Lab 1 Solutions

Problem 1. Which of the following functions are increasing? eventually nondecreasing? If you remember techniques from calculus, you can make use of those.

a. $f(x) = -x^2$
b. $f(x) = x^2 + 2x + 1$
c. $f(x) = x^3 + x$

Solution:

a. not increasing, not eventually nondecreasing

b. not increasing, eventually nondecreasing

c. increasing, eventaully nondecreasing.

Problem 2. Use the limit definitions of complexity classes given in class to decide whether each of the following is true or false, and in each case, prove your answer.

a. $4n^3 + n$ is $\Theta(n^3)$.
b. $\log n$ is $o(n)$.
c. 2^n is $\omega(n^2)$.
d. 2^n is $o(3^n)$.

Solution:

(a) $4n^3 + n$ is $\Theta(n^3)$.

Solution: Since both polynomials have the same degree, the result follows by the Polynomial Theorem.

(b) $\log n$ is $o(n)$.

Solution:

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\frac{\log e}{n}}{1} = \lim_{n\to\infty} \log e \cdot \frac{1}{n} = 0.$$

(c) 2^n is $\omega(n^2)$.

Solution:

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{2n}{2^n \cdot \ln 2} = \lim_{n\to\infty} \frac{2}{2^n \cdot \ln^2 2} = 0.$$

(d) 2^n is $o(3^n)$.

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

Problem 3. Show that for all $n > 4$, $2^n < n!$. Hint: Use induction.

Solution. For $n = 5$, this is obvious. Assume $n \geq 5$ and $2^n < n!$. We prove $2^{n+1} < (n+1)!$. Since $n \geq 5$, we have

$$\begin{aligned}
(n+1)! &= (n+1) \cdot n! \\
&> 2 \cdot n! \\
&> 2 \cdot 2^n \\
&= 2^{n+1}.
\end{aligned}$$

This completes the induction and the proof.