Skill Problem A: Log(n) Fibonacci

Below is an outline of an approach to computing the Fibonacci number F_n, for each natural number n, in log(n) time (running time measured in terms of *value* of n rather than *size* of n). Your task for this problem is to answer the question that is asked in each point of the outline below. The more questions you can answer correctly, the more points you will get. The problem is worth 9 points. Each part is worth 1.5 points.

As usual, the Fibonacci numbers are given by

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

1. [1.5 points] Use induction to prove the following.

$$\text{for each } n \geq 1,$$
$$F_n = [0, 1] \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   Here, a 1x1 matrix [y] is considered to be the same as the element y.

2. [1.5 points] Based on your answer to part 1, if you already know that the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

   can be computed in O(log n) time, why can you conclude that the value of F_n can also be computed in O(log n) time? Explain.

3. [1.5 points] Let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

   Suppose n is a power of 2 ($n = 2^m$ for some natural number m). Describe in English an algorithm for computing M^n in log(n) time. Explain clearly why this algorithm runs in O(log n) time.

4. [1.5 points] Give an O(log n) algorithm that accepts positive integers n and outputs the sequence of bits (as a String) that represents n in binary. Example: if n = 10, the algorithm would output "1010". Make sure to explain why your algorithm runs in O(log n) time.

5. [1.5 points] Using part 4, give an O(log n * log n) for computing M^n, for any natural number n. Be sure to prove that your algorithm has O(log n * log n) running time.

6. [1.5 points] Improve the algorithm of part 5 to obtain an O(log n) running time for computing M^n, for any natural number n.

What to turn in:

Submit your document that contains your solutions to all 6 parts of this problem. The best way is to turn in a Word document, but if some parts have to be handwritten, that will be acceptable.