one and only one process z such that $(z^T(t))$ is indistinguishable from $(z(t)1_{t \leq T})$. Now, we shall show that $z \in Y$. Obviously, z is RCLL and adapted. We only have to verify CV. Take sequences $(Z_n), (\sigma_n)$ as in definition of CV. The sequence (σ_n) is uniformly bounded by some constant T . So

$$\mathbb{E} z(\sigma_n) Z_n = \mathbb{E} z^T(\sigma_n) Z_n \rightarrow \mathbb{E} z^T(\sigma) Z = \mathbb{E} z(\sigma) Z,$$

since z^T is uniformly bounded. We explore boundedness of z^T to prove that z_n converges to z in $\sigma(Y, X)$ topology. It suffices to check that $\mathbb{E} z_n(\sigma) \gamma = \mathbb{E} z(\sigma) \gamma$ for any bounded stopping time σ and $\gamma \in L^1(\Omega, \mathcal{F}_\sigma, \mathbb{P})$, which results from bounded convergence theorem (bound is γM for some $M \in \mathbb{R}_+$). This completes the proof of condition C.

By theorem 2.1 NFL in J is equivalent to the existence of positive functional y in $Y_{++}^{\Gamma+}$ such that $y|_J \leq 0$. To finish the proof it suffices to show that $y \in Y_{++}^{\Gamma+}$ is equivalent to conditions i)-iii). In fact, i)-ii) is equivalent to $y \in Y$, since it is exactly the definition of Y . Equivalence of iii) and positivity of y , i.e. $y \in Y_{++}^{\Gamma+}$, can be shown in a similar way as in the proof of theorem 4.2. ■

7. Appendix A

We state here the most important results concerning spaces \mathcal{M} and $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$, which can be found in [7].

We start with a definition of a linear normed space M . Let

$$M = \left\{ \sum_{i=1}^N \alpha_i \delta_{t_i} : \text{ for some } N \in \mathbb{N}, (\alpha_i) \subset \mathbb{R}, (t_i) \subset \mathbb{R}_+ \right\}.$$

It is easy to see that M is actually a linear space. We shall equip it with a norm. First we denote by \mathcal{D} a set of bounded by 1 and Lipschitz continuous functions with constant 1, i.e.

$$\mathcal{D} = \{ f : \mathbb{R}_+ \rightarrow \mathbb{R} : \forall t, s \in \mathbb{R}_+ |f(t)| \leq 1, |f(t) - f(s)| \leq |t - s| \}.$$

Then we define a functional on M by

$$\|\mu\|_M = \sup \left\{ \left| \sum_{i=1}^N f(t_i) \alpha_i \right| : f \in \mathcal{D} \right\},$$

for μ having the representation $\mu = \sum_{i=1}^N \alpha_i \delta_{t_i}$.

LEMMA 7.1. $(M, \|\cdot\|_M)$ forms a normed linear space.

Once we have defined the space M we shall need to calculate norms of some simple elements.

LEMMA 7.2. Let $\alpha, \beta \geq 0, t, s \in \mathbb{R}_+, t \neq s$.

$$(1) \quad \|\alpha \delta_t + \beta \delta_s\|_M = \alpha + \beta$$

$$(2) \quad \|\alpha \delta_t - \beta \delta_s\|_M = \begin{cases} |\alpha - \beta| + |t - s|(\alpha \wedge \beta) & |t - s| \leq 2 \\ \alpha + \beta & |t - s| > 2 \end{cases}$$

Having the above lemma we can see that M is not complete. Consider a sequence $\mu_n = \sum_{i=1}^n \frac{1}{2^i} \delta_{2^i}$. It is a Cauchy sequence: for $n > m$ $\|\mu_n - \mu_m\|_M = \sum_{i=m}^n \frac{1}{2^i}$. But it does not have a limit in M .

DEFINITION 7.3. Denote by $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ the completion of M with the norm generated by $\|\cdot\|_M$.

We shall concentrate on the space \mathcal{M}' of continuous functionals on \mathcal{M} . We will see that there exists 1-1 correspondence between \mathcal{M}' and the set of all Lipschitz continuous bounded functions.

LEMMA 7.4. Let $\mu^* \in \mathcal{M}'$. A function $f(t) = \langle \mu^*, \delta_t \rangle$ is bounded and Lipschitz continuous i.e. there exists a constant C such that $|f(t)| \leq C$ and $|f(t) - f(s)| \leq C|t - s|$. In particular, we can take $C = \|\mu^*\|_{\mathcal{M}'}$.

LEMMA 7.5. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded and Lipschitz continuous function i.e. $|f(t)| \leq C$ and $|f(t) - f(s)| \leq C|t - s|$ for some constant C . Then there exists exactly one continuous linear functional μ^* on \mathcal{M} such that $\langle \mu^*, \delta_t \rangle = f(t)$. Moreover, $\|\mu^*\|_{\mathcal{M}'} \leq C$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a mapping defined on Ω with values in the Banach space \mathcal{M} . X is a simple random variable if it is measurable and admits a finite number of values, i.e. there exists $N \in \mathbb{N}$, sequence $(\mu_n)_{n=1, \dots, N} \subset \mathcal{M}$ and N disjoint measurable sets $(A_n)_{n=1, \dots, N}$ such that $A_n \in \mathcal{F}$, $\bigcup_{n=1}^N A_n = \Omega$ and $X = \sum_{n=1}^N \mu_n 1_{A_n}$. A mapping X is strongly measurable if there exists a sequence of simple random variables converging to X a.s. in the norm of \mathcal{M} .

DEFINITION 7.6. A space $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$ consists of all strongly measurable random variables X for which the functional

$$\|X\|_{L_{\mathbb{P}}^1(\Omega, \mathcal{M})} = \mathbb{E} \|X\|_{\mathcal{M}}$$

is finite.

LEMMA 7.7. ([10]) The space $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$ with the norm $\|\cdot\|_{L_{\mathbb{P}}^1(\Omega, \mathcal{M})}$ forms a Banach space.

LEMMA 7.8. Let τ be a non-negative random variable, $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $Y\delta_\tau \in L_{\mathbb{P}}^1(\Omega, \mathcal{M})$.

Proof. We will construct a sequence of simple random variables $X_n \in L_{\mathbb{P}}^1(\Omega, \mathcal{M})$ with the limit equal to $Y\delta_\tau$. Let Y_n be a sequence of simple random variables converging to Y a.s. Fix $n \in \mathbb{N}$. Set $A_k = \{\tau \in [\frac{kn}{n^2}, \frac{(k+1)n}{n^2})\}$ for $k = 0, \dots, (n^2 - 1)$. Put

$$X_n = \sum_{k=0}^{n^2-1} Y_n 1_{A_k} \delta_{\frac{kn}{n^2}}.$$

Then we have a pointwise convergence of X_n to $Y\delta_\tau$ from lemma 7.2. ■

Following Schwartz ([9]) we construct a dual space to $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$. A mapping $\Phi : \Omega \rightarrow \mathcal{M}'$, where \mathcal{M}' is dual to \mathcal{M} , is called $*$ -weakly measurable if for any $x \in \mathcal{M}$ the function $\omega \mapsto \langle \Phi(\omega), x \rangle$ is measurable as a function from Ω to \mathbb{R} . Let $\mathcal{L}^\infty(\Omega, \mathcal{M}')$ be a set of all $*$ -weakly measurable mappings for which the function $\Phi \mapsto \inf\{K \geq 0 : \|\Phi\|_{\mathcal{M}'} \leq K \text{ a.s.}\}$ is finite. We define an equivalence relation in the set $\mathcal{L}^\infty(\Omega, \mathcal{M}')$: $\Phi \sim \Psi$ if $\forall x \in \mathcal{M} \langle \Phi, x \rangle = \langle \Psi, x \rangle$ a.s.

DEFINITION 7.9. We introduce

$$L_*^\infty(\Omega, \mathcal{M}') = \mathcal{L}^\infty(\Omega, \mathcal{M}') / \sim$$

with the functional

$$\|\Phi\|_{L_*^\infty(\Omega, \mathcal{M}')} = \inf_{\phi \sim \Phi} \|\phi\|_{\mathcal{L}^\infty(\Omega, \mathcal{M}')} = \inf\{K \geq 0 : \|\Phi\|_{\mathcal{M}'} \leq K \text{ a.s.}\}.$$

THEOREM 7.10. ([9]) The space $L_*^\infty(\Omega, \mathcal{M}')$ with the functional $\|\cdot\|_{L_*^\infty(\Omega, \mathcal{M}')}$ forms a Banach space. Moreover, it is dual to $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$. Every $\Psi \in L_*^\infty(\Omega, \mathcal{M}')$ defines a linear functional on $L_{\mathbb{P}}^1(\Omega, \mathcal{M})$ as

$$L_{\mathbb{P}}^1(\Omega, \mathcal{M}) \ni X \mapsto \langle \Psi, X \rangle_{\langle L_*^\infty(\Omega, \mathcal{M}'), L_{\mathbb{P}}^1(\Omega, \mathcal{M}) \rangle} = \mathbb{E} \langle \Psi, X \rangle_{\langle \mathcal{M}', \mathcal{M} \rangle}.$$

The space $L_*^\infty(\Omega, \mathcal{M}')$ has a nice description in terms of stochastic processes.

THEOREM 7.11. ([7]) Any element $\Psi \in L_*^\infty(\Omega, \mathcal{M}')$ can be seen as a measurable process $y(t)(\omega) : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ (i.e. measurable with respect to the σ -field on $\mathbb{R}_+ \times \Omega$) defined as $y(t) = \langle \delta_t, \Psi \rangle_{\langle \mathcal{M}, \mathcal{M}' \rangle}$, unique up to indistinguishability. Moreover, its trajectories are bounded and Lipschitz continuous with constant $\|\Psi\|_{L_*^\infty(\Omega, \mathcal{M}')}$ for almost all $\omega \in \Omega$.

Proof. First note that $y(t)$ defined above is measurable for each $t \in \mathbb{R}_+$, since Ψ is $*$ -weakly measurable. For any $\omega \in \Omega$ the function $t \rightarrow \langle \delta_t, \Psi(\omega) \rangle_{\langle \mathcal{M}, \mathcal{M}' \rangle}$ is Lipschitz continuous and bounded with constant $\|\Psi(\omega)\|_{\mathcal{M}'}$. Then a simple argument shows that $y(t)_{t \in \mathbb{R}_+}$ is a measurable process (see for example Remark 1.14 in [5]).

Take Φ , another element from the abstraction class of Ψ and define $z(t) = \langle \delta_t, \Phi \rangle_{\langle \mathcal{M}, \mathcal{M}' \rangle}$. We know that for any $\mu \in \mathcal{M}$ we have $\langle \mu, \Psi \rangle_{\langle \mathcal{M}, \mathcal{M}' \rangle} = \langle \mu, \Phi \rangle_{\langle \mathcal{M}, \mathcal{M}' \rangle}$ a.s., therefore $z(t)$ is a modification of $y(t)$, but for continuous processes it is equivalent to indistinguishability.

The last assertion of the theorem results from the definition of the norm $\|\cdot\|_{L^\infty(\Omega, \mathcal{M})}$.

Now we shall prove the reverse implication: any function $y(t, \omega)$ satisfying the conditions stated in the theorem defines a linear functional in $L^1_{\mathbb{P}}(\Omega, \mathcal{M})$. Let H be a linear subspace of $L^1_{\mathbb{P}}(\Omega, \mathcal{M})$ spanned by random variables $1_A \delta_t$ for $A \in \mathcal{F}_t$, $t \in \mathbb{R}_+$. On H we set $\Psi(1_A \delta_t) = \mathbb{E} 1_A y(t)$. Linearity of this function is clear. We have only to show that Ψ is continuous. Let $Y \in H$. We can write $Y = \sum_{k=1}^K \alpha_k 1_{A_k} \delta_{t_k}$ for some K , $A_k \in \mathcal{F}_{t_k}$, $t_k \in \mathbb{R}_+$, $\alpha_k \in \mathbb{R}$, $k = 1, \dots, K$. Hence

$$\begin{aligned} \Psi(Y) &= \sum_{k=1}^K \alpha_k \Psi(1_{A_k} \delta_{t_k}) = \sum_{k=1}^K \alpha_k \mathbb{E}(1_{A_k} y(t_k)) \\ &= \mathbb{E} \sum_{k=1}^K \alpha_k 1_{A_k} y(t_k) = \int_{\Omega} \sum_{k=1}^K \alpha_k 1_{A_k}(\omega) y(t_k)(\omega) d\mathbb{P}(\omega). \end{aligned}$$

For almost all $\omega \in \Omega$, $y(t)(\omega)$ as a function of $t \in \mathbb{R}_+$ is Lipschitz continuous with some constant L and is bounded by L . Fix $\omega \in \Omega$. Then by lemma 7.5 $y(t)(\omega)$ defines a continuous linear functional on \mathcal{M} with norm M . Since $\sum_{k=1}^K \alpha_k 1_{A_k}(\omega) \delta_{t_k} \in \mathcal{M}$ we obtain

$$\left| \sum_{k=1}^K \alpha_k 1_{A_k}(\omega) y(t_k)(\omega) \right| \leq L \left\| \sum_{k=1}^K \alpha_k 1_{A_k}(\omega) \delta_{t_k} \right\|_{\mathcal{M}} = L \|Y(\omega)\|_{\mathcal{M}}.$$

Thus

$$\begin{aligned} |\Psi(Y)| &= \left| \int_{\Omega} \sum_{k=1}^K \alpha_k 1_{A_k}(\omega) y(t_k)(\omega) d\mathbb{P}(\omega) \right| \leq \int_{\Omega} \left| \sum_{k=1}^K \alpha_k 1_{A_k}(\omega) y(t_k)(\omega) \right| d\mathbb{P}(\omega) \\ &\leq \int_{\Omega} K \|Y(\omega)\|_{\mathcal{M}} d\mathbb{P}(\omega) = K \|Y\|_{L^1_{\mathbb{P}}(\Omega, \mathcal{M})} \end{aligned}$$

We extend Ψ to the whole of $L^1_{\mathbb{P}}(\Omega, \mathcal{M})$ as a continuous linear functional (see Yosida [10] IV.5.1). Observe that $\Psi(1_A \delta_{\tau}) = \mathbb{E} 1_A y(\tau)$ for any stopping time τ and $A \in \mathcal{F}_{\tau}$. Let τ_n be a sequence of stopping times admitting finite number of values and converging to τ almost surely. By lemma 7.2 $1_A \delta_{\tau_n} \xrightarrow{\mathcal{M}} 1_A \delta_{\tau}$ a.s. and from dominated convergence theorem $1_A \delta_{\tau_n} \rightarrow 1_A \delta_{\tau}$ in $L^1_{\mathbb{P}}(\Omega, \mathcal{M})$. Thus $\Psi(1_A \delta_{\tau_n}) \rightarrow \Psi(1_A \delta_{\tau})$. On the other hand $\mathbb{E} 1_A y(\tau_n) \rightarrow \mathbb{E} 1_A y(\tau)$ by the dominated convergence theorem ($y(t)$ is bounded by L). Then $\Psi(1_A \delta_{\tau}) = \mathbb{E} 1_A y(\tau)$. Similar argument shows that for any $\Theta \in L^1(\Omega, \mathcal{F}_{\tau}, \mathbb{P})$ we have $\Psi(\Theta \delta_{\tau}) = \mathbb{E} \Theta y(\tau)$. ■

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