# Finite Horizon Optimal Stopping of Discontinuous Functionals with Applications to Impulse Control with Delay*

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**Abstract**

We study finite horizon optimal stopping problems for continuous time Feller-Markov processes. The functional depends on time, state of the process and external parameters, and may exhibit discontinuities with respect to the time-variable. Both left and right-hand discontinuities are considered. We investigate the dependence of the value function on the parameters, initial state of the process and on the stopping horizon. We show that an optimal stopping time exists and can be explicitly constructed once the value function is computed. We demonstrate how to approximate the optimal stopping time by solutions to discrete-time problems. Our results are applied to the study of impulse control problems with finite time horizon, decision lag and execution delay.

**Keywords:** optimal stopping, Feller Markov process, discontinuous functional, impulse control, decision lag, execution delay

## 1. Introduction

The interest in optimal stopping and impulse control of continuous-time Markov processes has been continually fuelled by applications to such areas as finance, resource management or production scheduling. The foundations of optimal stopping has been a subject of intensive studies almost 3 decades ago. The mathematical framework was built in seminal papers by Bismut and Skalli [6], El Karoui [11], El Karoui et al. [12], Fakeev [14], and Mertens [17] with the extensions in El Karoui et al. [13]. Another topic sparking a lot of interest was the regularity of the value

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function. Bismut [7] applied methods of convex analysis. The time-discretization technique, used in the present paper, was explored in Mackevicius [15]. The penalty method, introduced by Stettner and Zabczyk [26], has been further extended in [27]. A survey of various results and approaches to optimal stopping for standard Markov processes can be found in [28].

Another group of papers was devoted to stopping of diffusion processes in which differential structure was strongly used to solve suitable variational inequalities. A predominant solution technique was pioneered in the classical monograph by Bensoussan and Lions [5], who studied a variational approach to stopping of non-degenerate diffusions where the cost/reward was described by a continuous function. Generalizations covered degeneracy of the diffusion (Menaldi [16]), removal of the discounting factor and relaxation of many assumptions regarding the functional and the coefficients of the diffusion (see Fleming, Soner [10]). Recently, Bassan and Ceci studied discontinuous functionals and more general family of processes ([1, 2]). They proved that optimal stopping of a lower/upper semi-continuous functional yields a lower/upper semi-continuous value function. They showed the existence of optimal stopping times under certain conditions, but they were not able to construct them explicitly.

This paper aims to provide a complementary view on optimal stopping and impulse control problems. Among various methods used to study the regularity of the optimal stopping value function the discrete time approach, pursued in this paper and rooted in probabilistic methods developed in Mackevicius [15], seems to be the most promising tool to study the time discontinuity of the functionals.

At the heart of our interest is the optimal stopping problem

$$v(x, T_1, T_2, b) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x\big\{F(\tau, X(\tau), b)\big\},$$

where $(X(t))$ is a Feller-Markov process and $b$ is a parameter. The properties of $v$ and the existence of optimal stopping times are well-known if $F$ is continuous and bounded: the value function is continuous and the optimal stopping time is given as the first hitting time of the set on which the value function coincides with $F$ (see [28] and the references therein). Our study is centered on functionals with the function $F$ having discontinuities with respect to the time variable. Such functionals appear naturally in the study of impulse control with decision lag (see Section 5). We demonstrate that certain kinds of discontinuities prevent existence of optimal stopping times, while others, even though the value function is discontinuous, have solutions in a standard Markovian form. We show how the discontinuities in $F$ are transferred into $v$. The results when $F$ is right-continuous with respect to the time variable are summarized in Theorems 3.1 and 3.10. The left-continuous case can be found in Theorem 4.2 and Corollary 4.3.

Our research complements the aforementioned papers on variational techniques in two dimensions. Firstly, the continuity properties of the value function are proved for the class of weakly Feller-Markov processes (this is a wide class of processes embracing, inter alia, Levy processes and diffusions) on locally compact separable spaces therefore providing a universal basis for the search of further smoothness results in far more technical realm of variational inequalities. Secondly, we provide explicit formulas for $\varepsilon$-optimal and optimal stopping times for discontinuous functionals. These results also benefit numerical methods for solution of stopping problems by variational methods by providing detailed estimates on the magnitude of discontinuities, their exact positions and the relation of the value function to optimal stopping rules.

The results of the present paper rely on an approximation of the continuous-time stopping problem with appropriately constructed discrete-time counterparts (see Theorem 3.1). This approach provides an alternative method for numerical computation of the value function. Its most exciting feature is, however, that approximations of an optimal stopping time can be efficiently computed; we prove that $\varepsilon$-optimal stopping times, approximating the optimal stopping time, can be constructed from optimal-stopping times for the related discrete-time systems, see the proof of Lemma 3.4. This proof is of its own interest as, to the best of our knowledge, it opens a new path to prove the existence and form of $\varepsilon$-optimal and optimal stopping times even in the well-studied case of continuous and bounded $F$.

The properties of weak Feller processes that enable our approach are collected in Section 2. We would like to turn reader's attention to Proposition 2.1, which states that the study of weak Feller processes can be limited, with high probability, to compact sets. We also show that the assumptions in the definition of weak Feller processes cannot be relaxed without surrendering properties of the value function and its relation to the optimal stopping time. The example, attached at the end of Subsection 3.1, demonstrates that if the semigroup is only assumed to map the space of continuous bounded functions into itself, the value function of the stopping problem with a continuous bounded $F$ may not be continuous and the optimal stopping time is not determined by the coincidence of $F$ with the value function.

Main application, as well as the motivation for the research presented in this paper, is the problem of finite-horizon impulse control in the presence of execution delay and decision lag, with many applications in financial markets and decision-making processes (regulatory delays, delayed data availability, liquidity risk, real-options, see [3, 8]). It appears that the discontinuities of the kind studied in this paper are natural when there is either delay in execution of impulses or decision lag. A simple version of the control problem when the execution delay equals the decision lag and the underlying process is a jump-diffusion is solved by Øksendal and Sulem [20]. They transform the problem to a sequence of no-delay optimal stopping problems by application of the variational techniques. Bruder and Pham [8] consider more general controls (the execution delay is a multiplicity of the decision lag) and a diffusion as the underlying. They prove, using variational approach, that there exists a solution and provide a sketch of a numerical algorithm. Different techniques are employed by Bayraktar and Egami [3] who give explicit formulas for optimal strategies if there is no decision lag (the execution of impulses might be delayed) and the set of admissible control strategies is restricted to threshold strategies. Our results, presented in Section 5, are closest in their spirit to [8]. However, in our setting the underlying process is weakly Feller on a locally compact separable state space and no relation between the length of decision lag and execution delay is imposed. We rephrase the problem as a finite system of optimal stopping problems which can be solved explicitly. We prove the existence and the form of an optimal control as well as point out the discontinuities in the value functions of the auxiliary optimal stopping problems. In our opinion, our method has several advantages compared to those used in the aforementioned papers. Firstly, our results hold for general weak Feller processes. Theorem 5.1 can be viewed as a universal tool to assess basic smoothness properties of the value function as well as the existence of optimal strategies. Secondly, our proofs address only the inherent difficulties of the control problem leaving aside the technicalities of the variational approach. This enables us to provide a detailed construction of an optimal strategy and a proof of its optimality. Finally, our system of auxiliary optimal stopping problems can suit as a basis for numerical solution: it can be split into separate stopping problems which, after smoothing (see Theorem 3.5), have representations in the form of variational inequalities as in [8].

The paper is organized as follows. Section 2 collects properties of weakly Feller processes. They are used to study, in Section 3, stopping problems for functionals with right-continuous dependence on time. Left-continuous functionals are dealt with in Section 4. Impulse control problem is formulated and solved in Section 5.

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# 2. Properties of weak Feller processes

Consider a standard Markov right continuous process $\big(X(t)\big)$ defined on a locally compact separable space $E$ endowed with a metric $\rho$ with respect to which every closed ball is compact. The Borel $\sigma$-algebra on $E$ is denoted by $\mathcal{E}$. It is assumed throughout this paper that $\big(X(t)\big)$ satisfies the weak Feller property:

$$P_t\, \mathcal{C}_0 \subseteq \mathcal{C}_0$$

where $\mathcal{C}_0$ is the space of continuous bounded functions $E \to \mathbb{R}$ vanishing in infinity, and $P_t$ is the transition semigroup of the process $\big(X(t)\big)$, i.e. $P_t h(x) = \mathbb{E}^{\,x}\left\{h\big(X(t)\big)\right\}$ for any bounded measurable $h : E \to \mathbb{R}$. Right continuity of $\big(X(t)\big)$ and Theorem T1, Chapter XIII in [18] implies that the semigroup $P_t$ satisfies the following uniform continuity property:

$$\forall\, f \in \mathcal{C}_0 \qquad \lim_{t \to 0+} P_t f = f \ \text{ in } \mathcal{C}_0. \tag{1}$$

A class of strong Feller processes, whose semigroup transforms measurable bounded functions into continuous functions, is often studied in connection with optimal stopping and control of non-degenerate (with uniformly elliptic generators) diffusions and is closely related to the existence of smooth density function. A class of weak Feller processes is considerably larger and consists of numerous stochastic processes commonly used in mathematical practice, as general as non-exploding diffusions and Levy processes.

Due to the weak Feller property the study of many optimal stopping problems can be restricted to compact state spaces. Indeed, the following proposition states that the process does not leave a compact ball around its initial point with arbitrarily large probability over a finite time. Let

$$\gamma_T(x,R) = \mathbb{P}^x \left\{\exists_{s\in[0,T]}\ \rho\big(x, X(s)\big) \geq R\right\}. \tag{2}$$

**PROPOSITION 2.1** *For any compact set $K \subseteq E$*

$$\sup_{x\in K} \gamma_T(x,R) \to 0 \tag{3}$$

*as $R \to \infty$.*

***Proof.*** The proof exploits ideas of Proposition 1 and Lemma 1 of [26]. Fix a compact set $K \subseteq E$. The proof consists of two steps:

Step 1. For each $\varepsilon > 0$ there are compact sets $L_1, L_2 \subseteq E$ such that $K \subseteq L_i$, $i = 1, 2$,

$$\inf_{x \in K} P_T 1_{\{L_1\}}(x) \geq 1 - \varepsilon \tag{4}$$

and

$$\sup_{x \notin L_2} \sup_{t \in [0,T]} P_t 1_{\{L_1\}}(x) \leq \varepsilon. \tag{5}$$

In fact, consider a family of continuous functions $g_K^n$ such that $\|g_K^n\| \leq 1$ ($\|\cdot\|$ stands for the supremum norm), $g_K^n(y) = 1$, for $y \in B(K, n)$, and $g_K^n(y) = 0$, for $y \notin B(K, n+1)$, where $B(K, n) := \{z \in E : \rho(z, K) \leq n\}$. These functions are in $\mathcal{C}_0$ and $g_K^n(x)$ converge point-wise to a constant function equal to 1 as $n \to \infty$. Due to the dominated convergence theorem the sequence $P_T g_K^n(x)$ converges also to 1. The sequence $P_T g_K^n$ is monotone since the operator $P_T$ is monotone and the sequence of functions $g_K^n$ is non-decreasing. By Dini's theorem (see [24, Thm 7.13]), $P_T g_K^n$ converges uniformly on compact sets to 1. This implies that there exists $n^*$ such that $P_T g_K^{n^*}(x) \geq 1 - \varepsilon$ for all $x \in K$. This completes the proof of (4) with $L_1 = B(K, n^* + 1)$ since $1_{\{L_1\}}(x) \geq g_K^{n^*}(x)$.

The mapping $(t, x) \mapsto P_t f(x)$ is continuous for any $f \in \mathcal{C}_0$. Indeed, letting $t_n \to t$ and $x_n \to x$ we have

$$\begin{aligned} |P_{t_n} f(x_n) - P_t f(x)| &\leq |P_{t_n} f(x_n) - P_t f(x_n)| + |P_t f(x_n) - P_t f(x)| \\ &\leq \sup_{y \in E} |P_{t_n} f(y) - P_t f(y)| + |P_t f(x_n) - P_t f(x)|. \end{aligned}$$

The second term tends to 0 from the continuity of $x \mapsto P_t f(x)$, while the convergence of the first term follows from (1).

Define $h(x) = \sup_{t \in [0,T]} P_t g_K^{n^*+1}(x)$. Above result yields that $h \in \mathcal{C}$, where $\mathcal{C}$ is the space of continuous bounded functions $E \to \mathbb{R}$. The proof that $h \in \mathcal{C}_0$ is performed by contradiction. Assume that there exists a sequence $(x_n)$ converging to infinity such that $h(x_i) \geq \delta > 0$ ($h$ is non-negative by definition). Let $(t_i)$ be such that $h(x_i) = P_{t_i} g_L^{n^*+1}(x_i)$, $i = 1, 2, \ldots$. Consider a subsequence $t_{i_j}$ converging to $t^*$. For large $j$ the following inequality holds:

$$\left| h(x_{i_j}) - P_{t^*} g_K^{n^*+1}(x_{i_j}) \right| = \left| P_{t_{i_j}} g_K^{n^*+1}(x_{i_j}) - P_{t^*} g_K^{n^*+1}(x_{i_j}) \right| \leq \frac{\delta}{2}.$$

On the other hand, $P_{t^*} g_K^{n^*+1}$ is in $\mathcal{C}_0$ (by the definition of weak Feller property), which implies that $\lim_{j \to \infty} P_{t^*} g_K^{n^*+1}(x_{i_j}) = 0$. This is a contradiction of the assumption $h(x_i) \geq \delta$.

Since $h$ is in $\mathcal{C}_0$ there exists $r$ such that $h(x) \leq \varepsilon$ for $x \notin B(K, r)$. This implies that $L_2 = B(K, r)$ satisfies (5) because $g_K^{n^*+1}(x) \geq 1_{\{L_1\}}(x)$.

Step 2. Let $\tau = \inf\{s \geq 0 : \rho(K, X(s)) \geq R\}$, where $R$ is such that $L_2 \subset B(K, R)$. For

$x \in K$, using (4) and (5) we have

$$\begin{aligned}
1-\varepsilon &\le \mathbb{P}^x\left\{X(T) \in L_1\right\} \\
&= \mathbb{P}^x\left\{X(T) \in L_1, \tau \le T\right\} + \mathbb{P}^x\left\{X(T) \in L_1, \tau > T\right\} \\
&\le \mathbb{E}^x\left\{1_{\tau \le T} P_{X(\tau)}\left\{X(T-\tau) \in L_1\right\}\right\} + \mathbb{P}^x\left\{\tau > T\right\} \\
&\le \varepsilon \mathbb{P}^x\left\{\tau \le T\right\} + \mathbb{P}^x\left\{\tau > T\right\} = 1 - \mathbb{P}^x\left\{\tau \le T\right\}(1-\varepsilon)
\end{aligned}$$

and therefore

$$\mathbb{P}^x\left\{\tau \le T\right\} \le \frac{\varepsilon}{1-\varepsilon},$$

which completes the proof. ■

**COROLLARY 2.2**

*i)* $P_t\mathcal{C} \subset \mathcal{C}$*, where* $\mathcal{C}$ *is the space of continuous bounded functions* $E \to \mathbb{R}$ *(the Feller property).*

*ii)* $\lim_{t\to 0} P_t f(x) = f(x)$ *uniformly on compact subsets of* $E$ *for* $f \in \mathcal{C}$*.*

***Proof.*** Let $f \in \mathcal{C}$ and $K \subseteq E$ be a compact set. By Proposition 2.1 there exists a sequence $r_n \to \infty$ such that

$$\sup_{x\in K} \gamma_t(x, r_n) \le 2^{-n}.$$

Define continuous functions $g_n : E \to \mathbb{R}$ satisfying the following properties: $0 \le g_n(x) \le 1$, $g_n(x) = 1$ for $x \in B(K, r_n)$, and $g_n(x) = 0$ for $x \notin B(K, r_n + 1)$. Functions $f_n(x) = f(x)g_n(x)$ are in $\mathcal{C}_0$. By the weak Feller property $P_t f_n(x)$ are continuous. The construction of $r_n$ yields

$$\sup_{x\in K} \left|P_t f(x) - P_t f_n(x)\right| \le \frac{\|f\|}{2^n}.$$

Therefore, $P_t f_n$ converges uniformly on K to $P_t f$, which implies that $P_t f$ is continuous on $K$. Arbitrariness of $K$ yields that $P_t f \in \mathcal{C}$.

To prove (ii) notice that for any $s \in [0, t]$

$$|P_s f(x) - f(x)| \le |P_s f(x) - P_s f_n(x)| + |P_s f_n(x) - f_n(x)| + |f_n(x) - f(x)|,$$

where $f_n$ is defined above. Therefore,

$$\sup_{x\in K} \left|P_s f(x) - f(x)\right| \le \frac{\|f\|}{2^n} + \|P_s f_n - f_n\|.$$

Since $f_n \in \mathcal{C}_0$, the second term converges to zero as $s \to 0$. The first term can be made arbitrarily small by the choice of $n$. This completes the proof. ■

**LEMMA 2.3** *Let $u : E \times E^B \to \mathbb{R}$ be a continuous bounded function, where $E^B$ is a locally compact separable space.*

*i) For any sequence $(x_k, b_k, d_k) \subseteq E \times E^B \times [0, \infty)$ converging to $(x, b, d)$ we have*

$$\lim_{k \to \infty} P_{d_k} u(x_k, b_k) = P_d u(x, b),$$

*where $P_d u(x, b) = \mathbb{E}^x \{u(X(d), b)\}$.*

*ii) For any $d^* \geq 0$ and compact sets $K \subseteq E$, $B \subseteq E^B$ we have*

$$\lim_{d \to d^*,\, d \geq 0} \sup_{x \in K} \sup_{b \in B} |P_d u(x, b) - P_{d^*} u(x, b)| = 0.$$

***Proof.*** (i) Let $K = \{x, x_1, x_2, \ldots\}$, $B = \{b, b_1, b_2, \ldots\}$. By Proposition 2.1 for any $\varepsilon > 0$ there exists a compact set $L \subset E$ such that

$$\sup_{x \in K} \mathbb{P}^x \Big( \exists_{s \in [0, d+1]} X(s) \notin L \Big) < \varepsilon.$$

Define a continuous function $g : E \to [0, 1]$ such that $g(x) = 1$ for $x \in L$, $g(x) = 0$ for $x \notin B(L, 1)$. The function $\bar{u}(x, b) = g(x) u(x, b)$ has a compact support for any fixed $b \in E^B$ and $|P_t u(x, b) - P_t \bar{u}(x, b)| \leq \varepsilon \|u\|$ for $(x, b) \in K \times B$. Therefore

$$\begin{aligned} \big|P_d u(x, b) - P_{d_k} u(x_k, b_k)\big| &\leq 2\varepsilon \|u\| + \big|P_d \bar{u}(x, b) - P_{d_k} \bar{u}(x_k, b_k)\big| \\ &\leq 2\varepsilon \|u\| + \big|P_d \bar{u}(x, b) - P_{d_k} \bar{u}(x_k, b)\big| + \big|P_{d_k} \bar{u}(x_k, b_k) - P_{d_k} \bar{u}(x_k, b)\big|. \end{aligned}$$

The second term converges to 0 by the Feller property (see Corollary 2.2). The third term converges to 0 by uniform continuity of $\bar{u}$ on $E \times B$.

(ii) Define $g$ and $\bar{u}$ as above and notice that

$$\big|P_d u(x, b) - P_{d^*} u(x, b)\big| \leq 2\varepsilon \|u\| + \big|P_d \bar{u}(x, b) - P_{d^*} \bar{u}(x, b)\big|.$$

By (1) the second term converges to 0 uniformly in $(x, b) \in K \times B$. ∎

The following lemma explores another aspect of continuity of weak Feller processes.

**LEMMA 2.4** *([9, Theorem 3.7]) For any compact set $K \subseteq E$ and any $\varepsilon, \delta > 0$ there is $h_0 > 0$ such that*

$$\sup_{0 \leq h \leq h_0} \sup_{x \in K} \mathbb{P}^x \{X(h) \notin B(x, \delta)\} < \varepsilon.$$

# 3. Optimal stopping of right-continuous functionals

This section studies optimal stopping problems of functionals that are discontinuous with respect to time. Functionals are right-continuous (the discontinuities are from the left), which complies with right-continuity of weakly Feller processes. The properties of value functions are explored and existence of optimal and $\varepsilon$-optimal stopping times is proved. It becomes evident that under certain conditions optimal stopping times may not exist. Subsection 3.1 provides a detailed study of a simple discontinuous functional. It is concluded by an example that shows that our assumption that the semigroup of the process $(X(t))$ maps $\mathcal{C}_0$ into itself cannot be relaxed to the mapping of continuous bounded functions into itself. In Subsection 3.2 we explore the properties of the stopping problem with a lower bound on the stopping times. Subsections 3.3 and 3.4 extend these results to multiple discontinuities and functionals with an integral term. The most advanced results of this section are collected in Theorems 3.8 and 3.10.

We make the following standing assumptions. The state space $E$ is locally compact with a metric $\rho$ with respect to which every closed ball is compact. The process $(X(t))$ is defined on the state space $E$ and satisfies the weak Feller property. The space of parameters $E^B$ is locally compact with the metric denoted by $\rho_B$.

## 3.1. Optimal stopping of a simple discontinuous functional

Fix $T^* \geq 0$ and let $f \in \mathcal{C}([0,T^*] \times E \times E^B)$, $g \in \mathcal{C}([0,T^*] \times E \times E^B)$. Define the functional

$$J(s,T,x,b,\tau) = \mathbb{E}^x \Big\{ 1_{\{\tau < T-s\}} f(s+\tau, X(\tau), b) + 1_{\{\tau \geq T-s\}} g(T, X(T-s), b) \Big\}, \tag{6}$$

where $T \in [0,T^*]$, $s \in [0,T]$, $x \in E$, $b \in E^B$ and $\tau \geq 0$. The problem is to maximize the functional over all stopping times $\tau$. Denote by $w(s,T,x,b)$ the corresponding value function:

$$w(s,T,x,b) = \sup_{\tau} J(s,T,x,b,\tau). \tag{7}$$

In the following theorem we study the continuity of $w$ and characterize optimal and $\varepsilon$-optimal stopping times. If the functions $f$ and $g$ do not coincide at the time $T-s$ the functional is discontinuous and an optimal stopping time may not exist.

**THEOREM 3.1**

*i) The function $w(s,T,x,b)$ is continuous and bounded on $\big\{(s,T,x,b) \in [0,T^*] \times [0,T^*] \times E \times E^B : s < T\big\}$ (there might be a discontinuity at $s = T$), and*

$$\lim_{s \to T-} w(s,T,x,b) = \max\big(f(T,x,b), g(T,x,b)\big)$$

*uniformly[1] in $(T,x,b) \in [0,T^*] \times K \times B$, for any compact $K \subseteq E$ and $B \subseteq E^B$.*

[1] The uniformity of convergence is understood as

$$\lim_{\delta \to 0+} \sup_{x \in K} \sup_{b \in B} \sup_{T \in [\delta, T^*]} \big| w(T-\delta,T,x,b) - \max(f(T,x,b), g(T,x,b)) \big| = 0.$$

*ii) For each $\varepsilon > 0$ and $s \in [0, T]$ the stopping time*

$$\tau_s^\varepsilon = \inf\left\{t \geq 0 : w(t+s, T, X(t), b) \leq F(t+s, X(t), b) + \varepsilon\right\}, \tag{8}$$

*where*

$$F(u, x, b) = \begin{cases} f(u, x, b), & u < T, \\ g(T, x, b), & u = T, \end{cases} \tag{9}$$

*is $\varepsilon$-optimal, i.e. $J(s, T, x, b, \tau_s^\varepsilon) \geq w(s, T, x, b) - \varepsilon$,*

*iii) If $g(T, x, b) \geq f(T, x, b)$ the function $w(s, T, x, b)$ is continuous on $\big\{(s, T, x, b) \in [0, T^*] \times [0, T^*] \times E \times E^B : s \leq T\big\}$ (there is no discontinuity at $s = T$) and the stopping time*

$$\tau_s = \inf\left\{t \geq 0 : w(t+s, T, X(t), b) \leq F(t+s, X(t), b)\right\} \tag{10}$$

*is optimal for the functional $J(s, T, x, b, \cdot)$. Moreover,*

$$\lim_{\varepsilon \to 0+} \tau_s^\varepsilon = \tau_s.$$

The proof of the above theorem consists of several lemmas. Let $\Delta_n(s, T) = \frac{T-s}{n}$ for $T \in [0, T^*]$ and $s \leq T$. Consider the following discretized stopping problem

$$w^n(s, T, x, b) = \sup_{\tau \in \mathcal{T}_{\Delta_n(s,T)}} \mathbb{E}^x \Big\{ 1_{\tau < T-s} f(\tau + s, X(\tau), b) \\ + 1_{\tau \geq T-s} g(T, X(T-s), b) \Big\}, \tag{11}$$

where $\mathcal{T}_{\Delta_n(s,T)}$ is the class of stopping times taking values in the set $\mathcal{H}^n(s, T) := \{0, \Delta_n(s, T), \ldots, n\Delta_n(s, T)\}$. The family of stopping problems $(w^n)$ can be decomposed into a sequence of simple maximization problems:

$$w^1(s, T, x, b) = \max\Big(f(s, x, b), P_{T-s}\, g(T, x, b)\Big),$$
$$w^{n+1}(s, T, x, b) = \max\Big(f(s, x, b), P_{\Delta_{n+1}(s,T)} w^n(s + \Delta_{n+1}(s, T), T, x, b)\Big), \quad n = 1, 2, \ldots,$$

where $P_t w(s, T, x, b) = \mathbb{E}^x \left\{ w\big(s, T, X(t), b\big)\right\}$. Indeed, $w^n(s + \Delta_{n+1}(s, T), T, x, b)$ is a value function for the problem in which stopping is allowed in the moments

$$\big\{0, \Delta_n\big(s + \Delta_{n+1}(s, T), T\big), \ldots, n\Delta_n\big(s + \Delta_{n+1}(s, T), T\big)\big\},$$

which simplifies to $\{0, \Delta_{n+1}(s, T), \ldots, n\Delta_{n+1}(s, T)\}$.

Let $\mathcal{D} = \big\{(s, T, x, b) \in [0, T^*] \times [0, T^*] \times E \times E^B : s < T\big\}$. Notice that the difference $\overline{\mathcal{D}} \setminus \mathcal{D}$ consists of the points of the form $(T, T, x, b)$. The following lemma explores continuity properties of the value functions $(w^n)$ and their extensions to $\overline{\mathcal{D}}$. Notice that $w^n$ may be discontinuous at $\overline{\mathcal{D}} \setminus \mathcal{D}$ (take e.g. $f = 1$ and $g = 0$).

**LEMMA 3.2** *Functions $w^n(s, T, x, b)$ are continuous and bounded on $\mathcal{D}$. Their restrictions to $\mathcal{D}$ have unique continuous extensions $\bar{w}^n$ to functions on $\overline{\mathcal{D}}$ that satisfy*

$$\bar{w}^n(T, T, x, b) = \max\big(f(T, x, b), g(T, x, b)\big).$$

***Proof.*** Boundedness of $w^n$ follows directly from the boundedness of the functional. Continuity on $\mathcal{D}$ is proved via induction. The function $w^1$ is continuous as a maximum of continuous functions. For $w^n$, $n > 1$, it suffices to show the continuity of $P_{\Delta_{n+1}(s,T)} w^n(s, T, x, b)$ under the assumption that $w^n(s, T, x, b)$ is continuous and bounded on $\mathcal{D}$. This follows from Lemma 2.3, which states that

$$\lim_{k\to\infty} P_{d_k} w^n(s_k, T_k, x_k, b_k) = P_d w^n(s, T, x, b),$$

where $d_k = \Delta_{n+1}(s_k, T_k)$, $d = \Delta_{n+1}(s, T)$, and $\big((s_k, T_k, x_k, b_k)\big)_{k\geq 1} \subset \mathcal{D}$ is a sequence converging to $(s, T, x, b) \in \mathcal{D}$. This completes the proof of continuity and boundedness of $w^n$ on $\mathcal{D}$.

Consider the following auxiliary maximization problem: for $T \in (0, T^*]$ and $n = 1, 2, \ldots$

$$v^n(s, T, x, b) = \max\big(f(s, x, b), P_{\Delta_n(s,T)} h\big(s+\Delta_n(s, T), T, x, b\big)\big), \quad (s, x, b) \in [0, T]\times E\times E^B, \tag{12}$$

where $h : [0, T^*] \times [0, T^*] \times E \times E^B \to \mathbb{R}$ is a bounded continuous function. Lemma 2.3 implies that $P_{\Delta_n(s,T)} h(s + \Delta_n(s, T), T, x, b)$ converges to $h(T, T, x, b)$ as $s \to T$ uniformly in $n = 1, 2, \ldots$ and $(T, x, b) \in [0, T^*] \times K \times B$ for compact $K \subseteq E$, $B \subseteq E^B$. Uniform convergence of $f(s, x, b)$ to $f(T, x, b)$ for $(T, x, b) \in [0, T^*] \times K \times B$ follows from the uniform continuity of $f$ on compact sets. Finally, we have

$$\lim_{s\to T-} v^n(s, T, x, b) = \max\big(f(T, x, b), h(T, T, x, b)\big) \tag{13}$$

uniformly in $T \in [0, T^*]$, $x \in K$, $b \in B$ and $n = 1, 2, \ldots$.

Existence of the continuous extension $\bar{w}^n$ follows from the following result: for any compact subset $K \subseteq E$, $B \subseteq E^B$

$$\lim_{s\to T-} w^n(s, T, x, b) = \max\big(f(T, x, b), g(T, x, b)\big) \tag{14}$$

uniformly on $(T, x, b) \in [0, T^*] \times K \times B$. The proof of the limit (14) is performed by induction. Fix compact sets $K \subseteq E$, $B \subseteq E^B$. The value function $w^1(s, T, x, b)$ can be written as the maximization problem (12) with $h(s, T, x, b) = g(T, x, b)$. Hence, $\lim_{s\to T-} w^1(s, T, x, b) = \max\big(f(T, x, b), g(T, x, b)\big)$ uniformly in $(T, x, b) \in [0, T^*] \times K \times B$. Next, assume that the convergence (14) holds for $w^n$. The value function $w^{n+1}$ on $\mathcal{D}$ has the form (12) with

$$h(s, T, x, b) = \begin{cases} w^n(s, T, x, b), & s < T, \\ f \vee g(T, x, b), & s \geq T. \end{cases}$$

The choice of $h$ for $s \geq T$ does not affect $v^{n+1}$ on $s < T$, which explains the arbitrary choice of $h$ above. Since $w^n$ satisfies (14), $h$ is continuous. Application of (13) yields that the limit property (14) is satisfied by $w_{n+1}$. Therefore,

$$\bar{w}^n(s,T,x,b) = \begin{cases} w^n(s,T,x,b), & s < T, \\ f \vee g(T,x,b), & s = T. \end{cases}$$

■

The following lemma provides an estimate of the error for the approximation of $w(s,T,x,b)$ by $w^n(s,T,x,b)$ on the set $\mathcal{D}$. The estimate is one-sided as $w^n(s,T,x,b) \le w(s,T,x,b)$ by construction: it represents the optimization of the same functional but on a restricted set of stopping times. The value functions $w$ and $w^n$ are identical on the set $\overline{\mathcal{D}} \setminus \mathcal{D}$: $w(T,T,x,b) = g(T,x,b) = w^n(T,T,x,b)$.

**LEMMA 3.3** *For every compact set $K \subseteq E$, $B \subseteq E^B$ and $\varepsilon > 0$ there exists $n_0$ such that for $n \ge n_0$*

$$\sup_{x\in K}\sup_{b\in B}\sup_{T\in[0,T^*]}\sup_{s\in[0,T)}\big(w(s,T,x,b) - w^n(s,T,x,b)\big) \le \varepsilon\big(4 + 11\|f\| + 3\|g\|\big).$$

***Proof.*** By Proposition 2.1 there exists a compact set $L \subseteq E$ such that

$$\sup_{x\in K}\mathbb{P}^x\{X(s) \notin L \text{ for some } s \in [0,T^*]\} < \varepsilon.$$

Functions $f, g$ are uniformly continuous on $(s,x,b) \in [0,T^*] \times L \times B$, so there exists $\delta > 0$ such that

$$\sup_{b\in B}\sup_{x\in L}\sup_{y\in B(x,\delta)}\sup_{t,s\in[0,T^*],\ |t-s|\le\delta} |f(s,x,b) - f(t,y,b)| + |g(s,x,b) - g(t,y,b)| < \varepsilon.$$

By Lemma 2.4 there is $h_0 > 0$ such that

$$\sup_{0\le h\le h_0}\sup_{x\in L}\mathbb{P}^x\{X(h) \notin B(x,\delta)\} < \varepsilon.$$

Set $n_0 = T^*/(h_0 \wedge \delta)$. For $n \ge n_0$ we have $\Delta_n(s,T) \le h_0 \wedge \delta$ which enables us to use the estimates formulated above.

Fix $T \in [0,T^*]$, $s \in [0,T)$, $b \in B$ and $x \in K$. We have

$$\begin{aligned} &w(s,T,x,b) - \tilde{w}^n(s,T,x,b) \\ &\le \sup_{0\le\tau\le T-s}\mathbb{E}^x\Big\{1_{\{\tau<T-s\}}\, f(s+\tau, X(\tau), b) + 1_{\{\tau\ge T-s\}}\, g(T, X(T-s), b) \\ &\qquad\qquad - 1_{\{\hat\tau<T-s\}}\, f(s+\hat\tau, X(\hat\tau), b) - 1_{\{\hat\tau\ge T-s\}}\, f \vee g(T, X(T-s), b)\Big\}, \end{aligned}$$

where $\hat\tau$ is a stopping time derived from $\tau$ in the following way:

$$\hat\tau = \inf\{t \in \{0, \Delta_n(s,T), \dots, n\Delta_n(s,T)\} : t \ge \tau\},$$

and $\tilde{w}^n$ is the value function of an auxiliary discrete optimal stopping problem

$$\tilde{w}^n(s,T,x,b) = \sup_{\tau\in\mathcal{T}_{\Delta_n(s,T)}} \mathbb{E}^x\Big\{1_{\tau<T-s} f(\tau+s, X(\tau), b) \\ + 1_{\tau\geq T-s} f\vee g(T, X(T-s), b)\Big\}.$$

The difference between $w$ and $\tilde{w}^n$ can be bounded in the following way:

$$\begin{aligned} &w(s,T,x,b) - \tilde{w}^n(s,T,x,b) \\ &\leq \quad \sup_{0\leq\tau\leq T-s} \mathbb{E}^x\Big\{1_{\{\tau\leq T-s-\Delta_n(s,T)\}} \big(f(s+\tau, X(\tau), b) - f(s+\hat{\tau}, X(\hat{\tau}), b)\big)\Big\} \\ &\quad + \sup_{0\leq\tau\leq T-s} \mathbb{E}^x\Big\{1_{\{T-s-\Delta_n(s,T)<\tau<T-s\}} \big(f(s+\tau, X(\tau), b) - f\vee g(T, X(T-s), b)\big)\Big\} \\ &\quad + \sup_{0\leq\tau\leq T-s} \mathbb{E}^x\Big\{1_{\{\tau\geq T-s\}} \big(g(T, X(T-s), b) - f\vee g(T, X(T-s), b)\big)\Big\}. \end{aligned}$$

Assume $n \geq n_0$. Consider the first term. By the strong Markov property of $X(t)$ and the results summarized at the beginning of the proof we have

$$\begin{aligned} &\mathbb{E}^x\Big\{1_{\{\tau\leq T-s-\Delta_n(s,T)\}} \big(f(s+\tau, X(\tau), b) - f(s+\hat{\tau}, X(\hat{\tau}), b)\big)\Big\} \\ &\quad = \mathbb{E}^x\Big\{1_{\{\tau\leq T-s-\Delta_n(s,T)\}} \mathbb{E}^{X(\tau)}\big\{f(s+\tau, X(0), b) - f(s+\hat{\tau}, X(\hat{\tau}-\tau), b)\big|\mathcal{F}_\tau\big\}\Big\} \\ &\quad \leq 2\|f\|\,\mathbb{P}^x\Big\{\big\{X(\tau)\notin L\big\} \text{ or } \big\{X(\tau)\in L,\ X(\hat{\tau})\notin B(X(\tau),\delta)\big\}\Big\} + \varepsilon \leq 4\varepsilon\|f\| + \varepsilon. \end{aligned}$$

The second term is dominated by

$$\sup_{0\leq\tau\leq T-s} \mathbb{E}^x\Big\{1_{\{T-s-\Delta_n(s,T)<\tau<T-s\}} \big(f(s+\tau, X(\tau), b) - f(T, X(T-s), b)\big)\Big\}$$

since $-(f\vee g) \leq -f$ and the estimation as above can be used. The third term is non-positive. Consequently $w(s,T,x,b) - \tilde{w}^n(s,T,x,b) \leq 2\varepsilon(1+4\|f\|)$.

The next step of the proof is to show the relation between $\tilde{w}^n$ and $w^n$. Obviously, $\tilde{w}^n$ dominates $w^n$. The results summarized at the beginning of the proof imply for $y\in L$ and $n\geq n_0$ the following inequalities:

$$\begin{aligned} &\mathbb{E}^y\big\{f\vee g\big(T, X(\Delta_n(s,T)), b\big)\big\} \leq f\vee g(T-\Delta_n(s,T), y, b) + \varepsilon(1+\|f\vee g\|), \\ &\mathbb{E}^y\big\{g\big(T, X(\Delta_n(s,T)), b\big)\big\} \geq g(T-\Delta_n(s,T), y, b) - \varepsilon(1+\|g\|). \end{aligned}$$

These inequalities drive the following estimates for the value functions $\tilde{w}^1$ and $w^1$ on $y\in L$:

$$\begin{aligned} &\tilde{w}^1\big(T-\Delta_n(s,T), T, y, b\big) \leq f\vee g\big(T-\Delta_n(s,T), y, b\big) + \varepsilon(1+\|f\vee g\|), \\ &w^1\big(T-\Delta_n(s,T), T, y, b\big) \geq f\vee g\big(T-\Delta_n(s,T), y, b\big) - \varepsilon(1+\|g\|). \end{aligned}$$

Hence, the difference $(\tilde{w}^1 - w^1)\big(T - \Delta_n(s,T), y, b\big)$ is bounded by $2\varepsilon + \varepsilon\|f\| + 2\varepsilon\|g\|$ for $y \in L$.

Now we reduce the task of bounding $\tilde{w}^n - w^n$ to the estimation of the difference $\tilde{w}^1 - w^1$:

$$
\begin{aligned}
&\tilde{w}^n(s,T,x,b) - w^n(s,T,x,b) \\
&\leq \sup_{\tau \in \mathcal{T}_{\Delta_n(s,T)}} \mathbb{E}^x \Big\{ 1_{\tau < T - s - \Delta_n(s,T)} f(\tau + s, X(\tau), b) \\
&\qquad\qquad\qquad + 1_{\tau \geq T - s - \Delta_n(s,T)} \tilde{w}^1 \big(T - \Delta_n(s,T), T, X(T - s - \Delta_n(s,T)), b\big) \\
&\quad - 1_{\tau < T - s - \Delta_n(s,T)} f(\tau + s, X(\tau), b) \\
&\qquad\qquad\qquad - 1_{\tau \geq T - s - \Delta_n(s,T)} w^1 \big(T - \Delta_n(s,T), T, X(T - s - \Delta_n(s,T)), b\big) \Big\} \\
&\leq \mathbb{E}^x \Big\{ (\tilde{w}^1 - w^1)\big(T - \Delta_n(s,T), T, X(T - s - \Delta_n(s,T)), b\big) \Big\}.
\end{aligned}
$$

Inserting the bound for $\tilde{w}^1 - w^1$ we obtain

$$
\begin{aligned}
&\tilde{w}^n(s,T,x,b) - w^n(s,T,x,b) \\
&\leq \|\tilde{w}^1 - w^1\| \, \mathbb{P}^x \big\{ X(T - s - \Delta_n(s,T)) \notin L \big\} \\
&\qquad\qquad + (2\varepsilon + \varepsilon\|f\| + 2\varepsilon\|g\|) \, \mathbb{P}^x \big\{ X(T - s - \Delta_n(s,T)) \in L \big\} \\
&\leq \varepsilon(2\|f\| + \|g\|) + (2\varepsilon + \varepsilon\|f\| + 2\varepsilon\|g\|) \leq 2\varepsilon + 3\varepsilon\|f\| + 3\varepsilon\|g\|.
\end{aligned}
$$

To complete the proof combine this estimate with the bound for the difference $w - \tilde{w}^n$.

■

Lemma 3.3 implies that $w(s,T,x,b)$ is continuous on $\mathcal{D}$. Since the approximation is uniform in $(s,T,x,b) \in \mathcal{D} \cap [0,T^*] \times [0,T^*] \times K \times B$ for any compact set $K \subseteq E$, $B \subseteq E^B$ we have

$$
\lim_{s \to T-} w(s,T,x,b) = \max\big(f(T,x,b), g(T,x,b)\big)
$$

uniformly in $x \in K$, $b \in B$ and $T \in [0,T^*]$, which completes the proof of Theorem 3.1(i).

The form of an $\varepsilon$-optimal stopping time follows from Lemma 3.4. A general theory cannot be applied because of the discontinuity of the functional. To the best of our knowledge the proof of the optimality of the stopping time presented below is original even in the standard case of continuous functionals.

**LEMMA 3.4** *For each $\varepsilon > 0$, $s \in [0,T]$ the stopping time*

$$
\tau_s^\varepsilon = \inf\{t \geq 0 : w(s+t, T, X(t), b) \leq F(s+t, X(t), b) + \varepsilon\}
$$

*is $\varepsilon$-optimal, i.e. $J(s,T,x,\tau_s^\varepsilon, b) \geq w(s,T,x,b) - \varepsilon$.*

***Proof.*** Fix $b \in E^B$ and $T \in [0,T^*]$. Consider the discretization (11). Functions $w^n$ satisfy the following supermartingale property:

$$
\mathbb{E}^x \big\{ w^n(s+t', T, X(t'), b) \big| \mathcal{F}_t \big\} \leq w^n(s+t, T, X(t), b)
$$

for $t, t' \in \mathcal{H}^n(s,T)$, $t \le t'$. Take arbitrary $0 \le t \le t' \le T - s$ and two non-increasing sequences $(t_n), (t'_n)$ converging to $t$, $t'$ such that $t_n \le t'_n$ and $t_n, t'_n \in \mathcal{H}^n(s,T)$. The supermartingale property of $w^n$ implies that

$$\mathbb{E}^x\big\{w^n(s+t'_n, T, X(t'_n), b)\big|\mathcal{F}_{t_n}\big\} \le w^n(s+t_n, T, X(t_n), b).$$

Due to the right-continuity of $t \mapsto X(t)$ and the convergence of $w^n$ to $w$ (see Lemma 3.3) we have $\lim_{n\to\infty} w^n(s+t_n, T, X(t_n), b) = w(s+t, T, X(t), b)$. The right-continuity of the filtration $(\mathcal{F}_t)$ and the dominated convergence theorem imply that

$$\lim_{n\to\infty} \mathbb{E}^x\big\{w^n(s+t'_n, T, X(t'_n), b)\big|\mathcal{F}_{t_n}\big\} = \mathbb{E}^x\big\{w(s+t', T, X(t'), b)\big|\mathcal{F}_t\big\}.$$

Hence $t \mapsto w(s+t, T, X(t), b)$ is a right continuous supermartingale.

By Proposition 2.1 and Lemma 3.3 for any $k$ there exist a compact set $L^k \subseteq E$ and a positive integer $n_k$ such that

$$\mathbb{P}^x\big(X(t) \in L^k \quad \forall\, t \in [0,T]\big) \ge 1 - \frac{1}{k}, \tag{15}$$

$$w(s,T,x,b) \le w^{n_k}(s,T,x,b) + \frac{1}{k}, \qquad \forall x \in L^k, \quad s \in [0,T]. \tag{16}$$

The optimal stopping time for $w^{n_k}$ is given by

$$\tau_s^k = \inf\big\{t \in \mathcal{H}^{n_k}(s,T): \quad w^{n_k}(s+t, T, X(t), b) \le F(s+t, X(t), b)\big\},$$

where $F$ is defined in (9). Clearly

$$\mathbb{E}^x w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b) = w^{n_k}(s,T,x,b).$$

Furthermore, we have

$$\begin{aligned}
&\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} \\
&\qquad = \mathbb{E}^x\Big\{w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\Big\} \\
&\qquad\qquad + \mathbb{E}^x\Big\{1_{\{\tau_s^\varepsilon \le \tau_s^k\}}\Big(w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) - w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\Big)\Big\} \\
&\qquad\qquad + \mathbb{E}^x\Big\{1_{\{\tau_s^\varepsilon > \tau_s^k\}}\Big(w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) - w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\Big)\Big\}.
\end{aligned}$$

The first term is equal to $w^{n_k}(s,T,x,b)$. The second term is non-negative since by the supermartingale property of $w$ and the domination of $w^{n_k}$ by $w$ we have

$$\mathbb{E}^x\Big\{1_{\{\tau_s^\varepsilon \le \tau_s^k\}}\, w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\Big\} \ge \mathbb{E}^x\big\{1_{\{\tau_s^\varepsilon \le \tau_s^k\}}\, w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\big\}.$$

The third term is bounded from below by $-2\|F\|\mathbb{P}^x(\tau_s^\varepsilon > \tau_s^k)$. Inequalities (15) and (16) imply $\mathbb{P}^x(\tau_s^\varepsilon > \tau_s^k) \le 1/k$ for $k \ge 1/\varepsilon$. Consequently,

$$\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} \ge w^{n_k}(s,T,x,b) - \frac{2\|F\|}{k} \ge w(s,T,x,b) - \frac{2\|F\|+1}{k}.$$

Letting $k \to \infty$ we obtain $\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} \geq w(s,T,x,b)$. The converse inequality follows directly from the supermartingale property of $w$. Therefore

$$\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} = w(s,T,x,b). \tag{17}$$

By the right-continuity of the process $X(t)$, the continuity of $(t,x) \mapsto F(t,x,b)$ for $(t,x) \in [0, T-s) \times E$ and the fact that $w(T,T,x,b) = F(T,x,b)$ we have that

$$w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) \leq F(s+\tau_s^\varepsilon, X(\tau_s^\varepsilon), b) + \varepsilon. \tag{18}$$

Taking the expectation of both sides of (18) and using (17) we finally obtain

$$w(s,T,x,b) \leq \mathbb{E}^x\big\{F(s+\tau_s^\varepsilon, X(\tau_s^\varepsilon), b)\big\} + \varepsilon. \tag{19}$$

■

Assume now that $g(T,x,b) \geq f(T,x,b)$ for all $(T,x,b) \in [0,T^*] \times E \times E^B$. The value function $w$ is continuous on its whole domain $\overline{\mathcal{D}}$ accordingly to the statement (i) of the present theorem. The stopping time $\tau_s$ is well-defined. We prove its optimality by showing that it can be approximated by $\tau_s^\varepsilon$ as $\varepsilon \to 0^+$. First notice that $\tau_s^\varepsilon \leq \tau_s$. As the sequence $(\tau_s^\varepsilon)_{\varepsilon>0}$ is increasing as $\varepsilon$ decreases to 0 there exists $\tau_s^0 = \lim_{\varepsilon\to 0^+} \tau_s^\varepsilon$ with the property $\tau_s^0 \leq \tau_s$. By Theorem 3.13 of [9] the process $X(t)$ is quasi-left continuous, i.e. $X(\tau_s^\varepsilon) \to X(\tau_s^0)$ a.s. Continuity of $w$ and $F$ ($F(u,x,b)$ may have an upward jump as $u$ tends to $T$) yields, almost surely,

$$\begin{aligned}
&\lim_{\varepsilon\to 0+} w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) = w(s+\tau_s^0, T, X(\tau_s^0), b),\\
&\lim_{\varepsilon\to 0+} F(s+\tau_s^\varepsilon, X(\tau_s^\varepsilon), b) \leq F(s+\tau_s^0, X(\tau_s^0), b).
\end{aligned}$$

Therefore, by (18)

$$w(s+\tau_s^0, T, X(\tau_s^0), b) \leq F(s+\tau_s^0, X(\tau_s^0), b).$$

Combining this result with the trivially satisfied opposite inequality we obtain

$$w(s+\tau_s^0, T, X(\tau_s^0)) = F(s+\tau_s^0, X(\tau_s^0)).$$

Consequently $\tau_s^0 = \tau_s$ a.s. By the dominated convergence theorem applied to (19) we have $w(s,T,x,b) \leq \mathbb{E}^x F(s+\tau_s, X(\tau_s), b)$, which proves optimality of $\tau_s$. The proof of Theorem 3.1 is complete.

Under the assumptions of Theorem 3.1(iii) the optimal stopping problem with the discontinuous functional (6) can be transformed into a stopping problem with a continuous functional. Define

$$r(s,T,x,b) = \mathbb{E}^x g\big(T, X(T-s), b\big), \qquad (s,T,x,b) \in [0,T^*] \times [0,T^*] \times E \times E^B, \quad s \leq T.$$

The Feller property of $X(t)$ implies the function $r$ is continuous.

**THEOREM 3.5** *Assuming that $g(T,x,b) \geq f(T,x,b)$, the value function $w(s,T,x,b)$ has the following representation:*

$$w(s,T,x,b) = \sup_{\tau \leq T-s} \mathbb{E}^x \Big\{ f\big(s+\tau, X(\tau), b\big) \vee r\big(s+\tau, T, X(\tau), b\big) \Big\}.$$

*The optimal stopping time for the above functional,*

$$\tau_s^* = \inf\{t \in [0, T-s] : w\big(s+t, T, X(t), b\big) = f\big(s+t, X(t), b\big) \vee r\big(s+t, T, X(t), b\big)\},$$

*defines an optimal stopping for the functional (6) by*

$$\tau_s' = \begin{cases} \tau_s^*, & w\big(s+\tau_s^*, T, X(\tau_s^*), b\big) = f\big(s+\tau_s^*, X(\tau_s^*), b\big), \\ T-s, & w\big(s+\tau_s^*, T, X(\tau_s^*), b\big) > f\big(s+\tau_s^*, X(\tau_s^*), b\big). \end{cases}$$

***Proof.*** Consider a discrete stopping problem

$$v^n(s,T,x,b) = \sup_{\tau \in \mathcal{T}_{\Delta_n(s,T)}} \mathbb{E}^x \big\{ f(s+\tau, X(\tau), b) \vee r(s+\tau, T, X(\tau), b) \big\}. \tag{20}$$

We shall prove by induction that $v^n$ is identical to $w^n$ introduced in the proof of Theorem 3.1. Noting $g(T,x,b) = r(T,T,x,b) \geq f(T,x,b)$ we have

$$\begin{aligned} v^1(s,T,x,b) &= \max\Big( f(s,x,b) \vee r(s,T,x,b), P_{T-s}(f \vee r(\cdot, T, \cdot, \cdot))(T,x,b) \Big) \\ &= \max\Big( f(s,x,b), r(s,T,x,b), P_{T-s}g(T,x,b) \Big) \\ &= \max\Big( f(s,x,b), P_{T-s}g(T,x,b) \Big) = w^1(s,T,x,b). \end{aligned}$$

Given the inductive assumption $v^n = w^n$ we have:

$$\begin{aligned} v^{n+1}(s,T,x,b) &= \max\Big( f(s,x,b) \vee r(s,T,x,b), P_{\Delta_{n+1}(s,T)} v^n(s + \Delta_{n+1}(s,T), T, x, b) \Big) \\ &= \max\Big( f(s,x,b), P_{\Delta_{n+1}(s,T)} r(s + \Delta_{n+1}(s,T), T, x, b), \\ &\qquad\qquad P_{\Delta_{n+1}(s,T)} v^n(s + \Delta_{n+1}(s,T), T, x, b) \Big) \\ &= \max\Big( f(s,x,b), P_{\Delta_{n+1}(s,T)} v^n(s + \Delta_{n+1}(s,T), T, x, b) \Big) \\ &= \max\Big( f(s,x,b), P_{\Delta_{n+1}(s,T)} w^n(s + \Delta_{n+1}(s,T), T, x, b) \Big) \\ &= w^{n+1}(s,T,x,b). \end{aligned}$$

The third equality results from the observation $v^n(s,T,x,b) \geq r(s,T,x,b)$. Lemma 3.3 implies $v^n$ converges to the value function of the problem

$$(s,T,x,b) \mapsto \sup_{\tau \leq T-s} \mathbb{E}^x \Big\{ f\big(s+\tau, X(\tau), b\big) \vee r\big(s+\tau, T, X(\tau), b\big) \Big\}.$$

Due to $v^n = w^n$, this value function is equal to $w$.

The optimality of the stopping time $\tau_s^*$ follows from Theorem 3.1. Its relation to the optimal stopping time for the functional (6) is evident by the following identity:

$$\mathbb{E}^x\Big\{f\big(s+\tau_s^*, X(\tau_s^*), b\big) \vee r\big(s+\tau_s^*, T, X(\tau_s^*), b\big)\Big\} = \mathbb{E}^x\Big\{F\big(s+\tau_s', X(\tau_s'), b\big)\Big\},$$

where $F$ is defined as in the statement of Theorem 3.1. ■

The stopping time $\tau_s'$ constructed in Theorem 3.5 might not coincide with $\tau_s$ defined in (10). Indeed, take any weak Feller process $X(t)$ and define $f(s,x,b) = 2s \wedge 1$ and $g(s,x,b) = 1$ for any $s \geq 0$. Fix a stopping horizon $T = 1$. Simple computations show that $w(s,1,x,b) = 1$ and $r(s,1,x,b) = 1$. Hence $\tau_0^* = 0$ and $\tau_0' = 1$ whereas $\tau_0 = 0.5$.

Theorem 3.5 may appear at first sight as a shortcut to the proof of Theorem 3.1. The transformation of a discontinuous stopping problem into a continuous one is valid *only* under the assumption that $g(T,x,b) \geq f(T,x,b)$. If this assumption is not satisfied the relation between $\varepsilon$-optimal stopping times for these two problems is unclear. The transformed problem has an optimal solution while the original one can only be approximated by $\varepsilon$-optimal times. It can be however shown that the value function of the transformed problem is identical on the set $\mathcal{D}$ to the value function of the original one.

Theorem 3.1 implies the following standard result. The methods of proof are different than usually used, especially in the case of $\varepsilon$-optimal strategies.

**COROLLARY 3.6** *The value function of a standard optimal stopping problem*

$$w(s,T,x) = \sup_{\tau \leq T-s} \mathbb{E}^x\Big\{f\big(s+\tau, X(\tau)\big)\Big\}$$

*is continuous and bounded for a continuous bounded $f$. Optimal stopping time is given by* $\tau = \inf\{t \geq 0 : w(s+t, T, X(t)) \leq f(s+t, X(t))\}$.

The following example (which is a slight modification of an example from [26]) shows that the assumption $P_t\,\mathcal{C}_0 \subseteq \mathcal{C}_0$ can not be replaced by $P_t\,\mathcal{C} \subseteq \mathcal{C}$.

**Example**. Let $E = E_0 \cup E_1$, with $E_0 = \big\{(0,1), (0,\frac{1}{2}), \ldots, (0,\frac{1}{n}), \ldots, (0,0)\big\}$, $E_1 = \{(1,0), (2,0), \ldots (n,0), \ldots\}$ with the topology induced by $\mathbb{R}^2$. Define a Markov process in the following fashion. The state $(0,0)$ is absorbing. The process starting from $(0,\frac{1}{n})$, after an independent exponentially distributed time with parameter 1, is shifted to the state $(n,0)$ and then after an independent exponentially distributed time with parameter $n^2$ is shifted to $(0,\frac{1}{n+1})$. One can check that such a process is Markov with a transition operator $P_t$ satisfying $P_t\,\mathcal{C} \subseteq \mathcal{C}$. Let $f(s,x) = 0$ for $x \in E_0$ and $f(s,x) = 1$ for $x \in E_1$. Then $w(s,T,(0,\frac{1}{n})) = 1 - e^{-(T-s)}$ and $w(s,T,(0,0)) = 0$ which means that the value function is discontinuous in $(0,0)$.

## 3.2. Constrained optimal stopping of a simple discontinuous functional

Consider the following optimal stopping problem: for $0 \leq T_1 \leq T_2 \leq T^*$

$$\tilde{w}(T_1,T_2,x,b) = \sup_{\tau \geq T_1} \mathbb{E}^x\Big\{1_{\{\tau < T_2\}}\, f(\tau, X(\tau), b) + 1_{\{\tau \geq T_2\}}\; g(T_2, X(T_2), b)\Big\}. \tag{21}$$

The difference between this problem and the problem studied in Subsection 3.1 lies only in the set of stopping times over which the optimization is performed. In (21) they are bounded from below by $T_1$ whereas in (7) they are unrestricted. One can expect some similarities in the optimal control strategies and in the properties of the value functions of these stopping problems. This issue is explored in the following proposition:

**PROPOSITION 3.7**

*i) The value function $\tilde{w}$ has the following representation*

$$\tilde{w}(T_1, T_2, x, b) = \mathbb{E}^x w\big(T_1, T_2, X(T_1), b\big),$$

*where $w$ is the value function defined by* (7).

*ii) The function $\tilde{w}$ is continuous and bounded on $\mathcal{D}_1 = \{(T_1, T_2, x, b) \in [0, T^*] \times [0, T^*] \times E \times E^B : T_1 < T_2\}$ and*

$$\lim_{T_1 \to T_2-} \tilde{w}(T_1, T_2, x, b) = f \vee g(T_2, x, b)$$

*uniformly in $T_2 \in [0, T^*]$ and $(x, b)$ in compact subsets of $E \times E^B$.*

*iii) An $\varepsilon$-optimal stopping time is given by*

$$\tau^\varepsilon = \inf\left\{t \geq T_1 : w(t, T_2, X(t), b) \leq F(t, X(t), b) + \varepsilon\right\}, \tag{22}$$

*where $F(u, x, b) = f(u, x, b)$ for $u < T_2$ and $F(T_2, x, b) = g(T_2, x)$.*

*iv) If $g(T_2, x, b) \geq f(T_2, x, b)$, for $(x, b) \in E \times E^B$, the function $\tilde{w}$ is continuous on its domain $\overline{\mathcal{D}}_1$, and the optimal stopping time is given by*

$$\tau = \inf\left\{t \geq T_1 : w(t, T_2, X(t), b) \leq F(t, X(t), b)\right\} \tag{23}$$

*with*

$$\lim_{\varepsilon \to 0^+} \tau^\varepsilon = \tau.$$

***Proof.*** Fix $T_1 \leq T_2 \leq T^*$. Let $\Delta_n = \frac{T_2 - T_1}{n}$. Consider a discretized stopping problem:

$$\tilde{w}^n(T_1, T_2, x, b) = \sup_{\tau \in \tilde{\mathcal{T}}_{\Delta_n(T_1, T_2)}} \mathbb{E}^x\left\{1_{\tau < T_2} f(\tau, X(\tau), b) + 1_{\tau \geq T_2} f \vee g(T_2, X(T_2), b)\right\},$$

where $\tilde{\mathcal{T}}_{\Delta_n(T_1, T_2)}$ denotes a set of all stopping times with values in $\big\{T_1 + k\Delta_n : k = 0, 1, \ldots, n\big\}$. The above supremum can be written as

$$\tilde{w}^n(T_1, T_2, x, b) = \mathbb{E}^x w^n\big(T_1, T_2, X(T_1), b\big),$$

where $w^n(s, T, x, b)$ is the value function of the problem (11). An argument analogous to the one in the proof of Lemma 3.3 extends this relation to the value functions $\tilde{w}$ and $w$. Corollary 2.2 implies (ii) and the first part of assertion (iv). The form of optimal stopping times and convergence of $\tau^\varepsilon$ to $\tau$ can be proved identically as in Theorem 3.1. ■

## 3.3. Optimal stopping of a functional with multiple discontinuities

The purpose of this subsection is to extend the results of previous sections to functionals with multiple discontinuities. Consider the following parametrized optimal stopping problem

$$w(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x F\big(s+\tau, X(\tau), b\big), \qquad (s,T,x,b) \in \Delta \times E \times E^B, \tag{24}$$

where $F : [0,T^*] \times E \times E^B \to \mathbb{R}$ is a bounded function and $\Delta = \{(s,T) \in [0,T^*] \times [0,T^*] : s \le T\}$. Notice that the role of $T$ is different than in (6): it only limits the set of stopping times over which the optimization is performed and does not affect the functional.

**THEOREM 3.8** *Assume for a compact set $B \subseteq E^B$ the function $F$ has the following decomposition:*

$$F(t,x,b) = \sum_{i=0}^{N^*} 1_{\{t \in [t_i(b), t_{i+1}(b))\}}\, f_i(t,x,b) + 1_{\{t=T^*\}}\, f_{N^*+1}(T^*,x,b), \qquad (t,x,b) \in [0,T^*] \times E \times B, \tag{25}$$

*where*

- $N^* \ge 0$ *is a number depending on* $B$,
- $t_0, t_1, \dots, t_{N^*+1} : B \to [0,T^*]$ *is a sequence of continuous functions such that* $t_0(b) \le \dots \le t_{N^*+1}(b)$, $t_0 \equiv 0$ *and* $t_{N^*+1} \equiv T^*$,
- $f_0, f_1, \dots, f_{N^*+1} : [0,T^*] \times E \times B \to \mathbb{R}$ *is a sequence of continuous bounded functions such that*
$$f_i(t_{i+1}(b),x,b) \le f_{i+1}(t_{i+1}(b),x,b), \qquad i = 0,\dots,N^*.$$

*The value function $w(s,T,x,b)$ has the following decomposition for $b \in B$:*

$$w(s,T,x,b) = \sum_{i=0}^{N^*} 1_{\{T \in [t_i(b), t_{i+1}(b))\}}\, w_i(s,T,x,b) + 1_{\{T=T^*\}}\, w_{N^*+1}(s,T^*,x,b), \qquad (s,T,x,b) \in \Delta \times E \times B, \tag{26}$$

*where $w_0, \dots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ are continuous bounded functions. Moreover,*

$$0 \le w(s,t_i(b),x,b) - w(s,t_i(b)-,x,b) \le \sup_{y \in E} \big(F(t_i(b),y,b) - F(t_i(b)-,y,b)\big), \qquad (x,b) \in E \times B, \quad s < t_i(b), \quad i = 1,\dots,N^*+1. \tag{27}$$

The assertion (27) can be rewritten in terms of the functions $(f_i)_{i=0,\ldots,N^*}$ as follows:

$$0 \le w_{i+1}(s, t_{i+1}(b), x, b) - w_i(s, t_{i+1}(b), x, b) \le \sup_{y\in E} \big(f_{i+1}(t_{i+1}(b), y, b) - f_i(t_{i+1}(b), y, b)\big),$$
$$(x, b) \in E \times B, \quad s < t_{i+1}(b), \quad i = 0, \ldots, N^*.$$

***Proof of Theorem 3.8.*** It is sufficient to prove the theorem for $T < T^*$ since the case $T = T^*$ can be easily reduced to the above case by a suitable extension of the time horizon $T^*$ and the functions $f_i$.

Fix a compact set $B \subseteq E^B$ and the decomposition of $F$: an integer $N^*$, functions $t_0, \ldots, t_{N^*+1}$ and $f_0, \ldots, f_{N^*+1}$. Let $N \in \{0, \ldots, N^*\}$. For $b \in B$ and $T \in [t_N(b), t_{N+1}(b))$ the value function $w(s, T, x, b)$ is defined as

$$w(s,T,x,b) = \sup_{\tau\le T-s} \mathbb{E}^x\bigg\{ \sum_{i=0}^{N-1} 1_{\{t_i \le s+\tau < t_{i+1}(b)\}}\, f_i\big(s+\tau, X(\tau), b\big)$$
$$+ 1_{\{t_N(b)\le s+\tau\le T\}}\, f_N\big(s+\tau, X(\tau), b\big)\bigg\}.$$

Consider a sequence of value functions:

$$u_N(s,T,x,b) = \sup_{\tau\le T-s} \mathbb{E}^x\big\{f_N\big(s+\tau, X(\tau), b\big)\big\},$$
$$u_i(s,T,x,b) = \sup_{\tau\le t_{i+1}(b)-s} \mathbb{E}^x\Big\{1_{\{\tau < t_{i+1}(b)-s\}}\, f_i\big(s+\tau, X(\tau), b\big)$$
$$+ 1_{\{\tau = t_{i+1}(b)-s\}}\, u_{i+1}\big(t_{i+1}(b), T, X(t_{i+1}(b)-s), b\big)\Big\}$$
$$i = 0, \ldots, N-1.$$

Domains of functions $u_i$, denoted by $\mathcal{D}_i$, $i = 0, \ldots, N$, are as follows:

$$\mathcal{D}_N = \big\{(s,T,x,b) \in \Delta\times E\times B : T \in [t_N(b), t_{N+1}(b)],\ s \in [t_N(b), T]\big\},$$
$$\mathcal{D}_i = \big\{(s,T,x,b) \in \Delta\times E\times B : T \in [t_N(b), t_{N+1}(b)],\ s \in [t_i(b), t_{i+1}(b)]\big\},$$
$$i = 0, \ldots, N-1.$$

For convenience the domains include $T = t_{N+1}(b)$.

By Theorem 3.1 the function $u_N(s, T, x, b)$ is continuous on $\mathcal{D}_N$. Continuity of $u_i$, $i = N-1, \ldots, 0$ is proved by a backward induction. The definition of $u_i$, $i \le N-1$, already has the form (6): the function $f_i(x, t, b)$ is continuous and $f_i(t_{i+1}(b), x, b) \le u_{i+1}(t_{i+1}(b), T, x, b)$ as $f_i(t_{i+1}(b), x, b) \le f_{i+1}(t_{i+1}(b), x, b) \le u_{i+1}(t_{i+1}(b), T, x, b)$. Theorem 3.1 implies, inductively, that $u_i$ is continuous on $\mathcal{D}_i$. The functions $u_i$ and $u_{i+1}$ are identical on $\mathcal{D}_i \cap \mathcal{D}_{i+1}$, so the function

$$v_N(s,T,x,b) = \sum_{i=0}^{N} 1_{\{s\in[t_i(b),t_{i+1}(b))\cap[0,T]\}}\ u_i(s,T,x,b) \tag{28}$$

is continuous on $\mathcal{D} = \big\{(s,T,x,b) \in \Delta \times E \times B : T \in [t_N(b), t_{N+1}(b)]\big\}$. Due to the Bellman principle (it can be proved by discretization as in Theorem 3.1) the function $u_i(s,T,x,b)$ is the value function of the optimal stopping problem starting at $s$, i.e. for $(s,T,x,b) \in \mathcal{D}_i$, $T < t_{N+1}(b)$

$$u_i(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x F\big(s+\tau, X(\tau), b\big).$$

Therefore, $v_N$ coincides with $w$ on the set $\mathcal{D} \cap \{(s,T,x,b) : T < t_{N+1}(b)\}$. Since $\mathcal{D}$ is a closed set, $v_N$ can be trivially extended as a continuous bounded function to the domain $\Delta \times E \times B$. This extension satisfies all the conditions of the function $w_N$ in the representation (26).

Let $\eta = \sup_{y \in E} \big(F\big(t_{N+1}(b), y, b\big) - F\big(t_{N+1}(b)-, y, b\big)\big)$. We have

$$\begin{aligned}
&w(s, t_{N+1}(b), x, b) \\
&= \sup_{\tau \le t_{N+1}(b)-s} \mathbb{E}^x \bigg\{ \sum_{i=0}^{N} 1_{\{t_i(b) \le s+\tau < t_{i+1}(b)\}}\, f_i\big(s+\tau, X(\tau), b\big) \\
&\qquad\qquad\qquad\qquad + 1_{\{s+\tau \ge t_{N+1}(b)\}}\, f_{N+1}\big(t_{N+1}(b), X(t_{N+1}), b\big) \bigg\} \\
&\le \sup_{\tau \le t_{N+1}(b)-s} \mathbb{E}^x \bigg\{ \sum_{i=0}^{N} 1_{\{t_i(b) \le s+\tau < t_{i+1}(b)\}}\, f_i\big(s+\tau, X(\tau), b\big) \\
&\qquad\qquad\qquad\qquad + 1_{\{s+\tau \ge t_{N+1}(b)\}}\, f_N\big(t_{N+1}(b), X(t_{N+1}(b)), b\big) \bigg\} + \eta \\
&= w\big(s, t_{N+1}(b)-, x, b\big) + \eta.
\end{aligned}$$

The last equality follows from the continuity of $w_N$ and its coincidence with $w$ for $T < t_{N+1}(b)$. This implies (27) for $i = N+1$. ■

**COROLLARY 3.9** *An optimal stopping time for the problem (24) is given by the formula*

$$\tau_s = \inf\big\{t \ge 0 : \quad \big(s+t, X(t)\big) \in I(T,b)\big\},$$

*where*

$$I(T,b) = \big\{(t,x) \in [0,T] \times E : \quad w(t,T,x,b) \le F(t,x,b)\big\}.$$

***Proof.*** Recalling the argument at the beginning of the proof of Theorem 3.8, we can assume that $T < T^*$. Fix $b, T$ and the decomposition (26) of $w$. By Theorem 3.1 an optimal stopping time is given by $\tau_s = \inf\big\{t \ge 0 : \quad \big(s+t, X(t)\big) \in \tilde{I}(T,b)\big\}$ with the stopping region

$$\tilde{I}(T,b) = \bigcup_{i=0,\dots,N^*} \big\{(t,x) \in [t_i(b), t_{i+1}(b)) \times E : \quad u_i(t,T,x,b) \le F(t,x,b)\big\}.$$

Functions $(u_i)_{i=0,\dots,N^*}$ are defined in the proof of Theorem 3.8. Due to (28), the set $\tilde{I}(T,b)$ coincides with $I(T,b)$ in the statement of the corollary. ■

## 3.4. Constrained parametrized optimal stopping with an integral term

The setting of the preceding section is extended to functionals with an integral term. Let $F : [0,T^*] \times E \times E^B \to \mathbb{R}$ be a bounded function and $f : [0,T^*] \times E \times E^B \to \mathbb{R}$ be a continuous bounded function. Consider the following optimal stopping problem:

$$w(T_1,T_2,x,b) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x \bigg\{ \int_0^\tau f\big(s, X(s), b\big) ds + F\big(\tau, X(\tau), b\big) \bigg\},$$
$$(T_1,T_2,x,b) \in \Delta \times E \times E^B, \quad (29)$$

where $\Delta$ is the set of admissible time constraints, i.e $\Delta = \{(T_1,T_2) \in [0,T^*]^2 : T_1 \le T_2\}$.

**THEOREM 3.10** *Assume for a compact set $B \subseteq E^B$ the function $F$ has the decomposition* (25)*. For $b \in B$ the value function $w(T_1,T_2,x,b)$ can be written as*

$$w(T_1,T_2,x,b) = \sum_{i=0}^{N^*} 1_{\{T_2 \in [t_i(b), t_{i+1}(b))\}}\, w_i(T_1,T_2,x,b) + 1_{\{T_2 = T^*\}}\, w_{N^*+1}(T_1,T^*,x,b),$$
$$(T_1,T_2,x,b) \in \Delta \times E \times B, \quad (30)$$

*where $w_0,\dots,w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ are continuous bounded functions. The discontinuities of $w$ are bounded as follows:*

$$0 \le w(T_1, t_i(b), x, b) - w(T_1, t_i(b)-, x, b) \le \sup_{y \in E} \big( F(t_i(b), y, b) - F(t_i(b)-, y, b) \big),$$
$$T_1 < t_i(b), \quad (x,b) \in E \times B,$$

*for $i = 1, \dots, N^*+1$. Moreover, there exists an optimal stopping time for every $x$, $T_1$, $T_2$ and $b$.*

***Proof.*** Notice that for a stopping time $\tau$

$$\mathbb{E}^x \Big\{ \int_0^\tau f(s, X(s), b) ds \Big\} = H(0,x,b) - \mathbb{E}^x \big\{ H(\tau, X(\tau), b) \big\},$$

where

$$H(t,x,b) = \mathbb{E}^x \Big\{ \int_0^{T^*-t} f(t+s, X(s), b) ds \Big\}, \qquad t \in [0,T^*], \quad x \in E, \quad b \in E^B.$$

Due to Corollary 2.2 the function $H$ is continuous and bounded.

Above observation drives the following reformulation of the functional (29):

$$w(T_1,T_2,x,b) = H(0,x,b) + \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x \bigg\{ -H\big(\tau, X(\tau), b\big) + F\big(\tau, X(\tau), b\big) \bigg\}.$$

The continuity of $H$ implies that the discontinuities of $\tilde{F}(t,x,b) = F(t,x,b) - H(t,x,b)$ and $F(t,x,b)$ are identical.

The arguments as in Proposition 3.7 imply that

$$w(T_1, T_2, x, b) = H(0, x, b) + \mathbb{E}^x \tilde{w}(T_1, T_2, X(T_1), b),$$

where

$$\tilde{w}(s, T, x, b) = \sup_{\tau \le T-s} \mathbb{E}^x \tilde{F}(s+\tau, X(\tau), b).$$

By virtue of Theorem 3.8 the function $\tilde{w}$ has the decomposition:

$$\tilde{w}(s,T,x,b) = \sum_{i=0}^{N^*} 1_{\{T \in [t_i(b), t_{i+1}(b))\}}\, \tilde{w}_i(s,T,x,b) + 1_{\{T=T^*\}}\, \tilde{w}_{N^*+1}(s,T^*,x,b),$$
$$(s,T,x,b) \in \Delta \times E \times B,$$

where $\tilde{w}_0, \ldots, \tilde{w}_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ are continuous bounded functions that satisfy (27). Then

$$w_i(T_1, T_2, x, b) = H(0,x,b) + \mathbb{E}^x \tilde{w}(T_1, T_2, X(T_1), b), \qquad i = 0, \ldots, N^*+1,$$

are continuous, bounded (by Corollary 2.2) and satisfy the assertions of the present theorem. ■

**COROLLARY 3.11** *Using notation from the proof, an optimal stopping time is given by*

$$\tau = \inf \big\{ t \ge T_1 : \quad \tilde{w}(t, T_2, X(t), b) \le \tilde{F}(t, X(t), b) \big\},$$

*for* $b \in B$, $T_1, T_2 \in \Delta$.

***Proof.*** Analogous to the proof of Corollary 3.9. ■

# 4. Optimal stopping of left-continuous functionals

This section explores properties of value functions of optimal stopping problems with left-continuous functionals. The main difficulty arising here stems from the fact that the functional is itself left-continuous whereas the process $X(t)$ is right-continuous. It prevents the application of the most natural discretization technique as in the previous section. The problem, however, can be reformulated in a way that permits the use of the results for right-continuous functionals.

Consider a parametrized optimal stopping problem

$$w(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x \Big\{ 1_{\{\tau \le t_1(b) - s\}}\, f_1\big(s+\tau, X(\tau), b\big)$$
$$+ 1_{\{\tau > t_1(b)-s\}}\, v\big(t_1(b) \vee s, T, X((t_1(b)-s) \vee 0), b\big) \Big\},$$

where $t_1 : E^B \to \mathbb{R}$ is continuous, $f_1 : [0, T^*] \times E \times E^B \to \mathbb{R}$ and $v : \Delta \times E \times E^B \to \mathbb{R}$ are continuous and bounded. Notice the peculiarity of the functional. The function $v$ is evaluated at a fixed time $(t_1(b) - s) \vee 0$ in contract to the standard policy of the evaluation at $\tau$. This construction is motivated by the presumption that $v$ is the value function of a stopping problem and the evaluation at $(t_1(b) - s) \vee 0$ is optimal.

**LEMMA 4.1** *Assume $f_1(t_1(b), x, b) \geq v(t_1(b), t_1(b), x, b)$. The value function has the decomposition*

$$w(s, T, x, b) = 1_{\{s \leq t_1(b)\}}\, w_1\big(s, T, x, b\big) + 1_{\{s > t_1(b)\}}\, v\big(s, T, x, b\big) \tag{31}$$

*for continuous bounded functions $w_i : \Delta \times E \times E^B \to \mathbb{R}$ satisfying*

$$0 \leq w_1\big(t_1(b), T, x, b\big) - v\big(t_1(b), T, x, b\big) \leq \big(f_1(t_1(b), x, b) - v(t_1(b), T, x, b)\big) \vee 0. \tag{32}$$

*An optimal stopping time is $\tau_s = 0$ for $s > t_1(b)$ and*

$$\tau_s = \inf \big\{ t \in [0, t_1(b) - s] : w(s + t, T, X(t), b) \leq f_1(s + t, X(t), b) \big\} \wedge (T - s), \tag{33}$$

*for $s \leq t_1(b)$ with the convention* $\inf \emptyset = \infty$.

***Proof.*** For $s \leq t_1(b)$ define the following auxiliary value function

$$\begin{aligned}\tilde{w}(s, T, x, b) = \sup_{\tau \leq T - s} \mathbb{E}^x \Big\{ & 1_{\{\tau < t_1(b)\}}\, f_1\big(s + \tau, X(\tau), b\big) \\ & + 1_{\{\tau \geq t_1(b)\}}\, f_1\big(t_1(b), X(t_1(b) - s), b\big) \vee v\big(t_1(b), T, X(t_1(b) - s), b\big) \Big\}.\end{aligned}$$

This value function dominates $w$ since the functional in $\tilde{w}$ dominates the functional in $w$ pointwise. Theorem 3.8 implies the value function $\tilde{w}$ has the form

$$\tilde{w}(s, T, x, b) = 1_{\{T < t_1(b)\}}\, \tilde{w}_1(s, T, x, b) + 1_{\{T \geq t_1(b)\}}\, \tilde{w}_2(s, T, x, b),$$

for continuous bounded $\tilde{w}_1, \tilde{w}_2$ and there exists an optimal stopping time $\tilde{\tau}_s$ given by

$$\tilde{\tau}_s = \inf \big\{ t \in [0, T - s] : \tilde{w}(s + t, T, X(t), b) \leq \tilde{F}(s + t, T, X(t), b) \big\}$$

for $\tilde{F}(t, T, x, b) = 1_{\{t < t_1(b)\}}\, f_1(t, x, b) + 1_{\{t \geq t_1(b)\}}\, f_1(t_1(b), x, b) \vee v(t_1(b), T, x, b)$ (to be absolutely precise in the application of Theorem 3.8 the variable $T$ has to be doubled: as a terminal time for stopping and as an additional parameter due to its appearance in $v$). By (27) we have

$$\begin{aligned}0 \leq \tilde{w}_2(s, t_1(b), x, b) - \tilde{w}_1(s, t_1(b), x, b) \\ \leq \sup_{y \in E} \big\{ \big( v(t_1(b), t_1(b), y, b) - f_1(t_1(b), x, b) \big) \big\} \vee 0 = 0,\end{aligned}$$

where the last equality results from the assumption $v(t_1(b), t_1(b), y, b) \leq f_1(t_1(b), x, b)$. This implies $\tilde{w}$ is continuous.

For fixed $T, x, b$ and $s \le t_1(b)$ define a stopping time

$$\tau_s = \begin{cases} \tilde{\tau}_s, & \tilde{\tau}_s < t_1(b) - s, \\ t_1(b) - s, & \tilde{\tau}_s \ge t_1(b) - s \quad \text{and} \\ & \qquad f_1\big(t_1(b), X(t_1(b) - s), b\big) \ge v\big(t_1(b), T, X(t_1(b) - s), b\big), \\ T - s, & \tilde{\tau}_s \ge t_1(b) - s \quad \text{and} \\ & \qquad f_1\big(t_1(b), X(t_1(b) - s), b\big) < v\big(t_1(b), T, X(t_1(b) - s), b\big). \end{cases}$$

This stopping time is identical to the one defined in (33). It is also optimal for $w$ as a verification below shows. First notice

$$w(s, T, x, b) \ge \mathbb{E}^x \Big\{ 1_{\{\tau_s \le t_1(b)\}} f_1\big(s+\tau_s, X(\tau_s), b\big) + 1_{\{\tau_s > t_1(b)\}} v\big(t_1(b), T, X(t_1(b) - s), b\big) \Big\}.$$

For $t_1(b) \neq T$ the right-hand side equals to $\tilde{w}(s, T, x, b)$. The assumption $f_1(t_1(b), x, b) \ge v(t_1(b), t_1(b), x, b)$ extends this result to $T = t_1(b)$. Therefore, $\tilde{w}(s, T, x, b)$ coincides with $w(s, T, x, b)$ for $s \le t_1(b)$ and we put $w_1 = \tilde{w}$.

Above arguments do not hold for $s > t_1(b)$ as there might be a strict inequality between $\tilde{w}$ and $w$. However, the stopping problem becomes trivial since $w(s, T, x, b) = v(s, T, x, b)$ on $s > t_1(b)$. An optimal stopping time is $\tau_s = 0$.

Inequalities (32) follow from the following indentity:

$$w_1\big(t_1(b), T, x, b\big) = \tilde{w}_1\big(t_1(b), T, x, b\big) = f_1\big(t_1(b), x, b\big) \vee v\big(t_1(b), T, x, b\big).$$

∎

The optimal stopping time for $w(s, T, x, b)$ cannot be written in the standard form

$$\inf\{t \in [0, T - s] : \quad w\big(s + t, T, X(t), b\big) \le F\big(s + t, T, X(t), b\big)\},$$

where

$$F(t, T, x, b) = 1_{\{t \le t_1(b)\}} f_1\big(t, x, b\big) + 1_{\{t > t_1(b)\}} v\big(t, T, x, b\big),$$

because the process $t \mapsto F(s + t, T, X(t), b)$ might not be right-continuous.

Now we turn our attention towards a parametrized optimal stopping problem with multiple discontinuities

$$w(s, T, x, b) = \sup_{\tau \le T - s} \mathbb{E}^x F\big(s + \tau, X(\tau), b\big), \qquad (s, T, x, b) \in \Delta \times E \times B, \tag{34}$$

where $F : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a bounded function.

**THEOREM 4.2** *Assume that for a compact set $B \subseteq E^B$ the function $F$ has the following decomposition:*

$$F(t, x, b) = 1_{\{t=0\}} f_0(0, x, b) + \sum_{i=1}^{N^*+1} 1_{\{t \in (t_{i-1}(b), t_i(b)]\}} f_i(t, x, b), \qquad (t, x, b) \in [0, T^*] \times E \times B, \tag{35}$$

*where*

- $N^* \geq 0$ *is a number depending on* $B$,
- $t_0, t_1, \ldots, t_{N^*+1} : B \to [0, T^*]$ *is a sequence of continuous functions such that* $t_0(b) \leq \cdots \leq t_{N^*+1}(b)$, $t_0 \equiv 0$ *and* $t_{N^*+1} \equiv T^*$,
- $f_0, f_1, \ldots, f_{N^*+1} : [0, T^*] \times E \times B \to \mathbb{R}$ *is a sequence of continuous bounded functions such that*
$$f_i(t_i(b), x, b) \geq f_{i+1}(t_i(b), x, b), \qquad i = 0, \ldots, N^*. \tag{36}$$

*The value function* $w(s, T, x, b)$ *has the following decomposition for* $b \in B$:

$$w(s,T,x,b) = 1_{\{s=0\}}\, w_0(0,T,x,b) + \sum_{i=1}^{N^*+1} 1_{\{s \in (t_{i-1}(b), t_i(b)]\}}\, w_i(s,T,x,b), \qquad (s,T,x,b) \in \Delta \times E \times B,$$

*where* $w_0, \ldots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ *are continuous bounded functions. Moreover,*

$$0 \leq w_i(t_i(b),T,x,b) - w_{i+1}(t_i(b),T,x,b) \leq f_i(t_i(b),x,b) - f_{i+1}(t_i(b),x,b), \qquad T \geq t_i(b), \quad (x,b) \in E \times B, \quad i = 0, \ldots, N^*. \tag{37}$$

*An optimal stopping time is given by the formula*

$$\tau_s = \inf\{t \in [0, T-s] : \quad w\big(s+t, T, X(t), b\big) \leq F\big(s+t, X(t), b\big)\}. \tag{38}$$

The last assertion of the theorem provides a relation between jumps of the functions $w$ and $F$. It can be rewritten as

$$0 \leq w(t_i(b),T,x,b) - w(t_i(b)+,T,x,b) \leq F(t_i(b),y,b) - F(t_i(b)+,y,b), \qquad i = 0, \ldots, N^*, \quad (x,b) \in E \times B.$$

This implies that the stopping time $\tau_s$ defined in the theorem is well-defined and the infimum is attained.

***Proof.*** Fix a compact set $B \subseteq E^B$ and a decomposition (35). Define the value functions

$$v^{N^*+1}(s,T,x,b) = \sup_{\tau \leq T-s} \mathbb{E}^x \Big\{ f_{N^*+1}(s+\tau, X(\tau), b) \Big\},$$

and for $i = N^*, \ldots, 0$,

$$v^i(s,T,x,b) = \sup_{\tau \leq T-s} \mathbb{E}^x \Big\{ 1_{\{s+\tau \leq t_i(b)\}}\, f_i(s+\tau, X(\tau), b) + 1_{\{s+\tau > t_i(b)\}}\, v^{i+1}(t_i(b) \vee s, T, X((t_i(b)-s) \vee 0), b) \Big\}.$$

By Theorem 3.1 the function $v^{N^*+1}$ is continuous, bounded and coincides with $w$ for $s > t_{N^*}(b)$.

Consider the following inductive hypotheses for $v^i$, $0 \in \{1, \ldots, N^*\}$:

i) $v^i$ coincides with $w$ on the set $\mathcal{D}^i = \{(s,T,x,b) \in \Delta \times E \times B : s > t_{i-1}(b)\}$, where $t_{-1} \equiv -\infty$,

ii) $v^i$ has a decomposition

$$v^i(s,T,x,b) = 1_{\{s \le t_i(b)\}}\, v_1^i(s,T,x,b) + 1_{\{s > t_i(b)\}}\, v^{i+1}(s,T,x,b) \tag{39}$$

for a continuous bounded function $v_1^i : \Delta \times E \times B \to \mathbb{R}$.

First, we show that these hypotheses are satisfied for $v^{N^*}$. By the definition of $v^{N^*+1}$ and inequalities (36) we have

$$v^{N^*+1}\big(t_{N^*}(b), t_{N^*}(b), x, b\big) = f_{N^*+1}\big(t_{N^*}(b), x, b\big) \le f_{N^*}\big(t_{N^*}(b), x, b\big).$$

The assumptions of Lemma 4.1 are satisfied and $v^{N^*}$ has the following representation:

$$v^{N^*}(s,T,x,b) = 1_{\{s \le t_{N^*}(b)\}}\, v_1^{N^*}(s,T,x,b) + 1_{\{s > t_{N^*}(b)\}}\, v^{N^*+1}(s,T,x,b)$$

for a continuous bounded function $v_1^{N^*} : \Delta \times E \times B \to \mathbb{R}$. Due to the Bellman principle $v^{N^*}$ coincides with $w$ on the set $\{(s,T,x,b) \in \Delta \times E \times B : s > t_{N^*-1}(b)\}$.

Assume the inductive hypotheses for $v^{i+1}$. Define an auxiliary stopping problem

$$\begin{aligned}\tilde v^i(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x \Big\{ & 1_{\{s+\tau \le t_i(b)\}}\, f_i(s+\tau, X(\tau), b) \\ & + 1_{\{s+\tau > t_i(b)\}}\, v_1^{i+1}(t_i(b) \vee s, T, X((t_i(b)-s) \vee 0), b) \Big\},\end{aligned}$$

where $v_1^{i+1}$ is the function from decomposition (39). We infer from the definition of $v^{i+1}$ and the inequalities (36) that

$$v_1^{i+1}\big(t_i(b), t_i(b), x, b\big) = f_{i+1}\big(t_i(b), x, b\big) \le f_i\big(t_i(b), x, b\big).$$

Lemma 4.1 implies $\tilde v^i$ can be written as:

$$\tilde v^i(s,T,x,b) = 1_{\{s \le t_i(b)\}}\, v_1^i(s,T,x,b) + 1_{\{s > t_i(b)\}}\, v_1^{i+1}(s,T,x,b)$$

for a continuous bounded function $v_1^i : \Delta \times E \times B \to \mathbb{R}$. The value function $\tilde v^i$ coincides with $v^i$ for $s \le t_{i+1}(b)$. By the Bellman principle the function $v^i$ agrees with $w$ on the set $\{(s,T,x,b) \in \Delta \times E \times B : s > t_{i-1}(b)\}$. This completes the proof of the hypotheses (i)-(ii) for $v^i$.

Put $w_{N^*+1} = v^{N^*+1}$ and $w_i = v_1^i$ for $i = 0, \ldots, N^*$. This definition is justified by conditions (i)-(ii), for $i \le N^*$ and by the construction of $v^{N^*+1}$.

Inequalities (37) are justified by (32) in Lemma 4.1:

$$\begin{aligned}w_i\big(t_i(b),T,x,b\big) - w_{i+1}\big(t_i(b),T,x,b\big) &= v_1^i\big(t_i(b),T,x,b\big) - v_1^{i+1}\big(t_i(b),T,x,b\big) \\ &\le \big(f_i\big(t_i(b),x,b\big) - v_1^{i+1}\big(t_i(b),T,x,b\big)\big) \vee 0 \le f_i\big(t_i(b),x,b\big) - f_{i+1}\big(t_i(b),x,b\big).\end{aligned}$$

An optimal stopping time can be extracted from optimal stopping times $\tau_s^i$ for the partial value functions $v^0, \ldots, v^{N^*+1}$. Fix $s, T, x, b$ and let $i$ be such that $s \in (t_{i-1}(b), t_i(b)]$, with the convention $t_{-1} \equiv -\infty$. The procedure is as follows. If $\tau_s^i \le t_i(b)$ it is optimal to stop at $\tau_s^i$. Otherwise, the control is handed over to the level $i+1$. Since $f_i$ dominates $f_{i+1}$ at $t = t_i(b)$ the inequality $\tau_s^i > t_i(b)$ implies $\tau_{t_i(b)}^{i+1} > t_i(b)$. Again, it is optimal to stop at $\tau_{t_i(b)}^{i+1}$ if $\tau_{t_i(b)}^{i+1} \le t_{i+1}(b)$, and to continue to the level $i+2$ if $\tau_{t_i(b)}^{i+1} > t_{i+1}(b)$. This routine is repeated until the terminal time $T$ is reached.

Thanks to the representation (33) of the stopping times $\tau_\cdot^k$, $k = i, \ldots, N^*$, the stopping time offered by the above procedure can be written as

$$
\begin{aligned}
&\inf\{t \in [s, t_i(b) \wedge T] : \quad w\big(t, T, X(t-s), b\big) \le f_i\big(t, X(t-s), b\big)\} \\
&\wedge \inf\{t \in (t_i(b) \wedge T, t_{i+1}(b) \wedge T] : \quad w\big(t, T, X(t-s), b\big) \le f_{i+1}\big(t, X(t-s), b\big)\} \\
&\ldots \\
&\wedge \inf\{t \in (t_{N^*}(b) \wedge T, t_{N^*+1}(b) \wedge T] : \quad w\big(t, T, X(t-s), b\big) \le f_{N^*+1}\big(t, X(t-s), b\big)\}
\end{aligned}
$$

with the convention $\inf \emptyset = \infty$. Above expression simplifies to the formula (38) in the statement of the theorem. ■

An analogous argument as in the proof of the Theorem 3.10 extends the above result to functionals with an integral term and a restricted stopping region:

**COROLLARY 4.3** *Assume $F : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a bounded function and $f : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a continuous bounded function. Consider an optimal stopping problem:*

$$
w(T_1, T_2, x, b) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x \bigg\{ \int_0^\tau f\big(s, X(s), b\big) ds + F\big(\tau, X(\tau), b\big) \bigg\}, \qquad (T_1, T_2, x, b) \in \Delta \times E \times E^B. \quad (40)
$$

*If $F$ has the decomposition* (35) *for a compact set $B \subseteq E^B$, the value function $w(T_1, T_2, x, b)$ has, for $b \in B$, the following decomposition:*

$$
w(T_1, T_2, x, b) = 1_{\{T_1 = 0\}}\, w_0(0, T_2, x, b) + \sum_{i=1}^{N^*+1} 1_{\{T_1 \in (t_{i-1}(b), t_i(b)]\}}\, w_i(T_1, T_2, x, b), \qquad (T_1, T_2, x, b) \in \Delta \times E \times B, \quad (41)
$$

*where $w_0, \ldots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ are continuous bounded functions. The discontinuities of $w$ are bounded as follows:*

$$
0 \le w(t_i(b), T_2, x, b) - w(t_i(b)+, T_2, x, b) \le F(t_i(b), x, b) - F(t_i(b)+, x, b), \qquad T_2 > t_i(b), \quad (x, b) \in E \times B,
$$

*for $i = 0, \ldots, N^* + 1$. Moreover, there exists an optimal stopping time for every $x$, $T_1$, $T_2$ and $b$.*

# 5. Impulse control with decision lag and execution delay

The theory of optimal stopping of left-continuous functionals is nicely illustrated by its application to impulse control problems. As in the previous sections a Markov process $\big(X(t)\big)$ is defined on a locally compact separable metric space $(E, \mathcal{E})$ and satisfies the weak Feller property. Now it is controlled using impulses. Impulsive strategy is a sequence of pairs $(\tau_i, \xi_i)$, where $(\tau_i)$ are stopping times with respect to the history $(\mathcal{F}_t)$ and variables $\xi_i$ are $\mathcal{F}_{\tau_i}$-measurable. The pair $(\tau_i, \xi_i)$ is interpreted in the following way: at the moment $\tau_i + \Delta$ the process $X_t$ is shifted to the state given by $\Gamma\big(X_-(\tau_i + \Delta), \xi_i\big)$, where $X_-(\tau_i + \Delta)$ represents the state of the process strictly before the exercise of the impulse (the process does not have to be left-continuous so this value may not coincide with the left-hand limit of the controlled process). A deterministic $\Delta \geq 0$ imposes a delay in the execution of the impulse. We write $\Pi = \big((\tau_1, \xi_1), (\tau_2, \xi_2), \dots\big)$ and denote such controlled process by $\big(X^\Pi(t)\big)$. Notice that the filtration $(\mathcal{F}_t)$ depends on the control and on the initial state of the process $(X^\Pi(t))$.

There are two time points related to an impulse $(\tau_i, \xi_i)$. At $\tau_i$, called the *ordering time*, a decision is made upon the action $\xi_i$. It is then *executed* at time $\tau_i + \Delta$. This naming convention will be used throughout this section.

Let $h \geq 0$ and $\Theta$ be a given compact set. The set of admissible controls $\mathcal{A}(x)$ consists of impulsive strategies $\Pi = \big((\tau_1, \xi_1), (\tau_2, \xi_2), \dots\big)$ being a sequence of stopping times $\tau_i$ such that $\tau_{i+1} \geq \tau_i + h$, for $i = 1, 2, \dots$, and $\mathcal{F}_{\tau_i}$ measurable control variables $\xi_i$ taking values in the set $\Theta$. Value $h \geq 0$ has the meaning of a decision lag, i.e. it is the minimal time gap separating ordering times. If a new impulse $(\tau_i, \xi_i)$ is ordered at the moment when a pending impulse $(\tau_k, \xi_k)$ is scheduled to be executed (i.e. when $\tau_i = \tau_k + \Delta$), the decision about $\xi_i$ is made *after* the shift of $X^\Pi$ determined by $\xi_k$.

A mathematically precise construction of the probability space on which the controlled process is defined can be found in a seminal paper by Robin [22] and his thesis [23]. Let $\big(X_k^\Pi(t)\big)$ be a sequence of processes defined inductively in the following way:

$$
\begin{aligned}
X_0^\Pi(t) &= X(t), \quad t \geq 0,\\
X_{i+1}^\Pi(t) &= \begin{cases} X_i^\Pi(t), & t \leq \tau_{i+1} + \Delta,\\ X^{\Gamma\left(X_i^\Pi(\tau_{i+1}+\Delta), \xi_{i+1}\right)}(t - \tau_{i+1} - \Delta), & t > \tau_{i+1} + \Delta, \end{cases}
\end{aligned}
$$

where $\big(X^\mu(t)\big)$ denotes a process starting from an initial distribution $\mu$; in the above formula

$$X^{\Gamma\left(X_i^\Pi(\tau_{i+1}+\Delta), \xi_{i+1}\right)}(0) \sim \Gamma\big(X_i^\Pi(\tau_{i+1} + \Delta), \xi_{i+1}\big).$$

Intuitive meaning of $X_i^\Pi(t)$ is that this is a process controlled by first $i$ impulses, i.e. by $\Pi_i = \big((\tau_1, \pi_1), \dots, (\tau_i, \xi_i)\big)$. The controlled process $X^\Pi(t)$ can be composed of the segments $X_i^\Pi(t)$ in the following way:

$$X^\Pi(t) = 1_{\{t \leq \tau_1\}}\, X_0^\Pi(t) + \sum_{i=1}^\infty 1_{\{\tau_i < t \leq \tau_{i+1}\}}\, X_i^\Pi(t). \tag{42}$$

Consider an optimal control problem with a finite horizon $T > 0$ and a functional given by

$$J(x, \Pi, T) = \mathbb{E}^x \Big\{ \int_0^T e^{-\alpha s} f\big(t, X^\Pi(t)\big) dt + e^{-\alpha T} g\big(X^\Pi(T)\big) \\ + \sum_{i=1}^{\infty} 1_{\{\tau_i + \Delta \le T\}} \, e^{-\alpha(\tau_i + \Delta)} c\big(X^\Pi_{i-1}(\tau_i + \Delta), \xi_i\big) \Big\}, \quad (43)$$

where $\alpha \geq 0$ is a discount factor, $f$ measures a running reward (cost), $g$ is a terminal reward (cost) and $c$ is the cost for impulses. Although the probability measure with respect to which the expectation in (43) is computed depends on the control $\Pi$ we omit this dependence in the notation.

Our goal is to find the value function

$$v(x) = \sup_{\Pi \in \mathcal{A}(x)} J(x, \Pi, T)$$

and an admissible strategy $\Pi^* \in \mathcal{A}(x)$, called the *optimal strategy*, for which the supremum is attained.

We make the following standing assumptions:

(A1) Functions $c : E \times \Theta \to \mathbb{R}$, $f : [0, T] \times E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are continuous and bounded.

(A2) The function $\Gamma : E \times \Theta \to E$ is continuous.

The main result of this section is summarized in the theorem below.

**THEOREM 5.1** *Assume (A1)-(A2) and $h > 0$.*

*i) The value function $v(x)$ is continuous and bounded.*

*ii) For every $x \in E$ there exists an optimal strategy, i.e. an admissible strategy in $\mathcal{A}(x)$, for which the maximum is attained.*

Theorem 5.1generalizes and complements several existing results on optimal control with and without delay ([1, 2, 8, 19, 20]). Its formulation, suggesting a standard approach in solving optimal control problems, is misleading. The controlled process $(X^\Pi_t)$ is no longer Markovian due to the accumulation of pending impulses. An approach, suggested in [8], leads via a system of optimal stopping problems of Markovian type. Our solution is influenced by this idea, but differs from [8] in many points. Our setting is much more general as we only assume the underlying process to be defined on a locally compact separable state space and to satisfy the weak Feller property. Our proofs benefit from the discretization techniques which do not rely on a convenient form of the infinitesimal generator of the underlying Markov process. In contrast, existing results employ formulations via partial differential equations and are often limited by technical assumptions arising from the theory of PDEs.

In Subsection 5.1 we develop a system of optimal stopping problems possessing certain Markovian properties. The stopping problems comprising the system have discontinuous functionals. These discontinuities come naturally as a result of the decision lag $h$ and delay $\Delta$ limiting admissible strategies and their execution. Stopping techniques developed in previous sections enable us to solve the discontinuous stopping problems and prove the existence and form of optimal strategies in full detail. This part of the development is pursued in Subsection 5.2. The proof of Theorem 5.1 is located in Subsection 5.3. It is followed by a discussion of the relation of our findings to the existing results.

## 5.1. Reduction to optimal stopping problems

As it has been pointed out the controlled process $(X_t^\Pi)$ is no longer Markovian due to the accumulation of pending impulses. Our solution is based on a decomposition of the optimal control problem into an infinite dimensional system of optimal stopping problems. For $n \geq 0$ denote by

$$v_i^n(x, s, d, \pi) : E \times [0, T^*] \times [0, h] \times ([0, \Delta] \times \Theta)^i \to \mathbb{R}, \qquad i = 0, \ldots, n,$$

the value function for the maximization of the functional (43) under the conditions described by the parameters:

- $n$ is the maximum number of impulses that can be ordered, $n \geq 0$,
- the first new impulse can be ordered after at least $d$ units of time, $d \in [0, h]$,
- $x$ is a starting point for the process $(X(t))$, $x \in E$,
- $s$ denotes the time until maturity $T$, so the optimization horizon is $s$, $s \in [0, T]$,
- $i$ is the number of pending impulses (stored in $\pi$), $i \geq 0$,
- $\pi$ consists of $i$ pairs $((\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))$, where $\xi_k$ is the action, $\delta_k$ is the time until the execution of the action $\xi_k$ and $\delta_1 < \cdots < \delta_i \leq s$.

The role of $s$ in the parameters of $(v_i^n)$ is different than in previous sections: it denotes the time until maturity $T$. This choice is motivated by two observations. Firstly, it allows us to skip the maturity $T$ in the parameters of $v_i^n$ and reduces the dimension of the problem. Secondly, all the points of discontinuities of the above value functions are naturally expressed relative to the distance to the maturity $T$ (see Theorem 5.3).

To simplify the notation, define an operator

$$Mv_i^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_{i-1}, \xi_{i-1})\big) = \sup_{\xi \in \Theta} v_i^n\big(x, s, h, (\delta_1, \xi_1), \ldots, (\delta_{i-1}, \xi_{i-1}), (\Delta, \xi)\big).$$

The following standard result holds:

**LEMMA 5.2** *The operator $M$ maps a continuous bounded function into a continuous bounded function.*

We first explore the relations between the value functions $v_i^n$ when no new impulses are allowed, i.e. $n = 0$. Functions $v_i^0$ satisfy the following equations:

$$
\begin{aligned}
&v_i^0\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
&= \mathbb{E}^x \bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T - s + u, X(u)) du + e^{-\alpha\delta_1} c(X(\delta_1), \xi_1) \\
&\qquad + e^{-\alpha\delta_1} v_{i-1}^0\big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, (d - \delta_1) \vee 0, (\delta_2 - \delta_1, \xi_2), \ldots, (\delta_i - \delta_1, \xi_i)\big) \bigg\},
\end{aligned} \tag{44}
$$

and

$$
v_0^0(x, s, d) = \mathbb{E}^x \bigg\{ \int_0^{s} e^{-\alpha u} f(T - s + u, X(u)) du + e^{-\alpha s} g(X(s)) \bigg\}. \tag{45}
$$

If $n > 0$ and $i > 0$, the value function $v_i^n$ is separately defined on three subsets of the parameter space:

i) $s - \Delta < d$: no impulse can be ordered because the time between possible decision about an impulse and the maturity is shorter than the delay of the execution $\Delta$. This is based on the assumption that all pending impulses are executed before or at the maturity. Impulses ordered after the moment $s - \Delta$ do not affect the value of the functional.

ii) $s - \Delta \geq d$ and $\delta_1 < d$: it is possible to order a new impulse, but a pending impulse $(\delta_1, \xi_1)$ is executed before a new one can be ordered,

iii) $s - \Delta \geq d$ and $\delta_1 \geq d$: it is possible to order a new impulse before the execution of a pending impulse $(\delta_1, \xi_1)$.

In (i) and (ii) no impulses can be ordered before $\delta_1$. The value functions can be written as

$$
\begin{aligned}
v_i^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) &= \mathbb{E}^x \bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T - s + u, X(u)) du + e^{-\alpha\delta_1} c(X(\delta_1), \xi_1) \\
&+ e^{-\alpha\delta_1} v_{i-1}^n\Big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, (d - \delta_1) \vee 0, (\delta_2 - \delta_1, \xi_2), \ldots, (\delta_i - \delta_1, \xi_i)\Big) \bigg\}.
\end{aligned} \tag{46}
$$

We divide (iii) into three subcases:
a) $\delta_1 \leq s - \Delta$ (by the conditions in (iii) we have $d \leq \delta_1$)

$$
\begin{aligned}
v_i^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) &= \sup_{d \leq \tau \leq \delta_1} \mathbb{E}^x \bigg\{ \int_0^{\tau} e^{-\alpha u} f(T - s + u, X(u)) du \\
&+ 1_{\{\tau < \delta_1\}}\, e^{-\alpha\tau} M v_{i+1}^{n-1}\big(X(\tau), s - \tau, (\delta_1 - \tau, \xi_1), \ldots, (\delta_i - \tau, \xi_i)\big) \\
&+ 1_{\{\tau = \delta_1\}}\, e^{-\alpha\delta_1} \Big( c(X(\delta_1), \xi_1) \\
&\qquad + v_{i-1}^n\Big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \ldots, (\delta_i - \delta_1, \xi_i)\Big)\Big) \bigg\},
\end{aligned} \tag{47}
$$

b) $\delta_1 > s - \Delta > 0$ (by the conditions in (iii) we have $d \leq s - \Delta$)

$$
\begin{aligned}
v_i^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) = \sup_{d \leq \tau \leq s-\Delta} \mathbb{E}^{\,x}\bigg\{ &\int_0^\tau e^{-\alpha u} f(T - s + u, X(u))du \\
&+ 1_{\{\tau < s-\Delta\}}\, e^{-\alpha\tau} M v_{i+1}^{n-1}\big(X(\tau), s - \tau, (\delta_1 - \tau, \xi_1), \ldots, (\delta_i - \tau, \xi_i)\big) \\
&+ 1_{\{\tau = s-\Delta\}}\, e^{-\alpha(s-\Delta)} v_i^n\Big(\big(X(s-\Delta), \Delta, 0, (\delta_1 - s + \Delta, \xi_1), \ldots, (\delta_i - s + \Delta, \xi_i)\big)\Big)\bigg\},
\end{aligned}
\tag{48}
$$

c) $\delta_1 > s - \Delta = 0$ (by the conditions in (iii) we have $d = 0$)

$$
\begin{aligned}
&v_i^n\big(x, \Delta, 0, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
&\quad = \max\Bigg( M v_{i+1}^{n-1}\big(x, \Delta, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big), \\
&\qquad\qquad \mathbb{E}^{\,x}\bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T - s + u, X(u))du + e^{-\alpha\delta_1} c(X(\delta_1), \xi_1) \\
&\qquad\qquad\qquad + e^{-\alpha\delta_1} v_{i-1}^n\big(\Gamma(X(\delta_1), \xi_1), \Delta - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \ldots, (\delta_i - \delta_1, \xi_i)\big)\bigg\}\Bigg).
\end{aligned}
\tag{49}
$$

Formula (49) has a clear meaning. When time until maturity equals $\Delta$ there are only two choices: either to order an impulse immediately (it will be executed at $T$; no more impulses can be ordered afterwards) or to execute only the pending impulses.

If $n > 0$ and $i = 0$, there are two possibilities:

i) $d > s - \Delta$: no more impulses can be ordered

$$
v_0^n\big(x, s, d\big) = v_0^0\big(x, s, d\big), \tag{50}
$$

ii) $d \leq s - \Delta$: a new impulse can be ordered

$$
\begin{aligned}
v_0^n\big(x, s, d\big) = \sup_{d \leq \tau \leq s-\Delta} \mathbb{E}^{\,x}\bigg\{ &\int_0^\tau e^{-\alpha u} f(T - s + u, X(u))du \\
&+ e^{-\alpha\tau} \max\Big( M v_1^{n-1}\big(X(\tau), s - \tau\big), v_0^0\big(X(\tau), s - \tau, 0\big)\Big)\bigg\}.
\end{aligned}
\tag{51}
$$

The relations developed above are heuristic. In what follows we show that there is a unique solution $(v_i^n)$ to the system of equations (44) - (51) and $v_i^n(x, s, d, \pi)$ is the optimal value of the cost functional $J$ with initial condition $(x, T - s)$, a new impulse order allowed after $d$ units of time, $i$ impulses in the memory $\pi$ and at most $n$ new impulse orders.

## 5.2. Solution to the system of optimal stopping problems

It is an inherent property of our model that functions $v_i^n$ may not be continuous. They are however piecewise continuous, which is one of the findings of the theorem below. Using results from previous sections we are able to prove that the stopping problems (47), (48) and (51) have optimal solutions. These solutions are the building blocks of the optimal control for the problem (43).

**THEOREM 5.3** *Assume (A1)-(A2) and $h > 0$. There is a unique solution $(v_i^n)$ to the system of equations* (44)-(51)*. The functions $v_i^n$ are the value functions for the functional $J$ with $i$ impulses in the memory and at most $n$ new impulse orders allowed. Furthermore*

*i) Functions $v_i^0$ are bounded and continuous with respect to all arguments.*

*ii) For $n > 0$ the functions $v_i^n$ have the following decomposition:*

$$\begin{aligned} v_i^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &\\ = 1_{\{s \geq d+\Delta+Nh\}}\, u_{i,N+1}^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &\\ + \sum_{m=1}^{N} 1_{\{s \in [d+\Delta+(m-1)h, d+\Delta+mh)\}}\, u_{i,m}^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &\\ + 1_{\{s < d+\Delta\}}\, u_{i,0}^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big), & \end{aligned} \tag{52}$$

*where $N = \max\{m : T - \Delta - mh \geq 0\}$, the functions $u_{i,0}^n, u_{i,1}^n, \dots, u_{i,N+1}^n : E \times [0, T] \times [0, h] \times ([0, \Delta] \times E^\xi)^i \to \mathbb{R}$ are continuous, bounded and*

$$\begin{aligned} u_{i,m}^n\big(x, s, s-\Delta-mh, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &\\ \leq u_{i,m+1}^n\big(x, s, s-\Delta-mh, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big), &\\ m = 0, \dots, N. & \end{aligned} \tag{53}$$

*iii) All optimal stopping problems used in the construction of $(v_i^n)$ have solutions, i.e. there exists stopping times for which the suprema are attained.*

***Proof of Theorem 5.3.*** The functions $v_i^0$, $i \geq 0$, are uniquely determined by equations (44)-(45). They are the value functions for the functional $J$ with no future orders allowed. Lemma 2.3 implies $v_0^0$ is continuous and bounded. Further, an inductive argument shows $v_i^0$, $i \geq 1$, are continuous and bounded. The inductive step follows from Lemma 2.3 or, directly, from Corollary 4.3 with $T_1 = T_2 = \delta_1$.

The rest of the proof relies on the induction with respect to the ordering $\preccurlyeq$ on the set of indices $(n, i) \in \{0, 1, \dots, N\} \times \{0, 1, \dots, N\}$ defined as follows:

$$(n', i') \preccurlyeq (n, i) \qquad \text{if} \qquad n' < n, \quad \text{or} \quad (n' = n \quad \text{and} \quad i' \leq i). \tag{54}$$

First we prove that the system of equations (44)-(51) defines functions $v_i^n$ in an explicit way. It is clearly true for $v_0^i$. Assume $v_{i'}^{n'}$ is defined for all $(n', i') \preccurlyeq (n, i)$ such that $(n', i') \neq (n, i)$.

If $n > 0$ the equation (49) defines $v_i^n(x, \Delta, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))$ for $d = 0$. This is extended to arbitrary $d \in [0, h]$ via (46)-(48). For $n = 0$ equations (50)-(51) provide explicit formulas for $v_i^n$.

The proof of the continuity of $v_i^n$ follows by induction with respect to the ordering $\preccurlyeq$. Assertion (i) implies conditions (52)-(53) are satisfied for $n = 0$. Assume, as an inductive hypothesis, they are satisfied for all $(n', i') \preccurlyeq (n, i)$ such that $(n', i') \neq (n, i)$.

**Preliminary step ($n > 0$, $i = 0$):** If $d > s - \Delta$ the function $v_0^n$ coincides with $v_0^0$, which is continuous by assertion (i). Otherwise, $v_0^n$ is given by (51). It can be written equivalently as

$$v_0^n\big(x, s, d\big) = \sup_{d \leq \tau \leq s - \Delta} \mathbb{E}^x \bigg\{ \int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du + F_0^n\big(\tau, X(\tau), s\big) \bigg\},$$

where

$$F_0^n\big(t, x, s\big) = e^{-\alpha t} \max\Big( M v_1^{n-1}(x, s - t), v_0^0\big(x, s - t, 0\big) \Big).$$

By the inductive hypothesis (52) the function $F_0^n$ has the following decomposition:

$$\begin{aligned} F_0^n\big(t, x, s\big) = {} & 1_{\{t \leq s - \Delta - Nh\}}\, f_{0,N+1}^n\big(t, x, s\big) \\ & + \sum_{m=2}^N 1_{\{t \in (s - \Delta - mh, s - \Delta - (m-1)h]\}}\, f_{0,m}^n\big(t, x, s\big) \\ & + 1_{\{t > s - \Delta - h\}}\, f_{0,1}^n\big(t, x, s\big), \end{aligned}$$

where

$$f_{0,m}^n\big(t, x, s\big) \;=\; e^{-\alpha t} \max\Big( M u_{1,m-1}^{n-1}(x, s - t), v_0^0(x, s - t, 0) \Big), \qquad m \;=\; 1, \ldots, N + 1.$$

Lemma 5.2 with the set of parameters $E^B = [0, T]$, $b = (s)$ implies that $(f_{0,m}^n)$ are continuous. We infer from the inductive assumption (53) that

$$f_{0,m}^n\big(s - \Delta - mh, x, s\big) \leq f_{0,m+1}^n\big(s - \Delta - mh, x, s\big), \qquad m = 1, \ldots, N.$$

By virtue of Corollary 4.3 with the set of parameters $E^B = [0, T]$, $b = (s)$, the value function

$$w_0^n(T_1, T_2, x, s) = \sup_{T_1 \leq \tau \leq T_2} \mathbb{E}^x \bigg\{ \int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du + F_0^n\big(\tau, X(\tau), s\big)\bigg\}$$

has the decomposition

$$\begin{aligned} w_0^n\big(T_1, T_2, x, s\big) = {} & 1_{\{T_1 \leq s - \Delta - Nh\}}\, w_{0,N+1}^n\big(T_1, T_2, x, s\big) \\ & + \sum_{m=2}^N 1_{\{T_1 \in (s - \Delta - mh, s - \Delta - (m-1)h]\}}\, w_{0,m}^n\big(T_1, T_2, x, s\big) \\ & + 1_{\{T_1 > s - \Delta - h\}}\, w_{0,1}^n\big(T_1, T_2, x, s\big), \end{aligned}$$

with continuous functions $w^n_{0,1}, w^n_{0,2}, \dots, w^n_{0,N+1}$ such that

$$w^n_{0,m}(s-\Delta-mh, T_2, x, s) \le w^n_{0,m+1}(s-\Delta-mh, T_2, x, s), \qquad m = 1, \dots, N.$$

Comparing with (51), we obtain $v^n_0(x,s,d) = w^n_0(d, s-\Delta, x, s)$ for $d \le s-\Delta$.

We summarize the results on $v^n_0$:

$$v^n_0(x,s,d) = \begin{cases} v^0_0(x,s,d), & d > s-\Delta, \\ w^n_0(d, s-\Delta, x, s), & d \le s-\Delta. \end{cases}$$

Decomposition (52) of $v^n_0$ is thus given by

$$\begin{aligned} u^n_{0,0}(x,s,d) &= v^0_0(x,s,d), \\ u^n_{0,m}(x,s,d) &= w^n_{0,m}(d, s-\Delta, x, s), \qquad m = 1, \dots, N+1. \end{aligned}$$

Inequalities (53) for $m = 1, \dots, N$ result from those for $w^n_{0,m}$. The relation for $m = 0$, $u^n_{0,0}(x,s,s-\Delta) \le u^n_{0,1}(x,s,s-\Delta)$, directly follows from

$$v^0_0(x,s,s-\Delta) \le w^n_0\big(s-\Delta, s-\Delta, x, s\big).$$

Having proved the assertions of theorem for $i = 0$, i.e. when there are no pending impulses, we turn our attention to the case $n > 0$, $i > 0$. Value functions $v^n_i$ were defined on three disjoint subsets of parameters. We will first consider them separately and merge the results at the end of the proof.

**Case (i) and (ii):** We infer from the representation (46) and inductive assumption (52) that in case (i), i.e. for $s - \Delta < d$,

$$v^n_i\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) = \hat g^n_{i,0}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big),$$

where

$$\begin{aligned} \hat g^n_{i,0}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) &= \mathbb{E}^x\bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T-s+u, X(u))du + e^{-\alpha\delta_1} c(X(\delta_1),\xi_1) \\ &+ e^{-\alpha\delta_1} u^n_{i-1,0}\big(\Gamma(X(\delta_1),\xi_1), s-\delta_1, (d-\delta_1)\vee 0, (\delta_2-\delta_1,\xi_2),\dots,(\delta_i-\delta_1,\xi_i)\big)\bigg\}, \end{aligned}$$

and in the case (ii), i.e. for $s - \Delta \ge d > \delta_1$,

$$\begin{aligned} v^n_i\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) &= 1_{\{s \ge d+\Delta+Nh\}}\, \hat g^n_{i,N+1}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) \\ &+ \sum_{m=1}^N 1_{\{s\in[d+\Delta+(m-1)h, d+\Delta+mh)\}}\, \hat g^n_{i,m}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big), \end{aligned}$$

where, for $m = 1, \dots, N+1$,

$$\begin{aligned} \hat g^n_{i,m}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) &= \mathbb{E}^x\bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T-s+u, X(u))du + e^{-\alpha\delta_1} c(X(\delta_1),\xi_1) \\ &+ e^{-\alpha\delta_1} u^n_{i-1,m}\big(\Gamma(X(\delta_1),\xi_1), s-\delta_1, d-\delta_1, (\delta_2-\delta_1,\xi_2),\dots,(\delta_i-\delta_1,\xi_i)\big)\bigg\}. \end{aligned}$$

Lemma 2.3 implies the continuity of $\hat{g}^n_{i,m}$, $m = 0, \dots, N+1$ (the set of parameters is $E^B = [0,T] \times [0,h] \times ([0,\Delta] \times \Theta)^i$, $b = (s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))$). The semigroup of the process $X(t)$ is monotonous, i.e. maps non-negative functions into non-negative ones. This, together with the assumption (53), proves that, for $\delta_1 < s - \Delta$,

$$\hat{g}^n_{i,m}\big(x, s, s-\Delta-mh, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \le \hat{g}^n_{i,m+1}\big(x, s, s-\Delta-mh, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big),$$
$$m = 0, \dots N. \tag{55}$$

Above results can also be obtained via Corollary 4.3.

**Case (iii):** We will use a shorthand notation $D = (s - \Delta) \wedge \delta_1$. Formulas (47)-(49) can be equivalently written as

$$\begin{aligned} v^n_i\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) = \sup_{d \le \tau \le D} \mathbb{E}^x \bigg\{ \int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du \qquad (56) \\ + F^n_i\big(\tau, X(\tau), s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \bigg\}, \end{aligned}$$

where

$$\begin{aligned} &F^n_i\big(t, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\ &\quad = 1_{\{t<D\}}\ e^{-\alpha t} M v^{n-1}_{i+1}\big(x, s-t, (\delta_1 - t, \xi_1), \dots, (\delta_i - t, \xi_i)\big) \\ &\quad + 1_{\{t=D\}}\ e^{-\alpha D} \max \Big( M v^{n-1}_{i+1}\big(x, s-D, (\delta_1 - D, \xi_1), \dots, (\delta_i - D, \xi_i)\big), \\ &\qquad\qquad h^n_i\big(x, s-D, (\delta_1 - D, \xi_1), \dots, (\delta_i - D, \xi_i)\big) \Big), \end{aligned}$$

and

$$\begin{aligned} &h^n_i\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\ &\quad = 1_{\{\delta_1 \le s-\Delta\}} \Big( c\big(x, \xi_1\big) + v^n_{i-1}\big(\Gamma(x, \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i)\big) \Big) \\ &\quad + 1_{\{\delta_1 > s-\Delta\}}\, e^{-\alpha\delta_1}\ \mathbb{E}^x \Big\{ c\big(X(\delta_1), \xi_1\big) \\ &\qquad + v^n_{i-1}\big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i)\big) \Big\}. \end{aligned}$$

Inductive hypotheses, monotonicity of the operator $M$ and Lemma 5.2 imply that $M v^{n-1}_{i+1}$ has a decomposition of the type (52)-(53) with the functions $M u^{n-1}_{i+1,m}$, $m = 0, \dots, N+1$. The functional $F$ is therefore left-continuous for $t < D$ (in the notation of Section 4). Left-continuity clearly fails at $t = D$. Due to the decomposition of $v_{i-1}^n$ we have

$$\begin{aligned}
h_i^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
&= 1_{\{\delta_1 \le s-\Delta-Nh\}}\ h_{i,N+1}^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
&\quad + \sum_{m=1}^{N-1} 1_{\{\delta_1 \in (s-\Delta-mh, s-\Delta-(m-1)h]\}}\ h_{i,m}^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
&\quad + 1_{\{\delta_1 > s-\Delta\}}\ h_{i,0}^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big),
\end{aligned}$$

with

$$\begin{aligned}
h_{i,0}^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) &= e^{-\alpha\delta_1}\, \mathbb{E}^x\Big\{c\big(X(\delta_1), \xi_1\big) \\
&\quad + u_{i-1,0}^n\big(\Gamma(X(\delta_1), \xi_1), s-\delta_1, 0, (\delta_2-\delta_1, \xi_2), \ldots, (\delta_i-\delta_1, \xi_i)\big)\Big\}, \\
h_{i,m}^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) &= c(x, \xi_1) \\
&\quad + u_{i-1,m}^n\big(\Gamma(x, \xi_1), s-\delta_1, 0, (\delta_2-\delta_1, \xi_2), \ldots, (\delta_i-\delta_1, \xi_i))\big), \quad m = 1, \ldots, N+1.
\end{aligned}$$

Functions $h_{i,m}^n$, $m = 0, \ldots, M+1$, are continuous and bounded by Lemma 2.3.

Thanks to the decomposition of $h_i^n$ the function $F_i^n$ can be written as

$$\begin{aligned}
F_i^n\big(t, x, (s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))\big) \\
&= 1_{\{t \le s-\Delta-Nh\}}\ f_{i,N+1}^n\big(t, x, (s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))\big) \\
&\quad + \sum_{m=1}^{N} 1_{\{t \in (s-\Delta-mh, s-\Delta-(m-1)h]\}}\ f_{i,m}^n\big(t, x, (s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))\big) \\
&\quad + 1_{\{t > s-\Delta\}}\ f_{i,0}^n\big(t, x, (s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))\big),
\end{aligned}$$

with

$$\begin{aligned}
f_{i,m}^n\big(t, x, (s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))\big) \\
&= 1_{\{t < \delta_1 \wedge (s-\Delta)\}}\ e^{-\alpha t}\ M u_{i+1,m}^{n-1}\big(x, s-t, (\delta_1-t, \xi_1), \ldots, (\delta_i-t, \xi_i)\big) \\
&\quad + 1_{\{t \ge \delta_1 \wedge (s-\Delta)\}}\, e^{-\alpha t} \max\Bigg(M u_{i+1,m}^{n-1}\big(x, s-t, (\delta_1-t, \xi_1), \ldots, (\delta_i-t, \xi_i)\big), \\
&\qquad h_{i,m}^n\big(x, (s-t, (\delta_1-t, \xi_1), \ldots, (\delta_i-t, \xi_i))\big)\Bigg).
\end{aligned}$$

Functions $f_{i,m}^n$ are not continuous; they can have an upward jump at $t = (s-\Delta)\wedge\delta_1$. Combination of Corollary 4.3 with Theorem 3.10 (the set of parameters is $E^B = [0, T] \times ([0, \Delta] \times \Theta)^i$, $b = (s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)))$ implies that for $T_2 \ge \delta_1 \wedge (s-\Delta)$ the function $w_i^n$ defined as

$$\begin{aligned}
w_i^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x\Big\{ \int_0^\tau e^{-\alpha u} f(T-s+u, X(u))du \\
+ F_i^n\big(\tau, X(\tau), s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big)\Big\},
\end{aligned}$$

has the following decomposition

$$
\begin{aligned}
& w_i^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
& = 1_{\{T_1 \le s-\Delta-Nh\}}\ w_{i,N+1}^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
& \qquad\qquad + \sum_{m=1}^{N} 1_{\{T_1 \in (s-\Delta-mh, s-\Delta-(m-1)h]\}}\ w_{i,m}^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
& \qquad\qquad + 1_{\{T_1 > s-\Delta\}}\ w_{i,0}^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big)
\end{aligned}
$$

with continuous bounded functions $w_{i,m}^n : [0,T]^2 \times E \times [0,T] \times [([0,\Delta] \times \Theta)^i \to \mathbb{R}$ satisfying the following set of inequalities for $m = 0, \ldots, N$

$$
\begin{aligned}
& w_{i,m}^n\big(s-\Delta-mh, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
& \qquad\qquad \le w_{i,m+1}^n\big(s-\Delta-mh, T_2, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big).
\end{aligned} \tag{57}
$$

Notice that $v_i^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) = w\big(d, \delta_1 \wedge (s-\Delta), x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big)$ on $d \le \delta_1 \wedge (s-\Delta)$.

**Final step:** The results derived above are used to obtain (52)-(53) for $v_i^n$. Findings in case (i), $s-\Delta < d$, imply $u_{i,0}^n = \hat{g}_{i,0}^n$. Functions $u_{i,m}^n$, for $m > 0$, are defined through cases (ii) and (iii). Indeed, on $s - \Delta \ge d$ we have

$$
\begin{aligned}
u_{i,m}^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) &= 1_{\{\delta_1 < d\}}\, \hat{g}_{i,m}^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\
&\quad + 1_{\{\delta_1 \ge d\}}\, w_{i,m}^n\big(d, \delta_1 \wedge (s-\Delta), x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big).
\end{aligned}
$$

Continuity of $u_{i,m}^n$ can only be violated at $d = \delta_1$. It is however not the case because

$$
\hat{g}_{i,m}^n\big(x, s, \delta_1, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) = w_{i,m}^n\big(\delta_1, \delta_1, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big)
$$

on $s - \Delta \ge d$. The function $u_{i,m}^n$ can be extended in a continuous way to its whole domain, i.e. $s \ge 0$.

The inequalities (55) and (57) imply (53) for $m = 1, \ldots, N$. Since

$$
w_{i,0}^n\big(s-\Delta, s-\Delta, x, s, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \ge \hat{g}_{i,0}^n\big(x, s, s-\Delta, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big).
$$

inequalities (55) and (57) justify (53) for $m = 0$ as well. Bellman principle and the existence of solutions to all considered optimal stopping problems imply $v_i^n$ is the value function for the functional $J$ with $i$ impulses in the memory and at most $n$ future impulse orders. ■

**REMARK 5.4** *It might be tempting to use the technique pioneered in Theorem 3.5 to remove the discontinuity of $F_i^n$ in equation* (56) *at $t = D$ in the following fashion. Define for $t \in [0, D]$,* *where we write $D$ for $(s-\Delta)\wedge\delta_1$,*

$$r_i^n\big(t,x,s,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big)$$
$$=\mathbb{E}^x\bigg\{e^{-\alpha(D-t)}\max\Big(Mv_{i+1}^{n-1}\big(X(D-t),s-D,(\delta_1-D,\xi_1),\dots,(\delta_i-D,\xi_i)\big),$$
$$h_i^n\big(X(D-t),s-D,(\delta_1-D,\xi_1),\dots,(\delta_i-D,\xi_i)\big)\Big)\bigg\},$$

*and*

$$\tilde{F}_i^n\big(t,x,s,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big)$$
$$=e^{-\alpha t}\max\Big(Mv_{i+1}^{n-1}\big(X(t),s-t,(\delta_1-t,\xi_1),\dots,(\delta_i-t,\xi_i)\big),$$
$$r_i^n\big(t,X(t),s,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big)\Big).$$

*The value function in* (56) *can be equivalently written with $\tilde{F}_i^n$ in place of $F_i^n$. The intuition standing behind this result comes from Theorem 3.5. Formal justification goes via time-discretization and an analogous but more laborious proof than that of Theorem 3.5.*

*However promising it looks, the approach proposed above does not benefit our problem. The decomposition of $r_i^n$ depends on $D$ and the points of discontinuity do not coincide with those in* (52)*. This leads to multiplication of the number of discontinuities and requires further steps to prove the properties of $v_i^n$.*

## 5.3. Main theorem and remarks

***Proof of Theorem 5.1.*** Due to a non-zero decision-lag $h$, the maximum number of impulses on the interval $[0,T]$ is bounded by $N=\lceil T/h\rceil$. Therefore, $v(x)=v_0^N(x,0,0)$, which by Theorem 5.3 is continuous. An optimal strategy can be constructed from the solutions to the stopping problems considered in the proof of Theorem 5.3. These optimal stopping times exist by Corollary 4.3. Impulse sizes are determined by appropriate suprema, e.g.

$$\sup_{\xi\in\Theta}u_{i+1,m}^{n-1}\big(x,s-t,h,(\delta_1-t,\xi_1),\dots,(\delta_i-t,\xi_i),(\Delta,\xi)\big).$$

Due to the compactness of $\Theta$ and continuity of $u_{i+1,m}^{n-1}$ with respect to $\xi$ a measurable selector exists. ■

The discontinuities of the value functions solving the system of optimal stopping problems (44)-(51) are due to the delay $\Delta>0$ and the decision lag $h>0$. If both quantities coincide, $\Delta=h$, the optimal control problem can be reformulated as a sequence of no-delay optimal stopping problems. Øksendal and Sulem [20] studied such a problem with a jump-diffusion as the underlying process $(X(t))$ and a random time horizon defined as the first exit time from a given open set. The very idea of their approach can be accommodated in our general setting with a finite horizon and yield analogous results.

Bruder and Pham [8] consider controls where the execution delay is a multiplicity of the decision lag, i.e. $\Delta = mh$. This assumption is crucial for their method of solution because it allows to divide the time between the ordering and execution of the impulse into $m$ intervals of the length $h$ on which only one impulse can be ordered. We relaxed this condition in the present paper. It forced the introduction of parameter $d$ in the functions $v_i^n$ as well as the construction of a new system of optimal stopping problems (see Subsection 5.1).

Our paper can be naturally extended in two directions. The first one is the removal of the decision lag $h$. It should, intuitively, smooth out the resulting system of optimal stopping problems leaving only one discontinuity at $s = \Delta$. On the other hand, when $h = 0$ it is possible to have strategies leading to an infinite number of pending impulses, which has two consequences: the system of optimal stopping problems is truly infinite and its solution might not result in a valid control policy (the resulting sequence of stopping times can have an accumulation point smaller than the ordering horizon $T - \Delta$).

The second extension of the paper is into infinite horizon functionals. It requires the introduction of discounting and the removal of the final payoff $g$. A simple example of such a problem is studied by Bar-Ilan and Sulem [4] in the realm of inventory models. Our intuition says that such infinite horizon models can be solved via an infinite system of optimal stopping problems with continuous functionals.

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