From Discrete to Continuous Time Evolutionary Finance Models*

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Abstract

This paper aims to open a new avenue for research in continuous-time financial market models with endogenous prices and heterogenous investors. The main result is the derivation of the limit of a discrete-time evolutionary stock market model as the length of the time period tends to zero. The resulting explicit model in continuous time generalizes the workhorse model of mathematical finance by introducing asset prices that are driven by the market interaction of investors following self-financing trading strategies. Our approach also offers a numerical scheme for the simulation of the continuous-time model.

1 Introduction

Research in evolutionary finance models, which are employed to study the wealth dynamics driven by the market interaction of investment strategies, has seen tremendous progress in the last few years. This approach has shed light on optimal long term investment strategies and the dynamics of asset prices in incomplete markets; see Evstigneev, Hens and Schenk-Hoppé [7] for a survey of the current state of the art in evolutionary finance.

All of the existing models in this theory are formulated in discrete-time with a fixed length of the time period. This approach is very suitable, for instance, in the study of repeated investment in betting markets, in which a bet is placed and no further action is taken until the outcome is

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revealed. This description is less appropriate for modeling stock markets because they exhibit marked variations in the frequencies of asset payoffs, arrival of information and trade. This main weakness of discrete-time finance models can be overcome by moving to a continuous-time framework.

In this paper we derive the limit of the evolutionary finance stock market model by Evstigneev, Hens and Schenk-Hoppé [8, 9]. To this end their discrete-time model is generalized to accommodate time periods of arbitrary length whilst maintaining consistency of investment strategies and asset payoffs over time. When the length of the time period tends to zero, this model converges to a continuous-time limit. The resulting dynamics has an explicit representation as a system of random differential equations. It turns out that the limit model generalizes the classical continuous-time dynamics of self-financing portfolios by introducing market interaction of investors. We justify this interpretation by providing an alternative derivation of the model within the standard continuous-time mathematical finance framework; see for instance the textbooks by Björk [3] or Shreve [18]. Our findings highlight the potential of dynamic economic theory, combined with evolutionary ideas, for mathematical finance.

The existence of continuous-time limits for discrete-time agent-based finance models with finitely many traders (see e.g. Chiarella, Dieci and He [5] and Hommes and Wagener [10]), by-and-large, is an open problem. One exception is the paper by Buchmann and Weber [4] who derive the continuous-time limit of a betting market model à la Kelly [11]. This framework, however, rules out capital gains and therefore cannot appropriately model stock markets. There is a strand of literature studying continuous-time models inspired by locally interacting particle models in physics. These limit models are deterministic (though can display chaotic dynamics) and describe the market dynamics on an aggregate level only, see e.g. Lux [12]. Their derivation usually rests on quite restrictive assumption on the types of traders in the market. A truly stochastic agent-based model with chartists and fundamentalists in continuous time is presented in Chiarella, He and Zheng [6]; but their approach relies on ad hoc assumptions on agent behavior and the sluggish adjustment of asset prices. Rheinlaender and Steinkamp [15] consider a closely related model in continuous time which is a stochastic extension of Zeeman's [19] one-dimensional deterministic model.

The results presented in this paper are useful from a practical finance perspective because they provide a numerical scheme for the simulation of the continuous-time dynamics. We prove that this dynamics is uniformly approximated (on finite time intervals) by the discrete-time model with small time steps. The approximation error decreases linearly with the length of the time step. This method possesses an advantage over standard numerical techniques since it approximates the continuous-time wealth dynamics without violating market clearing at any point in time.

In mathematical finance discrete-time approximations of continuous-time models have a long tradition, see Prigent [17] for the state of the art in approximation theory for financial markets. Most models however do not take into account the market interaction of investors. For instance, the market impact of trades (and its implications for optimal behavior of large investors) is usually studied for a single trader facing an exogenous price impact function (e.g. Bank and Baum [2]).

The paper is organized in two main parts aiming to make its main findings accessible to non-specialist readers. The first part is devoted to the presentation and discussion of the discrete-time model and its continuous-time limit without going into technical details. The proofs and more intricate mathematical aspects are contained in the second part.

Section 2 presents a version of the discrete-time model [8, 9] with arbitrarily small time steps and time-dependent trading strategies. A heuristic derivation of its continuous-time limit is given in Section 3. Section 4 develops a model in continuous time with market interaction of heterogenous investors from the classical continuous-time dynamics of self-financing portfolios. Section 5 contains the main result on the convergence of the dynamics of the discrete-time evolutionary stock market model to the continuous-time model. The proofs are presented in Section 6. Section 7 discusses the proof of the main result under weaker conditions. Section 8 concludes. Auxiliary results are collected in an appendix.

2 Discrete-time evolutionary model

This section introduces a discrete-time evolutionary stock market model with time periods of arbitrary length. The model extends the approach presented in Evstigneev, Hens and Schenk-Hoppé [8, 9].

Given the length $\nu > 0$ of a time period, time is discrete and proceeds through the set $t_n^\nu = n\nu$, $n = 0, 1, 2, ...$ There are K assets (stocks) with a random dividend intensity $\delta(t) = (\delta_1(t), ..., \delta_K(t))$, $t \geq 0$, with $\delta_k(t) \geq 0$ for $k = 1, ..., K$. Although the dependence on the random event $\omega \in \Omega$ is suppressed in this notation, the process $\delta(t)$ depends on two arguments and is assumed to be measurable with respect to the product σ-algebra.

Each asset is in positive net supply of one. The total dividend paid by asset k at time t_{n+1}^ν (to the investors who hold the asset over the time period $[t_n^\nu, t_{n+1}^\nu)$) is given by

$$D_{\nu,k}(t_{n+1}^\nu) = \int_{t_n^\nu}^{t_{n+1}^\nu} \delta_k(s)\, ds.$$

To simplify the presentation, we will assume that the sum of the dividend intensities is constant and, without further loss of generality, set this value equal to one, i.e. $\bar{\delta}(t) = \sum_{k=1}^K \delta_k(t) = 1$. The case of a time-dependent process $\bar{\delta}(t)$ is discussed in detail in Section 7.

There are I investors with initial wealth $V_\nu^i(0) \geq 0$, $i = 1, ..., I$, such that $\bar{V}_\nu(0) = \sum_{i=1}^I V_\nu^i(0) > 0$. Each investor is represented by a (possibly random) trading strategy $\lambda^i(t_n^\nu) = (\lambda_1^i(t_n^\nu), ..., \lambda_K^i(t_n^\nu))$, $n \geq 0$. It is assumed that $\sum_{k=1}^K \lambda_k^i(t_n^\nu) = 1$ and $\lambda_k^i(t_n^\nu) > 0$ for all $k = 1, ..., K$. The component $\lambda_k^i(t_n^\nu)$ describes the investor's budget share invested in asset k at the point in time t_n^ν.

Every investor consumes a constant fraction, $c\nu$, of his wealth in every period with the remainder being invested in assets. The constant $c > 0$ (with $c\nu < 1$) is the same for all investors and represents the intensity of consumption. The monetary value of investor i's investment in asset k is given by $(1 - c\nu)\lambda_k^i(t_n^\nu)V_\nu(t_n^\nu)$.

The discrete-time dynamics is defined by a map describing the evolution of the vector of investors' wealth between two consecutive points in time. The wealth of investor i, $i = 1, ..., I$, evolves as

$$V_\nu^i(t_{n+1}^\nu) = \sum_{k=1}^K \theta_{\nu,k}^i(t_n^\nu)\big[S_{\nu,k}(t_{n+1}^\nu) + D_{\nu,k}(t_{n+1}^\nu)\big] \tag{1}$$

with portfolio

$$\theta_{\nu,k}^i(t_n^\nu) = \frac{(1 - c\nu)\lambda_k^i(t_n^\nu)V_\nu^i(t_n^\nu)}{S_{\nu,k}(t_n^\nu)}, \qquad k = 1, \ldots, K. \tag{2}$$

The quantity $\theta_{\nu,k}^i(t_n^\nu)$ represents the number of shares of asset k owned by investor i at the beginning of the period $[t_n^\nu, t_{n+1}^\nu)$. The market for asset k clears if the total number of shares owned by investors is equal to 1: $\sum_{i=1}^I \theta_{\nu,k}^i(t_n^\nu) = 1$. The asset price $S_{\nu,k}(t_n^\nu)$ is then given by

$$S_{\nu,k}(t_n^\nu) = (1 - c\nu)\langle\lambda_k(t_n^\nu), V_\nu(t_n^\nu)\rangle. \tag{3}$$

Here $\langle x, y\rangle = \sum_{i=1}^I x^i y^i$ denotes the scalar product.

Inserting (2) and (3) into (1) yields,

$$V_\nu^i(t_{n+1}^\nu) = \sum_{k=1}^K \frac{\lambda_k^i(t_n^\nu)V_\nu^i(t_n^\nu)}{\langle\lambda_k(t_n^\nu), V_\nu(t_n^\nu)\rangle}\Big[(1 - c\nu)\langle\lambda_k(t_{n+1}^\nu), V_\nu(t_{n+1}^\nu)\rangle + D_{\nu,k}(t_{n+1}^\nu)\Big], \tag{4}$$

with $i = 1, ..., I$ or, equivalently, in vector notation

$$V_\nu(t_{n+1}^\nu) = \Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))\left[(1 - c\nu)\Lambda(t_{n+1}^\nu)V_\nu(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu)\right], \tag{5}$$

where, for each $t = t_n^\nu$, the matrix $\Lambda(t) \in \mathbb{R}^{K\times I}$ is given by $\Lambda_{ki}(t) = \lambda_k^i(t)$ and

$$\Theta_{ik}(\Lambda, V) = \frac{\Lambda_{ki}V^i}{(\Lambda V)_k}. \tag{6}$$

In this model investment strategies are the 'primitives' while asset prices and portfolios are endogenous. The evolutionary view to this model is explained in detail in [7, 8] while an interpretation of the approach from the perspective of demand theory is provided in [9]. Indeed the models considered in these papers are obtained by setting, in (5), the length of the time period to $\nu = 1$.

Remark. The derivation of the corresponding model with an arbitrary, strictly increasing sequence of time points (t_n) is straightforward. The dynamics is given by (5) when replacing t_n^ν by t_n, t_{n+1}^ν by t_{n+1}, $c\nu$ by $c(t_{n+1}-t_n)$ and $D_\nu(t_{n+1}^\nu)$ by $\int_{t_n}^{t_{n+1}} \delta(s)ds$.

Define the sets

$$\mathcal{L} = \Big\{\Lambda \in (0,1]^{K\times I} : \sum_{k=1}^{K} \Lambda_{ki} = 1 \text{ for all } i\Big\}$$

and

$$\mathcal{D} = \Big\{V \in [0,\infty)^I : \sum_{i=1}^{I} V^i = \frac{1}{c}\Big\}.$$

Theorem 1 *Fix any $\nu > 0$ with $0 < c\nu < 1$. Suppose $\Lambda(t_n^\nu) \in \mathcal{L}$ for all n. Then, for every $V_\nu(t_n^\nu) \in \mathcal{D}$, there exists a unique $V_\nu(t_{n+1}^\nu) \in \mathcal{D}$ that satisfies the discrete-time dynamics* (5)*. Therefore every initial value $V_\nu(0) \in \mathcal{D}$ generates, via* (5)*, a sample path $V_\nu(t_n^\nu) \in \mathcal{D}$, $n = 0, 1, 2, ...$*

This result ensures that the wealth dynamics (5) is well-defined for every length ν of time periods. It might be surprising that the dynamics is considered only on the set $\mathcal{D}$. However summation of (4) over $i = 1, ..., I$ yields

$$\bar{V}_\nu(t_n^\nu) = \sum_{i=1}^{I} V_\nu^i(t_n^\nu) = \frac{1}{c\nu}\sum_{k=1}^{K} D_{\nu,k}(t_{n+1}^\nu) = \frac{1}{c}. \tag{7}$$

(This identity says the market for the consumption good clears.) The remark after the proof of Theorem 1 (see Section 6.3) implies that any non-negative initial wealth distribution with a strictly positive sum is mapped into $\mathcal{D}$ after the first time step. Thus the restriction to the set $\mathcal{D}$ is without loss of generality.

This paper studies the limit of the sample paths of the discrete-time model (5) as $\nu \to 0$. Our main finding is that this limit is the solution of a continuous-time model which is a random differential equation. The most surprising property of this limit model, which is derived in the next two sections, is that it possesses a natural interpretation as the wealth dynamics of self-financing strategies with market interaction. The limit model corresponds to the workhorse model of financial mathematics (e.g. Björk [3] or Shreve [18]) but extends the usual wealth dynamics by having endogenous asset prices and heterogenous investors.

3 Heuristic derivation of the limit model

It is instructive to present first a short-cut leading to the limit of the discrete-time model (5) as $\nu \to 0$. While the continuous-time model obtained is correct, this approach is heuristic. The proper mathematical proof of the approximation property of sample paths has to proceed differently, see Section 5. The derivation presented here might also provide a valuable shortcut for classroom presentations.

The study of the discrete-time model's limit behavior requires a consistent treatment of different trading frequencies. This frequency (given by the inverse of the length ν of time periods) increases as $\nu \to 0$. We assume that investors describe their benchmark budget shares $\Lambda(t)$ at every time moment $t \geq 0$. Portfolios are rebalanced however only at the discrete points in time t_n^ν, $n = 0, 1, \ldots$. The smaller ν, the higher the rebalancing frequency and, therefore, the better the tracking of the benchmark strategy. In the limit the benchmark is perfectly matched.

Using that $V_\nu(t) = \Theta(\Lambda(t), V_\nu(t))\, \Lambda(t)\, V_\nu(t)$ for all $t = t_n^\nu$, one can rewrite (5) as

$$
\begin{aligned}
V_\nu(t_{n+1}^\nu) - V_\nu(t_n^\nu) \ &= \\
&\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu)) \Big[(1 - c\nu) \big[\Lambda(t_{n+1}^\nu) V_\nu(t_{n+1}^\nu) - \Lambda(t_n^\nu) V_\nu(t_n^\nu) \big] + D_\nu(t_{n+1}^\nu) \Big] \\
&- c\nu\, V_\nu(t_n^\nu).
\end{aligned} \tag{8}
$$

This representation allows to express the change in each investor's wealth from time t_n^ν to t_{n+1}^ν as the sum of (from left to right) changes in the asset prices, income from asset payoffs and consumption expenditure.

Suppose the function $t \mapsto \Lambda(t)$ is differentiable. Then dividing both sides of (8) by $\nu > 0$ and letting $\nu \to 0$ gives a differential equation

$$
dV(t) = \Theta(\Lambda(t), V(t)) \Big[d\left(\Lambda(t) V(t) \right) + \delta(t) dt \Big] - cV(t) dt. \tag{9}
$$

The dynamics described by (9), which is an implicit differential equation, is in continuous time. Analogously to the discrete-time case, marginal changes in an investor's wealth stem from price changes, asset payoffs and consumption.

Unfortunately the derivation is as incomplete (to avoid saying 'wrong') as it is brief. One cannot conclude that the sample paths, generated by the discrete-time system (8), converge, as $\nu \to 0$, to the solution of the differential equation (9) describing the continuous-time system.

In this paper we follow a route akin to the convergence of numerical schemes to prove that the sample paths of (9) actually are the limit of the sample paths of the discrete-time model (5) as the length of the time periods, ν, tends to zero. This approach relies on deriving a bound on the difference between the paths of continuous and discrete-time models and showing that this bound tends to zero on any compact time interval as $\nu \to 0$. The proof that the limit of the discrete-time model is actually described by the solution to (9) is provided in Section 5.

Before turning to the technical part of the paper, which is concerned with the proofs, we prepare the ground for the interpretation of the continuous-time system (9). This is done by presenting an alternative derivation of the limit model which is based on the wealth dynamics of self-financing strategies with market interaction in continuous time.

4 Continuous-time evolutionary model

This section derives an evolutionary stock market model in continuous time from the wealth dynamics of an investor with a self-financing strategy. The main generalization is the incorporation of the market interaction of traders by introducing endogenous prices in this standard framework of mathematical finance. It will turn out that this continuous-time evolutionary stock market model is identical to the heuristically obtained system (9).

We will use the term differential equation throughout this paper; but it is important point out that these equations are actually integral equations. All functions are assumed to be measurable with respect to the Borel sigma-algebra without further mentioning.

The wealth dynamics of a trader, say i, employing a self-financing strategy in a market with K assets is given by (see e.g. Björk [3] or Shreve [18])

$$dV^i(t) = \sum_{k=1}^{K} \theta_k^i(t) \big[dS_k(t) + \delta_k(t)dt \big] - dC^i(t). \tag{10}$$

Here $S_k(t)$ is the price, $\theta_k^i(t)$ is the investor's portfolio, and $C^i(t)$ denotes the cumulative consumption of investor i. The dividend intensity $\delta_k(t) \geq 0$ satisfies $\bar{\delta}(t) = \sum_{k=1}^{K} \delta_k(t) = 1$.

The investor's portfolio can be written as

$$\theta_k^i(t) = \frac{\lambda_k^i(t) V^i(t)}{S_k(t)} \tag{11}$$

with some real-valued process $\lambda_k^i(t)$, provided $S_k(t) \neq 0$. As in the discrete-time model, $\lambda^i(t) = (\lambda_1^i(t), ..., \lambda_K^i(t))$ is the vector of investor i's budget shares.

Suppose there are I investors and each asset is in net supply of one. Then, assuming $\theta_k^1(t) + ... + \theta_k^I(t) = 1$, (11) implies $S_k(t) = \lambda_k^1(t)\, V^1(t) + ... + \lambda_k^I(t)\, V^I(t) = \langle \lambda_k(t), V(t) \rangle$. Suppose further that the consumption is proportional to wealth, $dC^i(t) = cV^i(t)dt$, with the same constant $c > 0$ for all the investors.

Summarizing, one obtains the dynamics

$$dV^i(t) = \sum_{k=1}^{K} \frac{\lambda_k^i(t)V^i(t)}{\langle \lambda_k(t), V(t) \rangle} \big[d\langle \lambda_k(t), V(t) \rangle + \delta_k(t)dt \big] - cV^i(t)dt \tag{12}$$

for $i = 1, ..., I$. In vector notation:

$$dV(t) = \Theta(\Lambda(t), V(t)) \big[d(\Lambda(t)V(t)) + \delta(t)dt \big] - cV(t)dt. \tag{13}$$

This dynamics corresponds to (9) which was obtained heuristically from the discrete-time model.

Define the set

$$\mathcal{L}'_M = \Big\{ L' \in \mathbb{R}^{K \times I} : \sum_{k=1}^{K} L'_{ki} = 0 \text{ for all } i \text{ and } \|L'\| \leq M \Big\}$$

with $\| \cdot \|$ denoting the Euclidean norm.

Theorem 2 *Suppose* $\Lambda(t) \in \mathcal{L}$ *for all* $t \geq 0$ *and there is a constant* $M \geq 0$ *such that* $\frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ *for all* $t \geq 0$. *Then the system* (13) *has a unique solution* $V(t) \in \mathcal{D}$ *for every initial value* $V(0) \in \mathcal{D}$. *The solution is continuous and global, i.e. defined for all* $t \geq 0$.

The assumption $\frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ means that $\Lambda(t)$ is differentiable with uniformly bounded derivative. This rules out arbitrarily fast changes in the investors' strategies. It is essentially a technical condition needed to interpret the dynamics (13) as a standard random differential equation.

5 Convergence result

This section presents the main result on the convergence of sample paths generated by the discrete-time model to that of the continuous-time model as the length ν of the time period tends to zero.

Fix an initial value $V(0) = V_\nu(0) \in \mathcal{D}$ for both models. Then Theorem 1 ensures, for any fixed $\nu > 0$, existence and uniqueness of a sample path $V_\nu(t_n^\nu)$, $n = 0, 1, 2, ...$, while Theorem 2 gives the corresponding result for the sample path $V(t)$, $t \geq 0$. Both sample paths live in the set $\mathcal{D}$.

Define the distance between two sample paths at time t_n^ν by

$$\alpha_n^\nu = \|V(t_n^\nu) - V_\nu(t_n^\nu)\|$$

with Euclidean norm $\| \cdot \|$. Our aim is to derive an upper bound on α_n^ν (independent of the dividend intensity process $\delta(t)$) that uniformly converges to zero as $\nu \to 0$ on compact time intervals of the form $[0, T]$.

The convergence result requires a slightly stronger assumption (discussed in detail after the theorem) on the investment strategies than is needed for the existence and uniqueness of solution. Define for every $\varepsilon > 0$ the set of fully ε-diversified strategies

$$\mathcal{L}_\varepsilon = \Big\{\Lambda \in [\varepsilon, 1]^{K\times I} : \quad \sum_{k=1}^{K} \Lambda_{ki} = 1 \quad \text{for all } i\Big\}.$$

Theorem 3 *Suppose there exist $\varepsilon, M > 0$ such that $\Lambda(t) \in \mathcal{L}_\varepsilon$ and $\frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ for all $t \geq 0$. Let $V(0) = V_\nu(0) \in \mathcal{D}$. Then, for every $T > 0$, there exists a constant $C_1 > 0$, depending on T, ε, M, but independent of $V(0)$, $\nu > 0$ and $(\delta(t))_{t\in[0,T]}$, such that $\alpha_n^\nu \leq C_1\nu$ for $n = 0, 1, ..., \lfloor T/\nu \rfloor$.*

This result implies that the continuous-time model (13) is *approximated* by the model (5). For small time steps the sample path generated by the discrete-time system is close to the continuous-time sample path. This approximation property shows that the random differential equation (13) is the correct limit model and that the heuristic derivation provides the right answer.

Despite the convergence of sample paths, the asymptotic dynamics of both, discrete- and continuous-time model can exhibit marked differences because convergence of sample paths is only proved on compact time intervals. Indeed the continuous-time model possesses very different long-term dynamics compared to its discrete-time counterpart as is illustrated in Palczewski and Schenk-Hoppé [14].

The condition of fully ε-diversified investment strategies, which bounds the budget shares strictly away from zero, is needed to ensure a minimal degree of 'niceness' in the behavior of sample paths. Under this condition, prices are bounded away from zero and, thus, dividend-yields are bounded away from infinity. This restricts the local movement of the vector of investors' wealth from being arbitrarily fast. Conditions with a similar spirit are standard in approximation results for numerical schemes of deterministic and stochastic differential equations. It is noteworthy that the convergence in Theorem 3 is uniform in $\delta(t)$. Viewed as a numerical approximation, the upper bound on the difference of the sample paths holds true for any potential realization of the dividend intensity.

Interestingly, the discrete-time model does not correspond to a standard numerical approximation scheme (such as Euler or Runge-Kutta). However, it can be written in the form of an explicit difference equation (with the increments of the state variable on one side only) as we show in the proof in Section 6.2. For numerical simulation purposes it might be of interest to implement and compare both schemes in terms of speed of convergence and numerical accuracy.

6 Proofs

We now turn to the technical part of the paper. The proofs of Theorems 2 and 3 require several preparatory steps and a number of auxiliary results. These steps are particularly useful in simulation studies that employ numerical implementations of either discrete- or continuous-time model. The proof of Theorem 1 (which is similar to [9]) is provided as well to avoid any loose ends.

Section 6.1 shows how to reduce the dimension of the dynamics. This is a necessary step to obtain, in Section 6.2, an explicit representation for the dynamics of continuous-time model. Sections 6.3 and 6.4 prove the existence and uniqueness of the solution to the discrete- resp. continuous-time system. The main result on the convergence of sample paths as $\nu \to 0$ is proved in Section 6.5.

6.1 Reduction of dimension

As already observed for the discrete-time model in (7), the sum of investors' wealth is equal to $1/c$, $c > 0$ the saving rate. The same holds true for the model with continuous time: simply take the sum of (12) over $i = 1, ..., I$ and use the assumption $\bar{\delta}(t) = 1$ for all $t \geq 0$. In both cases one finds

$$V^I(t) = \frac{1}{c} - \sum_{i=1}^{I-1} V^i(t). \tag{14}$$

The dimension of the system can therefore be reduced by one. While this step is optional for the discrete-time system, it turns out to be crucial for deriving an explicit differential equation for the model in continuous time.

Define

$$\hat{\mathcal{D}} = \Big\{ \hat{V} \in [0,\infty)^{I-1} : \sum_{i=1}^{I-1} \hat{V}^i \leq \frac{1}{c} \Big\}.$$

There is a one-to-one relation between elements in $\hat{\mathcal{D}}$ and $\mathcal{D}$ using (14). Given $V \in \mathcal{D}$, define $\hat{V} = (V^1, ..., V^{I-1})$, which is in $\hat{\mathcal{D}}$. Conversely, for $\hat{V} \in \hat{\mathcal{D}}$ define $V = (\hat{V}^1, ..., \hat{V}^{I-1}, (1/c) - \sum_{i=1}^{I-1} \hat{V}^i) \in \mathcal{D}$.

Consider the discrete-time dynamics (5). The representation of the corresponding system in terms of $\hat{V}$ is given by

$$\begin{aligned} \hat{V}_\nu(t_{n+1}^\nu) = \hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big) \bigg[&(1 - c\nu)\Big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu)\Big)\hat{V}_\nu(t_{n+1}^\nu) \\ &+ \frac{1 - c\nu}{c} \lambda^I(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu) \bigg] \end{aligned} \tag{15}$$

with function $\hat{\Theta} : \mathbb{R}^{I \times K} \times \mathbb{R}^{I-1} \to \mathbb{R}^{(I-1) \times K}$ given by

$$\hat{\Theta}_{ik}(\Lambda, \hat{V}) = \frac{\Lambda_{ki} \hat{V}^i}{\sum_{i=1}^{I-1} (\Lambda_{ki} - \Lambda_{kI}) \hat{V}^i + \Lambda_{kI}/c}. \tag{16}$$

$\hat{\Lambda}(t) \in \mathbb{R}^{K\times(I-1)}$ is the matrix obtained from $\Lambda(t)$ by omitting the last column, i.e. $\hat{\Lambda}_{ki}(t) = \lambda_k^i(t)$, $k = 1, ..., K$, $i = 1, ..., I-1$, and $\Lambda^I(t) \in \mathbb{R}^{K\times(I-1)}$ is a matrix with $I-1$ identical columns, each being equal to the vector $\lambda^I(t)$.

The continuous-time model (13) implies

$$d\hat{V}(t) = \hat{\Theta}\big(\Lambda(t), \hat{V}(t)\big)\Big[\big(\hat{\Lambda}(t) - \Lambda^I(t)\big)\, d\hat{V}(t) + \big(d[\hat{\Lambda}(t) - \Lambda^I(t)]\big)\, \hat{V}(t) + \frac{1}{c} d\lambda^I(t) + \delta(t)dt\Big] - c\hat{V}(t)dt. \tag{17}$$

In the derivation of this system one uses (14) and its implication $dV^I(t) = -\sum_{i=1}^{I-1} dV^i(t)$.

6.2 Explicit formulation of dynamics

The equations describing the dynamics of the two models are implicit rather than explicit. The latter is much more convenient in simulation studies as it helps to avoid a time-consuming search for a fixed point or, in economic jargon, a (temporary) equilibrium. The proof of existence and uniqueness of solution (as well as any regularity properties) similarly benefits from the availability of an explicit representation.

It is straightforward to write both models, discrete-time (15) and continuous-time (17), in semi-explicit form. One finds

$$\begin{aligned}\Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big)\Big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu)\Big)\Big]\, \hat{V}_\nu(t_{n+1}^\nu) \\ = \hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big)\left[\frac{1-c\nu}{c}\lambda^I(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu)\right]\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\Big[\mathrm{Id} - \hat{\Theta}(\Lambda(t), \hat{V}(t))(\hat{\Lambda}(t) - \Lambda^I(t))\Big]\, d\hat{V}(t) \\ &\quad = \hat{\Theta}(\Lambda(t), \hat{V}(t))\Big[\big(d[\hat{\Lambda}(t) - \Lambda^I(t)]\big)\hat{V}(t) + \frac{1}{c}d\lambda^I(t) + \delta(t)dt\Big] - c\hat{V}(t)dt.\end{aligned} \tag{19}$$

An explicit representation can be obtained only if the matrix on the far left of the equation is invertible. One has the following positive result ensuring that, indeed, both models can be expressed in explicit form.

Theorem 4 *Assume that all trading strategies are fully diversified, i.e. $\Lambda \in \mathcal{L}$. Then the matrix*

$$\mathrm{Id} - (1-\alpha)\hat{\Theta}(\Lambda, \hat{V})\,(\hat{\Lambda} - \Lambda^I)$$

is invertible for every $\hat{V} \in \hat{\mathcal{D}}$ and $\alpha \in [0, 1]$. (The matrices $\hat{\Theta}$, $\hat{\Lambda}$ and Λ^I are defined in Section 6.1.)

Proof of Theorem 4. Let $A = (1-\alpha)\hat{\Theta}(\Lambda, \hat{V})$. This matrix has only non-negative entries. The sum of entries in the k-th column is given by

$$\sum_{i=1}^{I-1} A_{ik} = (1-\alpha)\frac{\sum_{i=1}^{I-1}\Lambda_{ki}\hat{V}^i}{\sum_{i=1}^{I-1}(\Lambda_{ki}-\Lambda_{kI})\hat{V}^i + \Lambda_{kI}/c}$$
$$= (1-\alpha)\frac{\sum_{i=1}^{I-1}\Lambda_{ki}\hat{V}^i}{\sum_{i=1}^{I-1}\Lambda_{ki}\hat{V}^i + \Lambda_{kI}\big(1/c - \sum_{i=1}^{I-1}\hat{V}^i\big)}.$$

If $\alpha > 0$ the above sum is strictly less than 1. For $\alpha = 0$ and $\sum_{i=1}^{I-1}\hat{V}^i < 1/c$, the full diversification of the I-th investor's strategy (i.e. $\Lambda_{kI} > 0$ for all k) implies that the sum of entries in every column of A is strictly less than 1. In both cases, Lemma 4 (in the Appendix) asserts that the matrix $\mathrm{Id} - A(\hat{\Lambda} - \Lambda^I)$ is invertible.

The proof in the case of $\alpha = 0$ and $\sum_{i=1}^{I-1}\hat{V}^i = 1/c$ requires a different argument. Invertibility of the matrix $D = \mathrm{Id} - A(\hat{\Lambda} - \Lambda^I)$ is equivalent to D having full rank. By reordering rows, which does not change the rank, we can assume that the last row of the matrix A contains only strictly positive entries (i.e. by placing an investor with non-zero wealth in the last row which is possible because $\hat{V}^i > 0$ for at least one $i = 1, ..., I-1$). Notice that adding rows does not change the rank of a matrix. First, adding the sum of rows 1 to $I-2$ to row $I-1$ gives

$$\begin{pmatrix} D_{11} & \dots & D_{1,I-2} & D_{1,I-1} \\ D_{21} & \dots & D_{2,I-2} & D_{2,I-1} \\ \vdots & & \vdots & \vdots \\ D_{I-2,1} & \dots & D_{I-2,I-2} & D_{I-2,I-1} \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

Second, we subtract from each row j, $j = 1, ..., I-2$, the last row multiplied by $D_{j,I-1}$. This leads to a matrix that has zeros in the last entry in each row $j = 1, ..., I-2$. The resulting matrix is given by

$$\begin{pmatrix} D_{11} - D_{1,I-1} & \dots & D_{1,I-2} - D_{1,I-1} & 0 \\ D_{21} - D_{2,I-1} & \dots & D_{2,I-2} - D_{2,I-1} & 0 \\ \vdots & & \vdots & \vdots \\ D_{I-2,1} - D_{I-2,I-1} & \dots & D_{I-2,I-2} - D_{I-2,I-1} & 0 \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

The rank of the above matrix is equal to 1 plus the rank of the matrix $\tilde{D}$ where

$$\tilde{D} = \begin{pmatrix} D_{11} - D_{1,I-1} & \dots & D_{1,I-2} - D_{1,I-1} \\ D_{21} - D_{2,I-1} & \dots & D_{2,I-2} - D_{2,I-1} \\ \vdots & & \vdots \\ D_{I-2,1} - D_{I-2,I-1} & \dots & D_{I-2,I-2} - D_{I-2,I-1} \end{pmatrix}$$

The i, j entry in $\tilde{D}$ is given by

$$\begin{aligned}\tilde{D}_{ij} &= \mathbf{1}_{i=j} - \sum_{k=1}^{K} A_{ik}(\Lambda_{kj} - \Lambda_{kI}) + \sum_{k=1}^{K} A_{I-1,k}(\Lambda_{k,I-1} - \Lambda_{kI}) \\ &= \mathbf{1}_{i=j} - \sum_{k=1}^{K} A_{ik}(\Lambda_{kj} - \Lambda_{k,I-1}).\end{aligned}$$

In matrix notation,

$$\tilde{D} = \mathrm{Id} - \tilde{A}(\tilde{B} - \tilde{C}),$$

where $\tilde{A} \in \mathbb{R}^{(I-2)\times K}$ is given by the matrix A omitting the last row, $\tilde{B} \in \mathbb{R}^{K\times(I-2)}$ is the matrix $\hat{\Lambda}$ omitting the last column and $\tilde{C} \in \mathbb{R}^{K\times(I-2)}$ has all columns equal to $(\Lambda_{1,I-1}, ..., \Lambda_{K,I-1})$. Each column sum of $\tilde{A}$ is strictly less than 1 because the last row of the matrix A contains only strictly positive entries and the sum of entries in each column of A equals one. Lemma 4 (in the Appendix) implies that the matrix $\tilde{D}$ is invertible. Therefore D has full rank and is invertible. □

6.3 Existence and uniqueness: discrete-time model

This section contains the proofs of the results ensuring that the dynamics of the discrete-time model is well-defined.

Proof of Theorem 1. We follow closely the proof of an analogous result in [9]. Rewrite (5) as

$$\Big[\mathrm{Id} - (1-c\nu)\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))\Lambda(t_{n+1}^\nu)\Big] V_\nu(t_{n+1}^\nu) = \Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu)) D_\nu(t_{n+1}^\nu).$$

Lemma 4 (in the Appendix) implies that the matrix on the left-hand side is invertible provided that $V_\nu(t_n^\nu) \in \mathcal{D}$. This leads to an explicit form of this dynamics:

$$V_\nu(t_{n+1}^\nu) = \Big[\mathrm{Id} - (1-c\nu)\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))\Lambda(t_{n+1}^\nu)\Big]^{-1} \Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu)) D_\nu(t_{n+1}^\nu). \tag{20}$$

Lemma 4 also implies that the inverse matrix in (20) maps the non-negative orthant $[0, \infty)^I$ into itself. Since the vector $\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu)) D_\nu(t_{n+1}^\nu)$ has only non-negative coordinates, one finds that $V_\nu(t_{n+1}^\nu)$ is non-negative. Equation (7) further gives that the coordinates of $V_\nu(t_{n+1}^\nu)$ sum up to $1/c$. This implies $V_\nu(t_{n+1}^\nu) \in \mathcal{D}$. □

The assumption $V_\nu(t_n^\nu) \in \mathcal{D}$ is not necessary for the validity of the above proof because the dynamics (20) maps any non-negative vector $V_\nu(t_n^\nu)$ with $\bar{V}_\nu(t_n^\nu) > 0$ into an element of $\mathcal{D}$. This follows from the fact that the investors' portfolios $\Theta(\Lambda, V)$ do not change if the vector V is multiplied by a non-negative constant, i.e.

$$\Theta(\Lambda, \alpha V) = \Theta(\Lambda, V), \qquad \text{for all } \alpha > 0.$$

One can also work with the reduced version of discrete-time model. Theorem 4 ensures that (18) can be written in the explicit form

$$\hat{V}_\nu(t^\nu_{n+1}) = \Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n)\big)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\Big)\Big]^{-1} \\ \hat{\Theta}\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n)\big)\left[\frac{1-c\nu}{c}\lambda^I(t^\nu_{n+1}) + D_\nu(t^\nu_{n+1})\right]. \tag{21}$$

The mapping of $\hat{V}_\nu(t^\nu_n)$ into $\hat{V}_\nu(t^\nu_{n+1})$ is uniquely determined. It extends to $V_\nu(\cdot)$ by (14). The proof of non-negativity of $\hat{V}_\nu(t^\nu_{n+1})$ however is more complicated in this case.

6.4 Existence and uniqueness: continuous-time model

The proof of existence and uniqueness of the solution to the model in continuous time requires the application of standard results in random differential equation theory.

Theorem 4 ensures that the system (19) is described by the following explicit dynamics

$$d\hat{V}(t) = \Big[\mathrm{Id} - \hat{\Theta}(\Lambda(t), \hat{V}(t))(\hat{\Lambda}(t) - \hat{\Lambda}^I(t))\Big]^{-1} \\ \Big[\hat{\Theta}(\Lambda(t), \hat{V}(t))\Big(\big(d[\hat{\Lambda}(t) - \hat{\Lambda}^I(t)]\big)\hat{V}(t) + \frac{1}{c}d\lambda^I(t) + \delta(t)dt\Big) - c\hat{V}(t)dt\Big]. \tag{22}$$

Define the set of admissible dividend intensities

$$\mathcal{S} = \Big\{\delta \in [0,\infty)^K : \sum_{k=1}^{K}\delta_k = 1\Big\}$$

and the function $F : \mathcal{S} \times [0,1] \times \hat{\mathcal{D}} \times \mathcal{L} \times \mathcal{L}' \to \mathbb{R}$ as

$$F(\delta, \alpha, \hat{V}, \Lambda, \Lambda') = \Big[\mathrm{Id} - (1-\alpha)\hat{\Theta}(\Lambda, \hat{V})(\hat{\Lambda} - \hat{\Lambda}^I)\Big]^{-1} \\ \Big[\hat{\Theta}(\Lambda, \hat{V})\Big(\delta + (1-\alpha)(\hat{\Lambda}' - \hat{\Lambda}'^I)\hat{V} + \frac{1-\alpha}{c}\Lambda'_{\cdot,I}\Big) - c\hat{V}\Big]. \tag{23}$$

The last column of Λ' is written as $\Lambda'_{\cdot,I}$. The 'hat'-notation has the same meaning as in Section 6.1: $\hat{\Lambda} \in \mathbb{R}^{K\times I-1}$ denotes the matrix obtained from Λ by omitting the last column, whereas the matrix $\hat{\Lambda}^I \in \mathbb{R}^{K\times I-1}$ has $I-1$ identical columns, each being equal to the last column of Λ.

Lemma 1 *The function F is continuously differentiable on $\mathcal{S} \times [0,1] \times \hat{\mathcal{D}} \times \mathcal{L} \times \mathcal{L}'$.*

Proof. Theorem 4 implies that the matrix $\mathrm{Id} - (1-\alpha)\hat{\Theta}(\hat{V}, \Lambda)(\hat{\Lambda} - \Lambda^I)$ is invertible. The function F therefore is well-defined. Direct computation shows that F is continuously differentiable. □

Proof of Theorem 2. According to the assumptions of the theorem the function $\Lambda(t)$ has a derivative $\Lambda'(t) = \frac{\partial}{\partial t}\Lambda(t)$ for *every* t. This implies that integration with respect to $d\Lambda(t)$ can be substituted by the integration with respect to $\Lambda'(t)dt$, Rudin [16, p. 325]. The dynamics (22) therefore can be written in the more compact form

$$d\hat{V}(t) = F(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda'(t))dt \tag{24}$$

with the function F defined above.

Arnold [1, Theorem 2.2.1] ensures existence and uniqueness of the solution if the right-hand side of (22) is locally integrable as a function of time and locally Lipschitz continuous in the argument V. Lemma 1 ensures local Lipschitz continuity. Local integrability follows from boundedness of $\delta(t)$ and $\Lambda'(t)$. The solution is global because all components are non-negative and their sum is finite for all t (i.e. $\hat{\mathcal{D}}$ is invariant). □

6.5 Proof of convergence result

The proof of sample path convergence relies on the reduction of dimension, see Section 6.1, and a representation of the explicit dynamics using the function F defined in (23). Discrete- and continuous-time dynamics are compared at the points in time t_n^ν, $n = 0, 1, \ldots$.

In the continuous-time case the change of the wealth vector $\hat{V}(t)$ between two points in time t_n^ν and t_{n+1}^ν can be expressed as (using equation (24))

$$\hat{V}(t_{n+1}^\nu) - \hat{V}(t_n^\nu) = \int_0^\nu F\big(\delta(t_n^\nu + h), 0, \hat{V}(t_n^\nu + h), \Lambda(t_n^\nu + h), \Lambda'(t_n^\nu + h)\big)dh. \tag{25}$$

Here $\Lambda'(t) = \frac{\partial}{\partial t}\Lambda(t)$ denotes the matrix of marginal changes of the components of the investment strategies.

The proof of the approximation result calls for a representation of discrete-time system (21) that facilitates comparison with the continuous-time dynamics (25). The following reformulation will prove very useful:

$$\hat{V}_\nu(t_{n+1}^\nu) - \hat{V}_\nu(t_n^\nu) = \int_0^\nu F\big(\delta(t_n^\nu + h), c\nu, \hat{V}_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda'(t_n^\nu + h)\big)dh. \tag{26}$$

Its derivation is presented at the end of this subsection.

Proof of Theorem 3. Denote the Euclidean distance between the two sample paths of the models with reduced dimension derived in Section 6.1 by

$$\hat{\alpha}_n^\nu = \|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\|.$$

One has $(\alpha_n^\nu)^2 \le I(\hat\alpha_n^\nu)^2$ because

$$\begin{aligned}\|V(t_n^\nu) - V_\nu(t_n^\nu)\|^2 &= \|\hat V(t_n^\nu) - \hat V_\nu(t_n^\nu)\|^2 + \Big(\sum_{i=1}^{I-1} (\hat V^i(t_n^\nu) - \hat V_\nu^i(t_n^\nu))\Big)^2 \\ &\le I\|\hat V(t_n^\nu) - \hat V_\nu(t_n^\nu)\|^2,\end{aligned}$$

which implies that it suffices to obtain the convergence result for the 'hat'-system.

According to the assumptions in the theorem there are $\varepsilon, M > 0$ such that $\Lambda(t) \in \mathcal{L}_\varepsilon$ for all $t \ge 0$ and its derivative satisfies $\Lambda'(t) = \frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ for all $t \ge 0$. The function F is continuously differentiable by Lemma 1 and the set $\mathcal{S} \times [0,1] \times \mathcal{D} \times \mathcal{L}_\varepsilon \times \mathcal{L}'_M$ is compact. This implies existence of a constant C_2 such that for any $\delta \in \mathcal{S}$, $\alpha, \alpha_* \in [0,1]$, $\hat V, \hat V_* \in \hat{\mathcal{D}}$, $\Lambda, \Lambda_* \in \mathcal{L}_\varepsilon$ and $\Lambda' \in \mathcal{L}'_M$

$$\|F(\delta, \alpha, \hat V, \Lambda, \Lambda')\| \le C_2, \tag{27}$$

$$\begin{aligned}&\big\|F(\delta, \alpha, \hat V, \Lambda, \Lambda') - F(\delta, \alpha_*, \hat V_*, \Lambda_*, \Lambda')\big\| \\ &\qquad\qquad \le C_2\big(\|\hat V - \hat V_*\| + |\alpha - \alpha_*| + \|\Lambda - \Lambda_*\|\big).\end{aligned} \tag{28}$$

This result plays an important role in the derivation of estimates in the remainder of this proof.

Subtracting equation (26) for the discrete-time system $\hat V_\nu(t_{n+1}^\nu)$ from equation (25) for the continuous-time system $\hat V(t_{n+1}^\nu)$ and taking norms on both sides, yields

$$\begin{aligned}\hat\alpha_{n+1}^\nu \le \hat\alpha_n^\nu + \Bigg\| \int_0^\nu \Big[&F\big(\delta(t_n^\nu + h), 0, \hat V(t_n^\nu + h), \Lambda(t_n^\nu + h), \Lambda'(t_n^\nu + h)\big) \\ &- F\big(\delta(t_n^\nu + h), c\nu, \hat V_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda'(t_n^\nu + h)\big)\Big] dh \Bigg\|.\end{aligned}$$

This inequality remains valid if the norm is pulled inside the integral (due to Jensen's inequality). The estimate (28) implies that

$$\hat\alpha_{n+1}^\nu \le \hat\alpha_n^\nu + \int_0^\nu C_2\Big(\|\hat V(t_n^\nu + h) - \hat V_\nu(t_n^\nu)\| + \|\Lambda(t_n^\nu + h) - \Lambda(t_n^\nu)\| + c\nu\Big) dh.$$

Boundedness of the derivative $\Lambda'(t)$ gives

$$\|\Lambda(t_n^\nu + h) - \Lambda(t_n^\nu)\| \le Mh.$$

The triangle inequality and the relation (27) yield

$$\|\hat V(t_n^\nu + h) - \hat V_\nu(t_n^\nu)\| \le \|\hat V(t_n^\nu + h) - \hat V(t_n^\nu)\| + \|\hat V(t_n^\nu) - \hat V_\nu(t_n^\nu)\| \le C_2 h + \hat\alpha_n^\nu.$$

Inserting these estimates, provides the following upper bound on the approximation error at time t_{n+1}^{ν}:

$$\begin{aligned}
\hat{\alpha}_{n+1}^{\nu} &\leq \hat{\alpha}_{n}^{\nu} + \int_0^{\nu} C_2\Big(C_2 h + \hat{\alpha}_{n}^{\nu} + Mh + c\nu\Big)dh \\
&\leq \hat{\alpha}_{n}^{\nu} + \frac{1}{2}{C_2}^2\nu^2 + \nu C_2\hat{\alpha}_{n}^{\nu} + \frac{1}{2}C_2 M\nu^2 + \frac{1}{2}C_2 c\nu^2 \\
&= (1+\nu C_2)\hat{\alpha}_{n}^{\nu} + \frac{C_2}{2}\nu^2(C_2 + M + c).
\end{aligned}$$

At the initial time, $\hat{V}(0) = \hat{V}_{\nu}(0) \in \hat{\mathcal{D}}$. This implies $\hat{\alpha}_0^{\nu} = 0$. The Gronwall lemma gives

$$\hat{\alpha}_{n}^{\nu} \leq \frac{(1+\nu C_2)^n}{\nu C_2}\,\frac{C_2}{2}\nu^2(C_2+M+c) \leq e^{n\nu C_2}\frac{1}{2}(C_2+M+c)\,\nu,$$

where the second estimate uses the inequality $(1+a)^n \leq e^{na}$ for $a \geq 0$. If $n\nu \leq T$, the expression $e^{n\nu C_2}$ is bounded by e^{TC_2}. Thus, the constant C_1 in the assertion of the theorem is given by

$$C_1 = \frac{1}{2}e^{TC_2}(C_2+M+c).$$

This completes the proof. □

Derivation of equation (26). An investor's wealth $\hat{V}_{\nu}(t_n^{\nu})$, less his consumption, is fully invested in the available assets, i.e.

$$\hat{V}_{\nu}(t_n^{\nu}) = \frac{1}{1-c\nu}\hat{\Theta}\big(\Lambda(t_n^{\nu}), \hat{V}_{\nu}(t_n^{\nu})\big)S_{\nu,k}(t_n^{\nu}).$$

The market clearing condition (3) implies

$$\hat{V}_{\nu}(t_n^{\nu}) = \hat{\Theta}\big(\Lambda(t_n^{\nu}), \hat{V}_{\nu}(t_n^{\nu})\big)\left[\Big(\hat{\Lambda}(t_n^{\nu}) - \hat{\Lambda}^I(t_n^{\nu})\Big)\hat{V}_{\nu}(t_n^{\nu}) + \frac{1}{c}\lambda^I(t_n^{\nu})\right]$$

which, together with (15), yields

$$\begin{aligned}
\hat{V}_{\nu}(t_{n+1}^{\nu}) - \hat{V}_{\nu}(t_n^{\nu}) &= -c\nu\hat{V}_{\nu}(t_n^{\nu}) \\
&+ \hat{\Theta}(\cdot)\Big[(1-c\nu)\Big(\hat{\Lambda}(t_{n+1}^{\nu}) - \hat{\Lambda}^I(t_{n+1}^{\nu})\Big)\big(\hat{V}_{\nu}(t_{n+1}^{\nu}) - \hat{V}_{\nu}(t_n^{\nu})\big) \\
&\qquad + (1-c\nu)\Big(\hat{\Lambda}(t_{n+1}^{\nu}) - \hat{\Lambda}(t_n^{\nu}) - \hat{\Lambda}^I(t_{n+1}^{\nu}) + \hat{\Lambda}^I(t_n^{\nu})\Big)\hat{V}_{\nu}(t_n^{\nu}) \\
&\qquad + \frac{1-c\nu}{c}\big(\lambda^I(t_{n+1}^{\nu}) - \lambda^I(t_n^{\nu})\big) + D_{\nu}(t_{n+1}^{\nu})\Big],
\end{aligned}$$

where $\hat{\Theta}(\cdot)$ is used as a shorthand notation for $\hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big)$. This equation can be written in the explicit form

$$\hat{V}_\nu(t_{n+1}^\nu) - \hat{V}_\nu(t_n^\nu) = \Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}(\cdot)\big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu)\big)\Big]^{-1} \times$$

$$\times \Big[(1-c\nu)\hat{\Theta}(\cdot)\Big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}(t_n^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu) + \hat{\Lambda}^I(t_n^\nu)\Big)\hat{V}_\nu(t_n^\nu)$$

$$+ \frac{1-c\nu}{c}\hat{\Theta}(\cdot)\big(\lambda^I(t_{n+1}^\nu) - \lambda^I(t_n^\nu)\big) + \hat{\Theta}(\cdot)D_\nu(t_{n+1}^\nu) - c\nu\hat{V}_\nu(t_n^\nu)\Big]$$

which is equivalent to (26).

7 Time-dependent aggregate dividend intensity

This section discusses the validity of the above results under a weaker condition on the sum of the dividend intensities $\bar{\delta}(t) = \sum_{k=1}^K \delta_k(t)$. Assume that $\bar{\delta}(t)$ is strictly positive and differentiable. Denote its derivative by $\bar{\delta}'(t)$. Suppose also that the trading strategy $\Lambda(t)$ is differentiable—with its derivative denoted by $\Lambda'(t)$.

7.1 Discrete-time system

The derivation of the discrete-time dynamics in Section 2 is valid in this more general case. The aggregate wealth however is no longer constant: summation of (4) over $i = 1, ..., I$ gives $\bar{V}_\nu(t_n^\nu) = \bar{D}_\nu(t_n^\nu)/(c\nu)$, where

$$\bar{D}_\nu(t_n^\nu) = \sum_{k=1}^K D_\nu(t_n^\nu).$$

Theorem 1 cannot be directly applied to prove existence and uniqueness of solutions to the dynamics (5) with time-dependent aggregate dividend intensity $\bar{\delta}(t)$. However a minor modification of its proof (explained below) suffices to justify the following assertion.

Lemma 2 *Assume that $0 < c\nu < 1$ and $\Lambda(t_n^\nu) \in \mathcal{L}$ for all n. Any $V_\nu(0) \in [0,\infty)^I \setminus \{0\}$ generates a unique sequence $V_\nu(t_n^\nu)$, $n = 1, 2, ...,$ satisfying (5). For $n \geq 1$, the aggregate wealth $\bar{V}_\nu(t_n^\nu)$ is equal to $\bar{D}_\nu(t_n^\nu)/(c\nu)$.*

The reduction of dimension follows similarly to Section 6.1. Define

$$\hat{\mathcal{D}}_\nu(t_n^\nu) = \Big\{\hat{V} \in [0,\infty)^{I-1} : \sum_{i=1}^{I-1} \hat{V}^i \leq \frac{\bar{D}_\nu(t_n^\nu)}{cv}\Big\}.$$

The system (5) can be written in terms of $\hat{V}$:

$$\hat{V}_\nu(t^\nu_{n+1}) = \hat{\Theta}_\nu\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n), \bar{D}_\nu(t^\nu_n)\big)\Big[(1-c\nu)\big(\hat{\Lambda}(t^\nu_{n+1})-\hat{\Lambda}^I(t^\nu_{n+1})\big)\hat{V}_\nu(t^\nu_{n+1}) + \frac{1-c\nu}{c\nu}\lambda^I(t^\nu_{n+1})\bar{D}_\nu(t^\nu_{n+1}) + D_\nu(t^\nu_{n+1})\Big], \tag{29}$$

where the function $\hat{\Theta}_\nu : \mathbb{R}^{I\times K} \times \mathbb{R}^{I-1} \times (0,\infty) \to \mathbb{R}^{(I-1)\times K}$ is given by

$$\hat{\Theta}_{\nu,ik}(\Lambda, \hat{V}, \bar{D}) = \frac{\Lambda_{ki}\hat{V}^i}{\sum_{i=1}^{I-1}(\Lambda_{ki} - \Lambda_{kI})\hat{V}^i + \Lambda_{kI}\frac{\bar{D}}{c\nu}}.$$

The identity

$$\hat{V}_\nu(t^\nu_n) = \hat{\Theta}_\nu\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n), \bar{D}(t^\nu_n)\big)\Big[\big(\hat{\Lambda}(t^\nu_n)-\hat{\Lambda}^I(t^\nu_n)\big)\hat{V}_\nu(t^\nu_n) + \frac{1}{c}\lambda^I(t^\nu_n)\bar{D}_\nu(t^\nu_n)\Big]$$

inserted into (29) gives

$$\begin{aligned}\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) = &- c\nu\hat{V}_\nu(t^\nu_n) + \hat{\Theta}_\nu\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n), \bar{D}(t^\nu_n)\big) \times \\ &\times \Big[(1-c\nu)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\Big)\big(\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n)\big) \\ &\quad + (1-c\nu)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_{n+1}) + \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) \\ &\quad + \frac{1-c\nu}{c}\big(\lambda^I(t^\nu_{n+1})\bar{D}(t^\nu_{n+1}) - \lambda^I(t^\nu_n)\bar{D}(t^\nu_n)\big) + D_\nu(t^\nu_{n+1})\Big].\end{aligned}$$

This equation can be written in explicit form:

$$\begin{aligned}\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) = &\Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}_\nu(\cdot)\big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\big)\Big]^{-1}\times \\ &\times \Big[(1-c\nu)\hat{\Theta}_\nu(\cdot)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_{n+1}) + \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) \\ &\quad + \frac{1-c\nu}{c}\hat{\Theta}_\nu(\cdot)\big(\lambda^I(t^\nu_{n+1})\bar{D}(t^\nu_{n+1}) - \lambda^I(t^\nu_n)\bar{D}(t^\nu_n)\big) \\ &\quad + \hat{\Theta}_\nu(\cdot)D_\nu(t^\nu_{n+1}) - c\nu\hat{V}_\nu(t^\nu_n)\Big],\end{aligned}$$

where $\hat{\Theta}_\nu(\cdot)$ denotes $\hat{\Theta}_\nu\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n), \bar{D}_\nu(t^\nu_n)\big)$.

Define $F : \mathcal{S} \times [0,1] \times [0,\infty)^I \times \mathcal{L} \times \mathcal{L}' \times [0,\infty) \times \mathbb{R} \times (0,\infty) \to \mathbb{R}^{I-1}$ by

$$\begin{aligned}F(\delta, \alpha, \hat{V}, \Lambda, \Lambda', \bar{d}, \bar{d}', w) = &\Big[\mathrm{Id} - (1-\alpha)\hat{\Theta}(\Lambda, \hat{V}, w)(\hat{\Lambda} - \hat{\Lambda}^I)\Big]^{-1}\times \\ &\times \Big[\hat{\Theta}(\Lambda, \hat{V}, w)\Big(\delta + (1-\alpha)(\hat{\Lambda}' - \hat{\Lambda}'^I)\hat{V} + \frac{1-\alpha}{c}\big(\bar{d}\hat{\Lambda}'_{\cdot,I} + \bar{d}'\hat{\Lambda}_{\cdot,I}\big)\Big) - c\hat{V}\Big],\end{aligned} \tag{30}$$

where

$$\hat{\Theta}(\Lambda, \hat{V}, w) = \frac{\Lambda_{ki}\hat{V}^i}{\sum_{i=1}^{I-1}(\Lambda_{ki} - \Lambda_{kI})\hat{V}^i + \Lambda_{kI}w/c}.$$

Applying the same argumentation as in Lemma 1, one can prove that the function F is continuously differentiable on the set of non-negative vectors with $\sum_{i=1}^{I-1} \hat{V}^i \leq w/c$. The discrete-time dynamics has the following integral representation:

$$\begin{aligned}\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) = \int_0^\nu F\Big(&\delta(t^\nu_n + h), c\nu, \hat{V}_\nu(t^\nu_n), \Lambda(t^\nu_n), \Lambda'(t^\nu_n + h),\\ &\bar{\delta}(t^\nu_n + h), \bar{\delta}'(t^\nu_n + h), \bar{D}(t^\nu_n)/\nu\Big) dh.\end{aligned} \tag{31}$$

This formulation will be used in the comparison of discrete- and continuous-time sample paths.

7.2 Continuous-time system

The aggregate wealth at time t in the continuous-time model is given by $\bar{\delta}(t)/c$ (sum (12) over $i = 1, ..., I$). The dimension of the system (13) can be reduced because $V^I(t) = \bar{\delta}(t)/c - \sum_{i=1}^{I-1} V^i(t)$. Indeed,

$$\begin{aligned}d\hat{V}(t) = \hat{\Theta}\big(\Lambda(t), \hat{V}(t), \bar{\delta}(t)\big)\Big[&\big(\hat{\Lambda}(t) - \Lambda^I(t)\big)\, d\hat{V}(t) + \big(d[\hat{\Lambda}(t) - \Lambda^I(t)]\big)\, \hat{V}(t)\\ &+ \frac{1}{c} d\big[\bar{\delta}(t)\lambda^I(t)\big] + \delta(t)dt\Big] - c\hat{V}(t)dt.\end{aligned} \tag{32}$$

By virtue of Theorem 4, this equation can be transformed into an explicit form:

$$\begin{aligned}d\hat{V}(t) = &\Big[\mathrm{Id} - \hat{\Theta}(\Lambda(t), \hat{V}(t))(\hat{\Lambda}(t) - \hat{\Lambda}^I(t))\Big]^{-1} \times\\ &\times \Big[\hat{\Theta}(\Lambda(t), \hat{V}(t))\Big(\big(d[\hat{\Lambda}(t) - \hat{\Lambda}^I(t)]\big)\hat{V}(t) + \frac{1}{c} d\big[\bar{\delta}(t)\lambda^I(t)\big] + \delta(t)dt\Big)\\ &\qquad - c\hat{V}(t)dt\Big],\end{aligned} \tag{33}$$

which, in compact notation, becomes

$$d\hat{V}(t) = F\big(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda'(t), \bar{\delta}(t), \bar{\delta}'(t), \bar{\delta}(t)\big)dt.$$

Lemma 3 *Suppose there exists M such that $\Lambda'(t) \in \mathcal{L}'_M$ and $|\bar{\delta}'(t)| \leq M$. Any $\hat{V}(0) \in \hat{\mathcal{D}}(0)$ extends to a unique global solution to the system* (33). *This solution is continuous and satisfies $\hat{V}(t) \in \hat{\mathcal{D}}(t)$, where*

$$\hat{\mathcal{D}}(t) = \Big\{\hat{V} \in [0, \infty)^{I-1} : \quad \sum_{i=1}^{I-1} \hat{V}^i \leq \bar{\delta}(t)/c\Big\}.$$

The proof of this result is analogous to that of Theorem 2.

7.3 Convergence result

Theorem 5 *Suppose there exist* $\varepsilon, M > 0$ *such that* $\Lambda'(t) \in \mathcal{L}'_M$, $\bar{\delta}'(t) \leq M$, $\bar{\delta}(t) \geq \varepsilon$ *and* $\Lambda(t) \in \mathcal{L}_\varepsilon$. *Let* $V(0) = V_\nu(0) \in \hat{\mathcal{D}}(0)$. *For every* $T > 0$ *there exists a constant* $C_1 > 0$ *depending on* T, M, ε *and* $\bar{\delta}(0)$ *(but independent of* $V(0)$, $\nu > 0$ *and* $(\delta(t))_{t\in[0,T]}$*), such that* $\alpha_n^\nu \leq C_1\nu$ *for* $n = 0, 1, ..., \lfloor T/\nu \rfloor$.

The assumption $\bar{\delta}(t) \geq \varepsilon$ plays the same role as the condition of fully ε-diversification of trading strategy. It assures that the dividend yield is bounded away from infinity.

Proof of Theorem 5. As in the proof of Theorem 3 it suffices to obtain an estimate for $\hat{\alpha}_n^\nu = \|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\|$. Subtracting (33) from (31), taking norms on both sides and applying Jensen's inequality, one obtains

$$\hat{\alpha}_{n+1}^\nu \leq \hat{\alpha}_n^\nu + \int_{t_n^\nu}^{t_{n+1}^\nu} \Big\| F\big(\delta(t), c\nu, \hat{V}_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda'(t), \bar{\delta}(t), \bar{\delta}'(t), \bar{D}(t_n^\nu)/\nu\big) \\ - F\big(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda'(t), \bar{\delta}(t), \bar{\delta}'(t), \bar{\delta}(t)\big)\Big\| dt.$$

Under the assumptions of Theorem 5 the relevant domain of F is given by the compact set

$$\mathcal{A} = \Big\{(\delta, \alpha, \hat{V}, \Lambda, \Lambda', \bar{d}, \bar{d}', w) \in \mathcal{S} \times [0,1] \times [0,\infty)^I \times \mathcal{L}_\varepsilon \times \\ \times \mathcal{L}'_M \times [0,\infty) \times \mathbb{R} \times (0,\infty) : \sum_{i=1}^{I-1} \hat{V}^i \leq w, \text{ and } \varepsilon \leq w \leq \bar{\delta}(0) + TM\Big\}.$$

Continuous differentiability of F implies existence of a constant C_3 such that $\|F(a) - F(a')\| \leq C_3\|a - a'\|$ and $\|F(a)\| \leq C_3$ for any $a, a' \in \mathcal{A}$. Therefore

$$\hat{\alpha}_{n+1}^\nu \leq \hat{\alpha}_n^\nu + \int_{t_n^\nu}^{t_{n+1}^\nu} C_3\Big(c\nu + \|\hat{V}_\nu(t_n^\nu) - \hat{V}(t)\| \\ + \|\Lambda(t_n^\nu) - \Lambda(t)\| + |\bar{D}(t_n^\nu)/\nu - \bar{\delta}(t)|\Big) dt.$$

Boundedness of the derivative $\Lambda'(t)$ yields $\|\Lambda(t_n^\nu) - \Lambda(t)\| \leq M(t - t_n^\nu)$. The triangle inequality and boundeness of F on $\mathcal{A}$ imply

$$\|\hat{V}_\nu(t_n^\nu) - \hat{V}(t)\| \leq \hat{\alpha}_n^\nu + C_3(t - t_n^\nu).$$

By Jensen's inequality

$$|\bar{D}(t_n^\nu)/\nu - \bar{\delta}(t)| = \Big|\frac{1}{\nu}\int_{t_{n-1}^\nu}^{t_n^\nu} \bar{\delta}(s)ds - \bar{\delta}(t)\Big| \leq \frac{1}{\nu}\int_{t_{n-1}^\nu}^{t_n^\nu} |\bar{\delta}(s) - \bar{\delta}(t)|ds.$$

Since the function $\bar{\delta}(t)$ is differentiable for every $t \geq 0$, the Mean Value Theorem implies $|\bar{\delta}(s) - \bar{\delta}(t)| \leq M|t-s|$. Since $t \geq t_n^\nu$, one obtains

$$|\bar{D}(t_n^\nu)/\nu - \bar{\delta}(t)| \leq \frac{M}{\nu}\int_{t_{n-1}^\nu}^{t_n^\nu} (t-s)ds = \frac{M}{2}\nu + M(t - t_n^\nu).$$

Inserting these estimates and changing variables by setting $h = t - t_n^\nu$, provides the following estimate for $\hat{\alpha}_{n+1}^\nu$:

$$\begin{aligned}
\hat{\alpha}_{n+1}^\nu &\leq \hat{\alpha}_n^\nu + \int_0^\nu C_3\Big(c\nu + \hat{\alpha}_n^\nu + C_3 h + Mh + \frac{M}{2}\nu + Mh\Big)dh \\
&\leq \hat{\alpha}_n^\nu + C_3 c\nu^2 + C_3\nu\hat{\alpha}_n^\nu + \frac{(C_3)^2}{2}\nu^2 + C_3\frac{M}{2}\nu^2 + C_3\frac{M}{2}\nu^2 + C_3\frac{M}{2}\nu^2 \\
&= (1 + C_3\nu)\hat{\alpha}_n^\nu + \frac{C_3}{2}\nu^2(2c + C_3 + 3M).
\end{aligned}$$

The Gronwall lemma gives

$$\hat{\alpha}_n^\nu \leq \frac{(1+\nu C_3)^n}{\nu C_3}\frac{C_3}{2}\nu^2(2c + C_3 + 3M) \leq e^{n\nu C_3}\frac{1}{2}(2c + 3M + C_3)\,\nu,$$

where one has to use the inequality $(1+a)^n \leq e^{na}$ for $a \geq 0$ and the fact that $\hat{\alpha}_0^\nu = 0$. If $n\nu \leq T$, the expression $e^{n\nu C_3}$ is bounded by e^{TC_3}. Thus the constant C_1 in the assertion of the theorem is given by

$$C_1 = \frac{1}{2}e^{TC_3}(2c + C_3 + 3M).$$

This completes the proof. □

8 Conclusion

This paper marks the birth of continuous-time evolutionary finance by opening an avenue for the study of the wealth dynamics of interacting trading strategies (and the endogenous asset price dynamics it entails) in continuous time. We derived the continuous-time limit of the discrete-time evolutionary stock market model by Evstigneev, Hens and Schenk-Hoppé [8, 9] as the length of the time period tends to zero. This limit model has an explicit representation as a random dynamical system and possesses a meaningful interpretation from an economics and finance point of view. The continuous-time model extends the standard framework of mathematical finance by introducing (endogenous) stock prices which are driven by the market interaction of investors. An efficient numerical simulation of the limit model is possible because our approximation results provide an explicit scheme which converges uniformly on finite time intervals. Future research will focus on analytical and numerical studies of the continuous-time model's dynamics.

Appendix

Lemma 4 *Suppose*

(i) $A \in \mathbb{R}^{N\times K}$ *is a matrix with non-negative entries,* $A_{ij} \geq 0$*, and all column sums are strictly less than* 1*:*

$$\sum_{i=1}^{N} A_{ij} < 1 \quad \textit{for all } j = 1, \ldots, K;$$

(ii) $B, C \in \mathbb{R}^{K\times N}$ *are matrices with non-negative entries and all column sums equal to* 1*; and*

(iii) C *has identical columns.*

Then

- *the matrix* $\mathrm{Id} - AB$ *is invertible and its inverse maps the non-negative orthant into itself; and*
- *the matrix* $\mathrm{Id} - A(B - C)$ *is invertible.*

Proof. The matrix $D = \mathrm{Id} - AB$ has a column-dominant diagonal. Each diagonal entry strictly dominates the sum of absolute values of the remaining entries in the corresponding column:

$$D_{ii} > \sum_{j=1, j\neq i}^{N} |D_{ji}|, \qquad i = 1, \ldots, N. \tag{34}$$

Indeed, the (i, j) entry of the matrix C is given by

$$\mathbf{1}_{i=j} - \sum_{k=1}^{K} A_{ik} B_{kj}.$$

All off-diagonal entries are non-positive and the entries on the diagonal are non-negative. The condition (34) is equivalent to

$$\sum_{i=1}^{N} \sum_{k=1}^{K} A_{ik} B_{kj} < 1, \qquad j = 1, \ldots, N.$$

The following computation proves this inequality:

$$\sum_{i=1}^{N} \sum_{k=1}^{K} A_{ik} B_{kj} = \sum_{k=1}^{K} \Big(\sum_{i=1}^{N} A_{ik} \Big) B_{kj} < \sum_{k=1}^{K} B_{kj} = 1,$$

where the strict inequality follows from assumption (ii).

Property (34) implies that the matrix D is invertible and D^{-1} maps the non-negative orthant into itself (see Murata [13, Corollary, p. 22 and Theorem 23, p. 24]). Invertibility of $[\mathrm{Id} - A(B - C)]$ is equivalent to the invertibility of

$$[\mathrm{Id} - AB + AC]D^{-1} = \mathrm{Id} + ACD^{-1}.$$

It suffices to prove that $x = 0$ is the only solution to the linear equation

$$x = -ACD^{-1}x. \tag{35}$$

For any $y \in \mathbb{R}^N$, the particular form of the matrix C implies $ACy = b\bar{y}$, where

$$b = \Big[\sum_{k=1}^{K} A_{1k}C_{k1}, \dots, \sum_{k=1}^{K} A_{Nk}C_{k1}\Big]^T, \quad \text{and} \quad \bar{y} = \sum_{i=1}^{N} y_i.$$

The linear equation (35) therefore can only have solutions of the form $x = \beta b$ with $\beta \in \mathbb{R}$. All coordinates of b are non-negative because the matrixes A and C have non-negative entries. Assume that $x = \beta b$ is the solution to (35); with $\beta \neq 0$ and $b^i > 0$ for at least one $i = 1, ..., N$. The condition $\beta \neq 0$ implies that $b = -b\, \overline{D^{-1}b}$, where $\overline{D^{-1}b} = \sum_{i=1}^{N} (D^{-1}b)_i$. This further yields $\overline{D^{-1}b} = -1$. Since the matrix D^{-1} maps the non-negative orthant into itself, all coordinates of $D^{-1}b$ are non-negative and $\overline{D^{-1}b} \geq 0$—a contradiction. This implies that the only solution to (35) is $x = 0$, which proves the invertibility of the matrix $[\mathrm{Id} - A(B - C)]$. □

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