

## Chapter 2

# Design Tools for Stabilization

Recursive designs in this book are composed of simple basic steps. They are referred to as “backstepping designs” because they “step back” toward the control input starting with the scalar equation which is separated from it by the largest number of integrations.

After a brief review of Lyapunov stability, this chapter introduces basic backstepping tools, first for systems without uncertainty, and then for those with uncertainty. The most elementary integrator backstepping tool evolves into a recursive design for the class of “strict feedback” systems. An extension to block backstepping is made via a result for stabilization of cascade systems.

In the absence of uncertainties, backstepping can be used to force a nonlinear system to behave like a linear system in a new set of coordinates. However, this and other forms of “feedback linearization” require cancellation of nonlinearities, even those which are helpful for stabilization and tracking. A major advantage of backstepping is that it has the flexibility to avoid cancellations of useful nonlinearities and pursue the objectives of stabilization and tracking, rather than that of linearization.

The task of nonlinear design is much more challenging in the presence of uncertainty. When the uncertainty is *matched*, that is, when it appears in the same equation as the control, the design with “nonlinear damping” guarantees boundedness even when the bound on the uncertainty is not known. A more advanced backstepping tool is used to achieve boundedness in the case when the uncertainty is not matched, that is, when it appears before the control input. Backstepping with such a general form of bounded uncertainties with unknown bounds is a key tool used to achieve boundedness without adaptation. When the uncertainty is in the form of constant but unknown parameters, then a more suitable form of backstepping is adaptive backstepping, developed in Chapter 3.

## 2.1 Stability

### 2.1.1 Main stability theorems

For all control systems, and for adaptive control systems in particular, stability is the primary requirement. Stability concepts that are widely used in control theory are *Lyapunov stability* and *input-output stability*. Tools for analysis of both of these types of stability are summarized in Appendices A through D. The initial chapters of this book deal mostly with Lyapunov stability, which we now briefly review.<sup>1</sup>

**Lyapunov Stability.** To begin with, we remind the reader that Lyapunov stability, asymptotic stability, uniform stability, uniform asymptotic stability, etc., are properties not of a dynamic system as a whole, but rather of its individual solutions. Consider the time-varying system

$$\dot{x} = f(x, t), \quad (2.1)$$

where  $x \in \mathbb{R}^n$ , and  $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is piecewise continuous in  $t$  and locally Lipschitz in  $x$ . The solution of (2.1) which starts from the point  $x_0$  at time  $t_0 \geq 0$  is denoted as  $x(t; x_0, t_0)$  with  $x(t_0; x_0, t_0) = x_0$ . Lyapunov stability concepts describe continuity properties of  $x(t; x_0, t_0)$  with respect to  $x_0$ . If the initial condition  $x_0$  is perturbed to  $\tilde{x}_0$ , then, for stability, the resulting perturbed solution  $x(t; \tilde{x}_0, t_0)$  is required to stay close to  $x(t; x_0, t_0)$  for all  $t \geq t_0$ . In addition, for asymptotic stability, the error  $x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)$  is required to vanish as  $t \rightarrow \infty$ . So, the solution  $x(t; x_0, t_0)$  of (2.1) is

- *bounded*, if there exists a constant  $B(x_0, t_0) > 0$  such that

$$|x(t; x_0, t_0)| < B(x_0, t_0), \quad \forall t \geq t_0; \quad (2.2)$$

- *stable*, if for each  $\varepsilon > 0$  there exists a  $\delta(\varepsilon, t_0) > 0$  such that

$$|\tilde{x}_0 - x_0| < \delta \Rightarrow |x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)| < \varepsilon, \quad \forall t \geq t_0; \quad (2.3)$$

- *attractive*, if there exist an  $r(t_0) > 0$  and, for each  $\varepsilon > 0$ , a  $T(\varepsilon, t_0) > 0$  such that

$$|\tilde{x}_0 - x_0| < r \Rightarrow |x(t; \tilde{x}_0, t_0) - x(t; x_0, t_0)| < \varepsilon, \quad \forall t \geq t_0 + T; \quad (2.4)$$

- *asymptotically stable*, if it is stable and attractive; and
- *unstable*, if it is not stable.

<sup>1</sup>For a detailed treatment of the subject, the reader is referred to the book by Khalil [81].

The stability properties of  $x(t; x_0, t_0)$  in general depend on the initial time  $t_0$ . For different  $t_0$ , different values of  $B(x_0, t_0)$ ,  $\delta(\varepsilon, t_0)$ ,  $r(t_0)$ , and  $T(\varepsilon, t_0)$  may be needed to satisfy (2.2), (2.3) and (2.4). When these constants are independent of  $t_0$ , the corresponding properties are *uniform*.<sup>2</sup> For adaptive systems, *uniform stability* is more desirable than just stability. Even more desirable is *uniform asymptotic stability*, often shortened to UAS. The solution  $x(t; x_0, t_0)$  is UAS if it is *uniformly stable and uniformly attractive*, that is, if  $\delta(\varepsilon, t_0) = \delta(\varepsilon)$ ,  $r(t_0) = r$ , and  $T(\varepsilon, t_0) = T(\varepsilon)$  do not depend on  $t_0$ .

Some solutions of a given system may be stable and others unstable. In particular, (2.1) may have stable and unstable *equilibria*, that is, constant solutions  $x(t; x_e, t_0) \equiv x_e$  satisfying  $f(x_e, t) \equiv 0$ . If an equilibrium  $x_e$  is asymptotically stable, then it has a *region of attraction* — a set  $\Omega$  of initial states  $x_0$  such that  $x(t; x_0, t_0) \rightarrow x_e$  as  $t \rightarrow \infty$  for all  $x_0 \in \Omega$ .<sup>3</sup> In this book, the stability properties for which an estimate of the region of attraction is given are referred to as *regional*. Otherwise they are called *local*. When the region of attraction is the whole space  $\mathbb{R}^n$ , then the stability properties are *global*.

Any equilibrium under investigation can be translated to the origin by redefining the state  $x$  as  $z = x - x_e$ . Such a translation  $z = x - x(t; x_0, t_0)$  can be defined for any solution  $x(t; x_0, t_0)$  so that the solution under investigation can always be considered to be an equilibrium at the origin with a corresponding redefinition of  $f(x, t)$  into  $\tilde{f}(z, t)$  such that  $\tilde{f}(0, t) \equiv 0$ , namely:

$$\dot{z} = f(z + x(t; x_0, t_0), t) - f(x(t; x_0, t_0), t) \triangleq \tilde{f}(z, t). \quad (2.5)$$

Therefore, there is no loss of generality in standardizing the stability results for the zero solution  $z(t; 0, t_0) \equiv 0$ . In adaptive tracking problems, this zero solution is particularly meaningful when the state  $z$  represents the tracking error and its derivatives.

To be of practical interest, stability conditions must not require that we explicitly solve (2.1). The direct method of Lyapunov aims at determining the stability properties of  $x(t; x_0, t_0)$  from the properties of  $f(x, t)$  and its relationship with a positive definite function  $V(x, t)$ . For global results, this function must be radially unbounded, that is,  $V(x, t) \rightarrow \infty$  as  $|x| \rightarrow \infty$  uniformly in  $t$ . For simplicity, we will assume that the translation to the origin has been performed, that is,  $f(0, t) \equiv 0$ , and thus the solution under investigation is  $x \equiv 0$ .

Uniform asymptotic stability is a desirable property, because systems that possess it can deal better with perturbations and disturbances. We shall see that, in general, adaptive designs achieve less than uniform asymptotic stability. However, they achieve more than uniform stability because they force the tracking error to converge to zero. This key property is referred to as *regulation*

<sup>2</sup>Clearly, all properties are uniform if the system is time-invariant:  $\dot{x} = f(x)$ .

<sup>3</sup>When  $x_e$  is only stable, then the solutions starting in  $\Omega$  remain close to  $x_e$  in the sense of (2.3).

when the reference signal is constant, and *tracking* when it is a time-varying signal. For convergence analysis, a powerful tool is the following theorem due to LaSalle [110] and Yoshizawa [201]:

**Theorem 2.1 (LaSalle-Yoshizawa)** *Let  $x = 0$  be an equilibrium point of (2.1) and suppose  $f$  is locally Lipschitz in  $x$  uniformly in  $t$ . Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a continuously differentiable, positive definite and radially unbounded function  $V(x)$  such that*

$$\dot{V} = \frac{\partial V}{\partial x}(x)f(x, t) \leq -W(x) \leq 0, \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \quad (2.6)$$

where  $W$  is a continuous function. Then, all solutions of (2.1) are globally uniformly bounded and satisfy

$$\lim_{t \rightarrow \infty} W(x(t)) = 0. \quad (2.7)$$

In addition, if  $W(x)$  is positive definite, then the equilibrium  $x = 0$  is globally uniformly asymptotically stable (GUAS).

Because of its importance, a more general version of this theorem and its proof are included in Appendix A (Theorem A.8), along with a frequently used technical lemma due to Barbalat [155] (Lemma A.6). The LaSalle-Yoshizawa theorem is applicable to time-varying systems and allows us to establish convergence to the set  $E$  where  $W(x) = 0$ . For most of our design tasks, we will construct  $V(x)$  such that the set  $E$  consists solely of the trajectories which meet the tracking objective, that is, along which the tracking error is zero.

For the regulation task, the designed system is usually time-invariant,

$$\dot{x} = f(x), \quad (2.8)$$

in which case we are interested in its *invariant sets*. A set  $M$  is called an invariant set of (2.8) if any solution  $x(t)$  that belongs to  $M$  at some time instant  $t_1$  must belong to  $M$  for all future and past time:

$$x(t_1) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathbb{R}. \quad (2.9)$$

A set  $\Omega$  is *positively invariant* if this is true for all future time only:

$$x(t_1) \in \Omega \Rightarrow x(t) \in \Omega, \quad \forall t \geq t_1. \quad (2.10)$$

Can we guarantee convergence to a desired invariant set? A rewarding answer to this question is provided by LaSalle's Invariance Theorem and its asymptotic stability corollary:

**Theorem 2.2 (LaSalle)** *Let  $\Omega$  be a positively invariant set of (2.8). Let  $V : \Omega \rightarrow \mathbb{R}_+$  be a continuously differentiable function  $V(x)$  such that  $\dot{V}(x) \leq 0, \forall x \in \Omega$ . Let  $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$ , and let  $M$  be the largest invariant set contained in  $E$ . Then, every bounded solution  $x(t)$  starting in  $\Omega$  converges to  $M$  as  $t \rightarrow \infty$ .*

**Corollary 2.3 (Asymptotic Stability)** *Let  $x = 0$  be the only equilibrium of (2.8). Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a continuously differentiable, positive definite, radially unbounded function  $V(x)$  such that  $\dot{V}(x) \leq 0, \forall x \in \mathbb{R}^n$ . Let  $E = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$ , and suppose that no solution other than  $x(t) \equiv 0$  can stay forever in  $E$ . Then the origin is globally asymptotically stable (GAS).*

These invariance results will motivate us to closely examine the invariant subsets of  $E$ . As we shall see, the convergence properties of the designed system are stronger if the dimension of  $M$  is lower. In the most favorable case of asymptotic stability, the largest invariant subset  $M$  of  $E$  is just the origin  $x = 0$ . Our aim will thus be to render the dimension of  $M$  as low as possible.

**Input-to-State Stability.** Another stability concept which is used throughout the book is that of input-to-state stability (ISS), introduced by Sontag [173]. The system

$$\dot{x} = f(x, u) \quad (2.11)$$

is said to be *input-to-state stable (ISS)* if for any  $x(0)$  and for any input  $u(\cdot)$  continuous and bounded on  $[0, \infty)$  the solution exists for all  $t \geq 0$  and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma \left( \sup_{0 \leq \tau \leq t} |u(\tau)| \right), \quad \forall t \geq 0, \quad (2.12)$$

where  $\beta(s, t)$  and  $\gamma(s)$  are strictly increasing functions of  $s \in \mathbb{R}_+$  with  $\beta(0, t) = 0, \gamma(0) = 0$ , while  $\beta$  is a decreasing function of  $t$  with  $\lim_{t \rightarrow \infty} \beta(s, t) = 0, \forall s \in \mathbb{R}_+$ .

This definition of input-to-state stability is appropriate for nonlinear systems since it explicitly incorporates the effect of the initial conditions  $x(0)$ : (2.12) shows that the norm of the state  $x(t)$  depends not only on the input  $u(\tau)$ , but also includes an asymptotically decaying contribution from  $x(0)$ . A more extensive treatment of ISS is given in Appendix C.

### 2.1.2 Control Lyapunov functions (clf)

This book is about control *design*: Our objective is to create closed-loop systems with desirable stability properties, rather than analyze the properties of a given system. For this reason, we are interested in an extension of the Lyapunov function concept, called a *control Lyapunov function* (clf).

Suppose that our problem for the time-invariant system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad f(0, 0) = 0, \quad (2.13)$$

is to design a feedback control law  $\alpha(x)$  for the control variable  $u$  such that the equilibrium  $x = 0$  of the closed-loop system

$$\dot{x} = f(x, \alpha(x)) \quad (2.14)$$

is globally asymptotically stable. We can pick a function  $V(x)$  as a Lyapunov candidate, and require that its derivative along the solutions of (2.14) satisfy  $\dot{V}(x) \leq -W(x)$ , where  $W(x)$  is a positive definite function. We therefore need to find  $\alpha(x)$  to guarantee that for all  $x \in \mathbb{R}^n$

$$\frac{\partial V}{\partial x}(x) f(x, \alpha(x)) \leq -W(x). \quad (2.15)$$

This is a difficult task. A stabilizing control law for (2.13) may exist but we may fail to satisfy (2.15) because of a poor choice of  $V(x)$  and  $W(x)$ . A system for which a good choice of  $V(x)$  and  $W(x)$  exists is said to possess a clf. Let us make this notion precise.

**Definition 2.4** A smooth positive definite and radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called a control Lyapunov function (clf) for (2.13) if

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V}{\partial x}(x) f(x, u) \right\} < 0, \quad \forall x \neq 0. \quad (2.16)$$

The clf concept of Artstein [4] and Sontag [172] is a generalization of Lyapunov design results by Jacobson [54] and Judjevic and Quinn [62]. Artstein [4] showed that (2.16) is not only necessary, but also sufficient for the existence of a control law satisfying (2.15), that is, the existence of a clf is equivalent to global asymptotic stabilizability.

For systems affine in the control,

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \quad (2.17)$$

the clf inequality (2.15) becomes

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) \alpha(x) \leq -W(x). \quad (2.18)$$

If  $V(x)$  is a clf for (2.17), then a particular stabilizing control law  $\alpha(x)$ , smooth for all  $x \neq 0$ , is given by Sontag's formula [171]

$$u = \alpha_s(x) = \begin{cases} -\frac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\frac{\partial V}{\partial x} g}, & \frac{\partial V}{\partial x} g \neq 0 \\ 0, & \frac{\partial V}{\partial x} g = 0. \end{cases} \quad (2.19)$$

It should be noted that (2.18) can be satisfied only if

$$\frac{\partial V}{\partial x} g(x) = 0 \Rightarrow \frac{\partial V}{\partial x} f(x) < 0, \quad (2.20)$$

and that in this case (2.19) results in

$$W(x) = \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} > 0, \quad \forall x \neq 0. \quad (2.21)$$

A further characterization of a stabilizing control law  $\alpha(x)$  for (2.17) with a given clf  $V$  is that  $\alpha(x)$  is continuous at  $x = 0$  if and only if the clf satisfies the *small control property*: For each  $\varepsilon > 0$  there is a  $\delta(\varepsilon) > 0$  such that, if  $x \neq 0$  satisfies  $|x| < \delta$ , then there is some  $u$  with  $|u| < \varepsilon$  such that

$$\frac{\partial V}{\partial x} [f(x) + g(x)u] < 0. \quad (2.22)$$

The main deficiency of the clf concept as a design tool is that for most nonlinear systems a clf is not known. The task of finding an appropriate clf may be as complex as that of designing a stabilizing feedback law. For several important classes of nonlinear systems, we will solve these two tasks simultaneously using a *backstepping* procedure. To initiate this procedure we need to be able to find  $V(x)$  and  $\alpha(x)$  at least for scalar systems. Fortunately, for scalar systems,  $V(x) = \frac{1}{2}x^2$  is always a reasonable clf and the inequality (2.18) is easy to satisfy. This is illustrated by an example which also issues a warning that some designs may lead to a waste of control effort.

**Example 2.5** For the scalar system shown in Figure 2.1,

$$\dot{x} = \cos x - x^3 + u, \quad (2.23)$$

our task is to design a feedback control law which creates and globally stabilizes the equilibrium at  $x = 0$ . We will compare three different designs.

In a *feedback linearization* design, the control law

$$u = -\cos x + x^3 - x \quad (2.24)$$

cancels both nonlinearities ( $\cos x$  and  $-x^3$ ) and replaces them by  $-x$  so that the resulting feedback system is linear:  $\dot{x} = -x$ . Taking

$$V(x) = \frac{1}{2}x^2 \quad (2.25)$$

as a clf for (2.23), we see that the control law (2.24) satisfies the requirement (2.18) with  $W(x) = x^2$ , that is,  $\dot{V}(x) \leq -x^2$ . However, there is an obvious irrationality of this control law: It cancels not only  $\cos x$ , but also  $-x^3$ . For

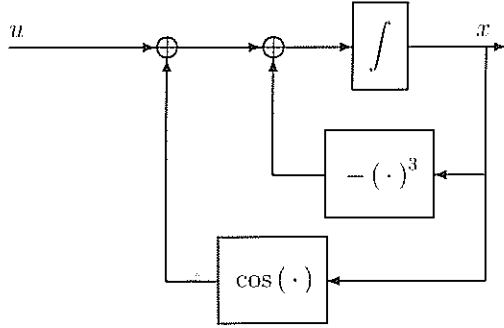


Figure 2.1: The block diagram of system (2.23).

stabilization at  $x = 0$ , the negative feedback term  $-x^3$  is helpful, especially for large values of  $x$ . On the other hand, the presence of  $x^3$  in the control law (2.24) is harmful: It leads to large magnitudes of  $u$  and may cause nonrobustness.

A more reasonable design is not to cancel  $-x^3$ . With  $V(x) = \frac{1}{2}x^2$  as before, we take  $W(x) = x^2 + x^4$ , so that the control law satisfying (2.18) becomes

$$u = -\cos x - x \triangleq \alpha(x). \quad (2.26)$$

In this case, the magnitude of  $u$  grows only linearly with  $|x|$ .

Finally, as our third control law we employ Sontag's formula (2.19). Since this formula is based on the assumption that  $f(0) = 0$ , we first cancel  $\cos x$  by introducing  $u = -\cos x + u_s$ . We again use  $V(x) = \frac{1}{2}x^2$  as our clf and evaluate  $\alpha_s(x)$  from (2.19) with  $f = -x^3$  and  $g = 1$ :

$$u_s = \alpha_s(x) = x^3 - x\sqrt{x^4 + 1}. \quad (2.27)$$

A remarkable property of (2.27) is that  $\alpha_s(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , which means that for large  $|x|$  the control law for  $u$  reduces to the term  $-\cos x$  required to place the equilibrium at  $x = 0$ . The rationale is clear: Except for the cancellation of  $\cos x$ , the control is inactive for large  $|x|$  because then the internal nonlinear feedback  $-x^3$  takes over and forces  $x$  towards zero. In this way the control effort is not wasted to achieve a property already present in the system. On the other hand, for small  $|x|$  we have  $\alpha_s \approx -x$ , which is the same as in the previous two control laws. It is easy to check that  $u = -\cos x + \alpha_s(x)$  satisfies (2.18) with  $W(x) = x^2\sqrt{x^4 + 1}$ . This control law is superior because it requires less control effort than the other two.  $\diamond$

**Example 2.6** The scalar system shown in Figure 2.2,

$$\dot{x} = x^3 + x^2 u, \quad (2.28)$$

is of interest because it is smoothly stabilizable in spite of the singularity at  $x = 0$ . We proceed with  $V(x) = \frac{1}{2}x^2$  and, because of the term  $x^3$ , we choose

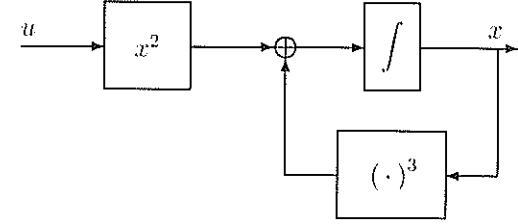


Figure 2.2: The block diagram of system (2.28).

$W(x) = c_1 x^4$  where  $c_1 > 0$ . Solving

$$\dot{V}(x, u) = x[x^3 + x^2 u] = -c_1 x^4 \quad (2.29)$$

for  $u$ , we obtain the control law

$$u = -(1 + c_1)x \triangleq \alpha(x), \quad (2.30)$$

which yields the globally asymptotically stable closed-loop system  $\dot{x} = -c_1 x^3$ .

In this case, Sontag's formula yields

$$u = \alpha_s(x) = -x(1 + \sqrt{1 + x^4}), \quad (2.31)$$

which satisfies (2.18) with  $W(x) = x^4\sqrt{1 + x^4}$ . Clearly, this control law requires more control effort than (2.30).  $\diamond$

For scalar systems the choice of a quadratic clf is obvious and the nonlinear design is straightforward: The harmful nonlinearities are cancelled by the control.

## 2.2 Backstepping

### 2.2.1 Integrator backstepping

The simplicity of scalar designs motivates us to use them as starting points of recursive designs for higher-order systems. Let us first construct clf's for second-order systems. We begin by augmenting the system (2.23) with an integrator:

$$\dot{x} = \cos x - x^3 + \xi \quad (2.32a)$$

$$\dot{\xi} = u. \quad (2.32b)$$

Let the design objective be the regulation of  $x(t)$ , that is,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for all  $x(0), \xi(0)$ . Of course,  $\xi(t)$  must remain bounded. From (2.32a), the only equilibrium with  $x = 0$  is at  $(x, \xi) = (0, -1)$ . We will meet our design

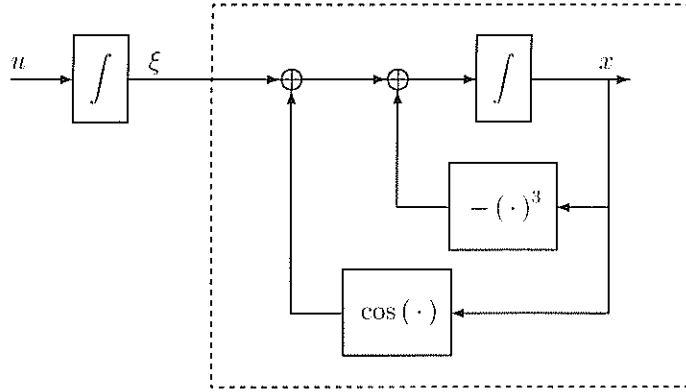


Figure 2.3: The block diagram of system (2.32).

objective by rendering this equilibrium GAS. In the block diagram in Figure 2.3 the scalar system (2.23) of Figure 2.1 appears in the dashed box. To construct a clf for (2.32) we will exploit the fact that a clf is known for its subsystem in the dashed box. Indeed, if  $\xi$  were the control input, then (2.32a) would be identical to (2.23), and the corresponding clf and control law would be  $V(x) = \frac{1}{2}x^2$  and  $\xi = -c_1x - \cos x$  (cf. (2.26)). Of course  $\xi$  is just a state variable and not the control. Nevertheless, as its “desired value” we prescribe

$$\xi_{\text{des}} = -c_1x - \cos x \triangleq \alpha(x). \quad (2.33)$$

Let  $z$  be the deviation of  $\xi$  from its desired value:

$$z = \xi - \xi_{\text{des}} = \xi - \alpha(x) = \xi + c_1x + \cos x. \quad (2.34)$$

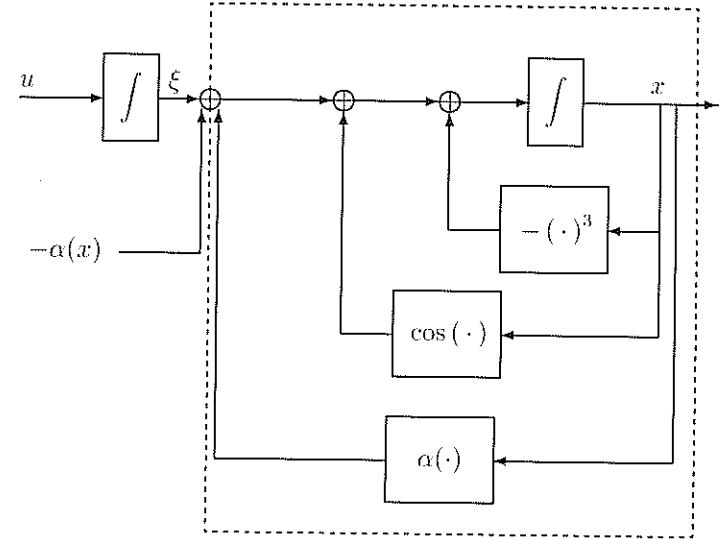
We call  $\xi$  a *virtual control*, and its desired value  $\alpha(x)$  a *stabilizing function*. The variable  $z$  is the corresponding *error variable*. Now we rewrite the system (2.32) in the  $(x, z)$ -coordinates in which it takes on a more convenient form, as illustrated in Figures 2.4 and 2.5. Starting from (2.32) and Figure 2.3, we add and subtract the stabilizing function  $\alpha(x)$  to the  $\dot{x}_1$ -equation as shown in Figure 2.4. Then we use  $\alpha(x)$  as the feedback control inside the dashed box and “backstep”  $-\alpha(x)$  through the integrator, as in Figure 2.5. In the new coordinates  $(x, z)$  the system is expressed as

$$\dot{x} = \cos x - x^3 + [\xi + c_1x + \cos x] - c_1x - \cos x = -c_1x - x^3 + z \quad (2.35a)$$

$$\dot{z} = \dot{\xi} - \dot{\alpha} = \dot{\xi} + (c_1 - \sin x)\dot{x} = u + (c_1 - \sin x)(-c_1x - x^3 + z). \quad (2.35b)$$

The first key feature of backstepping is that we don’t use a differentiator to implement the time derivative  $\dot{\alpha}$  in (2.35b); since  $\alpha(x)$  is a known function, it is easy to compute its time derivative analytically as

$$\dot{\alpha} = \frac{\partial \alpha}{\partial x} \dot{x} = -(c_1 - \sin x)(-c_1x - x^3 + z). \quad (2.36)$$

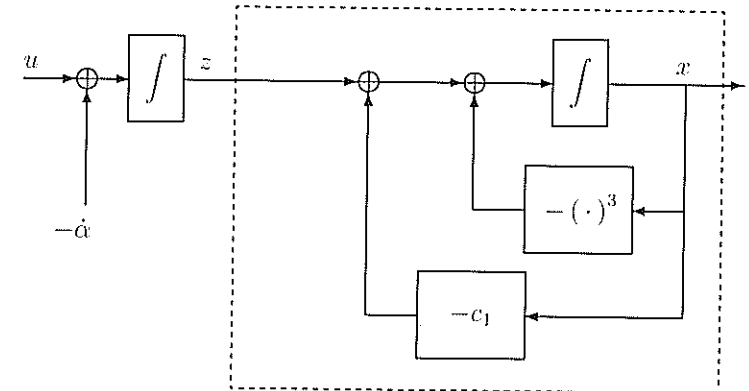
Figure 2.4: Introducing  $\alpha(x)$  as the desired value for  $\xi$ .

We now need to select a clf  $V_a$  for the system (2.32). Let us try to construct it by augmenting  $V(x)$  with a quadratic term in the error variable  $z$ :

$$V_a(x, \xi) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}(\xi + c_1x + \cos x)^2. \quad (2.37)$$

The derivative of  $V_a$  along the solutions of (2.35) is computed as

$$\begin{aligned} \dot{V}_a(x, z, u) &= x[-c_1x - x^3 + z] + z[u + (c_1 - \sin x)(-c_1x - x^3 + z)] \\ &= -c_1x^2 - x^4 + z[x + u + (c_1 - \sin x)(-c_1x - x^3 + z)]. \end{aligned} \quad (2.38)$$

Figure 2.5: Closing the feedback loop in the dashed box with  $+\alpha$  and “backstepping”  $-\alpha$  through the integrator.

As always, we let  $\dot{V}_a$  be an explicit function of  $u$  and design  $u$  to satisfy the clf inequality (2.18). For this reason, the cross-term  $xz$ , which is due to the presence of  $z$  in (2.35a), is grouped together with  $u$ . This is possible because  $u$  is also multiplied by  $z$  due to the chosen form of  $V_a$ . This is the second key feature of backstepping. Now we choose the control  $u$  to make  $\dot{V}_a$  negative definite in  $x$  and  $z$ . The simplest way to achieve this is to make the bracketed term in (2.38) equal to  $-c_2 z^2$ , where  $c_2 > 0$ :

$$\begin{aligned} u &= -c_2 z - x - (c_1 - \sin x) (-c_1 x - x^3 + z) \\ &= -c_2 (\xi + c_1 x + \cos x) - x - (c_1 - \sin x) (\xi + \cos x - x^3). \end{aligned} \quad (2.39)$$

With this control, the clf derivative is

$$\dot{V}_a = -c_1 x^2 - c_2 z^2, \quad (2.40)$$

which proves that in the  $(x, z)$  coordinates the equilibrium  $(0, 0)$  is GAS. In view of (2.34), the equilibrium  $(0, -1)$  in the  $(x, \xi)$  coordinates has the same property.

The resulting closed-loop system in the  $(x, z)$ -coordinates is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \quad (2.41)$$

Although written in a linear-like form, this system is nonlinear. An important structural property of this system is that its nonlinear “system matrix” is the sum of a *negative diagonal* and a *skew-symmetric* matrix function of  $x$ . This is the third key feature of backstepping, which will be extremely useful in other designs.

**Avoiding cancellations.** The above control law is not the best way to achieve negativity of  $\dot{V}_a$ , because it involves at least one unnecessary cancellation. A closer examination of (2.38) reveals that the term  $-z^2 \sin x$  need not be cancelled because it can be dominated by  $-c_2 z^2$ . A control law which avoids this cancellation is

$$u = -c_2 z - x - (c_1 - \sin x) (-c_1 x - x^3), \quad c_2 > c_1 + 1. \quad (2.42)$$

With this control, the clf derivative is

$$\dot{V}_a = -c_1 x^2 - x^4 - (c_2 - c_1 + \sin x) z^2. \quad (2.43)$$

Although more complicated than (2.40), this function is easily rendered negative definite by the choice  $c_2 > c_1 + 1$ . The resulting system in the  $(x, z)$  coordinates preserves its skew-symmetric form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 + c_1 - \sin x \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \quad (2.44)$$

The simplified control law (2.42) is an illustration of design flexibilities in satisfying the clf inequality  $\dot{V}_a \leq 0$  and at the same time avoiding unnecessary cancellations. In fact, more detailed calculations show that the control law can be further simplified to

$$u = -k_1 z - k_2 x^2 z, \quad (2.45)$$

with

$$k_1 > c_2 + c_1 + 1 + \frac{(c_1^2 + c_1 + 1)^2}{2c_1}, \quad k_2 \geq \frac{(c_1 + 1)^2}{4}. \quad (2.46)$$

Using this control we obtain

$$\dot{V}_a \leq -\frac{1}{2} c_1 x^2 - c_2 z^2. \quad (2.47)$$

Integrator backstepping as a general design tool is based on the following assumption:

**Assumption 2.7** Consider the system

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \quad (2.48)$$

where  $x \in \mathbb{R}^n$  is the state and  $u \in \mathbb{R}$  is the control input. There exist a continuously differentiable feedback control law

$$u = \alpha(x), \quad \alpha(0) = 0, \quad (2.49)$$

and a smooth, positive definite, radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\frac{\partial V}{\partial x}(x) [f(x) + g(x)\alpha(x)] \leq -W(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \quad (2.50)$$

where  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  is positive semidefinite.

Under this assumption, the control (2.49), applied to the system (2.48), guarantees global boundedness of  $x(t)$ , and, via the LaSalle-Yoshizawa theorem (Theorem 2.1), the regulation of  $W(x(t))$ :

$$\lim_{t \rightarrow \infty} W(x(t)) = 0. \quad (2.51)$$

A stronger convergence result is obtained using LaSalle's theorem (Theorem 2.2) with  $\Omega = \mathbb{R}^n$ :  $x(t)$  converges to the largest invariant set  $M$  contained in the set  $E = \{x \in \mathbb{R}^n \mid W(x) = 0\}$ . Clearly, if  $W(x)$  is positive definite, the control (2.49) renders  $x = 0$  the GAS equilibrium of (2.48).

**Lemma 2.8 (Integrator Backstepping)** Let the system (2.48) be augmented by an integrator:

$$\dot{x} = f(x) + g(x)\xi \quad (2.52a)$$

$$\dot{\xi} = u, \quad (2.52b)$$

and suppose that (2.52a) satisfies Assumption 2.7 with  $\xi \in \mathbb{R}$  as its control.

(i) If  $W(x)$  is positive definite, then

$$V_a(x, \xi) = V(x) + \frac{1}{2} [\xi - \alpha(x)]^2 \quad (2.53)$$

is a clf for the full system (2.52), that is, there exists a feedback control  $u = \alpha_a(x, \xi)$  which renders  $x = 0, \xi = 0$  the GAS equilibrium of (2.52). One such control is

$$u = -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)\xi] - \frac{\partial V}{\partial x}(x)g(x), \quad c > 0. \quad (2.54)$$

(ii) If  $W(x)$  is only positive semidefinite, then there exists a feedback control which renders  $\dot{V}_a \leq -W_a(x, \xi) \leq 0$ , such that  $W_a(x, \xi) > 0$  whenever  $W(x) > 0$  or  $\xi \neq \alpha(x)$ . This guarantees global boundedness and convergence of  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  to the largest invariant set  $M_a$  contained in the set  $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+1} \mid W(x) = 0, \xi = \alpha(x) \right\}$ .

**Proof.** Introducing the error variable

$$z = \xi - \alpha(x), \quad (2.55)$$

and differentiating<sup>4</sup> with respect to time, (2.52) is rewritten as

$$\dot{x} = f(x) + g(x)[\alpha(x) + z] \quad (2.56a)$$

$$\dot{z} = u - \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)(\alpha(x) + z)]. \quad (2.56b)$$

Using (2.50), the derivative of (2.53) along the solutions of (2.56) is

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x}(f + g\alpha + gz) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) \right] \\ &= \frac{\partial V}{\partial x}(f + g\alpha) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g \right] \\ &\leq -W(x) + z \left[ u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g \right], \end{aligned} \quad (2.57)$$

where the terms containing  $z$  as a factor have been grouped together. By the LaSalle-Yoshizawa theorem (Theorem 2.1), any choice of the control  $u$  which renders  $\dot{V}_a \leq -W_a(x, \xi) \leq -W(x)$ , with  $W_a$  positive definite in  $z = \xi - \alpha(x)$ , guarantees global boundedness of  $x, z$ , and  $\xi = z + \alpha(x)$ , and regulation of  $W(x(t))$  and  $z(t)$ . Furthermore, LaSalle's theorem (Theorem 2.2)

<sup>4</sup>Once again, note that the time derivative  $\dot{\alpha}$  in (2.56b) is implemented analytically without the need for a differentiator.

guarantees convergence of  $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$  to the largest invariant set contained in the set  $\left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{n+1} \mid W(x) = 0, z = 0 \right\}$ . Again, the simplest way to make  $\dot{V}_a$  negative definite in  $z$  is to choose the control (2.54), which renders the bracketed term in (2.57) equal to  $-cz$  and yields

$$\dot{V}_a \leq -W(x) - cz^2 \triangleq -W_a(x, \xi) \leq 0. \quad (2.58)$$

Clearly, if  $W(x)$  is positive definite, Theorem 2.1 guarantees the global asymptotic stability of  $x = 0, z = 0$ , which in turn implies that  $V_a(x, \xi)$  is a clf and  $x = 0, \xi = 0$  is the GAS equilibrium of (2.52).  $\square$

While the choice of control (2.54) is simple, this control may not be desirable because it involves cancellation of nonlinearities, some of which may be useful. As illustrated by (2.39) and (2.40), the requirement that  $\dot{V}_a$  in (2.57) be made negative by  $u$  allows considerable freedom in the choice of control law  $u = \alpha_a(x, \xi)$  such that

$$\dot{V}_a \leq -W(x) + z \left[ \alpha_a(x, \xi) - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g \right] = -W_a(x, \xi) \leq 0. \quad (2.59)$$

We stress that the main result of backstepping is not the specific form of the control law (2.54), but rather the construction of a Lyapunov function whose derivative can be made negative by a wide variety of control laws. In this way, the design of a stabilizing state-feedback controller is effectively reduced to satisfying the scalar inequality (2.59).

**Example 2.9** As a design tool, backstepping is less restrictive than feedback linearization. In some situations it can overcome singularities such as lack of controllability. This is illustrated by the system

$$\dot{x} = x\xi \quad (2.60a)$$

$$\dot{\xi} = u, \quad (2.60b)$$

which is uncontrollable at  $x = 0$ . Comparing with (2.52), we see that  $f(x) = 0, g(x) = x$ . Applying Lemma 2.8 with  $V(x) = \frac{1}{2}x^2$  we can choose

$$\alpha(x) = -x^2, \quad z = \xi - \alpha(x) = \xi + x^2, \quad (2.61)$$

so that  $W(x)$  in (2.50) is positive definite:  $W(x) = x^4$ . The substitution of (2.61) into (2.60) yields

$$\dot{x} = -x^3 + xz \quad (2.62a)$$

$$\dot{z} = u + 2x^2(z - x^2), \quad (2.62b)$$

and the derivative of  $V_a = \frac{1}{2}x^2 + \frac{1}{2}z^2$  is

$$\dot{V}_a = -x^4 + z(u + x^2 + 2x^2z - 2x^4). \quad (2.63)$$

The control (2.54) which renders  $\dot{V}_a = -x^4 - z^2$  is

$$u = -z - x^2 - 2x^2z + 2x^4 = -\xi - 2x^2 - 2x^2\xi. \quad (2.64)$$

The resulting system in the  $(x, \xi)$  coordinates is

$$\dot{x} = x\xi \quad (2.65a)$$

$$\dot{\xi} = -\xi - 2x^2 - 2x^2\xi, \quad (2.65b)$$

and its equilibrium  $(0, 0)$  is GAS.

A significant design flexibility of backstepping is in the choice of  $\alpha(x)$ . For the system (2.60), instead of (2.61) we can choose

$$\alpha(x) \equiv 0, \quad z = \xi, \quad (2.66)$$

so that  $W(x) \equiv 0$  is semidefinite and

$$V_a = \frac{1}{2}x^2 + \frac{1}{2}\xi^2. \quad (2.67)$$

The derivative of  $V_a$  along the solutions of (2.60) is

$$\dot{V}_a = x^2\xi + \xi u = \xi(u + x^2). \quad (2.68)$$

In this case the best we can do is to render  $\dot{V}_a$  negative semidefinite: The control

$$u = -\xi - x^2 \quad (2.69)$$

yields the closed-loop system

$$\dot{x} = x\xi \quad (2.70a)$$

$$\dot{\xi} = -\xi - x^2 \quad (2.70b)$$

and the Lyapunov derivative  $\dot{V}_a = -\xi^2$ . Then, Lemma 2.8(ii) guarantees that  $(x(t), \xi(t))$  is bounded and converges to the largest invariant set  $M_a$  of (2.70) contained in the set  $E_a$  where  $\xi = 0$ . But  $\xi(t) \equiv 0$  implies  $x(t) \equiv 0$ . Applying Corollary 2.3, we conclude that the equilibrium  $(0, 0)$  is GAS.

Comparing the two control laws (2.64) and (2.69) we see that the choice  $\alpha(x) \equiv 0$  simplified the control by eliminating the  $x^4$ -term. The design flexibility of backstepping will be further explored and exploited in the jet engine example of Section 2.4.  $\diamond$

Lemma 2.8 shows how to add a single integrator. This lemma can be repeatedly applied to add a whole chain of integrators.

**Corollary 2.10 (Chain of Integrators)** *Let the system (2.48) satisfying Assumption 2.7 with  $\alpha(x) = \alpha_0(x)$  be augmented by a chain of  $k$  integrators so that  $u$  is replaced by  $\xi_1$ , the state of the last integrator in the chain:*

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{k-1} &= \xi_k \\ \dot{\xi}_k &= u. \end{aligned} \quad (2.71)$$

*For this system, repeated application of Lemma 2.8 with  $\xi_1, \dots, \xi_k$  as virtual controls, results in the Lyapunov function*

$$V_a(x, \xi_1, \dots, \xi_k) = V(x) + \frac{1}{2} \sum_{i=1}^k [\xi_i - \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1})]^2. \quad (2.72)$$

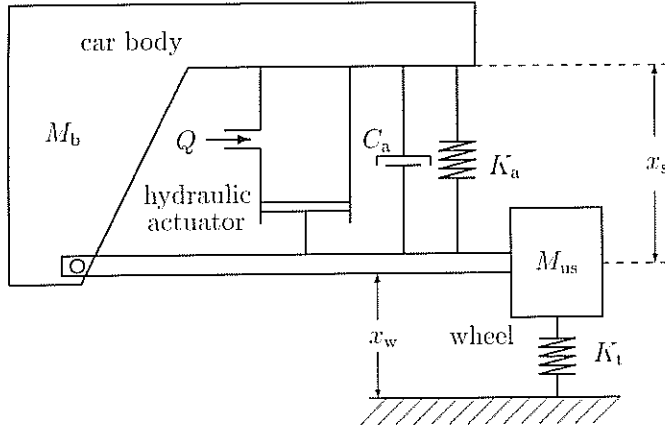
*Any choice of feedback control which renders  $\dot{V}_a \leq -W_a(x, \xi_1, \dots, \xi_k) \leq 0$ , with  $W_a(x, \xi_1, \dots, \xi_k) = 0$  only if  $W(x) = 0$  and  $\xi_i \neq \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1})$ ,  $i = 1, \dots, k$ , guarantees that  $[x^T(t), \xi_1(t), \dots, \xi_k(t)]^T$  is globally bounded and converges to the largest invariant set  $M_a$  contained in the set  $E_a = \left\{ [x^T, \xi_1, \dots, \xi_k]^T \in \mathbb{R}^{n+k} \mid W(x) = 0, \xi_i = \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1}), i = 1, \dots, k \right\}$ . Furthermore, if  $W(x)$  is positive definite, that is, if  $x = 0$  can be rendered GAS through  $\xi_1$ , then (2.72) is a clf for (2.71) and the equilibrium  $x = 0, \xi_1 = \dots = \xi_k = 0$  can be rendered GAS through  $u$ .*

### 2.2.2 Example: active suspension (parallel)

The previous section presented integrator backstepping as a design tool for nonlinear systems. However, even when dealing with *linear* systems, one may encounter situations where a *nonlinear* closed-loop response is desired. In those cases, backstepping can be used to design the corresponding nonlinear controller. We now illustrate this with an active suspension example.<sup>5</sup>

When designing vehicle suspensions, the dual objective is to minimize the vertical acceleration of the car body (for passenger comfort), and, at the same time, to maximize tire contact with the road surface (for handling). In recent years, manufacturers of passenger cars and off-road vehicles have been developing *active* suspension systems, with hydraulic actuators. Feedback control

<sup>5</sup>The active suspension examples of Sections 2.2.2 and 2.2.6 were suggested to us by Jim Winkelman and Doug Rhode of Ford Motor Company.



**Figure 2.6:** Quarter-car model for active suspension design with *parallel* connection of hydraulic actuator with passive spring/damper.

of the actuators improves both ride quality and handling performance, with the important secondary benefits of better braking and cornering because of reduced weight transfer.

Active suspension designs trade off ride quality and suspension travel. Hitting the suspension travel limits causes passenger discomfort as well as increased wear and tear on vehicle components. Hence, active suspensions should be designed to behave differently on smooth and rough roads. This can be achieved by introducing nonlinearities in the controller which make the suspension stiffer near its travel limits.

To see how backstepping can be used to design such a nonlinear controller, we consider the simplified quarter-car model in Figure 2.6, where  $Q$  is the fluid flow into the hydraulic actuator,  $x_s$  is the suspension travel, and the wheel is modeled as an unsprung mass  $M_{us}$  with a spring  $K_t$ . In the configuration of Figure 2.6, the hydraulic actuator is connected in parallel with a spring  $K_a$  and a damper  $C_a$ .

The fluid flow  $Q$  is adjusted by opening a current-controlled solenoid valve which can be modeled as a first-order linear system with the current  $i_v$  as input and the valve opening  $d_v$  as output:

$$\dot{d}_v = -c_v d_v + k_v i_v. \quad (2.73)$$

While the resulting flow  $Q$  is usually proportional to the product of the valve opening with the square root of the pressure differential across the valve, we will consider a more advanced valve, which effectively cancels the square-root nonlinearity and renders the current-to-flow dynamics linear:

$$\dot{Q} = -c_f Q + k_f i_v. \quad (2.74)$$

In the parallel configuration of Figure 2.6, neglecting leakage and compressibility, the flow  $Q$  is related to the suspension travel  $x_s$  through the equation

$$\dot{x}_s = \frac{1}{A} Q, \quad (2.75)$$

where  $A$  is the effective surface of the piston. To apply backstepping, we view the flow  $Q$  as the virtual control, and design for it a nonlinear stabilizing function  $\alpha(x_s)$  which will stiffen the suspension near its travel limits:

$$Q_{des} = \alpha(x_s) = -A [c_1 x_s + \kappa_1 x_s^3]. \quad (2.76)$$

The nonlinear term  $\kappa_1 x_s^3$  is negligible when  $|x_s|$  is small, but grows very fast when  $|x_s| > \kappa_1^{1/3}$ . With  $z = Q - \alpha(x_s)$  and (2.76), the equation (2.75) becomes

$$\dot{x}_s = -c_1 x_s - \kappa_1 x_s^3 + \frac{1}{A} z. \quad (2.77)$$

Using (2.74), we obtain

$$\dot{z} = -c_f Q + k_f i_v + (c_1 + 3\kappa_1 x_s^2) Q. \quad (2.78)$$

Applying Lemma 2.8, we obtain the control law for the current:

$$i_v = \frac{1}{k_f} \left[ c_f \alpha(x_s) - \frac{1}{A} x_s - (c_1 + 3\kappa_1 x_s^2) Q \right], \quad (2.79)$$

which results in the error system

$$\begin{aligned} \dot{x}_s &= -c_1 x_s - \kappa_1 x_s^3 + \frac{1}{A} z \\ \dot{z} &= -c_f z - \frac{1}{A} x_s. \end{aligned} \quad (2.80)$$

Clearly, the derivative of  $V(x_s, z) = \frac{1}{2}(x_s^2 + z^2)$  is negative definite:  $\dot{V} = -c_1 x_s^2 - \kappa_1 x_s^4 - c_f z^2$ . The nonlinear closed-loop system (2.80) is GAS and possesses the desired property of becoming “stiffer” as  $|x_s|$  becomes larger.

### 2.2.3 Feedback linearization and zero dynamics

One of the popular methods for nonlinear control design is *feedback linearization*, which employs a change of coordinates and feedback control to transform a nonlinear system into a system whose dynamics are linear (at least partially). A great deal of research has been devoted to this subject over the last two decades, as evidenced by the comprehensive books of Isidori [53] and Nijmeier and Van der Schaft [144] and the references therein. Since feedback linearization is not a goal pursued in this book, we only briefly review some

concepts needed for the remainder of the chapter. For maximum accessibility, we avoid the use of differential geometric notation.

Let us consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \\ y &= h(x), \quad y \in \mathbb{R} \end{aligned} \quad (2.81)$$

where  $f, g, h$  are smooth (that is, infinitely differentiable) vector functions. The derivative of the output  $y = h(x)$  is given by:

$$\dot{y} = \frac{\partial h}{\partial x}(x)f(x) + \frac{\partial h}{\partial x}(x)g(x)u. \quad (2.82)$$

If  $\frac{\partial h}{\partial x}(x_0)g(x_0) \neq 0$ , then the system (2.81) is said to have *relative degree one* at  $x_0$ .<sup>6</sup> In our terminology, this implies that the output  $y$  separated from the input  $u$  by one integrator only.

If  $\frac{\partial h}{\partial x}(x_0)g(x_0) = 0$ , there are two cases:

(i) If there exist points  $x$  arbitrarily close to  $x_0$  such that  $\frac{\partial h}{\partial x}(x)g(x) \neq 0$ , then (2.81) does not have a well-defined relative degree at  $x_0$ .

(ii) If there exists a neighborhood  $B_0$  of  $x_0$  such that  $\frac{\partial h}{\partial x}(x)g(x) = 0$  for all  $x \in B_0$ , then the relative degree of (2.81) at  $x_0$  may be well-defined.

In case (ii), we define

$$\psi_1(x) = h(x), \quad \psi_2(x) = \frac{\partial h}{\partial x}(x)f(x) \quad (2.83)$$

and compute the second derivative of  $y$ :

$$\ddot{y} = \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} f \right) f + \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} f \right) g u = \frac{\partial \psi_2}{\partial x}(x)f(x) + \frac{\partial \psi_2}{\partial x}(x)g(x)u. \quad (2.84)$$

If  $\frac{\partial \psi_2}{\partial x}(x_0)g(x_0) \neq 0$ , then (2.81) is said to have relative degree *two* at  $x_0$ .

If  $\frac{\partial \psi_2}{\partial x}(x)g(x) = 0$  in a neighborhood of  $x_0$ , then we continue the differentiation procedure.

**Definition 2.11** The system (2.81) is said to have relative degree  $\rho$  at the point  $x_0$  if there exists a neighborhood  $B_0$  of  $x_0$  on which

$$\frac{\partial \psi_1}{\partial x}(x)g(x) = \frac{\partial \psi_2}{\partial x}(x)g(x) = \dots = \frac{\partial \psi_{\rho-1}}{\partial x}(x)g(x) = 0 \quad (2.85)$$

$$\frac{\partial \psi_\rho}{\partial x}(x)g(x) \neq 0, \quad (2.86)$$

where

$$\psi_1(x) = h(x), \quad \psi_i(x) = \frac{\partial \psi_{i-1}}{\partial x}(x)f(x), \quad i = 2, \dots, \rho. \quad (2.87)$$

If (2.85) and (2.86) are valid for all  $x \in \mathbb{R}^n$ , then the relative degree of (2.81) is said to be globally defined.

<sup>6</sup>Note that since the functions are smooth,  $\frac{\partial h}{\partial x}(x_0)g(x_0) \neq 0$  implies that there exists a neighborhood of  $x_0$  on which  $\frac{\partial h}{\partial x}g(x) \neq 0$ .

Suppose now that (2.81) has relative degree  $\rho$  at  $x_0$ . Then we can use a change of coordinates and feedback control to locally transform this system into the *cascade connection* of a  $\rho$ -dimensional linear system and an  $(n - \rho)$ -dimensional nonlinear system. In particular, after differentiating  $\rho$  times the output  $y = h(x)$ , the control  $u$  appears:

$$\begin{aligned} y^{(\rho)} &= \underbrace{\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \dots \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} f \right) f \dots \right) f}_{\rho-1} \right) f(x) \\ &\quad + \underbrace{\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \dots \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} f \right) f \dots \right) f}_{\rho-1} \right) g(x)u}_{\rho-1} \\ &= \frac{\partial \psi_\rho}{\partial x}(x)f(x) + \frac{\partial \psi_\rho}{\partial x}(x)g(x)u. \end{aligned} \quad (2.88)$$

Since  $\frac{\partial \psi_\rho}{\partial x}g \neq 0$  in a neighborhood of  $x_0$ , we can linearize the input-output description of the system (2.81) using feedback to cancel the nonlinearities in (2.88):

$$u = \frac{1}{\frac{\partial \psi_\rho}{\partial x}(x)g(x)} \left[ -\frac{\partial \psi_\rho}{\partial x}(x)f(x) + v \right]. \quad (2.89)$$

Then the dynamics of  $y$  and its derivatives are governed by a chain of  $\rho$  integrators:  $y^{(\rho)} = v$ . Since our original system (2.81) has dimension  $n$ , we need to account for the remaining  $n - \rho$  states. Using differential geometric tools, it is easy to show that it is always possible to find  $n - \rho$  functions  $\psi_{\rho+1}(x), \dots, \psi_n(x)$  with  $\frac{\partial \psi_i}{\partial x}(x)g(x) = 0$ ,  $i = \rho + 1, \dots, n$  such that the change of coordinates

$$\begin{aligned} \zeta_1 &= y = h(x) = \psi_1(x), \quad \zeta_2 = \dot{y} = \psi_2(x), \dots, \zeta_\rho = y^{(\rho-1)} = \psi_\rho(x), \\ \zeta_{\rho+1} &= \psi_{\rho+1}(x), \dots, \zeta_n = \psi_n(x) \end{aligned} \quad (2.90)$$

is locally invertible and transforms, along with the feedback (2.89), the nonlinear system (2.81) into

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ &\vdots \\ \dot{\zeta}_{\rho-1} &= \zeta_\rho \\ \dot{\zeta}_\rho &= v \\ \dot{\zeta}_{\rho+1} &= \frac{\partial \psi_{\rho+1}}{\partial x}(x)f(x) = \phi_{\rho+1}(\zeta) \\ &\vdots \\ \dot{\zeta}_n &= \frac{\partial \psi_n}{\partial x}(x)f(x) = \phi_n(\zeta) \\ y &= \zeta_1. \end{aligned} \quad (2.91)$$

As a cascade connection of a chain of  $\rho$  integrators with an  $(n - \rho)$ -dimensional nonlinear system, this system is a special case of the cascade systems to which we will apply backstepping in the following sections.

The states  $\zeta_{\rho+1}, \dots, \zeta_n$  of the nonlinear subsystem in (2.91) have been rendered *unobservable* from the output  $y$  by the control (2.89). Hence, feedback linearization in this case is the nonlinear equivalent of placing  $n - \rho$  poles of a linear system at the origin and cancelling the  $\rho$  zeros with the remaining poles. Of course, to guarantee stability, the cancelled zeros must be stable. In the nonlinear case, using the new control input  $v$  to stabilize the linear subsystem of (2.91) does not guarantee stability of the whole system, unless the stability of the nonlinear part of (2.91) has been established separately.

When  $v$  is used to keep the output  $y$  equal to zero for all  $t > 0$ , that is, when  $\zeta_1 \equiv \dots \equiv \zeta_\rho \equiv 0$ , the dynamics of  $\zeta_{\rho+1}, \dots, \zeta_n$  are described by

$$\begin{aligned}\dot{\zeta}_{\rho+1} &= \phi_{\rho+1}(0, \dots, 0, \zeta_{\rho+1}, \dots, \zeta_n) \\ &\vdots \\ \dot{\zeta}_n &= \phi_n(0, \dots, 0, \zeta_{\rho+1}, \dots, \zeta_n).\end{aligned}\quad (2.92)$$

They are called the *zero dynamics* of (2.81), because they evolve on the subset of the state space on which the output of the system is identically zero. If the equilibrium at  $\zeta_{\rho+1} = \dots = \zeta_n = 0$  of the zero dynamics (2.92) is asymptotically stable, the system (2.81) is said to be *minimum phase*. With a slight abuse of notation, we will refer to the  $(\zeta_{\rho+1}, \dots, \zeta_n)$ -subsystem as the *zero dynamics subsystem* of (2.81), even when  $\zeta_1, \dots, \zeta_\rho$  are not zero.

In (2.81) the output  $y = h(x)$  is prespecified, possibly from a tracking objective, and the resulting cascade system is linear from the input  $v$  to the output  $y$ . This linearization process is usually called *input-output feedback linearization* [53]. If our goal is only to design a stabilizing controller, we may attempt to find an output with respect to which the relative degree is  $\rho = n$ . If such an output exists, the whole system is linearized without zero dynamics. This process is referred to as *full-state feedback linearization* [48, 49, 56, 178]. If such an output cannot be found, then we may look for an output which yields the highest relative degree, and thus results in a cascade whose linear subsystem has the highest dimension [117]. It is desirable that with respect to the chosen output the system be minimum phase. The importance of this property will be clear in the following sections which address problems of stabilization of cascade systems.

## 2.2.4 Stabilization of cascade systems

The Integrator Backstepping Lemma (Lemma 2.8) is a stabilization result for the cascade connection of a nonlinear system with an integrator. The nonlinear system, which is allowed to be unstable, is stabilized through the integrator.

We now consider cascade connections in which the nonlinear system is globally stable, but the input subsystem is more complex than just an integrator. We begin with the case where the input subsystem is linear:

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (2.93a)$$

$$\dot{\xi} = A\xi + bu, \quad y = h\xi, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \quad (2.93b)$$

We assume that when  $y = 0$  the nonlinear subsystem (2.93a) has a globally stable equilibrium at  $x = 0$ , and that an appropriate Lyapunov function  $V(x)$  is known:

$$\frac{\partial V}{\partial x}(x) f(x) \leq -W(x) \leq 0. \quad (2.94)$$

The problem is to stabilize the linear subsystem (2.93b) without destabilizing the nonlinear subsystem (2.93a), and, if possible, to achieve GAS of the equilibrium of (2.93) at  $(0, 0)$ . This problem is not solvable in general. Here it will be solved by requiring the input subsystem (2.93b) to have the following passivity property:

**Assumption 2.12** *The triple  $(A, b, h)$  is feedback positive real (FPR), that is, there exists a linear feedback transformation  $u = K\xi + v$  such that  $A + bK$  is Hurwitz and there are matrices  $P > 0$ ,  $Q \geq 0$  which satisfy*

$$(A + bK)^T P + P(A + bK) = -Q \quad (2.95a)$$

$$Pb = h^T. \quad (2.95b)$$

A sufficient condition for FPR is that there exists a feedback gain row vector  $K$  such that  $A + bK$  is Hurwitz, the transfer function  $Z(s) = h(sI - A - bK)^{-1}b$  is positive real (PR) (see Appendix D, Definition D.6), and the pair  $(A + bK, h)$  is observable. It should be noted from (2.95b) that the relative degree of PR transfer functions is one because  $b^T P b = h b > 0$ .

**Lemma 2.13 (Stabilization with FPR)** *Let  $V(x)$  be a Lyapunov function for (2.93a) satisfying (2.94). If the triple  $(A, b, h)$  is FPR, then a Lyapunov function for the cascade system (2.93) is*

$$V_a(x, \xi) = V(x) + \xi^T P \xi, \quad (2.96)$$

and the corresponding control law

$$u = \alpha_a(x, \xi) = K\xi - \frac{1}{2} \frac{\partial V}{\partial x} g(x) \quad (2.97)$$

guarantees that  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  is globally bounded and converges to the largest invariant set  $M_a$  contained in the set  $E_a = \left\{ \begin{bmatrix} \dot{x} \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, Q^{\frac{1}{2}} \xi = 0 \right\}$ .

If  $W(x)$  is positive definite, that is, if the nonlinear subsystem (2.93a) with  $y = 0$  has a globally asymptotically stable equilibrium at  $x = 0$ , then the equilibrium  $x = 0, \xi = 0$  is also GAS.

**Proof.** Using (2.94) and (2.95) and denoting  $u = K\xi + v$  with  $v = -\frac{1}{2}\frac{\partial V}{\partial x}g(x)$  from (2.97), the derivative of  $V_a(x, \xi)$  is

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x}(x)[f(x) + g(x)y] \\ &\quad + \xi^T P[(A + bK)\xi + bv] + [(A + bK)\xi + bv]^T P\xi \\ \text{by (2.94) and (2.95a)} &\leq -W(x) + \frac{\partial V}{\partial x}(x)g(x)y - \xi^T Q\xi + 2\xi^T Pbv \\ \text{by (2.95b)} &= -W(x) + \frac{\partial V}{\partial x}(x)g(x)y - \xi^T Q\xi + 2y \left[ -\frac{1}{2}\frac{\partial V}{\partial x}g(x) \right] \\ &= -W(x) - \xi^T Q\xi \leq 0. \end{aligned} \quad (2.98)$$

Since  $V_a$  is positive definite, radially unbounded and has a negative semidefinite derivative,  $x(t)$  and  $\xi(t)$  are globally bounded. Furthermore LaSalle's theorem (Theorem 2.2) guarantees convergence to the largest invariant set  $M_a$  in the set  $E_a$ .

If, in addition,  $W(x)$  is positive definite, then the global asymptotic stability of  $x = 0, \xi = 0$  is shown using Corollary 2.3. From the positive definiteness of  $W(x)$ , the set  $E_a$ , on which  $\dot{V}_a = 0$ , is given by  $E_a = \{(x, \xi) \mid x = 0, Q^{\frac{1}{2}}\xi = 0\}$ . Since  $V(x)$  is positive definite, it has a minimum at  $x = 0$ , and thus  $\frac{\partial V}{\partial x}(0) = 0$ . This implies that on the set  $E_a$  the control term  $v = -\frac{1}{2}\frac{\partial V}{\partial x}g(x)$  vanishes. Hence, on the set  $E_a$  the state  $\xi(t)$  satisfies

$$\dot{\xi} = (A + bK)\xi, \quad V_a(x, \xi) = \xi^T P\xi. \quad (2.99)$$

But  $V_a$  is constant on  $E_a$ , which means that  $\xi^T P\xi$  must be constant on  $E_a$ . Since  $A + bK$  is Hurwitz,  $\xi = 0$  is the only solution of  $\dot{\xi} = (A + bK)\xi$  that satisfies  $\xi^T P\xi = \text{constant}$ . Thus,  $\xi = 0$  on the largest invariant set contained in  $E_a$ . This implies that this invariant set  $M_a$  is just the equilibrium  $x = 0, \xi = 0$ , which, by Corollary 2.3, is GAS.  $\square$

The stabilizing control law (2.97) consists of two terms, one linear and one nonlinear. The purpose of the latter is to preserve the stability of the nonlinear subsystem.

**Example 2.14** For a comparison with backstepping, let us first reexamine the second-order system stabilized in Example 2.9:

$$\dot{x} = x\xi \quad (2.100a)$$

$$\dot{\xi} = u. \quad (2.100b)$$

In this system we have  $f(x) \equiv 0, g(x) = x, A = 0, B = 1$ , and  $y = \xi$ . Using  $V(x) = x^2$  we see from (2.94) that  $W(x) \equiv 0$ . The FPR condition is trivially satisfied and the stabilizing control (2.97) is

$$u = -k\xi - x^2, \quad k > 0. \quad (2.101)$$

With  $k = 1$  this is the same control law as (2.69) obtained by backstepping with  $\alpha(x) \equiv 0$ . We know from Example 2.9 that this control law achieves GAS of the equilibrium  $(x, \xi) = (0, 0)$ .

Next we consider a third-order system to which backstepping is not directly applicable:

$$\dot{x} = x(\xi_1 + \xi_2) \triangleq xy \quad (2.102a)$$

$$\dot{\xi}_1 = \xi_2 \quad (2.102b)$$

$$\dot{\xi}_2 = u. \quad (2.102c)$$

In this case  $b^T = [0 \ 1]$  and  $h = [1 \ 1]$ , so that the condition (2.95b) yields

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} p_{12} = 1 \\ p_{22} = 1. \end{matrix} \quad (2.103)$$

With this restriction on  $P$  and with  $K = [-k_1 \ -k_2]$ , (2.95a) results in

$$\begin{bmatrix} -2k_1 & p_{11} - k_1 - k_2 \\ p_{11} - k_1 - k_2 & 2 - 2k_2 \end{bmatrix} = - \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}. \quad (2.104)$$

For the simplest choice  $q_{11} = q_{22} = 1$  and  $q_{12} = 0$  we get  $k_1 = 0.5, k_2 = 1.5, p_1 = 2$ . Then the control law (2.97) is

$$u = -\frac{1}{2}\xi_1 - \frac{3}{2}\xi_2 - x^2. \quad (2.105)$$

The equilibrium  $(x, \xi_1, \xi_2) = (0, 0, 0)$  is GAS because  $Q = I$  is positive definite.  $\diamond$

**Example 2.15** Let us now consider a system in which  $\dot{x} = f(x)$  has a GAS equilibrium at  $x = 0$ :

$$\dot{x} = -x^3 - x^3(h_1\xi_1 + h_2\xi_2) \quad (2.106a)$$

$$\dot{\xi}_1 = \xi_2 \quad (2.106b)$$

$$\dot{\xi}_2 = u. \quad (2.106c)$$

When  $h_2 = 0$  this system is stabilizable by two steps of integrator backstepping as in Corollary 2.10. Thus, the case of interest is when  $h_2 \neq 0$  and  $h_1 h_2 \geq 0$ . This includes the case  $h_1 = 0$  when the transfer function  $\frac{h_2 s + h_1}{s^2}$  of the linear part is only *weak minimum phase* [163] because it has a zero at  $s = 0$ .

Choose a feedback  $u = -k_1\xi_1 - k_2\xi_2 + v$  with  $k_1, k_2 > 0$  which makes the polynomial  $q(s) = s^2 + k_2s + k_1$  Hurwitz, and denote  $p(s) = h_1 + h_2s$ . We can choose  $k_1 = a^2$  and  $k_2 = 2a$ , with  $a > \frac{h_1}{2h_2}$ , so that the transfer function  $Z(s) = p(s)/q(s)$  is positive real. We can then write (2.106a) as  $\dot{x} = f + gy$ ,  $f = g = -x^3$ . Clearly,  $x = 0$  is a GAS equilibrium for  $\dot{x} = f$ , so all the conditions of Lemma 2.13 are satisfied. Using  $V(x) = x^2$  in (2.97), we obtain the control law

$$u = -a^2\xi_1 - 2a\xi_2 + x^4. \quad (2.107)$$

The situation is quite different when  $h_1h_2 < 0$ , that is, when the transfer function  $Z(s) = p(s)/q(s)$  is nonminimum phase. Then, Lemma 2.13 does not apply. In fact, a detailed calculation given in [85, Example 3.4] shows that in this case the system *cannot* be globally stabilized.  $\diamond$

The FPR property is a passivity property. Its nonlinear counterpart will be employed in the stabilization of the nonlinear cascade

$$\dot{x} = f(x, \xi) + g(x, \xi)y, \quad f(0, \xi) = 0, \quad \forall \xi \in \mathbb{R}^q, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (2.108a)$$

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \quad (2.108b)$$

Our key assumption is that (2.108b) can be rendered *passive* or *strictly passive* (cf. Appendix D) via a feedback transformation  $u = k(\xi) + r(\xi)v$ .

**Definition 2.16** *The system*

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R} \quad (2.109)$$

*is said to be feedback passive (FP) if there exists a feedback transformation*

$$u = k(\xi) + r(\xi)v \quad (2.110)$$

*such that the resulting system  $\dot{\xi} = m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v$ ,  $y = h(\xi)$  is passive with a storage function  $U(\xi)$  which is positive definite and radially unbounded:*

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)). \quad (2.111)$$

*The system (2.109) is said to be feedback strictly passive (FSP) if the feedback (2.110) renders it strictly passive:*

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \psi(\xi(\sigma))d\sigma, \quad (2.112)$$

*where  $\psi(\cdot)$  is the positive definite dissipation rate.*

As in the linear case, FP systems of the form (2.109) must have relative degree one.

**Lemma 2.17 (Stabilization with Passivity)** *Let  $V(x)$  be a radially unbounded Lyapunov function for  $\dot{x} = f(x, \xi)$  satisfying*

$$\frac{\partial V}{\partial x}(x)f(x, \xi) \leq -W(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \quad \forall \xi \in \mathbb{R}, \quad (2.113)$$

*and let (2.108b) be FP as in Definition 2.16. Then, a Lyapunov function for the cascade system (2.108) is*

$$V_a(x, \xi) = V(x) + U(\xi), \quad (2.114)$$

*and the corresponding control law*

$$u = \alpha_a(x, \xi) = k(\xi) - r(\xi)\frac{\partial V}{\partial x}(x)g(x, \xi) \quad (2.115)$$

*guarantees that  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  is globally bounded and converges to the largest invariant set  $\bar{M}_a$  contained in the set  $\bar{E}_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0 \right\}$ . If (2.108b) is FSP, then (2.115) guarantees convergence to the largest invariant set  $M_a$  contained in the set  $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, \xi = 0 \right\}$ . Finally, if (2.108b) is FSP and  $W(x)$  is positive definite, that is, if  $\dot{x} = f(x, \xi)$  has a GAS equilibrium at  $x = 0$  uniformly in  $\xi$ , then the equilibrium  $x = 0, \xi = 0$  of (2.108) is also GAS.*

**Proof.** We first recall Theorem D.4 in Appendix D, which states that the negative feedback interconnection of two (strictly) passive systems is (strictly) passive, and of Lemma D.3, which states that a (strictly) passive system with a positive definite and radially unbounded storage function has a G(A)S equilibrium at  $x = 0$  when its external input is set to zero.

The closed-loop system (2.108) with the control (2.115) is

$$\begin{aligned} \dot{x} &= f(x, \xi) + g(x, \xi)y \\ \dot{\xi} &= m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v \\ y &= h(\xi), \quad v = -\frac{\partial V}{\partial x}(x)g(x, \xi). \end{aligned} \quad (2.116)$$

Let us now express (2.116) as the feedback interconnection of two passive systems  $\Sigma_1$  and  $\Sigma_2$ :

$$\Sigma_1 \quad \begin{cases} \dot{x} = f(x, \xi) + g(x, \xi)y \\ \eta = \frac{\partial V}{\partial x}(x)g(x, \xi) \end{cases} \quad (2.117a)$$

$$\Sigma_2 \quad \begin{cases} \dot{\xi} = m(\xi) + \beta(\xi)k(\xi) + \beta(\xi)r(\xi)v \\ y = h(\xi) \end{cases} \quad (2.117b)$$

$$v = -\eta. \quad (2.117c)$$

We already know that  $\Sigma_2$  is passive, since (2.111) is satisfied. To show that  $\Sigma_1$  is passive with storage function  $V(x)$ , we use (2.113):

$$\dot{V} = \frac{\partial V}{\partial x} (f + gy) \leq -W(x) + \frac{\partial V}{\partial x} gy = -W(x) + \eta y. \quad (2.118)$$

Integrating (2.118) on  $[0, t]$  we obtain

$$\int_0^t \eta(\sigma)y(\sigma)d\sigma \geq V(x(t)) - V(x(0)) + \int_0^t W(x(\sigma))d\sigma, \quad (2.119)$$

which shows that  $\Sigma_1$  is passive since  $W(x) \geq 0$ . From Theorem D.4 we conclude that (2.117) is passive with the positive definite and radially unbounded storage function  $V_a(x, \xi) = V(x) + U(\xi)$ . Lemma D.3 then states that  $x = 0, \xi = 0$  is a globally stable equilibrium of (2.116). To see that  $W(x) \rightarrow 0$  as  $t \rightarrow \infty$ , we differentiate (2.111) and combine the result with (2.113):

$$\begin{aligned} \dot{V}_a &= \dot{V} + \dot{U} \leq \frac{\partial V}{\partial x} (f + gy) + yv \\ &\leq -W(x) + \frac{\partial V}{\partial x} gy + yv = -W(x). \end{aligned} \quad (2.120)$$

Then, LaSalle's theorem (Theorem 2.2) guarantees convergence to the set  $\bar{M}_a$ . If (2.108b) is FSP, we replace (2.111) by (2.112). Then (2.120) becomes

$$\dot{V} \leq -W(x) - \psi(\xi), \quad (2.121)$$

which, since  $\psi(\xi)$  is positive definite, guarantees convergence to the set  $M_a$ . Finally, if  $W(x)$  is also positive definite, we conclude from (2.121) and Theorem 2.1 that  $x = 0, \xi = 0$  is GAS.  $\square$

**Example 2.18** Consider the cascade system:

$$\dot{x} = -x(1 + e^\xi) + x^3\xi^2 \quad (2.122a)$$

$$\dot{\xi} = \xi u. \quad (2.122b)$$

The choice of output  $y = \xi^2$  satisfies all the conditions of Lemma 2.17. First, (2.122b) is FSP: The feedback

$$u = -\xi^2 + v \quad (2.123)$$

results in  $\dot{\xi} = -\xi^3 + \xi v$ ,  $y = \xi^2$ , which is strictly passive with storage function  $U(\xi) = \frac{1}{2}\xi^2$ , since

$$\dot{U} = -\xi^4 + \xi^2 v = -\xi^4 + yv \quad (2.124)$$

implies that

$$\int_0^t y(\sigma)v(\sigma)d\sigma \geq U(\xi(t)) - U(\xi(0)) + \int_0^t \xi^4(\sigma)d\sigma. \quad (2.125)$$

Furthermore, (2.122a) can be represented in the form (2.108a) with

$$f(x, \xi) = -x(1 + e^\xi), \quad g(x, \xi) = x^3, \quad (2.126)$$

and (2.113) is satisfied with  $V(x) = \frac{1}{2}x^2$ ,  $W(x) = -x^2$ .

Applying Lemma 2.17, we conclude that the control

$$u = -\xi^2 - x^4 \quad (2.127)$$

guarantees GAS of  $x = 0, \xi = 0$ . Indeed, the derivative of the cdf  $V_a(x, \xi) = \frac{1}{2}(x^2 + \xi^2)$  is negative definite:

$$\dot{V}_a = -x^2(1 + e^\xi) + x^4\xi^2 - \xi^4 - x^4\xi^2 \leq -x^2 - \xi^4. \quad (2.128)$$

$\diamond$

**Remark 2.19** Lemmas 2.13 and 2.17 allow the zero dynamics of the input subsystem to be only stable rather than asymptotically stable. Such systems are said to be *weak minimum phase* [163]. This is illustrated in Example 2.15, where the cascade system (2.106) is stabilized even with  $h_1 = 0$ , that is, when the input subsystem (2.106a) has a simple zero at  $s = 0$ .  $\diamond$

## 2.2.5 Block backstepping with zero dynamics

Integrator backstepping (Lemma 2.8) is a recursive design tool. Now we want to develop a similar tool for feedback stabilization of a system augmented by a dynamic block more complicated than just an integrator. At first glance, it may appear that the cascade design in the preceding subsection provides us with such a tool. Not quite! The achievement of the cascade design is in being able to stabilize the input subsystem (2.93b) or (2.108b) *without destabilizing the original system*. What if the original system is not stable? Can we cascade it with a complicated input subsystem and still stabilize it in one step? We first show that this can be done with a linear input subsystem that is a minimum phase system with relative degree one. We then give a nonlinear extension of that result.

**Example 2.20** Let us start with an example in which we cascade the system (2.122) of Example 2.18 with a linear minimum phase system:

$$\begin{aligned} x\text{-subsystem} \quad & \begin{cases} \dot{x}_1 = -x_1(1 + e^{x_2}) + x_1^3 x_2^2 \\ \dot{x}_2 = x_2 y \end{cases} \\ \xi\text{-subsystem} \quad & \begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = u \\ y = \xi_1 + \xi_2 \end{cases} \end{aligned} \quad (2.129)$$

The transfer function of the input subsystem is  $\frac{s+1}{s^2}$  and its zero is at  $s = -1$ . One of its minimal realizations is

$$\dot{y} = y - \xi_1 + u \quad (2.130a)$$

$$\dot{\xi}_1 = -\xi_1 + y. \quad (2.130b)$$

Its zero dynamics, that is, the dynamics constrained by  $y(t) \equiv 0$ , are described by  $\dot{\xi}_1 = -\xi_1$ .

The cascade design of the preceding subsection is not applicable to (2.129) because the equilibrium  $x = 0$  of the  $x$ -subsystem with  $y = 0$  is unstable. To circumvent this obstacle, we first convert (2.130a) into an integrator via the feedback transformation

$$u = -y + \xi_1 + v, \quad (2.131)$$

where  $v$  is our new control variable. The system (2.129) is then rewritten as

$$\begin{aligned} \dot{x}_1 &= -x_1(1 + e^{x_2}) + x_1^3 x_2^2 \\ \dot{x}_2 &= x_2 y \\ \dot{y} &= v \\ \dot{\xi}_1 &= -\xi_1 + y. \end{aligned} \quad (2.132)$$

Now the subsystem consisting of the first three equations in (2.132) is in a form convenient for integrator backstepping. From Example 2.18 we already know that the  $x$ -subsystem can be stabilized with  $y$  as its virtual control (cf. (2.127)):

$$y_{\text{des}} = \alpha(x) = -x_1^4 - x_2^2. \quad (2.133)$$

The corresponding clf is  $V(x, \xi) = \frac{1}{2}(x_1^2 + x_2^2)$ . Hence, we can achieve stabilization and regulation of  $x_1, x_2, y$  by a direct application of Lemma 2.8. The resulting control law is

$$u = -(y + x_1^4 + x_2^2) - \xi_2 - 2x_2^2 y + 4x_1^4(1 + e^{x_2} - x_1^2 x_2^2) - x_2^2. \quad (2.134)$$

This design ignored the presence of the zero dynamics subsystem  $\dot{\xi}_1 = -\xi_1 + y$ . However, this subsystem is input-to-state stable (ISS) with respect to  $y$ , so that  $\xi_1$  is bounded because  $y$  is bounded, and moreover  $\lim_{t \rightarrow \infty} \xi_1(t) \rightarrow 0$  since  $\lim_{t \rightarrow \infty} y(t) \rightarrow 0$ .  $\diamond$

We now want to generalize the above example and formulate design tools which allow the original system to be unstable when  $y = 0$  and let us backstep more than a simple integrator at a time. Since we want to be able to apply these tools repeatedly, each lemma we formulate must guarantee for the cascade system all the properties assumed for the original system. As we will see, the constructed  $V_a(x, \xi)$  for the cascade system does not include the zero dynamics variables, but their boundedness is guaranteed by the boundedness of  $V_a$ . Hence, we must reformulate Assumption 2.7 to assume the same properties for the original system, by including the case when  $V(x)$  is not positive definite:

**Assumption 2.21** Suppose Assumption 2.7 is valid with  $V(x)$  positive semidefinite, and the closed-loop system (2.48) with the control (2.49) has the property that  $x(t)$  is bounded if  $V(x(t))$  is bounded.

Under this assumption, the control (2.49), applied to the system (2.48), guarantees not only global boundedness of  $x(t)$ , but also regulation of  $W(x(t))$ : From (2.50) we conclude that  $W(x(t))$  is integrable on  $[0, \infty)$  and uniformly continuous, and hence converges to zero by Lemma A.6. Furthermore, since all solutions  $x(t)$  are bounded, we can apply LaSalle's theorem (Theorem 2.2) to conclude that  $x(t)$  converges to the largest invariant set  $M$  contained in the set  $E = \{x \in \mathbb{R}^n \mid W(x) = 0\}$ . The following fact is easy to prove:

**Corollary 2.22** When Assumption 2.7 is replaced by Assumption 2.21, then the boundedness and convergence properties in part (ii) of Lemma 2.8 still hold.

**Lemma 2.23 (Linear Block Backstepping)** Consider the cascade system

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (2.135a)$$

$$\dot{\xi} = A\xi + bu, \quad y = h\xi, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}, \quad (2.135b)$$

where (2.135b) is a minimum phase system of relative degree one ( $hb \neq 0$ ). If (2.135a) satisfies Assumption 2.21 with  $y$  as its input, then there exists a feedback control which guarantees global boundedness and convergence of  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  to the largest invariant set  $M_a$  contained in the set  $E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, y = \alpha(x) \right\}$ . One choice for this control is

$$u = \frac{1}{hb} \left\{ -c(y - \alpha(x)) - hA\xi + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)y] - \frac{\partial V}{\partial x}(x)g(x) \right\}, \quad c > 0. \quad (2.136)$$

Moreover, if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, \xi = 0$  is GAS.

**Proof.** We recall from [164] that the relative-degree-one SISO linear system (2.135b) can be represented in the form

$$\dot{y} = hA\xi + hbu \quad (2.137a)$$

$$\dot{\zeta} = A_0\zeta + b_0y, \quad (2.137b)$$

where the eigenvalues of  $A_0$  are the (stable) zeros of the transfer function  $H(s) = h(sI - A)^{-1}b$  of the minimum phase system (2.135b). Using (2.137) and the feedback transformation

$$u = \frac{1}{hb}(v - hA\xi), \quad (2.138)$$

we rewrite (2.135) as follows:

$$\dot{x} = f(x) + g(x)y \quad (2.139a)$$

$$\dot{y} = v \quad (2.139b)$$

$$\dot{\zeta} = A_0\zeta + b_0y. \quad (2.139c)$$

We first ignore the zero dynamics (2.139c) and, using Corollary 2.22, apply Lemma 2.8 to (2.139a)–(2.139b) to achieve global boundedness of  $x$  and  $y$  and regulation of  $W(x(t))$  and  $y(t) - \alpha(x(t))$ . In view of (2.138) and (2.54), one choice of control is given by (2.136). Returning to (2.139c), we note that  $\zeta$  is bounded because  $y$  is bounded and  $A_0$  is Hurwitz. Thus,  $\xi$  is bounded. Since all solutions of (2.135) are bounded, we can apply LaSalle's theorem (Theorem 2.2) with  $\Omega = \mathbb{R}^{n+q}$  to conclude convergence to the set  $M_a$ .

From Lemma 2.8 we also know that if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, y = 0$  of (2.139a)–(2.139b), which is completely decoupled from (2.139c), is GAS. The fact that in this case the equilibrium  $x = 0, \xi = 0$  of the cascade system (2.135) is also GAS follows immediately from the following lemma:

**Lemma 2.24** *Consider the cascade system with  $\zeta \in \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$ :*

$$\dot{\zeta} = A_0\zeta + b_0y \quad (2.140a)$$

$$\dot{x} = f(x), \quad f(0) = 0 \quad (2.140b)$$

$$y = h(x), \quad h(0) = 0.$$

If (2.140b) is GAS and  $A_0$  is Hurwitz, then the equilibrium  $\zeta = 0, x = 0$  of the cascade (2.140) is GAS.

**Proof.** From the definition of GAS (Definition A.4 in Appendix A) we know that the GAS property of (2.140b) implies the existence of class  $\mathcal{KL}_\infty$  functions  $\beta$  and  $\beta_1$  such that

$$|x(t)| \leq \beta(|x(0)|, t), \quad |y(t)| \leq \beta_1(|x(0)|, t). \quad (2.141)$$

The solutions of (2.140a), on the other hand, are given by

$$\zeta(t) = e^{A_0 t} \zeta(0) + \int_0^t e^{A_0(t-\tau)} b_0 y(\tau) d\tau. \quad (2.142)$$

Since  $A_0$  is Hurwitz, we know that  $|e^{A_0 t}| \leq k_1 e^{-\alpha t}$ . Using this with (2.141) in (2.142), we obtain

$$|\zeta(t)| \leq |e^{A_0 t}| |\zeta(0)| + \int_0^t |e^{A_0(t-\tau)} b_0| |y(\tau)| d\tau$$

$$\begin{aligned} &\leq k_1 e^{-\alpha t} |\zeta(0)| + k_2 \int_0^t e^{-\alpha(t-\tau)} \beta_1(|x(0)|, \tau) d\tau \\ &\leq k_1 e^{-\alpha t} |\zeta(0)| + k_2 \sup_{0 \leq \tau \leq t/2} \beta_1(|x(0)|, \tau) \int_0^{t/2} e^{-\alpha(t-\tau)} d\tau \\ &\quad + k_2 \sup_{t/2 \leq \tau \leq t} \beta_1(|x(0)|, \tau) \int_{t/2}^t e^{-\alpha(t-\tau)} d\tau \\ &\leq k_1 e^{-\alpha t} |\zeta(0)| + k_2 \beta_1(|x(0)|, 0) \int_0^{t/2} e^{-\alpha(t-\tau)} d\tau \\ &\quad + k_2 \beta_1(|x(0)|, t/2) \int_{t/2}^t e^{-\alpha(t-\tau)} d\tau \\ &= k_1 e^{-\alpha t} |\zeta(0)| + \frac{k_2}{\alpha} \beta_1(|x(0)|, 0) e^{-\alpha t/2} (1 - e^{-\alpha t/2}) \\ &\quad + \frac{k_2}{\alpha} \beta_1(|x(0)|, t/2) (1 - e^{-\alpha t/2}) \\ &\leq k_1 e^{-\alpha t} |\zeta(0)| + \frac{k_2}{\alpha} \beta_1(|x(0)|, 0) e^{-\alpha t/2} + \frac{k_2}{\alpha} \beta_1(|x(0)|, t/2) \\ &\triangleq \beta_2 \left( \begin{bmatrix} \zeta(0) \\ x(0) \end{bmatrix}, t \right), \end{aligned} \quad (2.143)$$

where  $\beta_2$  is a class  $\mathcal{KL}_\infty$  function. Combining (2.141) with (2.143) proves that  $\zeta = 0, x = 0$  is GAS:

$$\left| \begin{bmatrix} \zeta(t) \\ x(t) \end{bmatrix} \right| \leq \beta_3 \left( \begin{bmatrix} \zeta(0) \\ x(0) \end{bmatrix}, t \right), \quad \beta_3 \in \mathcal{KL}_\infty. \quad (2.144)$$

□

Comparing Lemmas 2.13 and 2.23 we see that, instead of assuming global stability of  $x = 0$  when  $y = 0$ , Lemma 2.23 assumes only global stabilizability of  $x = 0$  through  $y$ . The corresponding assumptions on the input subsystems, however, reveal the price paid for this generalization: The minimum phase assumption of Lemma 2.23 is stronger than the FPR assumption of Lemma 2.13, which, as noted in Remark 2.19, allows some zeros to be on the imaginary axis, that is, to be weak minimum phase.

Let us now examine the cascade system

$$\dot{x} = -x(1 + e^{\xi_1}) + x^3 \xi_1^2 \quad (2.145a)$$

$$\dot{\xi}_1 = \xi_1 \xi_2^2 \quad (2.145b)$$

$$\dot{\xi}_2 = \xi_2 u. \quad (2.145c)$$

As we have already shown in Example 2.18, (2.145a)–(2.145b) is stabilizable through  $y = \xi_2^2$ , while (2.145c) with this output is FSP. However, if we try to stabilize the cascade (2.145), we run into difficulties because the relative degree of (2.145c) is not defined at  $\xi_2 = 0$ .

This example shows that we need to assume that the input subsystem

$$\dot{\xi} = m(\xi) + \beta(\xi)u, \quad y = h(\xi), \quad (2.146)$$

has a globally defined constant relative degree. For a nonlinear analog of Lemma 2.23, we also assume that the zero dynamics subsystem of (2.146) is ISS.

**Lemma 2.25 (Nonlinear Block Backstepping)** *Consider the cascade system:*

$$\dot{x} = f(x) + g(x)y, \quad f(0) = 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R} \quad (2.147a)$$

$$\dot{\xi} = m(x, \xi) + \beta(x, \xi)u, \quad y = h(\xi), \quad h(0) = 0, \quad \xi \in \mathbb{R}^q, \quad u \in \mathbb{R}. \quad (2.147b)$$

Assume that (2.147b) has globally defined and constant relative degree one uniformly in  $x$ , and that its zero dynamics subsystem is ISS with respect to  $x$  and  $y$  as its inputs. If (2.147a) satisfies Assumption 2.21 with  $y$  as its input, then there exists a feedback control which guarantees global boundedness and convergence of  $\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$  to the largest invariant set  $M_a$  contained in the set

$$E_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+q} \mid W(x) = 0, y = \alpha(x) \right\}. \text{ One particular choice is}$$

$$u = \left( \frac{\partial h}{\partial \xi}(\xi) \beta(x, \xi) \right)^{-1} \left\{ -c(y - \alpha(x)) - \frac{\partial h}{\partial \xi}(\xi) m(x, \xi) + \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)y] - \frac{\partial V}{\partial x}(x) g(x) \right\}, \quad c > 0. \quad (2.148)$$

Moreover, if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, \xi = 0$  is GAS.

**Proof.** Since the relative degree of the subsystem (2.147b) is globally defined and equal to one uniformly in  $x$ , there exists a global<sup>7</sup> change of coordinates of the form (2.90), in particular  $(y, \zeta) = (y, \phi(x, \xi))$  with  $\frac{\partial \phi}{\partial \xi} \beta \equiv 0$ , which transforms (2.147b) into

$$\dot{y} = \frac{\partial h}{\partial \xi}(\xi) m(x, \xi) + \frac{\partial h}{\partial \xi}(\xi) \beta(x, \xi) u \triangleq f_1(x, y, \zeta) + g_1(x, y, \zeta) u \quad (2.149a)$$

$$\dot{\zeta} = \frac{\partial \phi}{\partial x}(x, \xi) [f(x) + g(x)y] + \frac{\partial \phi}{\partial \xi}(x, \xi) m(x, \xi) \triangleq \Phi(\zeta, x, y). \quad (2.149b)$$

<sup>7</sup>This change of coordinates is a global diffeomorphism under additional conditions of connectedness and completeness [13].

We now consider the cascade system consisting of (2.147a) and (2.149a). If we linearize (2.149a) with the feedback given by (2.89),

$$u = \left( \frac{\partial h}{\partial \xi} \beta \right)^{-1} \left( v - \frac{\partial h}{\partial \xi} m \right), \quad (2.150)$$

we obtain  $\dot{y} = v$ . Then we can apply Lemma 2.8, with  $v$  as the new control input, to guarantee global boundedness of  $x$  and  $y$  and regulation of  $W(x(t))$  and  $y(t) - \alpha(x(t))$ . From (2.149b) and the ISS assumption on the zero dynamics,  $\zeta$  is also bounded, and thus  $\xi$  and  $u$  are bounded. Since all solutions of (2.147) are bounded, we can apply LaSalle's theorem (Theorem 2.2) with  $\Omega = \mathbb{R}^{n+q}$  to conclude convergence to the set  $M_a$ . Combining (2.150) with (2.54), we see that a particular choice of control is given by (2.148).

From Lemma 2.8 we also know that if  $V(x)$  and  $W(x)$  are positive definite, then the equilibrium  $x = 0, y = 0$  of (2.147a) and (2.149a), which is completely decoupled from (2.149b), is GAS. The fact that in this case the equilibrium  $x = 0, \xi = 0$  of the cascade system (2.147) is also GAS follows from Lemma C.4 by noting that the state  $(x, y)$  of the GAS system (2.147a) and (2.149a) is the input of the ISS system (2.149b).  $\square$

Lemma 2.25 relaxes the global stability assumption of Lemma 2.17 to global stabilizability of  $x = 0$  through  $y$ . As in the case of Lemmas 2.13 and 2.23, however, the price paid for this generalization is the strengthening of the FP assumption of Lemma 2.17 to the ISS assumption of Lemma 2.25.

The following example illustrates the use of block backstepping as a design tool.

### 2.2.6 Example: active suspension (series)

We return to the active suspension example, but in contrast to the parallel configuration of Figure 2.6, we now work with the quarter-car model of Figure 2.7, where the hydraulic actuator is connected in series with the spring/damper system.<sup>8</sup>

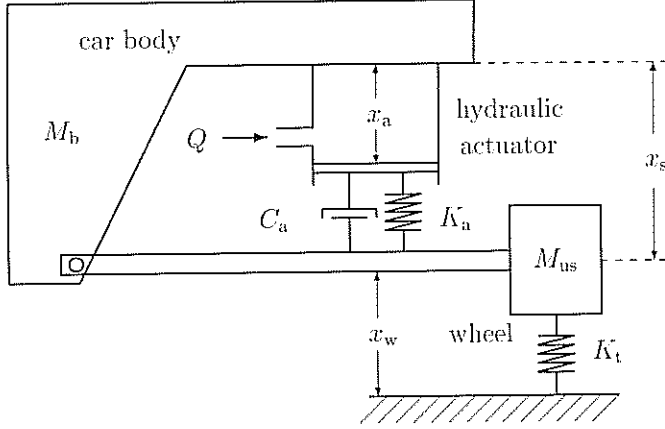
In this configuration, the piston position  $x_a$  is determined from the equation

$$\dot{x}_a = \frac{1}{A} Q. \quad (2.151)$$

Again, the control objective is to stiffen the suspension near its travel limits. Neglecting the wheel acceleration  $\ddot{x}_w$ , the acceleration of the car body is equal to the suspension acceleration  $\ddot{x}_s$  and is given by

$$M_b \ddot{x}_s = -K_a(x_s - x_a) - C_a(\dot{x}_s - \dot{x}_a). \quad (2.152)$$

<sup>8</sup>Both of these configurations are currently used in active suspension research and design.



**Figure 2.7:** Quarter-car model with *series* connection of hydraulic actuator with passive spring/damper.

Combining (2.151) and (2.152), we obtain the following suspension equation:

$$\ddot{x}_s = -\frac{K_a}{M_b}(x_s - x_a) - \frac{C_a}{M_b}\left(\dot{x}_s - \frac{1}{A}Q\right). \quad (2.153)$$

For the flow  $Q$ , we consider again the linear equation (2.74):

$$\dot{Q} = -c_f Q + k_f i_v. \quad (2.154)$$

The system composed of (2.151), (2.153), and (2.154) is a linear system which can be rewritten in the form of (2.135) with  $x_1 = x_s$ ,  $x_2 = \dot{x}_s$ ,  $\xi_1 = x_a$ ,  $\xi_2 = Q$ ,  $u = i_v$ :

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K_a}{M_b}x_1 - \frac{C_a}{M_b}x_2 + y \\ \dot{\xi}_1 &= \frac{1}{A}\xi_2 \\ \dot{\xi}_2 &= -c_f\xi_2 + k_f u \\ y &= \frac{K_a}{M_b}\xi_1 + \frac{C_a}{M_b A}\xi_2. \end{aligned} \quad (2.155)$$

Clearly, the assumptions of Lemma 2.23 are satisfied, since the  $(\xi_1, \xi_2)$ -subsystem is minimum phase and its relative degree is one. It should be noted that the assumptions of Lemma 2.13 are also satisfied, since the  $(x_1, x_2)$ -subsystem is GAS when  $y = 0$ , and the  $(\xi_1, \xi_2)$ -subsystem is FPR. Thus, if the objective were just stabilization, Lemma 2.13 would be applicable.

We first rewrite the system (2.155) in the form (2.139) with  $\zeta = \xi_1$ :

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K_a}{M_b}x_1 - \frac{C_a}{M_b}x_2 + y \\ \dot{y} &= \left(\frac{K_a}{M_b A} - \frac{C_a c_f}{M_b A}\right)\xi_2 + \frac{C_a k_f}{M_b A}u \\ \dot{\zeta} &= -\frac{K_a}{C_a}\zeta + \frac{M_b}{C_a}y. \end{aligned} \quad (2.156)$$

Next, we apply Lemma 2.8 to the  $(x_1, x_2)$ -subsystem: We view  $x_2$  as the virtual control and design for it the stabilizing function

$$\alpha_1(x_1) = -c_1 x_1 - \kappa_1 x_1^3, \quad (2.157)$$

which, with  $z_1 = x_2 - \alpha_1(x_1)$ , yields

$$\dot{z}_1 = -c_1 x_1 - \kappa_1 x_1^3 + z_1. \quad (2.158)$$

Using (2.155), we obtain

$$\dot{z}_1 = -\frac{K_a}{M_b}x_1 - \frac{C_a}{M_b}x_2 + y + (c_1 + 3\kappa_1 x_1^2)x_2. \quad (2.159)$$

Applying Lemma 2.23, we view  $y$  as the virtual control in (2.159) and design for it the stabilizing function

$$\alpha_2(x_1, x_2, \xi_1, \xi_2) = -c_2 z_1 - x_1 - (c_1 + 3\kappa_1 x_1^2)x_2 + \frac{K_a}{M_b}x_1 + \frac{C_a}{M_b}x_2, \quad (2.160)$$

which results in

$$\begin{aligned} \dot{z}_1 &= -c_1 x_1 - \kappa_1 x_1^3 + z_1 \\ \dot{z}_2 &= -c_2 z_1 - x_1 + z_2, \end{aligned} \quad (2.161)$$

where  $z_2 = y - \alpha_2$ . Using (2.160) and (2.155), we obtain

$$\begin{aligned} \dot{z}_2 &= \left(\frac{K_a}{M_b A} - \frac{C_a c_f}{M_b A}\right)\xi_2 + \frac{C_a k_f}{M_b A}u + \left(1 - \frac{K_a}{M_b} + 6\kappa_1 x_1 x_2\right)x_2 \\ &\quad + c_2(-c_2 z_1 - x_1 + z_2) + \left(c_1 + 3\kappa_1 x_1^2 - \frac{C_a}{M_b}\right)\left(-\frac{K_a}{M_b}x_1 - \frac{C_a}{M_b}x_2 + y\right). \end{aligned} \quad (2.162)$$

The control law for the current  $i_v$  then becomes

$$\begin{aligned} u &= \frac{M_b A}{C_a k_f} \left[ -c_3 z_2 - z_1 - c_2(-c_2 z_1 - x_1 + z_2) - \left(1 - \frac{K_a}{M_b} + 6\kappa_1 x_1 x_2\right)x_2 \right. \\ &\quad \left. - \left(c_1 + 3\kappa_1 x_1^2 - \frac{C_a}{M_b}\right)\left(-\frac{K_a}{M_b}x_1 - \frac{C_a}{M_b}x_2 + y\right) \right] - \left(\frac{K_a}{C_a k_f} - \frac{c_f}{k_f}\right)\xi_2. \end{aligned} \quad (2.163)$$

The resulting closed-loop system is nonlinear and is guaranteed to be GAS by Lemma 2.23:

$$\begin{aligned}\dot{x}_1 &= -c_1 x_1 - \kappa_1 x_1^3 + z_1 \\ \dot{z}_1 &= -c_2 z_1 - x_1 + z_2 \\ \dot{z}_2 &= -c_3 z_2 - z_1 \\ \dot{\zeta} &= -\frac{K_a}{C_a} \zeta + \frac{M_b}{C_a} (z_2 + \alpha_2) .\end{aligned}\quad (2.164)$$

The design objective of preventing large excursions of  $x_1$  is achieved by the presence of the term  $-\kappa_1 x_1^3$  in the first equation.

## 2.3 Recursive Design Procedures

Backstepping tools will now be employed to form systematic design procedures for general classes of nonlinear systems. In increasing order of complexity, the classes considered are strict-feedback systems, pure-feedback systems, and block-strict-feedback systems.

### 2.3.1 Strict-feedback systems

Nonlinear *strict-feedback systems* are of the form

$$\begin{aligned}\dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= f_1(x, \xi_1) + g_1(x, \xi_1)\xi_2 \\ \dot{\xi}_2 &= f_2(x, \xi_1, \xi_2) + g_2(x, \xi_1, \xi_2)\xi_3 \\ &\vdots \\ \dot{\xi}_{k-1} &= f_{k-1}(x, \xi_1, \dots, \xi_{k-1}) + g_{k-1}(x, \xi_1, \dots, \xi_{k-1})\xi_k \\ \dot{\xi}_k &= f_k(x, \xi_1, \dots, \xi_k) + g_k(x, \xi_1, \dots, \xi_k)u ,\end{aligned}\quad (2.165)$$

where  $x \in \mathbb{R}^n$  and  $\xi_1, \dots, \xi_k$  are scalars. The reason for referring to the  $\xi$ -subsystem as “strict-feedback” is that the nonlinearities  $f_i, g_i$  in the  $\xi_i$ -equation ( $i = 1, \dots, k$ ) depend only on  $x, \xi_1, \dots, \xi_i$ , that is, on state variables that are “fed back.”

The  $x$ -subsystem satisfies Assumption 2.7 with  $\xi_1$  as its control input. Our recursive design starts with the subsystem

$$\begin{aligned}\dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= f_1(x, \xi_1) + g_1(x, \xi_1)\xi_2 .\end{aligned}\quad (2.166)$$

If  $f_1 \equiv 0$  and  $g_1 \equiv 1$ , the Integrator Backstepping Lemma 2.8 would be directly applicable to (2.166), treating  $\xi_2$  as the control. In the presence of  $f_1(x, \xi_1)$  and

$g_1(x, \xi_1)$ , we proceed in the same way by constructing  $V_1(x, \xi_1)$  for (2.166) as

$$V_1(x, \xi_1) = V(x) + \frac{1}{2} [\xi_1 - \alpha(x)]^2 , \quad (2.167)$$

where  $\alpha(x)$  is a stabilizing feedback that satisfies (2.50) for the  $x$ -subsystem. Such intermediate control laws will be called *stabilizing functions*. To find a stabilizing function  $\alpha_1(x, \xi_1)$  for  $\xi_2$ , the virtual control in (2.166), we need to make the derivative of  $V_1$  nonpositive when  $\xi_2 = \alpha_1$ :

$$\begin{aligned}\dot{V}_1 &\leq -W(x) + [\xi_1 - \alpha(x)] \left\{ \frac{\partial V}{\partial x}(x)g(x) + f_1(x, \xi_1) + g_1(x, \xi_1)\xi_2 \right. \\ &\quad \left. - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi_1] \right\} \\ &= -W(x) + [\xi_1 - \alpha(x)] \left\{ \frac{\partial V}{\partial x}(x)g(x) + f_1(x, \xi_1) + g_1(x, \xi_1)\alpha_1(x, \xi_1) \right. \\ &\quad \left. + g_1(x, \xi_1)[\xi_2 - \alpha_1(x, \xi_1)] - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi_1] \right\} \\ &= -W_1(x, \xi_1) + \frac{\partial V_1}{\partial \xi_1}(x, \xi_1)g_1(x, \xi_1)[\xi_2 - \alpha_1(x, \xi_1)] ,\end{aligned}\quad (2.168)$$

where  $W_1(x, \xi_1) > 0$  when  $W(x) > 0$  or  $\xi_1 \neq \alpha(x)$ . If  $g_1(x, \xi_1) \neq 0$  for all  $x$  and  $\xi_1$ , one choice for  $\alpha_1$  is

$$\begin{aligned}\alpha_1(x, \xi_1) &= \frac{1}{g_1(x, \xi_1)} \left\{ -c_1 [\xi_1 - \alpha(x)] - \frac{\partial V}{\partial x}(x)g(x) - f_1(x, \xi_1) \right. \\ &\quad \left. + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi_1] \right\} ,\end{aligned}\quad (2.169)$$

with  $c_1 > 0$ , which yields  $W_1(x, \xi_1) = W(x) + c_1 [\xi_1 - \alpha(x)]^2$ . However, as we pointed out before, many other, possibly better, choices for  $\alpha_1$  are available, even if  $g_1(x, \xi_1) = 0$  at some points.

With  $\alpha_1(x, \xi_1)$  determined, our next step is to augment (2.166) with the  $\dot{\xi}_2$ -equation from (2.165). In a compact notation, we obtain

$$\begin{aligned}\dot{X}_1 &= F_1(X_1) + G_1(X_1)\xi_2 \\ \dot{\xi}_2 &= f_2(X_1, \xi_2) + g_2(X_1, \xi_2)\xi_3 ,\end{aligned}\quad (2.170)$$

where  $f_2(X_1, \xi_2), g_2(X_1, \xi_2)$  stand for  $f_2(x, \xi_1, \xi_2), g_2(x, \xi_1, \xi_2)$  and

$$X_1 = \begin{bmatrix} x \\ \xi_1 \end{bmatrix}, \quad F_1(X_1) = \begin{bmatrix} f(x) + g(x)\xi_1 \\ f_1(x, \xi_1) \end{bmatrix}, \quad G_1(X_1) = \begin{bmatrix} 0 \\ g_1(x, \xi_1) \end{bmatrix}. \quad (2.171)$$

The structure of (2.170) is identical to that of (2.166). We therefore repeat the same step by introducing

$$\begin{aligned} V_2(X_1, \xi_2) &= V_1(X_1) + \frac{1}{2} [\xi_2 - \alpha_1(X_1)]^2 \\ &= V(x) + \frac{1}{2} \sum_{i=1}^2 [\xi_i - \alpha_{i-1}(X_{i-1})]^2, \end{aligned} \quad (2.172)$$

where, for notational convenience, we use  $X_0 = x$  and  $\alpha_0(X_0) = \alpha(x)$ . The stabilizing function  $\alpha_2(X_2)$ ,  $X_2^T = [X_1^T \ \xi_2]^T$ , for the virtual control  $\xi_3$  is then determined to render

$$\dot{V}_2 \leq -W_2(X_1, \xi_2) + \frac{\partial V_2}{\partial \xi_2}(X_1, \xi_2) g_2(X_1, \xi_2) [\xi_3 - \alpha_2(X_2)], \quad (2.173)$$

with  $W_2(X_1, \xi_2) > 0$  when  $W_1(x, \xi_1) > 0$  or  $\xi_2 \neq \alpha_1(X_1)$ .

It is clear that this procedure will terminate at the  $k$ th step, at which the whole system (2.165) is to be stabilized by the actual control  $u$ . In our compact notation, (2.165) is rewritten as

$$\begin{aligned} \dot{X}_{k-1} &= F_{k-1}(X_{k-1}) + G_{k-1}(X_{k-1})\xi_k \\ \dot{\xi}_k &= f_k(X_{k-1}, \xi_k) + g_k(X_{k-1}, \xi_k)u, \end{aligned} \quad (2.174)$$

where

$$\begin{aligned} X_{k-1} &= \begin{bmatrix} X_{k-2} \\ \xi_{k-1} \end{bmatrix}, \quad F_{k-1}(X_{k-1}) = \begin{bmatrix} F_{k-2}(X_{k-2}) + G_{k-2}(X_{k-2})\xi_{k-1} \\ f_{k-1}(X_{k-2}, \xi_{k-1}) \end{bmatrix} \\ G_{k-1}(X_{k-1}) &= \begin{bmatrix} 0 \\ g_{k-1}(X_{k-2}, \xi_{k-1}) \end{bmatrix}. \end{aligned} \quad (2.175)$$

Once again, this is in the form of (2.166) and (2.170), and the Lyapunov function for (2.174) is

$$\begin{aligned} V_k(x, \xi_1, \dots, \xi_k) &= V_{k-1}(X_{k-1}) + \frac{1}{2} [\xi_k - \alpha_{k-1}(X_{k-1})]^2 \\ &= V(x) + \frac{1}{2} \sum_{i=1}^k [\xi_i - \alpha_{i-1}(X_{i-1})]^2. \end{aligned} \quad (2.176)$$

The meaning of the stabilizing functions  $\alpha_i(X_i)$  designed as intermediate control laws, is now clearer from (2.176). These would-be control laws are, in fact, the tools to construct a Lyapunov function for the strict-feedback system (2.165). The function  $V_k$  in (2.176) is indeed a Lyapunov function, because  $u$  can be chosen to make  $\dot{V}_k \leq -W_k \leq 0$ , with  $W_k > 0$  when  $W_{k-1} > 0$  or

$\xi_k \neq \alpha_{k-1}$ :

$$\begin{aligned} \dot{V}_k &= \dot{V}_{k-1} + (\xi_k - \alpha_{k-1}) \left[ f_k + g_k u - \frac{\partial \alpha_{k-1}}{\partial X_{k-1}} (F_{k-1} + G_{k-1}\xi_k) \right] \\ &\leq -W_{k-1}(X_{k-2}, \xi_{k-1}) + \frac{\partial V_{k-1}}{\partial \xi_{k-1}} g_{k-1} (\xi_k - \alpha_{k-1}) \\ &\quad + (\xi_k - \alpha_{k-1}) \left[ f_k + g_k u - \frac{\partial \alpha_{k-1}}{\partial X_{k-1}} (F_{k-1} + G_{k-1}\xi_k) \right] \\ &= -W_{k-1}(X_{k-2}, \xi_{k-1}) + (\xi_k - \alpha_{k-1}) \left[ \frac{\partial V_{k-1}}{\partial \xi_{k-1}} g_{k-1} + f_k + g_k u \right. \\ &\quad \left. - \frac{\partial \alpha_{k-1}}{\partial X_{k-1}} (F_{k-1} + G_{k-1}\xi_k) \right] \\ &\leq -W_k(X_{k-1}, \xi_k) \leq 0. \end{aligned} \quad (2.177)$$

If the nonsingularity condition

$$g_k(x, \xi_1, \dots, \xi_k) \neq 0, \quad \forall x \in \mathbb{R}^n, \quad \forall \xi_i \in \mathbb{R}, \quad i = 1, \dots, k, \quad (2.178)$$

is satisfied, then the simplest choice for  $u$  is

$$u = \frac{1}{g_k} \left[ -c_k(\xi_k - \alpha_{k-1}) - \frac{\partial V_{k-1}}{\partial \xi_{k-1}} g_{k-1} - f_k + \frac{\partial \alpha_{k-1}}{\partial X_{k-1}} (F_{k-1} + G_{k-1}\xi_k) \right], \quad (2.179)$$

with  $c_k > 0$ , which yields  $W_k = W_{k-1} + c_k(\xi_k - \alpha_{k-1})^2$ . We reiterate that a more desirable control law with less cancellations may also be found, even if the condition (2.178) is violated at some points.

### 2.3.2 Pure-feedback systems

A more general class of “triangular” systems comprises *pure-feedback systems*:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= f_1(x, \xi_1, \xi_2) \\ \dot{\xi}_2 &= f_2(x, \xi_1, \xi_2, \xi_3) \\ &\vdots \\ \dot{\xi}_{k-1} &= f_{k-1}(x, \xi_1, \dots, \xi_k) \\ \dot{\xi}_k &= f_k(x, \xi_1, \dots, \xi_k, u), \end{aligned} \quad (2.180)$$

where  $\xi_i \in \mathbb{R}$ . The  $x$ -subsystem again satisfies Assumption 2.7.

Compared with the strict-feedback systems of (2.165), pure-feedback systems lack the affine appearance of the variables  $\xi_k$ , to be used as virtual controls, and of the actual control  $u$  itself. Our recursive procedure starts with

the subsystem

$$\begin{aligned}\dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= f_1(x, \xi_1, \xi_2),\end{aligned}\quad (2.181)$$

in which  $\xi_2$  is the virtual control. Our Lyapunov function for (2.181) is again

$$V_1(x, \xi_1) = V(x) + \frac{1}{2} [\xi_1 - \alpha(x)]^2, \quad (2.182)$$

and its derivative  $\dot{V}_1$  is to be rendered nonpositive by  $\xi_2 = \alpha_1(x, \xi_1)$ :

$$\begin{aligned}\dot{V}_1 &\leq -W(x) + [\xi_1 - \alpha(x)] \left\{ \frac{\partial V}{\partial x}(x)g(x) + f_1(x, \xi_1, \xi_2) \right. \\ &\quad \left. - \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)\xi_1] \right\} \\ &= -W(x) + [\xi_1 - \alpha(x)] \left\{ \frac{\partial V}{\partial x}(x)g(x) + f_1(x, \xi_1, \alpha_1(x, \xi_1)) \right. \\ &\quad \left. + \bar{f}_1(x, \xi_1, \xi_2) [\xi_2 - \alpha_1(x, \xi_1)] - \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)\xi_1] \right\} \\ &= -W_1(x, \xi_1) + \frac{\partial V_1}{\partial \xi_1}(x, \xi_1) \bar{f}_1(x, \xi_1, \xi_2) [\xi_2 - \alpha_1(x, \xi_1)],\end{aligned}\quad (2.183)$$

where  $W_1(x, \xi_1) > 0$  when  $W(x) > 0$  or  $\xi_1 \neq \alpha(x)$ , and the function  $\bar{f}_1$  is smooth if  $f_1$  is smooth:

$$f_1(x, \xi_1, \xi_2) - f_1(x, \xi_1, \alpha_1(x, \xi_1)) = \bar{f}_1(x, \xi_1, \xi_2) [\xi_2 - \alpha_1(x, \xi_1)]. \quad (2.184)$$

Previously, when  $g_1 \neq 0$ , a simple choice for  $\alpha_1$  was to render  $W_1(x, \xi_1) = W(x) + c_1 [\xi_1 - \alpha(x)]^2$ . If we attempt to do the same in (2.183), we would have to solve for  $\alpha_1$  the following equation:

$$f_1(x, \xi_1, \alpha_1(x, \xi_1)) = -c_1 [\xi_1 - \alpha(x)] + \frac{\partial \alpha}{\partial x}(x) [f(x) + g(x)\xi_1]. \quad (2.185)$$

By the Implicit Function Theorem (see, e.g., [81]), a necessary condition for the solvability of (2.185) with respect to  $\alpha_1$  is

$$\frac{\partial f_1}{\partial \xi_2}(x, \xi_1, \xi_2) \neq 0, \quad \forall (x, \xi_1, \xi_2) \in \mathbb{R}^{n+2}. \quad (2.186)$$

This is quite restrictive and unnecessary for our ability to find an  $\alpha_1$  that satisfies (2.183). Even when (2.186) is violated at a set of points, such a smooth stabilizing function  $\alpha_1(x, \xi_1)$  may exist. We therefore proceed assuming that such an  $\alpha_1$  has been found.

At the next step we augment the subsystem (2.181) by one more equation from (2.180)

$$\begin{aligned}\dot{X}_1 &= F_1(X_1, \xi_2) \\ \dot{\xi}_2 &= f_2(X_1, \xi_2, \xi_3),\end{aligned}\quad (2.187)$$

where  $f_2(X_1, \xi_2, \xi_3)$  stands for  $f_2(x, \xi_1, \xi_2, \xi_3)$  and

$$X_1 = \begin{bmatrix} x \\ \xi_1 \end{bmatrix}, \quad F_1(X_1, \xi_2) = \begin{bmatrix} f(x) + g(x)\xi_1 \\ f_1(x, \xi_1, \xi_2) \end{bmatrix}. \quad (2.188)$$

We use  $\xi_3$  as the virtual control to stabilize (2.187) with respect to the Lyapunov function

$$\begin{aligned}V_2(X_1, \xi_2) &= V_1(X_1) + \frac{1}{2} [\xi_2 - \alpha_1(X_1)]^2 \\ &= V(x) + \frac{1}{2} \sum_{i=1}^2 [\xi_i - \alpha_{i-1}(X_{i-1})]^2,\end{aligned}\quad (2.189)$$

with  $X_0 = x$  and  $\alpha_0(X_0) = \alpha(x)$ . Our task is to find a stabilizing function  $\alpha_2(X_2)$ ,  $X_2^T = [X_1^T \quad \xi_2]^T$ , to yield

$$\dot{V}_2 \leq -W_2(X_1, \xi_2) + \frac{\partial V_2}{\partial \xi_2}(X_1, \xi_2) \bar{f}_2(X_1, \xi_2, \xi_3) [\xi_3 - \alpha_2(X_2)], \quad (2.190)$$

with  $\bar{f}_2(X_1, \xi_2, \xi_3) = f_2(X_1, \xi_2, \xi_3) - f_2(X_1, \xi_2, \alpha_2(X_1, \xi_2))$ , and  $W_2(X_1, \xi_2) > 0$  when  $W_1(x, \xi_1) > 0$  or  $\xi_2 \neq \alpha_1(X_1)$ . Once again, if we want  $W_2(X_1, \xi_2) = W_1(x, \xi_1) + c_2 [\xi_2 - \alpha_1(X_1)]^2$ , we need

$$\frac{\partial f_2}{\partial \xi_3}(X_1, \xi_2, \xi_3) \neq 0, \quad \forall X_1 \in \mathbb{R}^{n+1}, \quad \forall \xi_2 \in \mathbb{R}, \quad \forall \xi_3 \in \mathbb{R}. \quad (2.191)$$

In most situations we would avoid this requirement by directly finding  $\alpha_2(X_2)$  to satisfy the inequality (2.190).

Proceeding in the same fashion, in the  $k$ th step we arrive at the actual control  $u$  in

$$\begin{aligned}\dot{X}_{k-1} &= F_{k-1}(X_{k-1}, \xi_k) \\ \dot{\xi}_k &= f_k(X_{k-1}, \xi_k, u),\end{aligned}\quad (2.192)$$

where

$$X_{k-1} = \begin{bmatrix} X_{k-2} \\ \xi_{k-1} \end{bmatrix}, \quad F_{k-1}(X_{k-1}, \xi_k) = \begin{bmatrix} F_{k-2}(X_{k-2}, \xi_{k-1}) \\ f_{k-1}(X_{k-2}, \xi_{k-1}, \xi_k) \end{bmatrix}. \quad (2.193)$$

For the system (2.192) we use the Lyapunov function

$$\begin{aligned}V_k(x, \xi_1, \dots, \xi_k) &= V_{k-1}(X_{k-1}) + \frac{1}{2} [\xi_k - \alpha_{k-1}(X_{k-1})]^2 \\ &= V(x) + \frac{1}{2} \sum_{i=1}^k [\xi_i - \alpha_{i-1}(X_{i-1})]^2.\end{aligned}\quad (2.194)$$

The design is completed by finding a control law

$$u = \alpha_k(x, \xi_1, \dots, \xi_k) \quad (2.195)$$

which makes  $\dot{V}_k \leq -W_k \leq 0$ , with  $W_k > 0$  when  $W_{k-1} > 0$  or  $\xi_k \neq \alpha_{k-1}$ :

$$\begin{aligned} \dot{V}_k &= \dot{V}_{k-1} + (\xi_k - \alpha_{k-1}) \left[ f_k - \frac{\partial \alpha_{k-1}}{\partial X_{k-1}} F_{k-1} \right] \\ &\leq -W_{k-1}(X_{k-2}, \xi_{k-1}) + \frac{\partial V_{k-1}}{\partial \xi_{k-1}} \bar{f}_{k-1}(\xi_k - \alpha_{k-1}) \\ &\quad + (\xi_k - \alpha_{k-1}) \left[ f_k(X_{k-1}, \xi_k, u) - \frac{\partial \alpha_{k-1}}{\partial X_{k-1}} F_{k-1} \right] \\ &= -W_k(X_{k-1}, \xi_k) \leq 0. \end{aligned} \quad (2.196)$$

Once again, under the condition

$$\frac{\partial f_k}{\partial u}(X_{k-1}, \xi_k, u) \neq 0, \quad \forall X_{k-1} \in \mathbb{R}^{n+k-1}, \quad \forall \xi_k \in \mathbb{R}, \quad \forall u \in \mathbb{R}, \quad (2.197)$$

we would be able to find  $u = \alpha_k$  to yield  $W_k = W_{k-1} + c_k [\xi_k - \alpha_{k-1}]^2$ . However, not only can (2.196) often be satisfied even if (2.197) is violated, but even when this condition is satisfied, we may prefer a different choice of  $W_k$ .

### 2.3.3 Block-strict-feedback systems

Lemma 2.25 can also be applied repeatedly to design controllers for nonlinear systems which can be transformed, by a change of coordinates, into the *block-strict-feedback form*:

$$\begin{aligned} \dot{x} &= f(x) + g(x)y_1 \\ \dot{\chi}_1 &= \bar{f}_1(x, \chi_1) + \bar{g}_1(x, \chi_1)y_2 \\ y_1 &= h_1(\chi_1) \\ \dot{\chi}_2 &= \bar{f}_2(x, \chi_1, \chi_2) + \bar{g}_2(x, \chi_1, \chi_2)y_3 \\ y_2 &= h_2(\chi_2) \\ &\vdots \\ \dot{\chi}_i &= \bar{f}_i(x, \chi_1, \dots, \chi_i) + \bar{g}_i(x, \chi_1, \dots, \chi_i)y_{i+1} \\ y_i &= h_i(\chi_i) \\ &\vdots \\ \dot{\chi}_{k-1} &= \bar{f}_{k-1}(x, \chi_1, \dots, \chi_{k-1}) + \bar{g}_{k-1}(x, \chi_1, \dots, \chi_{k-1})y_k \\ y_{k-1} &= h_{k-1}(\chi_{k-1}) \\ \dot{\chi}_k &= \bar{f}_k(x, \chi) + \bar{g}_k(x, \chi)u \\ y_k &= h_k(\chi_k), \end{aligned} \quad (2.198)$$

where each of the  $k$  subsystems with state  $\chi_i \in \mathbb{R}^{n_i}$ , output  $y_i \in \mathbb{R}$ , and input  $y_{i+1}$  (for convenience we denote  $y_{k+1} \equiv u$ ) satisfies the following conditions:

(BSF-1) its relative degree is one uniformly in  $x, \chi_1, \dots, \chi_{i-1}$ , and

(BSF-2) its zero dynamics subsystem is ISS with respect to  $x, \chi_1, \dots, \chi_{i-1}, y_i$ .

Under conditions (BSF-1) and (BSF-2), the system (2.198) can be transformed into a form reminiscent of the strict-feedback form (2.165). In particular, (BSF-1) is equivalent to

$$\frac{\partial h_i}{\partial \chi_i} \bar{g}_i \neq 0, \quad \forall \chi_1 \in \mathbb{R}^{n_1}, \dots, \forall \chi_i \in \mathbb{R}^{n_i}, \quad i = 1, \dots, k. \quad (2.199)$$

This means that for each  $\chi_i$ -subsystem in (2.198) there exists a global change of coordinates  $(y_i, \zeta_i) = (h_i(\chi_i), \phi_i(x, \chi_1, \dots, \chi_i))$ , with  $\frac{\partial \phi_i}{\partial \chi_i} \bar{g}_i \equiv 0$ , which transforms it into the normal form (2.149):

$$\begin{aligned} \dot{y}_i &= \frac{\partial h_i}{\partial \chi_i}(\chi_i) [\bar{f}_i(x, \chi_1, \dots, \chi_i) + \bar{g}_i(x, \chi_1, \dots, \chi_i)y_{i+1}] \\ &\triangleq f_i(x, y_1, \zeta_1, \dots, y_i, \zeta_i) + g_i(x, y_1, \zeta_1, \dots, y_i, \zeta_i)y_{i+1} \end{aligned} \quad (2.200a)$$

$$\begin{aligned} \dot{\zeta}_i &= \sum_{j=1}^{i-1} \frac{\partial \phi_i}{\partial \chi_j}(x, \chi_1, \dots, \chi_i) [\bar{f}_j(x, \chi_1, \dots, \chi_j) + \bar{g}_j(x, \chi_1, \dots, \chi_j)y_{j+1}] \\ &\quad + \frac{\partial \phi_i}{\partial \chi_i}(x, \chi_1, \dots, \chi_i) f_i(x, \chi_1, \dots, \chi_i) \\ &\triangleq \bar{\Phi}_i(x, \chi_1, \dots, \chi_{i-1}, y_i, \zeta_i) \\ &\triangleq \bar{\Phi}_i(x, y_1, \zeta_1, \dots, y_{i-1}, \zeta_{i-1}, y_i, \zeta_i). \end{aligned} \quad (2.200b)$$

With this change of coordinates, (2.198) is transformed into

$$\begin{aligned} \dot{x} &= f(x) + g(x)y_1 \\ \dot{y}_1 &= f_1(x, y_1, \zeta_1) + g_1(x, y_1, \zeta_1)y_2 \\ \dot{y}_2 &= f_2(x, y_1, \zeta_1, y_2, \zeta_2) + g_2(x, y_1, \zeta_1, y_2, \zeta_2)y_3 \\ &\vdots \\ \dot{y}_{k-1} &= f_{k-1}(x, y_1, \zeta_1, \dots, y_{k-1}, \zeta_{k-1}) + g_{k-1}(x, y_1, \zeta_1, \dots, y_{k-1}, \zeta_{k-1})y_k \\ \dot{y}_k &= f_k(x, y_1, \zeta_1, \dots, y_k, \zeta_k) + g_k(x, y_1, \zeta_1, \dots, y_k, \zeta_k)u \\ \dot{\zeta}_1 &= \bar{\Phi}_1(x, y_1, \zeta_1) \\ &\vdots \\ \dot{\zeta}_k &= \bar{\Phi}_k(x, y_1, \zeta_1, \dots, y_{k-1}, \zeta_{k-1}, y_k, \zeta_k). \end{aligned} \quad (2.201)$$

If the zero dynamics variables  $\zeta_1, \dots, \zeta_k$  were not present, (2.201) would be identical to the strict-feedback form (2.165) with  $\xi_i$  replaced by  $y_i$ . Hence, the design procedure of Section 2.3.1 can be applied *mutatis mutandis* to (2.201). The presence of  $\zeta_i$  generates a few new terms and requires a modification of the proof of boundedness as well. First the boundedness of  $x(t), y_1(t), \dots, y_k(t)$  and the regulation of  $W(x(t)), y_1(t) - \alpha(x(t)), \dots, y_k(t) - \alpha_{k-1}(x(t), y_1(t), \zeta_1(t), \dots, y_k(t), \zeta_k(t))$  is established via a Lyapunov-like argument using the function  $V_k(x, y_1, \dots, y_k) = V(x) + \frac{1}{2} \sum_{i=1}^k [y_i - \alpha_{i-1}(x, y_1, \zeta_1, \dots, y_{i-1}, \zeta_{i-1})]^2$ . This implies that  $y_1$  is bounded. Then, the boundedness of  $y_2, \dots, y_k, \zeta_1, \dots, \zeta_k$  and  $u$  (and, consequently, the boundedness of  $\chi_1, \dots, \chi_k$ ) is established via an induction argument for  $i = 2, \dots, k+1$  (with  $y_{k+1} \equiv u$ ): If  $y_1, \dots, y_{i-1}$  and  $\zeta_1, \dots, \zeta_{i-2}$  are bounded, the ISS stability assumption on the  $\zeta_{i-1}$ -subsystem guarantees that  $\zeta_{i-1}$  is bounded. This implies that  $\alpha_i(x, y_1, \zeta_1, \dots, y_{i-1}, \zeta_{i-1})$  is bounded, which implies that  $y_i$  is also bounded. This completes the induction argument and shows that  $\chi_1, \dots, \chi_k, u$  are bounded, since they can be expressed as smooth functions of  $x, y_1, \dots, y_k, \zeta_1, \dots, \zeta_k$ .

## 2.4 Design Flexibility: Jet Engine Example

We have presented several backstepping, cascade, and block-backstepping design tools and procedures which make the design of nonlinear controllers systematic. We stress, however, that ‘systematic’ means neither rigid nor dogmatic. Following the same principles, various modifications of the tools and procedures are possible. The recursive construction of control Lyapunov functions is flexible, and so is the choice of stabilizing functions.

At present there are no specific optimality criteria to help us select the best member of the backstepping controller family. However, there are certain applications-oriented guidelines which in most cases will lead to a simpler and more robust controller.

It is clear from the design procedures that the complexity of the controller increases with the number of recursive steps. Much can be gained if the number of steps can be reduced. It is also desirable to satisfy the  $\dot{V}$ -inequalities with as few cancellations as possible. Exact cancellations can rarely be implemented and cancellation errors may lead to nonrobustness. Additional analysis may be required to identify useful nonlinearities and avoid their cancellation. For this purpose, a more flexible construction of control Lyapunov functions can be employed. Some of these guidelines are now illustrated on a design example of major practical interest.<sup>9</sup>

<sup>9</sup>This section is based on Krstić and Kokotović [105].

### 2.4.1 Jet engine stall and surge

Jet engine compression systems (Figure 2.8) have recently become the subject of intensive control studies aimed at understanding and preventing two types of instability: *rotating stall* and *surge*. Rotating stall manifests itself as a region of severely reduced flow that rotates at a fraction of the rotor speed. Surge is an axisymmetric pumping oscillation which can cause flameout and engine damage.

The simplest model<sup>10</sup> that describes these instabilities is a three-state Galerkin approximation of the nonlinear PDE model by Moore and Greitzer [137]. This model exhibits bifurcations analyzed by McCaughan [128], and was used by Liaw and Abed [113] for a nonlinear feedback control design. Control designs with experimental verifications are reported in Paduano et al. [149] and Eveker and Nett [34].

We will design a feedback controller for the three-state model

$$\dot{\Phi} = -\Psi + \Psi_C(\Phi) - 3\Phi R \quad (2.202)$$

$$\dot{\Psi} = \frac{1}{\beta^2} (\Phi - \Phi_T) \quad (2.203)$$

$$\dot{R} = \sigma R (1 - \Phi^2 - R), \quad R(0) \geq 0, \quad (2.204)$$

where  $\Phi$  is the mass flow,  $\Psi$  is the pressure rise,  $R \geq 0$  is the normalized stall cell squared amplitude,  $\Phi_T$  is the mass flow through the throttle, and  $\sigma$  and  $\beta$  are constant positive parameters. The compressor and throttle characteristics,  $\Psi_C(\Phi)$  and  $\Phi_T(\Psi)$ , analyzed in [128], are:

$$\Psi_C(\Phi) = \Psi_{C0} + 1 + \frac{3}{2}\Phi - \frac{1}{2}\Phi^3 \quad (2.205)$$

$$\Psi = \frac{1}{\gamma} (1 + \Phi_T(\Psi))^2, \quad (2.206)$$

where  $\Psi_{C0}$  is a constant, and  $\gamma$  can be changed by varying the throttle opening. The characteristic  $\Psi_C(\Phi)$  has its maximum at  $\Phi = 1$ . The no-stall equilibria have  $R = 0$  and they cannot be stable for  $\Phi^2 < 1$ , as can be easily seen from (2.204). The pairs of no-stall equilibrium values  $\Phi^e, \Psi^e$  are given by the intersections of the characteristics  $\Psi_C(\Phi)$  and  $\Phi_T(\Psi)$  in (2.205) and (2.206).

Using the flow through the throttle  $\Phi_T$  as a control input, our objective is to stabilize the equilibrium  $R^e = 0, \Phi^e = 1, \Psi^e = \Psi_C(\Phi^e) = \Psi_{C0} + 2$ . We translate the origin to the desired equilibrium,  $\phi = \Phi - 1, \psi = \Psi - \Psi_{C0} - 2$ , and let the control variable be

$$u = \frac{1}{\beta^2} (\Phi_T - 1 - \phi). \quad (2.207)$$

<sup>10</sup>The control problem for this jet engine model has been brought to our attention by Carl Nett of United Technologies Research Center and Jim Paduano of MIT.

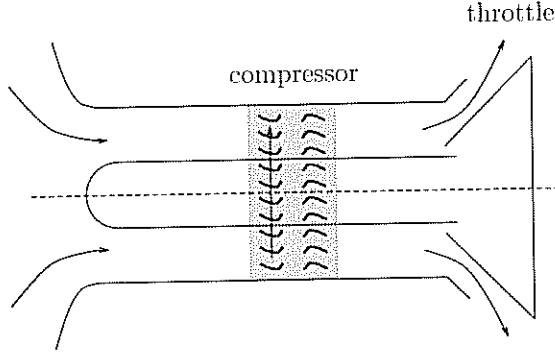


Figure 2.8: Compression system.

The model (2.202)–(2.204) is then rewritten as

$$\dot{R} = -\sigma R^2 - \sigma R(2\phi + \phi^2) \quad (2.208)$$

$$\dot{\phi} = -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \quad (2.209)$$

$$\dot{\psi} = -u. \quad (2.210)$$

These equations are reordered to reveal the similarity with the pure-feedback form (2.180). The only discrepancy is in the first equation which is not affine in the second variable  $\phi$ . Here is our chance to avoid being dogmatic. The only reason the  $\dot{x}$ -equation in (2.180) was assumed to be affine in  $\xi_1$  was to simplify the statement of the stabilizability condition (2.50) in Assumption 2.7. The stabilizability of the  $\dot{R}$ -equation (2.210) is obvious by inspection. A stabilizing virtual control is simply  $\phi = \alpha(R) = 0$  which yields  $\dot{R} = -\sigma R^2$ . Since physically  $R \geq 0$ , this means that  $R(t) \rightarrow 0$ . We can now use  $V_2 = \frac{1}{2}R^2 + \frac{1}{2}\phi^2$  and proceed to the second step of backstepping. This design, which is left to the reader, is completed in three steps.

## 2.4.2 A two-step design

When at the first step of a three-step design the stabilizing function  $\alpha_1$  is zero, this suggests that a simpler two-step design is possible. We therefore consider that the initial subsystem consists of the first two equations (2.208) and (2.209). This subsystem is cascaded with the integrator (2.210) and the whole system can be viewed as being in the cascade form (2.147) where  $(R, \phi)$  is  $x$  and  $\psi$  is  $\xi$ . Following Lemma 2.25, we need to satisfy Assumption 2.21 using  $\psi$  to stabilize the  $(R, \phi)$ -subsystem. This can be done using the semidefinite  $V_1 = \phi^2$  because the  $\dot{R}$ -equation is ISS as can be seen from (2.208):

$$\begin{aligned} \dot{R} &\leq -\sigma R^2 + 2\sigma R|\phi| - \sigma\phi^2 R \\ &\leq -\frac{\sigma}{2}R^2 - \frac{\sigma}{2}R(R - 4|\phi|), \end{aligned} \quad (2.211)$$

which implies that when  $R(t) > 4|\phi(t)|$ ,  $R(t)$  decays faster than the solution of  $\dot{w} = -\frac{\sigma}{2}w^2$ . Hence, an upper bound for  $R(t)$  is

$$R(t) \leq \frac{R(0)}{1 + R(0)\sigma t/2} + 4 \sup_{0 \leq \tau \leq t} |\phi(\tau)|. \quad (2.212)$$

Clearly,  $R$  is bounded if  $\phi$  is bounded, and  $R \rightarrow 0$  if  $\phi \rightarrow 0$ .

For  $\psi$  as the virtual control of the  $\dot{\phi}$ -equation we choose

$$\alpha(\phi, R) = c_1\phi - \frac{3}{2}\phi^2 - 3R. \quad (2.213)$$

With this choice we have avoided cancellation of the useful nonlinearities  $-\frac{1}{2}\phi^3$  and  $-3R\phi$ . Substituting  $\tilde{\psi} = \psi - \alpha(\phi, R)$  in (2.209), we get

$$\dot{\phi} = -c_1\phi - \frac{1}{2}\phi^3 - 3R\phi - \tilde{\psi}. \quad (2.214)$$

At the second step we differentiate  $\tilde{\psi} = \psi - \alpha(\phi, R) = -u - \frac{\partial \alpha}{\partial \phi}\dot{\phi} - \frac{\partial \alpha}{\partial R}\dot{R}$  and obtain

$$\dot{\tilde{\psi}} = -u - (c_1 - 3\phi) \left( -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3\phi R - 3R \right) + 3\sigma R(-2\phi - \phi^2 - R). \quad (2.215)$$

The control law is now chosen to render the derivative of  $V_2 = \phi^2 + \tilde{\psi}^2$  negative definite:

$$\begin{aligned} u &= c_2\tilde{\psi} - \phi - (c_1 - 3\phi) \left( -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3\phi R - 3R \right) \\ &\quad + 3\sigma R(-2\phi - \phi^2 - R). \end{aligned} \quad (2.216)$$

Note that in the  $(R, \phi, \tilde{\psi})$ -space the Lyapunov function  $V_2$  is only positive semidefinite, which is allowed by Assumption 2.21 and Lemma 2.25. In contrast, for the three-step design outlined in the preceding subsection, the final ctf would be  $V_2 = R^2 + \phi^2 + \tilde{\psi}^2$  and its derivative would involve terms from the  $\dot{R}$ -equation. This would make the control law more complicated than (2.216).

With the control (2.216) the resulting feedback system is

$$\begin{aligned} \dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} &= -\left(c_1 + \frac{1}{2}\phi^2 + 3R\right)\phi - \tilde{\psi} \\ \dot{\tilde{\psi}} &= \phi - c_2\tilde{\psi}. \end{aligned} \quad (2.217)$$

The equilibrium  $(\phi, \tilde{\psi}) = 0$  of the  $(\phi, \tilde{\psi})$ -subsystem is GAS for all  $R \geq 0$ . In addition,  $R(t) \rightarrow 0$  because of the ISS-property (2.212). This means that surge and stall are suppressed within the region of validity of this jet engine model.

### 2.4.3 Avoiding cancellations

Although we have already avoided cancellation of several useful nonlinearities, a further systematic simplification of the above controller is possible by a better choice of  $\alpha$  and a more flexible construction of the control Lyapunov function  $V_2$ . We illustrate this possibility on the no-stall ( $R = 0$ ) part of the jet engine model (2.208)–(2.210), rewritten here as

$$\dot{\phi} = -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \quad (2.218)$$

$$\dot{\psi} = -u. \quad (2.219)$$

To design a stabilizing function  $\alpha(\phi)$  for  $\psi$  in (2.218) with respect to  $V_1(\phi) = \frac{1}{2}\phi^2$ , we examine the inequality  $\dot{V}_1 < 0$ , that is,

$$\phi \left[ -\alpha(\phi) - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \right] < 0, \quad \forall \phi \neq 0. \quad (2.220)$$

We already noticed that  $-\frac{1}{2}\phi^3$  enhances this inequality and we did not cancel it in (2.213). But if we go further and rewrite (2.220) as

$$\phi \left[ -\alpha_1(\phi) - \frac{1}{2} \left( \phi + \frac{3}{2} \right)^2 \phi + \frac{9}{8}\phi \right] < 0, \quad \forall \phi \neq 0, \quad (2.221)$$

we recognize that  $-\frac{1}{2} \left( \phi + \frac{3}{2} \right)^2 \phi$  is also useful and should not be cancelled. Therefore,  $\alpha(\phi)$  is chosen to be linear

$$\alpha(\phi) = \left( c_1 + \frac{9}{8} \right) \phi, \quad c_1 > 0. \quad (2.222)$$

The derivative of  $V_1(\phi) = \frac{1}{2}\phi^2$  is then

$$\begin{aligned} \dot{V}_1 &= -c_1\phi^2 - \frac{1}{2} \left( \phi + \frac{3}{2} \right)^2 \phi^2 - \phi\tilde{\psi} \\ &\leq -c_1\phi^2 - \phi\tilde{\psi}, \end{aligned} \quad (2.223)$$

where  $\tilde{\psi} = \psi - \alpha(\phi)$ .

In the second step we denote  $c_0 = c_1 + \frac{9}{8}$ , and by differentiating  $\tilde{\psi} = \psi - c_0\phi$  we get

$$\dot{\tilde{\psi}} = -u + c_0 \left( \psi + \frac{3}{2}\phi^2 + \frac{1}{2}\phi^3 \right). \quad (2.224)$$

Proceeding as usual, our choice for the Lyapunov function  $V_2$  would be quadratic,  $V_2(\phi, \psi) = V_1(\phi) + \frac{1}{2}\tilde{\psi}^2 = \frac{1}{2}\phi^2 + \frac{1}{2}\tilde{\psi}^2$ , resulting in  $\dot{V}_2 \leq -c_1\phi^2 +$

$\tilde{\psi} \left[ -u + c_0 \left( \psi + \frac{3}{2}\phi^2 + \frac{1}{2}\phi^3 \right) - \phi \right]$ . To satisfy the inequality  $\dot{V}_2 < 0$ , the control law would have to cancel the nonlinearities  $c_0\frac{3}{2}\phi^2$  and  $c_0\frac{1}{2}\phi^3$ . We will now illustrate the construction of a more flexible Lyapunov function

$$V_2 = V_1 + F(V_1) + \frac{1}{2}\tilde{\psi}^2, \quad (2.225)$$

where  $F(\cdot)$  is yet to be selected as a continuously differentiable, nonnegative, and increasing function,  $\frac{dF(V_1)}{dV_1} \geq 0$ . In view of (2.223) and (2.224), the derivative of (2.225) satisfies

$$\begin{aligned} \dot{V}_2 &\leq -c_1\phi^2 - \phi\tilde{\psi} + \frac{dF(V_1)}{dV_1} \left( -c_1\phi^2 - \phi\tilde{\psi} \right) \\ &\quad + \tilde{\psi} \left( -u + c_0\psi + \frac{3c_0}{2}\phi^2 + \frac{c_0}{2}\phi^3 \right). \end{aligned} \quad (2.226)$$

After collecting all the terms with  $\tilde{\psi}$ , we get

$$\begin{aligned} \dot{V}_2 &\leq -c_1\phi^2 - c_1\phi^2 \frac{dF(V_1)}{dV_1} \\ &\quad + \tilde{\psi} \left( -u - \phi - \frac{dF(V_1)}{dV_1} \phi + c_0\psi + \frac{3c_0}{2}\phi^2 + \frac{c_0}{2}\phi^3 \right). \end{aligned} \quad (2.227)$$

In addition to the choice of a control law for  $u$ , we now have the freedom to choose  $\frac{dF(V_1)}{dV_1}$ . With this choice we will avoid the cancellation of the cubic term  $\frac{c_0}{2}\phi^3$  by  $u$ . We simply select  $\frac{dF(V_1)}{dV_1}$  to eliminate  $\frac{c_0}{2}\phi^3$ :

$$\frac{dF(V_1)}{dV_1} = \frac{c_0}{2}\phi^2 = c_0V_1. \quad (2.228)$$

This yields

$$F(V_1) = \frac{c_0}{2}V_1^2 = \frac{c_0}{8}\phi^4, \quad (2.229)$$

and the resulting Lyapunov function (2.225) is nonquadratic:

$$V_2(\phi, \psi) = \frac{1}{2}\phi^2 + \frac{c_0}{8}\phi^4 + \frac{1}{2}(\psi - c_0\phi)^2. \quad (2.230)$$

Substituting (2.228) into (2.227), we arrive at

$$\dot{V}_2 \leq -c_1\phi^2 - c_1\frac{c_0}{2}\phi^4 + \tilde{\psi} \left( -u - \phi + c_0\psi + \frac{3c_0}{2}\phi^2 \right). \quad (2.231)$$

The design could now be finished by selecting a control  $u$  which cancels  $\frac{3c_0}{2}\phi^2$ . However, even this cancellation can be avoided because of the strong stabilizing term  $c_1\frac{c_0}{2}\phi^4$  in (2.231). Completing squares,  $\frac{3c_0}{2}\phi^2\tilde{\psi} \leq c_1\frac{c_0}{2}\phi^4 + \frac{9c_0}{8c_1}\tilde{\psi}^2$ , we get

$$\dot{V}_2 \leq -c_1\phi^2 + \tilde{\psi} \left( -u - \phi + c_0\psi + \frac{9c_0}{8c_1}\tilde{\psi} \right). \quad (2.232)$$

Hence our control law can be selected to be linear,

$$u = \phi + c_0\psi + \left(c_2 + \frac{9c_0}{8c_1}\right)\bar{\psi}, \quad c_2 > 0, \quad (2.233)$$

and yield

$$\dot{V}_2 \leq -c_1\phi^2 - c_2\bar{\psi}^2. \quad (2.234)$$

This proves that the equilibrium  $\phi = 0, \psi = 0$  is globally asymptotically stable. Denoting

$$k_1 = 1 + c_2c_0 + \frac{9c_0^2}{8c_1}, \quad k_2 = c_2 + c_0 + \frac{9c_0}{8c_1}, \quad (2.235)$$

we rewrite (2.233) in the more compact form

$$u = -k_1\phi + k_2\psi \quad (2.236)$$

and obtain the closed-loop system

$$\dot{\phi} = -\frac{1}{2}\phi^3 - \frac{3}{2}\phi^2 - \psi \quad (2.237)$$

$$\dot{\psi} = k_1\phi - k_2\psi. \quad (2.238)$$

For comparison, we also derive a feedback linearizing controller,

$$u = k_1\phi + \left(k_2 - 3\phi - \frac{3}{2}\phi^2\right)\left(\psi + \frac{3}{2}\phi^2 + \frac{1}{2}\phi^3\right), \quad (2.239)$$

which makes the system (2.218), (2.219) appear linear in the coordinates  $\chi_1 = \phi$  and  $\chi_2 = \dot{\phi}$ . The controller simplification achieved with backstepping is impressive: While the linearizing control (2.239) grows as  $\phi^5$  and  $\psi\phi^2$ , the backstepping controller (2.236) is linear. The improvement over the control law (2.216) in Section 2.4 is also significant: (2.216) grows as  $\phi^4$  and  $\psi\phi$  because the quadratic nonlinearity was cancelled at the first step, so the cancellation could not be avoided at the second step.

In the remainder of the book we will not assume the presence of useful nonlinearities. However, it should always be understood that whenever such additional information is available, backstepping designs should incorporate it.

## 2.5 Stabilization with Uncertainty

The full power of backstepping is exhibited in the presence of uncertain nonlinearities and unknown parameters, because for such applications no other systematic design procedure exists. We now begin the study of such design problems which are the main subject of this book. The first of the design tools that will be used to counteract uncertainty is *nonlinear damping*.

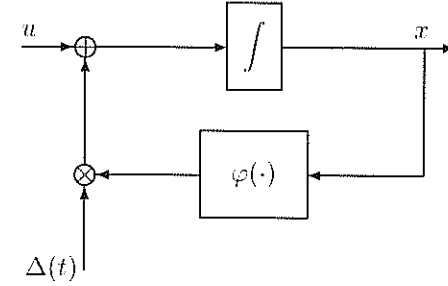


Figure 2.9: A system with “matched” uncertainty  $\Delta(t)$ .

### 2.5.1 Nonlinear damping

We introduce nonlinear damping for systems with “matched” uncertainty, in which both the uncertainty and the control appear in the same equation. The simplest example is the scalar nonlinear system depicted in Figure 2.9:

$$\dot{x} = u + \varphi(x)\Delta(t), \quad (2.240)$$

where  $\varphi(x)$  is a known smooth nonlinearity, and  $\Delta(t)$  is a bounded function of  $t$ . Let us first examine the case when  $\Delta(t)$  is an exponentially decaying disturbance:

$$\Delta(t) = \Delta(0)e^{-kt}. \quad (2.241)$$

Can such an innocent-looking uncertainty cause harm? One might be tempted to ignore it and use the linear control  $u = -cx$ , which results in the closed-loop system

$$\dot{x} = -cx + \varphi(x)\Delta(0)e^{-kt}. \quad (2.242)$$

While this design may be satisfactory when  $\varphi(x)$  is bounded by a constant or a linear function of  $x$ , it is inadequate if  $\varphi(x)$  is allowed to be any smooth nonlinear function. For example, when  $\varphi(x) = x^2$  we have

$$\dot{x} = -cx + x^2\Delta(0)e^{-kt}. \quad (2.243)$$

As we saw in Chapter 1, equations (1.29)–(1.32), the solution  $x(t)$  of this system can be calculated explicitly using the change of variable  $w = 1/x$ :

$$\dot{w} = -\frac{1}{x^2}\dot{x} = c\frac{1}{x} - \Delta(0)e^{-kt} = cw - \Delta(0)e^{-kt}, \quad (2.244)$$

which yields

$$w(t) = \left[w(0) - \frac{\Delta(0)}{c+k}\right]e^{ct} + \frac{\Delta(0)}{c+k}e^{-kt}. \quad (2.245)$$

The substitution  $w = 1/x$  gives

$$x(t) = \frac{x(0)(c+k)}{[c+k - \Delta(0)x(0)]e^{ct} + \Delta(0)x(0)e^{-kt}}. \quad (2.246)$$

From (2.246) we see that the behavior of the closed-loop system (2.243) depends critically on the initial conditions  $\Delta(0), x(0)$ :

- (i) If  $\Delta(0)x(0) < c + k$ , the solutions  $x(t)$  are bounded and converge asymptotically to zero.
- (ii) The situation changes dramatically when  $\Delta(0)x(0) > c + k > 0$ . The solutions  $x(t)$  which start from such initial conditions not only diverge to infinity, but do so in *finite time*:

$$x(t) \rightarrow \infty \text{ as } t \rightarrow t_f = \frac{1}{c+k} \ln \left\{ \frac{\Delta(0)x(0)}{\Delta(0)x(0) - (c+k)} \right\}. \quad (2.247)$$

Note that this finite escape cannot be eliminated by making  $c$  larger: For any values of  $c$  and  $k$  and for any nonzero value of  $\Delta(0)$  there exist initial conditions  $x(0)$  which satisfy the inequality  $\Delta(0)x(0) > c + k$ . This example shows that in a nonlinear system, neglecting the effects of exponentially decaying disturbances or nonzero initial conditions can be catastrophic.

To overcome this problem and guarantee that  $x(t)$  will remain bounded for all bounded  $\Delta(t)$  and for all  $x(0)$ , we augment the control law  $u = -cx$  with a *nonlinear damping term*  $-s(x)x$ :

$$u = -cx - s(x)x. \quad (2.248)$$

We design  $s(x)$  for the system (2.240), using the quadratic function  $V(x) = \frac{1}{2}x^2$  whose derivative is

$$\begin{aligned} \dot{V} &= xu + x\varphi(x)\Delta(t) \\ &= -cx^2 - x^2s(x) + x\varphi(x)\Delta(t). \end{aligned} \quad (2.249)$$

The objective of guaranteeing global boundedness of solutions can be equivalently expressed as rendering  $\dot{V}$  negative outside a compact region. This is achieved with the choice

$$s(x) = \kappa x^2(x), \quad \kappa > 0, \quad (2.250)$$

which yields the control

$$u = -cx - \kappa x \varphi^2(x) \quad (2.251)$$

and the derivative

$$\begin{aligned} \dot{V} &= -cx^2 - \kappa x^2 \varphi^2(x) + x\varphi(x)\Delta(t) \\ &= -cx^2 - \kappa \left[ x\varphi(x) - \frac{\Delta(t)}{2\kappa} \right]^2 + \frac{\Delta^2(t)}{4\kappa} \\ &\leq -cx^2 + \frac{\Delta^2(t)}{4\kappa}. \end{aligned} \quad (2.252)$$

Comparing (2.249) with (2.252) we see that the nonlinear damping term (2.250) is chosen to allow the completion of squares in (2.252). In more complicated situations we can use *Young's Inequality*, which, in a simplified form, states that if the constants  $p > 1$  and  $q > 1$  are such that  $(p-1)(q-1) = 1$ , then for all  $\varepsilon > 0$  and all  $(x, y) \in \mathbb{R}^2$  we have

$$xy \leq \frac{\varepsilon^p}{p} |x|^p + \frac{1}{q\varepsilon^q} |y|^q. \quad (2.253)$$

Choosing  $p = q = 2$  and  $\varepsilon^2 = 2\kappa$ , (2.253) becomes

$$xy \leq \kappa x^2 + \frac{1}{4\kappa} y^2, \quad (2.254)$$

which is the inequality used in (2.252):

$$x\varphi(x)\Delta(t) \leq \kappa x^2 \varphi^2(x) + \frac{\Delta^2(t)}{4\kappa}. \quad (2.255)$$

**Global boundedness and convergence.** Returning to (2.252), we see that  $\dot{V}$  is negative whenever  $|x(t)| \geq \frac{\Delta(t)}{2\sqrt{\kappa c}}$ . Since  $\Delta(t)$  is a bounded disturbance, we conclude that  $\dot{V}$  is negative outside the compact residual set

$$\mathcal{R} = \left\{ x : |x| \leq \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}} \right\}. \quad (2.256)$$

Recalling that  $V(x) = \frac{1}{2}x^2$ , we conclude that  $|x(t)|$  decreases whenever  $x(t)$  is outside the set  $\mathcal{R}$ , and hence  $x(t)$  is bounded:

$$\|x\|_\infty \leq \max \left\{ |x(0)|, \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}} \right\}. \quad (2.257)$$

Moreover, we can draw some conclusions about the asymptotic behavior of  $x(t)$ . Let us rewrite (2.252) as

$$\frac{d}{dt} \left( \frac{1}{2} x^2 \right) \leq -cx^2 + \frac{\Delta^2(t)}{4\kappa}. \quad (2.258)$$

To obtain explicit bounds on  $x(t)$ , we consider the signal  $x(t)e^{ct}$ . Using (2.258) we get

$$\begin{aligned} \frac{d}{dt} (x^2 e^{2ct}) &= \frac{d}{dt} (x^2) e^{2ct} + 2cx^2 e^{2ct} \\ &\leq -2cx^2 e^{2ct} + \frac{\Delta^2(t)}{2\kappa} e^{2ct} + 2cx^2 e^{2ct} \\ &= \frac{\Delta^2(t)}{2\kappa} e^{2ct}. \end{aligned} \quad (2.259)$$

Integrating both sides over the interval  $[0, t]$  yields

$$\begin{aligned} x^2(t)e^{2ct} &\leq x^2(0) + \int_0^t \frac{1}{2\kappa} \Delta^2(\tau) e^{2c\tau} d\tau \\ &\leq x^2(0) + \frac{1}{2\kappa} \left[ \sup_{0 \leq \tau \leq t} \Delta^2(\tau) \right] \int_0^t e^{2c\tau} d\tau \\ &= x^2(0) + \frac{1}{4\kappa c} \left[ \sup_{0 \leq \tau \leq t} \Delta^2(\tau) \right] (e^{2ct} - 1). \end{aligned} \quad (2.260)$$

Multiplying both sides with  $e^{-2ct}$  and using the fact that  $\sqrt{b^2 + c^2} \leq |b| + |c|$ , we obtain an explicit bound for  $x(t)$ :

$$\begin{aligned} |x(t)| &\leq |x(0)|e^{-ct} + \frac{1}{2\sqrt{\kappa c}} \left[ \sup_{0 \leq \tau \leq t} |\Delta(\tau)| \right] (1 - e^{-2ct})^{\frac{1}{2}} \\ &\leq |x(0)|e^{-ct} + \frac{1}{2\sqrt{\kappa c}} \left[ \sup_{0 \leq \tau \leq t} |\Delta(\tau)| \right]. \end{aligned} \quad (2.261)$$

Since  $\sup_{0 \leq \tau \leq t} |\Delta(\tau)| \leq \sup_{0 \leq \tau < \infty} |\Delta(\tau)| \triangleq \|\Delta\|_\infty$ , (2.261) leads to

$$|x(t)| \leq |x(0)|e^{-ct} + \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}}, \quad (2.262)$$

which shows that  $x(t)$  converges to the compact set  $\mathcal{R}$  defined in (2.256):

$$\lim_{t \rightarrow \infty} \text{dist} \{x(t), \mathcal{R}\} = 0. \quad (2.263)$$

We reiterate that these properties of boundedness (cf. (2.257)) and convergence (cf. (2.263)) are guaranteed for any bounded disturbance  $\Delta(t)$  and for any smooth nonlinearity  $\varphi(x)$ , including  $\varphi(x) = x^2$ . Furthermore, the nonlinear control law (2.251) does not assume knowledge of a bound on the disturbance, nor does it have to use large values for the gains  $\kappa$  and  $c$ . Indeed, the residual set  $\mathcal{R}$  defined in (2.256) is compact for any finite value of  $\|\Delta\|_\infty$  and for any positive values of  $\kappa$  and  $c$ . Hence, *global boundedness is guaranteed in the presence of bounded disturbances with unknown bounds, regardless of how small the gains  $\kappa$  and  $c$  are chosen*. While the size of  $\mathcal{R}$  cannot be estimated *a priori* if no bound for  $\|\Delta\|_\infty$  is given, it can be reduced *a posteriori* by increasing the values of  $\kappa$  and  $c$ .

This property is achieved by the “nonlinear damping” term  $-\kappa x \varphi^2(x)$  in (2.251), which renders the effective gain of (2.251) “selectively high.” When  $\kappa$  and  $c$  are chosen to be small, the gain is low around the origin, but it becomes high when  $x$  is in a region where  $\varphi(x)$  is large enough to make the term  $\kappa \varphi^2(x)$  large. If we interpret the nonlinearity  $\varphi(x)$ , which multiplies the disturbance  $\Delta(t)$ , as the “disturbance gain,” we see that the term  $-\kappa \varphi^2(x)$  causes the control gain to become large when the disturbance gain is large.

Finally, we should note that, if the disturbance  $\Delta(t)$  converges to zero in addition to being bounded, then the control (2.251) guarantees convergence of  $x(t)$  to zero in addition to global boundedness. To show this, let  $\bar{\Delta}(t)$  be a continuous nonnegative *monotonically decreasing* function such that  $|\Delta(t)| \leq \bar{\Delta}(t)$  and  $\lim_{t \rightarrow \infty} \bar{\Delta}(t) = 0$ . Then, starting with the first inequality from (2.260) and using an argument almost identical to the proof of Lemma 2.24, we obtain

$$|x(t)| \leq |x(0)|e^{-ct} + \frac{1}{2\sqrt{\kappa c}} \left( \bar{\Delta}(0)e^{-\frac{c}{2}t} + \bar{\Delta}(t/2) \right). \quad (2.264)$$

Since  $\lim_{t \rightarrow \infty} \bar{\Delta}(t/2) = 0$ , we see that  $\lim_{t \rightarrow \infty} x(t) = 0$ .

**ISS interpretation.** For interpreting the effect of the nonlinear damping term  $-\kappa x \varphi^2(x)$  in (2.251) from an input-output point of view, it is very convenient to use the concept of input-to-state stability: (cf. Appendix C) This  $\kappa$ -term renders the closed-loop system ISS with respect to the disturbance input  $\Delta(t)$ . To show that the ISS inequality (2.12) holds for our closed-loop system with  $u(\tau)$  replaced by the disturbance  $\Delta(\tau)$ , we repeat the argument that led from (2.259) to (2.261), this time integrating over the interval  $[t_0, t]$ . The result is

$$|x(t)| \leq |x(t_0)|e^{-c(t-t_0)} + \frac{1}{2\sqrt{\kappa c}} \left[ \sup_{t_0 \leq \tau \leq t} |\Delta(\tau)| \right], \quad (2.265)$$

which is identical to (2.12) with  $\beta(r, s) = re^{-cs}$ ,  $\gamma(r) = \frac{1}{2\sqrt{\kappa c}}r$ ,  $r = |x(t_0)|$  and  $s = t - t_0$ .

**Operator gain interpretation.** It is also convenient to interpret the effect of nonlinear damping from an operator point of view on the basis of (2.257) and Figure 2.10. For all initial conditions  $x(0)$  such that  $|x(0)| < \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}}$ , we obtain

$$\|x\|_\infty \leq \frac{1}{2\sqrt{\kappa c}} \|\Delta\|_\infty, \quad (2.266)$$

which shows that the nonlinear operator  $K$  mapping the disturbance  $\Delta(t)$  to the output  $x(t)$  is bounded, and its  $\mathcal{L}_\infty$ -induced gain is

$$\|K\|_{\infty \text{ ind}} \leq \frac{1}{2\sqrt{\kappa c}}. \quad (2.267)$$

The nonlinear damping term renders the operator  $K$  bounded for *any* positive values of  $\kappa$  and  $c$ . Note, however, that (2.266) does not provide a complete description of this operator because, unlike (2.257), it hides the effect of initial conditions, which can be quite dangerous for nonlinear systems.

The following lemma recapitulates the properties achieved with nonlinear damping as a design tool.

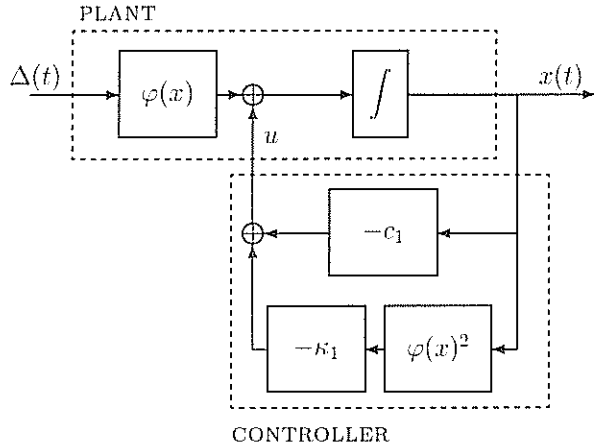


Figure 2.10: The bounded nonlinear operator  $K: \Delta(t) \rightarrow x(t)$ .

**Lemma 2.26 (Nonlinear Damping)** *Let the system (2.48) be perturbed as in*

$$\dot{x} = f(x) + g(x) [u + \varphi(x)^T \Delta(x, u, t)], \quad (2.268)$$

where  $\varphi(x)$  is a  $(p \times 1)$  vector of known smooth nonlinear functions, and  $\Delta(x, u, t)$  is a  $(p \times 1)$  vector of uncertain nonlinearities which are uniformly bounded for all values of  $x, u, t$ .

If Assumption 2.7 is satisfied with  $W(x)$  positive definite and radially unbounded, then the control

$$u = \alpha(x) - \kappa \frac{\partial V}{\partial x}(x) g(x) |\varphi(x)|^2, \quad \kappa > 0, \quad (2.269)$$

when applied to (2.268), renders the closed-loop system ISS with respect to the disturbance input  $\Delta(x, u, t)$  and hence guarantees global uniform boundedness of  $x(t)$  and convergence to the residual set

$$\mathcal{R} = \left\{ x : |x| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right) \right\}, \quad (2.270)$$

where  $\gamma_1, \gamma_2, \gamma_3$  are class- $\mathcal{K}_\infty$  functions such that<sup>11</sup>

$$\gamma_1(|x|) \leq V(x) \leq \gamma_2(|x|) \quad (2.271)$$

$$\gamma_3(|x|) \leq W(x). \quad (2.272)$$

<sup>11</sup>Since  $V(x)$  and  $W(x)$  are positive definite and radially unbounded and  $V(x)$  is smooth, there exist class- $\mathcal{K}_\infty$  functions  $\gamma_1, \gamma_2, \gamma_3$  satisfying (2.271) and (2.272).

**Proof.** The derivative of  $V(x)$  is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} [f + gu] + \frac{\partial V}{\partial x} g \varphi^T \Delta \\ \text{by (2.269)} &= \frac{\partial V}{\partial x} [f + g\alpha] - \kappa \left( \frac{\partial V}{\partial x} g \right)^2 |\varphi|^2 + \frac{\partial V}{\partial x} g \varphi^T \Delta \\ \text{by (2.50)} &\leq -W(x) - \kappa \left( \frac{\partial V}{\partial x} g \right)^2 |\varphi|^2 + \frac{\partial V}{\partial x} g \varphi^T \Delta \\ &\leq -W(x) - \kappa \left( \frac{\partial V}{\partial x} g \right)^2 |\varphi|^2 + \left| \frac{\partial V}{\partial x} g \right| |\varphi| \|\Delta\|_\infty \\ \text{by (2.254)} &\leq -W(x) + \frac{\|\Delta\|_\infty^2}{4\kappa}. \end{aligned} \quad (2.273)$$

From (2.273) it follows that  $\dot{V}$  is negative whenever  $W(x) > \frac{\|\Delta\|_\infty^2}{4\kappa}$ . Combining this with (2.272) we conclude that

$$|x(t)| > \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right) \Rightarrow \dot{V} < 0. \quad (2.274)$$

This means that if  $|x(0)| \leq \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right)$ , then

$$V(x(t)) \leq \gamma_2 \circ \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right), \quad (2.275)$$

which in turn implies that

$$|x(t)| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right). \quad (2.276)$$

If, on the other hand,  $|x(0)| > \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right)$ , then  $V(x(t)) \leq V(x(0))$ , which implies

$$|x(t)| \leq \gamma_1^{-1} \circ \gamma_2 (|x(0)|). \quad (2.277)$$

Combining (2.276) and (2.277) leads to the global uniform boundedness of  $x(t)$ :

$$\|x\|_\infty \leq \max \left\{ \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1} \left( \frac{\|\Delta\|_\infty^2}{4\kappa} \right), \gamma_1^{-1} \circ \gamma_2 (|x(0)|) \right\}, \quad (2.278)$$

while (2.274) and (2.271) prove the convergence of  $x(t)$  to the residual set defined in (2.270). Finally, the ISS property of the closed-loop system with respect to the disturbance input  $\Delta(x, u, t)$  follows from Theorem C.2.  $\square$

## 2.5.2 Backstepping with uncertainty

Lemma 2.26 deals with the case where the uncertainty is in the span of the control  $u$ , i.e., the *matching condition* is satisfied. Combining Lemma 2.26 with Lemma 2.8 allows us to go beyond the matching case, as the following example illustrates.

**Example 2.27** Consider the system

$$\dot{x} = \xi + x^2 \arctan \xi \Delta_0(t) \quad (2.279a)$$

$$\dot{\xi} = (1 + \xi^2)u + e^{x\xi} \Delta_0(t), \quad (2.279b)$$

where  $\Delta_0(t)$  is a bounded time-varying disturbance. Clearly, the uncertain terms in (2.279) are not in the span of the control  $u$ . Therefore, we will design a static nonlinear controller in two steps, combining nonlinear damping and backstepping.

**Step 1.** The starting point is equation (2.279a) and the choice of a virtual control variable. Clearly,  $\xi$  is the only choice. The fact that  $\xi$  is also present in the uncertain term does not present a problem, since it enters that term through the bounded function  $\arctan(\cdot)$ . In the notation of (2.268), we have

$$x^2 \arctan \xi \Delta_0(t) \triangleq x^2 \Delta_1(\xi, t) = \varphi_1(x) \Delta_1(\xi, t). \quad (2.280)$$

The uncertain nonlinearity  $\Delta_1(\xi, t)$  is bounded:

$$\|\Delta_1(\xi, t)\|_\infty = \|\Delta_0 \arctan \xi\|_\infty \leq \frac{\pi}{2} \|\Delta_0\|_\infty. \quad (2.281)$$

Hence, Lemma 2.26 can be used to design a stabilizing function for  $\xi$ . The unperturbed system in this case would be the integrator  $\dot{x} = \xi$ , for which a clf is given by  $V(x) = \frac{1}{2}x^2$  and the corresponding control is  $\alpha(x) = -c_1x$ . From (2.269) we have

$$\alpha_1(x) = -c_1x - \kappa_1 x \varphi_1^2(x), \quad (2.282)$$

which results in

$$\dot{x} = -c_1x + z - \kappa_1 x \varphi_1^2(x) + \varphi_1(x) \Delta_1(\xi, t), \quad (2.283)$$

with the error variable  $z$  defined as in Lemma 2.8:

$$z = \xi - \alpha_1(x). \quad (2.284)$$

The derivative of  $V(x)$  along (2.283) is

$$\begin{aligned} \dot{V} &= zx - c_1x^2 - \kappa_1x^2\varphi_1^2 + x^3 \arctan \xi \Delta_0(t) \\ \text{by (2.280)} &\leq zx - c_1x^2 - \kappa_1x^2\varphi_1^2 + |x\varphi_1(x)|\|\Delta_1\|_\infty \\ \text{by (2.254)} &= zx - c_1x^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1}, \end{aligned} \quad (2.285)$$

which confirms that if  $z \equiv 0$ , that is, if  $\xi$  were the actual control, then (2.282) would guarantee global uniform boundedness of  $x$ .

**Step 2.** Using the error variable  $z$  from (2.284), the system (2.279) is rewritten as

$$\dot{x} = -c_1x + z - \kappa_1x\varphi_1^2(x) + \varphi_1(x)\Delta_1(\xi, t) \quad (2.286a)$$

$$\begin{aligned} \dot{z} &= (1 + \xi^2)u + e^{x\xi}\Delta_0(t) - \frac{\partial\alpha_1}{\partial x} \left[ \xi + x^2 \arctan \xi \Delta_0(t) \right] \\ &= (1 + \xi^2)u - \frac{\partial\alpha_1}{\partial x} \xi + \left[ e^{x\xi} - \frac{\partial\alpha_1}{\partial x} x^2 \arctan \xi \right] \Delta_0(t), \end{aligned} \quad (2.286b)$$

where the partial  $\frac{\partial\alpha_1}{\partial x}$  is computed from (2.280) and (2.282):

$$\frac{\partial\alpha_1}{\partial x} = -c_1 - \kappa_1 \frac{\partial}{\partial x} [x\varphi_1^2(x)] = -c_1 - 5\kappa_1x^4. \quad (2.287)$$

If the  $\Delta_0(t)$ -term were not present in (2.286b), then Lemma 2.8 would dictate the Lyapunov function

$$V_2(x, \xi) = \frac{1}{2}x^2 + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}[\xi - \alpha_1(x)]^2 \quad (2.288)$$

and the following choice of control:

$$u = \bar{\alpha}(x, \xi) = \frac{1}{1 + \xi^2} \left[ -c_2z + \frac{\partial\alpha_1}{\partial x} \xi - x \right]. \quad (2.289)$$

To compensate for the presence of the  $\Delta_0(t)$ -term in (2.286b), Lemma 2.26 is used again. From (2.269) we obtain

$$u = \frac{1}{1 + \xi^2} \left\{ -c_2z + \frac{\partial\alpha_1}{\partial x} \xi - x - \kappa_2z \left[ e^{x\xi} - \frac{\partial\alpha_1}{\partial x} x^2 \arctan \xi \right]^2 \right\}, \quad (2.290)$$

which renders the derivative of  $V_2(x, \xi)$  negative outside a compact set, thus guaranteeing boundedness of  $x(t)$  and  $\xi(t)$ :

$$\begin{aligned} \dot{V}_2 &= \dot{V} + z\dot{z} \\ \text{by (2.285)} &\leq zx - c_1x^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} + z\dot{z} \\ \text{by (2.286b) and (2.290)} &= -c_1x^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} \\ &\quad + z \left\{ -c_2z - \kappa_2z \left[ e^{x\xi} - \frac{\partial\alpha_1}{\partial x} x^2 \arctan \xi \right]^2 \right. \\ &\quad \left. + \left[ e^{x\xi} - \frac{\partial\alpha_1}{\partial x} x^2 \arctan \xi \right] \Delta_0(t) \right\} \end{aligned}$$

$$\begin{aligned}
&\leq -c_1 x^2 - c_2 z^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} \\
&\quad - \kappa_2 z^2 \left[ e^{x\xi} - \frac{\partial\alpha_1}{\partial x} x^2 \arctan \xi \right]^2 \\
&\quad + |z| \left| e^{x\xi} - \frac{\partial\alpha_1}{\partial x} x^2 \arctan \xi \right| \|\Delta_0\|_\infty \\
\text{by (2.254)} \quad &\leq -c_1 x^2 - c_2 z^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} + \frac{\|\Delta_0\|_\infty^2}{4\kappa_2}. \quad (2.291)
\end{aligned}$$

◇

The combination of Lemmas 2.8 and 2.26, illustrated in the above example, is now formulated as another design tool.

**Lemma 2.28 (Boundedness via Backstepping)** *Consider the system*

$$\dot{x} = f(x) + g(x)u + F(x)\Delta_1(x, u, t), \quad (2.292)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ ,  $F(x)$  is an  $(n \times q)$  matrix of known smooth nonlinear functions, and  $\Delta_1(x, u, t)$  is a  $(q \times 1)$  vector of uncertain nonlinearities which are uniformly bounded for all values of  $x, u, t$ . Suppose that there exists a feedback control  $u = \alpha(x)$  that renders  $x(t)$  globally uniformly bounded, and that this is established via positive definite and radially unbounded functions  $V(x), W(x)$  and a constant  $b$ , such that

$$\frac{\partial V}{\partial x}(x) [f(x) + g(x)\alpha(x) + F(x)\Delta_1(x, u, t)] \leq -W(x) + b. \quad (2.293)$$

Now consider the augmented system

$$\dot{x} = f(x) + g(x)\xi + F(x)\Delta_1(x, u, t) \quad (2.294a)$$

$$\dot{\xi} = u + \varphi(x, \xi)^T \Delta_2(x, \xi, u, t), \quad (2.294b)$$

where  $\varphi(x, \xi)$  is a  $(p \times 1)$  vector of known smooth nonlinear functions, and  $\Delta_2(x, \xi, u, t)$  is a  $(p \times 1)$  vector of uncertain nonlinearities which are uniformly bounded for all values of  $x, \xi, u, t$ . For this system, the feedback control

$$\begin{aligned}
u = & -c[\xi - \alpha(x)] + \frac{\partial\alpha}{\partial x}(x) [f(x) + g(x)\xi] - \frac{\partial V}{\partial x}(x)g(x) \\
& - \kappa[\xi - \alpha(x)] \left\{ |\varphi(x, \xi)|^2 + \left| \frac{\partial\alpha}{\partial x}(x)F(x) \right|^2 \right\} \quad (2.295)
\end{aligned}$$

guarantees global uniform boundedness of  $x(t)$  and  $\xi(t)$  with any  $c > 0$  and  $\kappa > 0$ .

**Proof.** Using the error variable

$$z = \xi - \alpha(x), \quad (2.296)$$

the system (2.294) is rewritten as

$$\dot{x} = f(x) + g(x)[\alpha(x) + z] + F(x)\Delta_1(x, u, t) \quad (2.297a)$$

$$\begin{aligned}
\dot{z} = & u + \varphi(x, \xi)^T \Delta_2(x, \xi, u, t) \\
& - \frac{\partial\alpha}{\partial x}(x) [f(x) + g(x)\xi + F(x)\Delta_1(x, u, t)]. \quad (2.297b)
\end{aligned}$$

The derivative of

$$V_2(x, \xi) = V(x) + \frac{1}{2} [\xi - \alpha(x)]^2 = V(x) + \frac{1}{2} z^2 \quad (2.298)$$

along the solutions of (2.297) with the control (2.295) is

$$\begin{aligned}
\dot{V}_2 = & \frac{\partial V}{\partial x}(f + g\alpha + F\Delta_1) + \frac{\partial V}{\partial x}gz \\
& + z \left[ u + \varphi^T \Delta_2 - \frac{\partial\alpha}{\partial x}(f + g\xi + F\Delta_1) \right] \\
\leq & \frac{\partial V}{\partial x}(f + g\alpha + F\Delta_1) + z \left[ u - \frac{\partial\alpha}{\partial x}(f + g\xi) + \frac{\partial V}{\partial x}g \right] \\
& + z \left[ \varphi^T \Delta_2 - \frac{\partial\alpha}{\partial x}F\Delta_1^T \right] \\
\text{by (2.293)} \quad & \leq -W(x) + b + z \left[ u - \frac{\partial\alpha}{\partial x}(f + g\xi) + \frac{\partial V}{\partial x}g \right] \\
& + z \left[ \varphi^T \Delta_2 - \frac{\partial\alpha}{\partial x}F\Delta_1 \right] \\
\text{by (2.295)} \quad & = -W(x) + b - cz^2 - \kappa z^2 \left[ |\varphi|^2 + \left| \frac{\partial\alpha}{\partial x}F \right|^2 \right] \\
& + |z| |\varphi| \|\Delta_2\|_\infty + |z| \left| \frac{\partial\alpha}{\partial x}F \right| \|\Delta_1\|_\infty \\
\text{by (2.254)} \quad & = -W(x) - cz^2 + b + \frac{\|\Delta_1\|_\infty^2}{4\kappa} + \frac{\|\Delta_2\|_\infty^2}{4\kappa}. \quad (2.299)
\end{aligned}$$

The radial unboundedness of  $W(x)$  combined with (2.299) implies that  $\dot{V}_2$  is negative outside a compact set, which in turn implies that  $x(t)$  and  $\xi(t)$  are globally uniformly bounded. □

### 2.5.3 Robust strict-feedback systems

Just as we generalized Lemma 2.8 to strict-feedback systems in Section 2.3.1 and Lemma 2.25 to block-strict-feedback systems in Section 2.3.3, we can generalize Lemma 2.28 to broader classes of uncertain nonlinear systems.

We consider systems in the *robust strict-feedback form*:

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1^T(x_1)\Delta(x, u, t) \\ \dot{x}_2 &= x_3 + \varphi_2^T(x_1, x_2)\Delta(x, u, t) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}^T(x_1, \dots, x_{n-1})\Delta(x, u, t) \\ \dot{x}_n &= \beta(x)u + \varphi_n^T(x)\Delta(x, u, t),\end{aligned}\tag{2.300}$$

where  $\beta(x) \neq 0$ ,  $\forall x \in \mathbb{R}^n$ ,  $\varphi_i(x_1, \dots, x_i)$  is a  $(p \times 1)$  vector of known smooth nonlinear functions, and  $\Delta(x, u, t)$  is a  $(p \times 1)$  vector of uncertain nonlinearities which are *uniformly bounded* for all values of  $x, u, t$ .

**Corollary 2.29 (Robust Strict-Feedback Systems)** *The state  $x(t)$  of the system (2.300) will be globally uniformly bounded if the control is chosen as*

$$u = \frac{1}{\beta(x)} \alpha_n(x),\tag{2.301}$$

where the function  $\alpha_n(x)$  is defined by the following recursive expressions for  $i = 1, \dots, n$  (where we denote  $z_0 \equiv \alpha_0 \equiv 0$ ):

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_{i-1})\tag{2.302}$$

$$\alpha_i(x_1, \dots, x_i) = -c_i z_i - z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} - \kappa_i z_i \left| \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right|^2,\tag{2.303}$$

with  $c_i, \kappa_i$ ,  $i = 1, \dots, n$  positive design constants.

**Proof.** Using the definitions (2.302) and (2.303) and denoting  $x_0 \equiv \alpha_0 \equiv 0$ ,  $x_{n+1} \equiv \beta(x)u$ , the derivative of the error variable  $z_i$ ,  $i = 1, \dots, n$ , becomes

$$\begin{aligned}\dot{z}_i &= \dot{x}_i - \dot{\alpha}_{i-1}(x_1, \dots, x_{i-1}) \\ &= x_{i+1} + \varphi_i^T \Delta - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} (x_{j+1} + \varphi_j^T \Delta) \\ &= \alpha_i + z_{i+1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^T \Delta \\ &= -c_i z_i - z_{i-1} + z_{i+1} + \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^T \Delta - \kappa_i z_i \left| \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right|^2.\end{aligned}\tag{2.304}$$

The choice of control (2.301) guarantees that  $z_{n+1} \equiv 0$ . The closed-loop error system can therefore be expressed as

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 + z_2 + \varphi_1^T \Delta - \kappa_1 z_1 |\varphi_1|^2 \\ \dot{z}_2 &= -c_2 z_2 - z_1 + z_3 + \left( \varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1 \right)^T \Delta - \kappa_2 z_2 \left| \varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1 \right|^2 \\ &\vdots \\ \dot{z}_{n-1} &= -c_{n-1} z_{n-1} - z_{n-2} + z_n + \left( \varphi_{n-1} - \sum_{j=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial x_j} \varphi_j \right)^T \Delta \\ &\quad - \kappa_{n-1} z_{n-1} \left| \varphi_{n-1} - \sum_{j=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial x_j} \varphi_j \right|^2 \\ \dot{z}_n &= -c_n z_n - z_{n-1} + \left( \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j \right)^T \Delta - \kappa_n z_n \left| \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j \right|^2.\end{aligned}\tag{2.305}$$

Now we can use the quadratic Lyapunov function

$$V_n(z_1, \dots, z_n) = \frac{1}{2} \sum_{i=1}^n z_i^2\tag{2.306}$$

to prove global uniform boundedness. Indeed, the derivative of (2.306) along the solutions of (2.305) is

$$\begin{aligned}\dot{V}_n &= \sum_{i=1}^n \left[ -c_i z_i^2 + z_i \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^T \Delta - \kappa_i z_i^2 \left| \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right|^2 \right] \\ &\leq \sum_{i=1}^n \left[ -c_i z_i^2 + |z_i| \left| \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right| \|\Delta\|_\infty - \kappa_i z_i^2 \left| \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right|^2 \right] \\ &\leq \sum_{i=1}^n \left[ -c_i z_i^2 + \frac{\|\Delta\|_\infty^2}{4\kappa_i} \right].\end{aligned}\tag{2.307}$$

The last inequality implies that  $z(t)$  is globally uniformly bounded. But from (2.303) we see that, since the  $\alpha_i$ 's are smooth functions,  $x_i$  can be expressed as a smooth function of  $z_1, \dots, z_i$ :

$$x_1 = z_1, \quad x_i = z_i + \bar{\alpha}_{i-1}(z_1, \dots, z_{i-1}), \quad i = 2, \dots, n.\tag{2.308}$$

Hence,  $x(t)$  is globally uniformly bounded and, furthermore, converges to the compact residual set

$$\mathcal{R} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n c_i z_i^2 \leq \sum_{i=1}^n \frac{\|\Delta\|_\infty^2}{4\kappa_i} \right\},\tag{2.309}$$

whose size is unknown since the bound  $\|\Delta\|_\infty$  is unknown.  $\square$

## Notes and References

Integrator backstepping is an idea whose origins are difficult to trace, because of its simultaneous appearance, often implicit, in the works of Tsiniias [193], Koditschek [84], Byrnes and Isidori [12], and Sontag and Sussmann [175]. Kokotović and Sussmann [85] viewed the stabilization through an integrator as a special case of stabilization through an SPR transfer function, as in the early adaptive designs by Parks [150], Landau [109], and Narendra et al. [142]. This passivity view was extended to nonlinear cascades by Ortega [145] and Byrnes, Isidori, and Willems [14]. Integrator backstepping as a recursive design tool [85, Corollary 3.2] was employed in the cascade design of Saberi, Kokotović, and Sussmann [163] and further developed by Kanellakopoulos, Kokotović, and Morse [73]. The passivity aspect of this design was pointed out by Lozano, Brogliato, and Landau [116]. A tutorial overview of backstepping was given in the 1991 Bode lecture by Kokotović [88]. Among the current applications of backstepping are electric machines in the monograph of Dawson, Hu, and Burg [26] and steering and braking control by Chen and Tomizuka [17].

An important question, not addressed in the above references, is whether backstepping designs can incorporate optimization with respect to some meaningful cost functionals. It is clear that minimizing a partial cost functional at each step does not imply, and may in fact contradict, the overall optimality. A framework for a backstepping-like recursive optimization was proposed in a Russian language paper by Kolesnikov [89], which remained unnoticed in the English language literature.

All the above references apply backstepping to systems without uncertainty. For systems with uncertainty a robust nonlinear design was introduced for the matched case in the works of Corless and Leitmann [20] and Barmish, Corless, and Leitmann [6], and extended by various forms of “generalized matching conditions” [19, 20]. However, these extensions haven’t led to a discovery of backstepping with uncertainty. After the development of adaptive backstepping by Kanellakopoulos, Kokotović, and Morse [69, 87], backstepping with uncertainty was pursued in the works of Qu [160], Marino and Tomei [125], Freeman and Kokotović [37, 38], and Slotine and Hedrick [168]. Backstepping designs for systems with unmeasured or unmodeled dynamics have been developed by Praly and Jiang [159], Jiang, Teel, and Praly [61], Khalil [82], and Krstić, Sun, and Kokotović [104]. Recent advances in backstepping, which are beyond the scope of this book, are due to Coron and Praly [21] and Praly, d’Andréa-Novel, and Coron [158]. The restriction of backstepping to pure-feedback systems has motivated alternative designs applicable to other classes of nonlinear systems, such as those by Teel [187], Teel and Praly [192], Qu [161], and Mazenc and Praly [126].

## Chapter 3

### Adaptive Backstepping Design

The controllers designed in the preceding chapter guarantee that in the presence of uncertain bounded nonlinearities the closed-loop state remains bounded. In this chapter, and in the remainder of the book, the uncertainties are more specific. They consist of unknown constant parameters which appear linearly in the system equations. In the presence of such parametric uncertainties we will be able to achieve both boundedness of the closed-loop states and convergence of the tracking error to zero.

While all the controllers designed in Chapter 2 employ *static* feedback, the controllers in this chapter will, in addition, employ a form of nonlinear integral feedback. The underlying idea in the design of this *dynamic* part of feedback is *parameter estimation*. The dynamic part of the controller is designed as a *parameter update law* with which the static part is continuously *adapted* to new parameter estimates, hence its name: *Adaptive control law*. In using this traditional terminology, however, we should keep in mind that so conceived adaptive controllers are but one type of nonlinear dynamic feedback.

Adaptive controllers are dynamic and therefore more complex than the static controllers designed in Chapter 2. What is achieved with this additional complexity? As we will show in Section 3.1, an adaptive controller guarantees not only that the plant state  $x$  remains bounded, but also that it tends to a desired constant value (“regulation”) or asymptotically tracks a reference signal (“tracking”).

The first results leading to a new systematic design of adaptive controllers are presented in Section 3.2, which introduces *adaptive backstepping*. The recursive design procedure for *parametric strict-feedback systems* is then developed in Section 3.3.

In its basic form, the adaptive backstepping design employs *over-parametrization*, that is, more than one estimate per unknown parameter. This means that the dynamic part of the controller is not of minimal order. In Chapter 4, a more intricate backstepping procedure is developed—the tuning functions method—which employs the minimal number of parameter

estimates. The extended-matching design, presented in Section 3.4, is of interest as a transition between overparametrized and minimal-order designs. It also contains the first adaptive performance results.

### 3.1 Adaptation as Dynamic Feedback

The difference between a static and a dynamic (that is, adaptive) design will first be illustrated on the simplest nonlinear system:

$$\dot{x} = u + \theta\varphi(x). \quad (3.1)$$

This is the special case of the system (2.240), where the uncertainty  $\Delta(t)$  is the unknown constant parameter  $\theta$ .

Even if we do not know a bound for  $\theta$ , we can use Lemma 2.26 to design a static nonlinear controller which guarantees global boundedness of  $x(t)$ . The nonlinear damping design (2.251) applies also here. The corresponding static controller is

$$u = -cx - \kappa x\varphi^2(x), \quad (3.2)$$

and the resulting closed-loop system is of first order:

$$\dot{x} = -cx - \kappa x\varphi^2(x) + \theta\varphi(x). \quad (3.3)$$

According to (2.252), the derivative of  $V = \frac{1}{2}x^2$  satisfies

$$\dot{V} \leq -cx^2 + \frac{\theta^2}{4\kappa}, \quad (3.4)$$

which means that  $x(t)$  converges to the interval

$$|x| \leq \frac{|\theta|}{2\sqrt{\kappa c}}. \quad (3.5)$$

This interval can be reduced by increasing the gains  $\kappa$  and  $c$ , but  $x(t)$  will not converge to zero if  $\theta$  is a nonzero constant. Excessive increase of these gains enlarges the system bandwidth, which is undesirable. Our task is therefore to achieve  $\lim_{t \rightarrow \infty} x(t) = 0$  without increasing  $\kappa$  and  $c$ . In fact, we will first accomplish this task with  $\kappa = 0$ , and then use  $\kappa > 0$  for further improvement of transients.

To achieve regulation of  $x(t)$ , we design a dynamic feedback controller, that is, we employ adaptation.

If  $\theta$  were known, the control

$$u = -\theta\varphi(x) - c_1x, \quad c_1 > 0 \quad (3.6)$$

would render the derivative of  $V_0(x) = \frac{1}{2}x^2$  negative definite:  $\dot{V}_0 = -c_1x^2$ . Of course the control law (3.6) can not be implemented, since  $\theta$  is unknown.

Instead, one can employ its *certainty-equivalence* form in which  $\theta$  is replaced by an estimate  $\hat{\theta}$ :

$$u = -\hat{\theta}\varphi(x) - c_1x. \quad (3.7)$$

Substituting (3.7) into (3.6), we obtain

$$\dot{x} = -c_1x + \tilde{\theta}\varphi(x), \quad (3.8)$$

where  $\tilde{\theta}$  is the *parameter error*:

$$\tilde{\theta} = \theta - \hat{\theta}. \quad (3.9)$$

The derivative of  $V_0(x) = \frac{1}{2}x^2$  becomes

$$\dot{V}_0 = -c_1x^2 + \tilde{\theta}x\varphi(x). \quad (3.10)$$

Since the second term is indefinite and contains the unknown parameter error  $\tilde{\theta}$ , we can not conclude anything about the stability of (3.6). We make the controller dynamic with an update law for  $\hat{\theta}$ . To design this update law, we augment  $V_0$  with a quadratic term in the parameter error  $\tilde{\theta}$ ,

$$V_1(x, \tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\tilde{\theta}^2, \quad (3.11)$$

where  $\gamma > 0$  is the *adaptation gain*. The derivative of this function is

$$\begin{aligned} \dot{V}_1 &= x\dot{x} + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -c_1x^2 + \tilde{\theta}x\varphi(x) + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -c_1x^2 + \tilde{\theta} \left[ x\varphi(x) + \frac{1}{\gamma}\dot{\tilde{\theta}} \right]. \end{aligned} \quad (3.12)$$

The second term is still indefinite and contains  $\tilde{\theta}$  as a factor. However, the situation is much better than in (3.10), because we now have the dynamics of  $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$  at our disposal. With the appropriate choice of  $\dot{\hat{\theta}}$  we can cancel the indefinite term. Thus, we choose the update law

$$\dot{\hat{\theta}} = -\dot{\tilde{\theta}} = \gamma x\varphi(x), \quad (3.13)$$

which yields

$$\dot{V}_1 = -c_1x^2 \leq 0. \quad (3.14)$$

The resulting adaptive system consists of (3.1) with the control (3.7) and the update law (3.13), and is shown in Figure 3.1. In Figure 3.2, this system is redrawn in its closed-loop form consisting of (3.8) and (3.13), namely

$$\begin{aligned} \dot{x} &= -c_1x + \tilde{\theta}\varphi(x) \\ \dot{\tilde{\theta}} &= -\gamma x\varphi(x). \end{aligned} \quad (3.15)$$

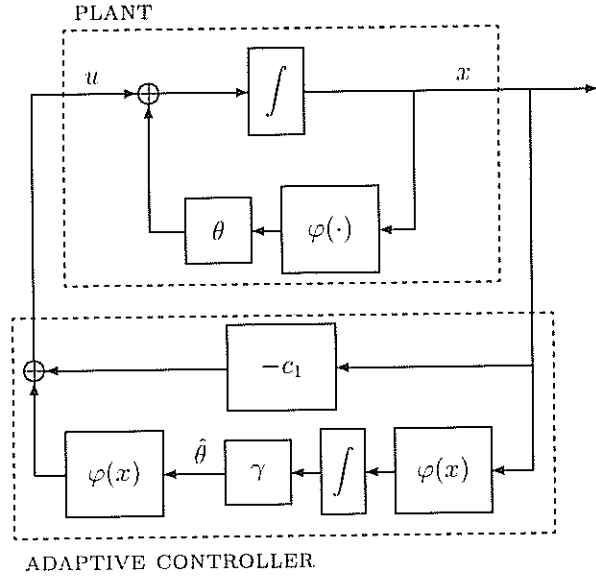


Figure 3.1: The closed-loop adaptive system (3.15).

Because  $\dot{V}_1 \leq 0$ , the equilibrium  $x = 0, \tilde{\theta} = 0$  of (3.15) is globally stable. In addition, the desired regulation property  $\lim_{t \rightarrow \infty} x(t) = 0$  follows from the LaSalle-Yoshizawa theorem (Theorem 2.1). The adaptive nonlinear controller which guarantees these properties is given by (3.8) and (3.13):

$$\begin{aligned} u &= -c_1 x - \hat{\theta} \varphi(x) \\ \dot{\hat{\theta}} &= \gamma x \varphi(x). \end{aligned} \quad (3.16)$$

One may think that the above adaptive design is so straightforward because (3.1) is a first-order system. In fact, this is due to the *matching condition*: The terms containing unknown parameters in (3.1) are in the span of the control, that is, they can be directly cancelled by  $u$  when  $\theta$  is known. To illustrate this point, let us consider the following second-order system, where again the uncertain term is “matched” by the control  $u$ :

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1) \\ \dot{x}_2 &= \theta \varphi_2(x) + u. \end{aligned} \quad (3.17)$$

If  $\theta$  were known, we would be able to apply Lemma 2.8: First view  $x_2$  as the virtual control, design the stabilizing function

$$\alpha_1(x_1) = -c_1 x_1 - \varphi_1(x_1), \quad (3.18)$$

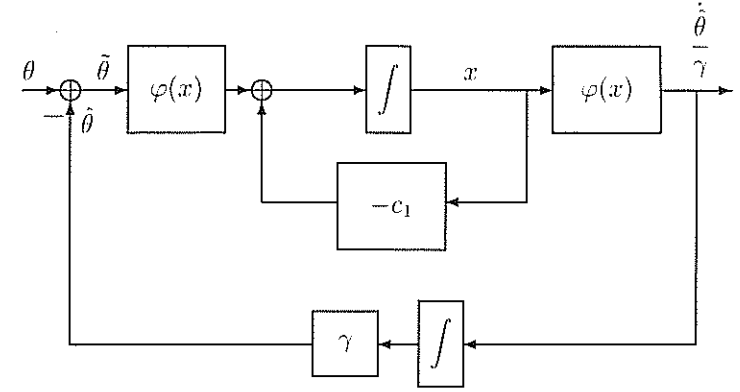


Figure 3.2: An equivalent representation of (3.15).

and then form the Lyapunov function

$$V_c(x) = \frac{1}{2} x_1^2 + \frac{1}{2} (x_2 - \alpha_1(x_1))^2, \quad (3.19)$$

whose derivative is rendered negative definite

$$\dot{V}_c(x) = -c_1 x_1^2 - c_2 (x_2 - \alpha_1)^2 \quad (3.20)$$

by the control

$$u = -c_2 (x_2 - \alpha_1) - x_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi_1) - \theta \varphi_2(x). \quad (3.21)$$

Since  $\theta$  is unknown, we again replace it with its estimate  $\hat{\theta}$  in (3.21) to obtain the adaptive control law:

$$u = -c_2 (x_2 - \alpha_1) - x_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi_1) - \hat{\theta} \varphi_2(x). \quad (3.22)$$

This results in the error system ( $z_1 = x_1, z_2 = x_2 - \alpha_1$ ):

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_2(x) \end{bmatrix} \tilde{\theta}. \quad (3.23)$$

Then we augment (3.20) with a quadratic term in the parameter error  $\tilde{\theta}$  to obtain the Lyapunov function:

$$V_1(z, \tilde{\theta}) = V_c + \frac{1}{2\gamma} \tilde{\theta}^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma} \tilde{\theta}^2. \quad (3.24)$$

Its derivative is

$$\dot{V}_1 = -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta} \left[ z_2 \varphi_2 - \frac{1}{\gamma} \dot{\tilde{\theta}} \right]. \quad (3.25)$$

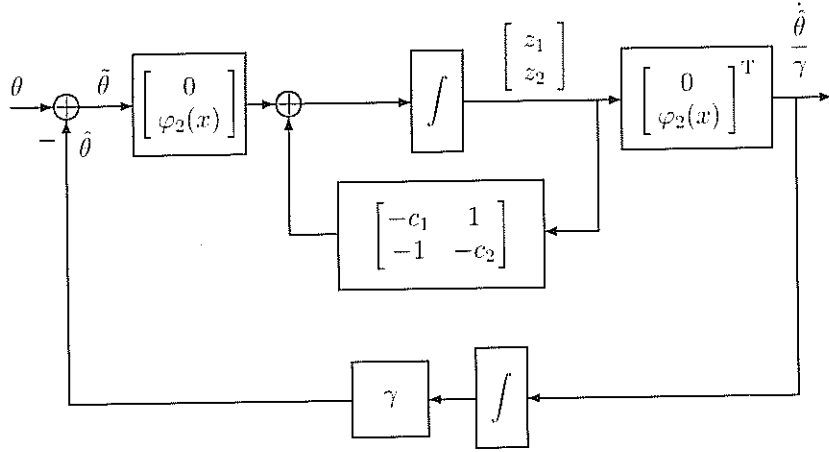


Figure 3.3: The closed-loop adaptive system (3.28).

The choice of update law

$$\dot{\hat{\theta}} = \gamma \varphi_2 z_2 \quad (3.26)$$

eliminates the  $\tilde{\theta}$ -term in (3.25) and renders the derivative of the Lyapunov function (3.24) nonpositive:

$$\dot{V}_1 = -c_1 z_1^2 - c_2 z_2^2 \leq 0. \quad (3.27)$$

This implies that the  $z = 0, \tilde{\theta} = 0$  equilibrium point of the closed-loop adaptive system consisting of (3.23) and (3.26) (see block diagram in Figure 3.3)

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_2(x) \end{bmatrix} \tilde{\theta} \\ \dot{\tilde{\theta}} &= -\gamma \begin{bmatrix} 0 & \varphi_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned} \quad (3.28)$$

is globally stable and, in addition,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

## 3.2 Adaptive Backstepping

### 3.2.1 Adaptive integrator backstepping

The adaptive design in the above examples was simple because of the matching: The parametric uncertainty was in the span of the control. We now move to the more general case of *extended matching*, where the parametric uncertainty enters the system one integrator before the control does:

$$\dot{x}_1 = x_2 + \theta \varphi(x_1) \quad (3.29a)$$

$$\dot{x}_2 = u. \quad (3.29b)$$

We use this example to introduce *adaptive backstepping*.

If  $\theta$  were known, we would apply Lemma 2.8 to design the stabilizing function for  $x_2$

$$\alpha_1(x_1, \theta) = -c_1 x_1 - \theta \varphi(x_1), \quad (3.30)$$

with the Lyapunov function

$$V_c(x, \theta) = \frac{1}{2} x_1^2 + \frac{1}{2} (x_2 - \alpha_1(x_1, \theta))^2, \quad (3.31)$$

whose derivative is rendered negative definite

$$\dot{V}_c(x, \theta) = -c_1 x_1^2 - c_2 (x_2 - \alpha_1(x, \theta))^2 \quad (3.32)$$

by the control

$$u = -c_2 (x_2 - \alpha_1(x_1, \theta)) - x_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \theta \varphi). \quad (3.33)$$

Since  $\theta$  is unknown and appears one equation before the control does, we can not apply Lemma 2.8 because the dependence of  $\alpha_1(x_1) = -c_1 x_1 - \theta \varphi(x_1)$  on the unknown parameter makes it impossible to continue the procedure. However, we can utilize the idea of integrator backstepping.

**Step 1.** If  $x_2$  were the control, an adaptive controller for (3.29a) would be given by (3.16):

$$\alpha_1(x_1, \vartheta_1) = -c_1 z_1 - \vartheta_1 \varphi(x_1) \quad (3.34)$$

$$\dot{\vartheta}_1 = \gamma z_1 \varphi(x_1). \quad (3.35)$$

In the above equations we have replaced the parameter estimate  $\hat{\theta}$  with the estimate  $\vartheta_1$ , which denotes the estimate generated in this design step. As we will see, there will be another estimate generated in the next step. With (3.34) and the new error variable  $z_2 = x_2 - \alpha_1$ , the  $\dot{z}_1$ -equation becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \vartheta_1) \varphi. \quad (3.36)$$

The derivative of the Lyapunov function

$$V_1(x, \vartheta_1) = \frac{1}{2} z_1^2 + \frac{1}{2\gamma} (\theta - \vartheta_1)^2 \quad (3.37)$$

along the solutions of (3.36) is

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 - \frac{1}{\gamma} (\theta - \vartheta_1) \dot{\vartheta}_1 \\ &= z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1) \left( \varphi_1 z_1 - \frac{1}{\gamma} \dot{\vartheta}_1 \right) \\ &= z_1 z_2 - c_1 z_1^2. \end{aligned} \quad (3.38)$$

**Step 2.** The derivative of  $z_2$  is now expressed as

$$\begin{aligned}\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 - \frac{\partial \alpha_1}{\partial \vartheta_1} \dot{\vartheta}_1.\end{aligned}$$

Substituting (3.29a) and the update law (3.35) results in

$$\begin{aligned}\dot{z}_2 &= u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \theta \varphi) - \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 - \theta \frac{\partial \alpha_1}{\partial x_1} \varphi.\end{aligned}\quad (3.39)$$

At this point we need to select a Lyapunov function and design  $u$  to render its derivative nonpositive. Our first attempt is the augmented Lyapunov function

$$V_2(z_1, z_2, \vartheta_1) = V_1(z_1, \vartheta_1) + \frac{1}{2} z_2^2,$$

whose derivative, using (3.38) and (3.39), is

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 \\ &= -c_1 z_1^2 + z_2 \left[ z_1 + u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 - \theta \frac{\partial \alpha_1}{\partial x_1} \varphi \right].\end{aligned}$$

The control  $u$  should now be able to cancel the indefinite terms in  $\dot{V}_2$ . To deal with the terms containing the unknown parameter  $\theta$ , we will try to employ the existing estimate  $\vartheta_1$ :

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 + \vartheta_1 \frac{\partial \alpha_1}{\partial x_1} \varphi.$$

From the resulting derivative

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 - (\theta - \vartheta_1) \frac{\partial \alpha_1}{\partial x_1} \varphi_1 z_2,$$

we see that we have no design freedom left to cancel the  $(\theta - \vartheta_1)$ -term. To overcome this difficulty, we replace  $\vartheta_1$  in the expression for  $u$  with a *new* estimate  $\vartheta_2$ :

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \vartheta_1} \gamma \varphi z_1 + \vartheta_2 \frac{\partial \alpha_1}{\partial x_1} \varphi. \quad (3.40)$$

With the choice (3.40), the  $\dot{z}_2$ -equation becomes

$$\dot{z}_2 = -c_2 z_2 - z_1 - (\theta - \vartheta_2) \frac{\partial \alpha_1}{\partial x_1} \varphi. \quad (3.41)$$

The presence of the new parameter estimate  $\vartheta_2$  suggests the following augmentation of the Lyapunov function:

$$\begin{aligned}V_2(z_1, z_2, \vartheta_1, \vartheta_2) &= V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma} (\theta - \vartheta_2)^2 \\ &= \frac{1}{2} (z_1^2 + z_2^2) + \frac{1}{2\gamma} [(\theta - \vartheta_1)^2 + (\theta - \vartheta_2)^2].\end{aligned}\quad (3.42)$$

The derivative of  $V_2$  is

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 - \frac{1}{\gamma} (\theta - \vartheta_2) \dot{\vartheta}_2 \\ &= z_1 z_2 - c_1 z_1^2 + z_2 \left[ -c_2 z_2 - z_1 - (\theta - \vartheta_2) \frac{\partial \alpha_1}{\partial x_1} \varphi \right] - \frac{1}{\gamma} (\theta - \vartheta_2) \dot{\vartheta}_2 \\ &= -c_1 z_1^2 - c_2 z_2^2 - (\theta - \vartheta_2) \left( \frac{\partial \alpha_1}{\partial x_1} \varphi + \frac{1}{\gamma} \dot{\vartheta}_2 \right).\end{aligned}\quad (3.43)$$

Now the  $(\theta - \vartheta_2)$ -term can be eliminated with the update law

$$\dot{\vartheta}_2 = -\gamma \frac{\partial \alpha_1}{\partial x_1} \varphi z_2, \quad (3.44)$$

which yields

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2. \quad (3.45)$$

The equations (3.41) and (3.44) along with (3.36) and (3.35) form the error system representation of the resulting closed-loop adaptive system:

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 + z_2 + (\theta - \vartheta_1) \varphi \\ \dot{z}_2 &= -c_2 z_2 - z_1 - (\theta - \vartheta_2) \frac{\partial \alpha_1}{\partial x_1} \varphi \\ \dot{\vartheta}_1 &= \gamma \varphi z_1 \\ \dot{\vartheta}_2 &= -\gamma \frac{\partial \alpha_1}{\partial x_1} \varphi z_2.\end{aligned}\quad (3.46)$$

The matrix form of this system,

$$\begin{aligned}\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi & 0 \\ 0 & -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \begin{bmatrix} \theta - \vartheta_1 \\ \theta - \vartheta_2 \end{bmatrix} \\ \frac{d}{dt} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} &= \gamma \begin{bmatrix} \varphi & 0 \\ 0 & -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},\end{aligned}\quad (3.47)$$

makes its properties more visible:

- The constant system matrix has negative terms along its diagonal, while its off-diagonal terms are skew-symmetric, and
- the matrix that multiplies the parameter errors in the  $\dot{z}$ -equation is used in the update laws for the parameter estimates.

The stability properties of (3.47) follow from (3.42) and (3.45): The LaSalle-Yoshizawa theorem (Theorem 2.1) establishes that  $z_1, z_2, \vartheta_1, \vartheta_2$  are bounded, and  $z \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $z_1 = x_1$ ,  $x_1$  is also bounded and converges to zero. The boundedness of  $x_2$  then follows from the boundedness of  $\alpha_1$  (defined in (3.34)) and the fact that  $x_2 = z_2 + \alpha_1$ . Using (3.40) we conclude that the control  $u$  is also bounded. Finally, we note that the regulation of  $z$  and  $x_1$  does *not* imply the regulation of  $x_2$ : From  $z_2 = x_2 - \alpha_1$  and (3.34) we see that  $x_2 + \vartheta_1 \varphi(0)$  will converge to zero. Thus,  $x_2$  is not guaranteed to converge to zero unless  $\varphi(0) = 0$ . However,  $x_2$  will converge to a constant value:

$$\lim_{t \rightarrow \infty} x_2 = -\theta \varphi(0) \triangleq x_2^e. \quad (3.48)$$

This can be seen from (3.29a): Since  $x_1$  and  $\dot{x}_1$  converge to zero, so does  $x_2 + \theta \varphi(0)$ .

With the above example we have illustrated the idea of adaptive backstepping. To formulate it as a design tool analogous to Lemma 2.8, we start with the assumption that an adaptive controller is known for an initial system.

**Assumption 3.1** Consider the system

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad (3.49)$$

where  $x \in \mathbb{R}^n$  is the state,  $\theta \in \mathbb{R}^p$  is a vector of unknown constant parameters, and  $u \in \mathbb{R}$  is the control input. There exist an adaptive controller

$$\begin{aligned} u &= \alpha(x, \vartheta) \\ \dot{\vartheta} &= T(x, \vartheta), \end{aligned} \quad (3.50)$$

with parameter estimate  $\vartheta \in \mathbb{R}^q$ , and a smooth function  $V(x, \vartheta) : \mathbb{R}^{(n+q)} \rightarrow \mathbb{R}$  which is positive definite and radially unbounded in the variables  $(x, \vartheta - \theta)$  such that for all  $(x, \vartheta) \in \mathbb{R}^{(n+q)}$ :

$$\frac{\partial V}{\partial x}(x, \vartheta) [f(x) + F(x)\theta + g(x)\alpha(x, \vartheta)] + \frac{\partial V}{\partial \vartheta}(x, \vartheta) T(x, \vartheta) \leq -W(x, \vartheta) \leq 0, \quad (3.51)$$

where  $W : \mathbb{R}^{n+q} \rightarrow \mathbb{R}$  is positive semidefinite.  $\square$

Under this assumption, the control (3.50), applied to the system (3.49), guarantees global boundedness of  $x(t), \vartheta(t)$  and, by the LaSalle-Yoshizawa theorem (Theorem 2.1), regulation of  $W(x(t), \vartheta(t))$ . Adaptive backstepping allows us to achieve the same properties for the augmented system.

**Lemma 3.2 (Adaptive Backstepping)** Let the system (3.49) be augmented by an integrator,

$$\dot{x} = f(x) + F(x)\theta + g(x)\xi \quad (3.52a)$$

$$\dot{\xi} = u, \quad (3.52b)$$

where  $\xi \in \mathbb{R}$ . Consider for this system the dynamic feedback controller

$$\begin{aligned} u &= -c(\xi - \alpha(x, \vartheta)) + \frac{\partial \alpha}{\partial x}(x, \vartheta) [f(x) + F(x)\bar{\vartheta} + g(x)\xi] \\ &\quad + \frac{\partial \alpha}{\partial \vartheta} T(x, \vartheta) - \frac{\partial V}{\partial x}(x, \vartheta) g(x), \quad c > 0 \end{aligned} \quad (3.53)$$

$$\dot{\vartheta} = T(x, \vartheta) \quad (3.54)$$

$$\dot{\bar{\vartheta}} = -\Gamma \left[ \frac{\partial \alpha}{\partial x}(x, \vartheta) F(x) \right]^T (\xi - \alpha(x, \vartheta)), \quad (3.55)$$

where  $\bar{\vartheta}$  is a new estimate of  $\theta$ ,  $\Gamma = \Gamma^T > 0$  is the adaptation gain matrix. Under Assumption 3.1, this adaptive controller guarantees global boundedness of  $x(t), \xi(t), \vartheta(t), \bar{\vartheta}(t)$  and regulation of  $W(x(t), \vartheta(t))$  and  $\xi(t) - \alpha(x(t), \vartheta(t))$ . These properties can be established with the Lyapunov function

$$V_a(x, \xi, \vartheta, \bar{\vartheta}) = V(x, \vartheta) + \frac{1}{2} [\xi - \alpha(x, \vartheta)]^2 + \frac{1}{2} (\theta - \bar{\vartheta})^T \Gamma^{-1} (\theta - \bar{\vartheta}). \quad (3.56)$$

**Proof.** With the error variable  $z = \xi - \alpha(x, \vartheta)$ , (3.52) is rewritten as

$$\dot{x} = f(x) + F(x)\theta + g(x) [\alpha(x, \vartheta) + z] \quad (3.57a)$$

$$\dot{z} = u - \frac{\partial \alpha}{\partial x}(x, \vartheta) [f(x) + F(x)\theta + g(x) (\alpha(x, \vartheta) + z)] - \frac{\partial \alpha}{\partial \vartheta} T(x, \vartheta). \quad (3.57b)$$

Note that in (3.57b) the derivative of  $\vartheta$  was replaced by the update law (3.54). Introducing a new parameter estimate  $\bar{\vartheta}$ , we augment the Lyapunov function:

$$V_a(x, \xi, \vartheta, \bar{\vartheta}) = V(x, \vartheta) + \frac{1}{2} z^2 + \frac{1}{2} (\theta - \bar{\vartheta})^T \Gamma^{-1} (\theta - \bar{\vartheta}). \quad (3.58)$$

Using (3.51), it is easy to show that the derivative of (3.58) satisfies

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x} (f + F\theta + g\alpha + gz) + \frac{\partial V}{\partial \vartheta} T \\ &\quad + z \left[ u - \frac{\partial \alpha}{\partial x} (f + F\theta + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta} T \right] - \dot{\bar{\vartheta}}^T \Gamma^{-1} (\theta - \bar{\vartheta}) \\ &= \frac{\partial V}{\partial x} (f + F\theta + g\alpha) + \frac{\partial V}{\partial \vartheta} T \\ &\quad + z \left[ u - \frac{\partial \alpha}{\partial x} (f + F\theta + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta} T + \frac{\partial V}{\partial x} g \right] - \dot{\bar{\vartheta}}^T \Gamma^{-1} (\theta - \bar{\vartheta}) \\ &\leq -W(x, \vartheta) + z \left[ u - \frac{\partial \alpha}{\partial x} (f + F\bar{\vartheta} + g(\alpha + z)) - \frac{\partial \alpha}{\partial \vartheta} T + \frac{\partial V}{\partial x} g \right] \\ &\quad - \left[ \frac{\partial \alpha}{\partial x} Fz + \dot{\bar{\vartheta}}^T \Gamma^{-1} \right] (\theta - \bar{\vartheta}). \end{aligned} \quad (3.59)$$

The  $(\theta - \bar{\vartheta})$ -term is now eliminated with the update law (cf. (3.55))

$$\dot{\bar{\vartheta}} = -\Gamma \left( \frac{\partial \alpha}{\partial x} F \right)^T z, \quad (3.60)$$

and the control (3.53) is chosen to make the bracketed term multiplying  $z$  in (3.59) equal to  $-cz$  (cf. (2.54)):

$$u = -cz + \frac{\partial \alpha}{\partial x} (f + F\bar{\vartheta} + g(\alpha + z)) + \frac{\partial \alpha}{\partial \vartheta} T - \frac{\partial V}{\partial x} g. \quad (3.61)$$

This results in the desired nonpositivity of  $\dot{V}_a$ :

$$\dot{V}_a \leq -W(x, \vartheta) - cz^2 \leq 0. \quad (3.62)$$

From (3.56) and (3.62) we conclude that  $V(x, \vartheta)$ ,  $\bar{\vartheta}$  and  $z$  are bounded. By Assumption 3.1, this means that  $x(t)$  and  $\vartheta(t)$  are bounded. Hence,  $\xi = z + \alpha(x, \vartheta)$  and  $u$  are bounded. By Theorem 2.1, the boundedness of all the signals combined with (3.62) proves the regulation of  $W(x(t), \vartheta(t))$  and  $z(t)$ .  $\square$

### 3.2.2 Adaptive block backstepping

We now extend the Adaptive Backstepping Lemma (Lemma 3.2) by augmenting the initial system with a relative-degree-one nonlinear system whose zero dynamics subsystem is ISS, just like we did in Chapter 2, Lemmas 2.8 and 2.25. The adaptive counterpart of Assumption 2.7 was Assumption 3.1. We now formulate the adaptive counterpart of Assumption 2.21, with analogous changes in the properties of  $V(x, \vartheta)$  from Assumption 3.1.

**Assumption 3.3** Suppose Assumption 3.1 is valid, but  $V(x, \vartheta)$  is only positive semidefinite, and the closed-loop system (3.49) with the adaptive controller (3.50) has the property that  $x(t)$  and  $\vartheta(t)$  are bounded if  $V(x(t), \vartheta(t))$  is bounded.  $\square$

Under this assumption, the control (3.50), applied to the system (3.49), guarantees global boundedness of  $x(t), \vartheta(t)$  and, by Lemma A.6, regulation of  $W(x(t), \vartheta(t))$ .

**Lemma 3.4 (Adaptive Block Backstepping)** Let the system (3.49) be augmented by a nonlinear system which is linear in the unknown parameter vector  $\theta$ ,

$$\dot{x} = f(x) + F(x)\theta + g(x)y \quad (3.63a)$$

$$\dot{\xi} = m(x, \xi) + M(x, \xi)\theta + \beta(x, \xi)u, \quad y = h(\xi), \quad (3.63b)$$

where  $\xi \in \mathbb{R}^q$ , and suppose that (3.63b) has relative degree one uniformly in  $x$  and that its zero dynamics subsystem is ISS with respect to  $y$  and  $x$ . Under Assumption 3.3, the feedback control

$$u = \left[ \frac{\partial h}{\partial \xi}(\xi)\beta(x, \xi) \right]^{-1} \left\{ -c(y - \alpha(x, \vartheta)) - \frac{\partial h}{\partial \xi}(\xi) [m(x, \xi) + M(x, \xi)\bar{\vartheta}] + \frac{\partial \alpha}{\partial x}(x, \vartheta) [f(x) + F(x)\bar{\vartheta} + g(x)y] + \frac{\partial \alpha}{\partial \vartheta} T(x, \vartheta) - \frac{\partial V}{\partial x}(x, \vartheta)g(x) \right\}, \quad (3.64)$$

with  $c > 0$  and  $\bar{\vartheta}$  a new estimate of  $\theta$ , along with the update laws

$$\dot{\bar{\vartheta}} = T(x, \vartheta) \quad (3.65)$$

$$\dot{\bar{\vartheta}} = \Gamma \left[ \frac{\partial h}{\partial \xi}(\xi)M(x, \xi) - \frac{\partial \alpha}{\partial x}(x, \vartheta)F(x) \right]^T (y - \alpha(x, \vartheta)), \quad (3.66)$$

with the adaptation gain matrix  $\Gamma = \Gamma^T > 0$ , guarantees global boundedness of  $x(t), \xi(t), \vartheta(t), \bar{\vartheta}(t)$  and regulation of  $W(x(t), \vartheta(t))$  and  $\xi(t) - \alpha(x(t), \vartheta(t))$ .

**Proof.** As in Lemma 2.25, we employ the change of coordinates  $(y, \zeta) = (h(\xi), \phi(x, \xi))$ , with  $\frac{\partial \phi}{\partial \xi}\beta \equiv 0$ , to transform (3.63b) into the normal form

$$\dot{y} = \frac{\partial h}{\partial \xi}(\xi) [m(x, \xi) + M(x, \xi)\theta + \beta(x, \xi)u] \quad (3.67a)$$

$$\begin{aligned} \dot{\zeta} &= \frac{\partial \phi}{\partial x}(x, \xi) [f(x) + F(x)\theta + g(x)y] + \frac{\partial \phi}{\partial \xi}(x, \xi) [m(x, \xi) + M(x, \xi)\theta] \\ &\triangleq \Phi_0(x, y, \zeta) + \Phi(x, y, \zeta)\theta. \end{aligned} \quad (3.67b)$$

Introducing a new parameter estimate  $\bar{\vartheta}$ , we use the feedback transformation

$$u = \left( \frac{\partial h}{\partial \xi}\beta \right)^{-1} \left\{ v - \frac{\partial h}{\partial \xi} [m + M\bar{\vartheta}] \right\} \quad (3.68)$$

to rewrite (3.63a) and (3.67a) as

$$\dot{x} = f(x) + F(x)\theta + g(x)y \quad (3.69a)$$

$$\dot{y} = v + \frac{\partial h}{\partial \xi}(\xi)M(x, \xi)(\theta - \bar{\vartheta}). \quad (3.69b)$$

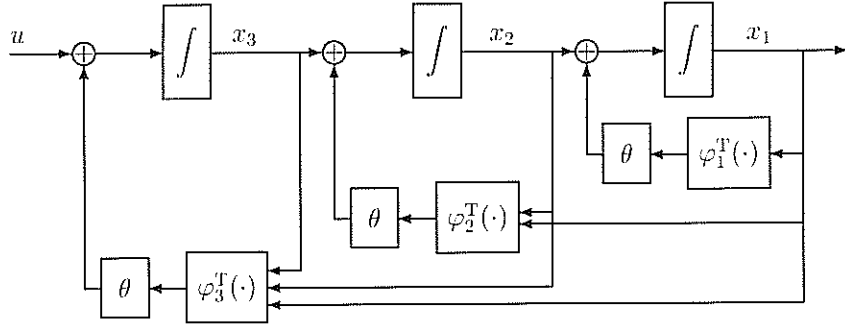
We now apply Lemma 3.1 to (3.69). The only difference between (3.69) and (3.52) is the presence of the additional parameter error term  $\frac{\partial h}{\partial \xi}M(\theta - \bar{\vartheta})$  in (3.69b). This term can be eliminated in  $\dot{V}_a$  by adding the term  $-\Gamma[\frac{\partial h}{\partial \xi}M]^T(y - \alpha)$  to the update law (3.55). Combining this modification with (3.68), we see that the resulting adaptive controller is given by (3.64)–(3.66). This guarantees the boundedness of  $x, \vartheta, \bar{\vartheta}, z$  and the regulation of  $W(x, \vartheta)$  and  $z$ . Hence,  $y = z + \alpha(x, \vartheta)$  is bounded. Then, from (3.67b) and the ISS property of the zero dynamics,  $\zeta$  is also bounded, and thus  $\xi$  and  $u$  are bounded.  $\square$

## 3.3 Recursive Design Procedures

### 3.3.1 Parametric strict-feedback systems

Through repeated application of Lemma 3.2, the backstepping design procedure is now generalized to nonlinear systems which can be transformed<sup>1</sup> into

<sup>1</sup>The coordinate-free characterization of these systems in terms of differential geometric conditions is given in Appendix G, Corollary G.15.



**Figure 3.4:** Block diagram of a third-order parametric strict-feedback system with  $\beta(x) = 1$ . The nonlinearities depend only on variables which are “fed back.”

the *parametric strict-feedback form*

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1^T(x_1)\theta \\ \dot{x}_2 &= x_3 + \varphi_2^T(x_1, x_2)\theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}^T(x_1, \dots, x_{n-1})\theta \\ \dot{x}_n &= \beta(x)u + \varphi_n^T(x)\theta,\end{aligned}\quad (3.70)$$

where  $\beta(x) \neq 0$  for all  $x \in \mathbb{R}^n$ . The reason for the name “parametric strict-feedback” can be deduced from the block diagram in Figure 3.4, where, except for the integrators, there are only feedback paths.

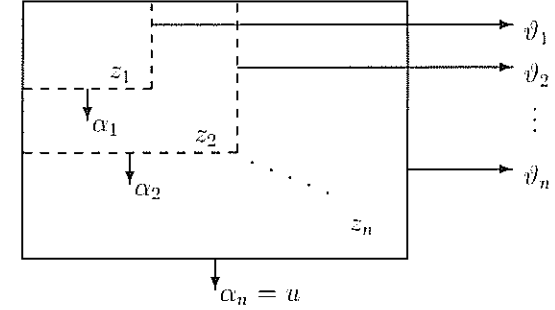
For systems in the form (3.70), the number of design steps required is equal to the degree  $n$  of the system. At each step, an error variable  $z_i$ , a stabilizing function  $\alpha_i$ , and a parameter estimate  $\vartheta_i$  are generated. As a result, if a system contains  $p$  unknown parameters, the overparametrized adaptive controller may employ as many as  $pn$  parameter estimates. A schematic representation of this design procedure is given in Figure 3.5, and the resulting expressions are summarized in the following theorem:

**Theorem 3.5 (Parametric Strict-Feedback Systems)** *For the system (3.70) with  $\beta(x) \neq 0$  for all  $x \in \mathbb{R}^n$ , consider the adaptive controller*

$$u = \frac{1}{\beta(x)}\alpha_n(x, \vartheta_1, \dots, \vartheta_n) \quad (3.71)$$

$$\dot{\vartheta}_i = \Gamma \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right) z_i, \quad i = 1, \dots, n, \quad (3.72)$$

where  $\vartheta_i \in \mathbb{R}^p$  are multiple estimates of  $\theta$ ,  $\Gamma = \Gamma^T > 0$  is the adaptation gain matrix, and the variables  $z_i$  and the stabilizing functions  $\alpha_i$ ,  $i = 1, \dots, n$ ,



**Figure 3.5:** The design procedure for overparametrized schemes. Each step generates an error variable  $z_i$ , a stabilizing function  $\alpha_i$ , and a *new* estimate  $\vartheta_i$  of the unknown parameter vector  $\theta$ .

are defined by the following recursive expressions (with  $c_i > 0$  being design constants, and  $z_0 \equiv \alpha_0 \equiv 0$  used for notational convenience):

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_{i-1}, \vartheta_1, \dots, \vartheta_{i-1}) \quad (3.73)$$

$$\begin{aligned}\alpha_i &= -c_i z_i - z_{i-1} - \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^T \vartheta_i \\ &\quad + \sum_{j=1}^{i-1} \left[ \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left( \varphi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k \right) z_j \right].\end{aligned} \quad (3.74)$$

This overparametrized adaptive controller guarantees global boundedness of  $x(t)$ ,  $\vartheta_1(t), \dots, \vartheta_n(t)$ , and regulation of  $x_1(t)$  and  $x_i(t) - x_i^e$ ,  $i = 2, \dots, n$ , where  $x_i^e = -\theta^T \varphi_{i-1}(0, x_2^e, \dots, x_{i-1}^e)$ .

**Proof.** Using the definitions (3.73), (3.74) and denoting  $x_0 \equiv \alpha_0 \equiv 0$ ,  $x_{n+1} \equiv \beta(x)u$ , the derivative of the error variable  $z_i$ ,  $i = 1, \dots, n$ , becomes

$$\begin{aligned}\dot{z}_i &= \dot{x}_i - \dot{\alpha}_{i-1}(x_1, \dots, x_{i-1}, \vartheta_1, \dots, \vartheta_{i-1}) \\ &= x_{i+1} + \varphi_i^T \theta - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \underbrace{\left( x_{j+1} + \varphi_j^T \theta \right)}_{\dot{x}_j} \\ &\quad - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \underbrace{\left( \varphi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k \right)}_{\dot{\vartheta}_j} z_j \\ &= \alpha_i + z_{i+1} + \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^T \theta\end{aligned}$$

$$\begin{aligned}
& -\sum_{j=1}^{i-1} \left[ \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left( \varphi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k \right) z_j \right] \\
& = -c_i z_i - z_{i-1} + z_{i+1} + \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right)^T (\theta - \vartheta_i). \quad (3.75)
\end{aligned}$$

The choice of control (3.71) guarantees that  $z_{n+1} \equiv 0$ . The closed-loop error system can therefore be expressed as

$$\begin{aligned}
\dot{z}_1 &= -c_1 z_1 + z_2 + \varphi_1^T (\theta - \vartheta_1) \\
\dot{z}_2 &= -c_2 z_2 - z_1 + z_3 + \left( \varphi_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi_1 \right)^T (\theta - \vartheta_2) \\
&\vdots \\
\dot{z}_{n-1} &= -c_{n-1} z_{n-1} - z_{n-2} + z_n + \left( \varphi_{n-1} - \sum_{j=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial x_j} \varphi_j \right)^T (\theta - \vartheta_{n-1}) \\
\dot{z}_n &= -c_n z_n - z_{n-1} + \left( \varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j \right)^T (\theta - \vartheta_n) \\
\dot{\vartheta}_i &= -\Gamma \left( \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right) z_i,
\end{aligned} \quad (3.76)$$

or, equivalently, in the matrix form

$$\begin{aligned}
\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & -c_{n-1} & 1 \\ 0 & \cdots & 0 & -1 & -c_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} \\
&+ \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & w_n \end{bmatrix}^T \begin{bmatrix} \theta - \vartheta_1 \\ \theta - \vartheta_2 \\ \vdots \\ \theta - \vartheta_n \end{bmatrix} \quad (3.77)
\end{aligned}$$

$$\frac{d}{dt} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_n \end{bmatrix} = \begin{bmatrix} \Gamma & 0 & \cdots & 0 \\ 0 & \Gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma \end{bmatrix} \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & w_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix},$$

where we have used the convenient notation

$$w_1 = \varphi_1, \quad w_i = \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j, \quad i = 2, \dots, n. \quad (3.78)$$

This system has two important properties: (i) The  $z$ -system matrix in (3.77) has negative diagonal and skew-symmetric off-diagonal terms, and (ii) the transpose of the matrix that multiplies the parameter errors in the  $\dot{z}$ -equation appears in the update law. This structure is a result of the design procedure, and it allows us to use the simple quadratic Lyapunov function

$$V_n(z_1, \dots, z_n, \vartheta_1, \dots, \vartheta_n) = \frac{1}{2} \sum_{i=1}^n [z_i^2 + (\theta - \vartheta_i)^T \Gamma^{-1} (\theta - \vartheta_i)] \quad (3.79)$$

to prove stability and regulation. Its derivative along the solutions of (3.77) is

$$\begin{aligned}
\dot{V}_n &= z^T \dot{z} - \sum_{i=1}^n \dot{\vartheta}_i^T \Gamma^{-1} (\theta - \vartheta_i) \\
&= \sum_{i=1}^n [-c_i z_i^2 + z_i w_i^T (\theta - \vartheta_i) - z_i w_i^T (\theta - \vartheta_i)] \\
&= -\sum_{i=1}^n c_i z_i^2. \quad (3.80)
\end{aligned}$$

The LaSalle-Yoshizawa theorem (Theorem 2.1) now guarantees the global uniform boundedness of  $z(t)$ ,  $\vartheta_1(t), \dots, \vartheta_n(t)$ , as well as the regulation of  $z(t)$ . Since  $z_1 = x_1$ , we see that  $x_1$  is also bounded and regulated. The boundedness of  $x_2, \dots, x_n$  then follows from the boundedness of  $\alpha_i$  (defined in (3.74)) and the fact that  $x_i = z_i + \alpha_{i-1}$ . Since  $x$  is bounded,  $\beta(x)$  is bounded away from zero. Combining this with (3.71) we conclude that the control  $u$  is also bounded. Finally, the regulation of  $x_i - x_i^*$  is concluded as follows: Since  $z_i(t)$ ,  $i = 1, \dots, n$  converge to zero,  $\dot{\vartheta}_i(t)$ ,  $i = 1, \dots, n$  also converge to zero and  $\dot{z}_i(t)$  is integrable over  $[0, \infty)$ . Furthermore, the boundedness of all the signals and their derivatives guarantees the boundedness of  $\ddot{z}_i(t)$  and hence the uniform continuity of  $\dot{z}_i(t)$ . From Lemma A.6, we conclude that  $\lim_{t \rightarrow \infty} \dot{z}_i(t) = 0$ ,  $i = 1, \dots, n$ . Since  $x$  can be expressed as a smooth vector function of  $z_1, \dots, z_n$  and  $\vartheta_1, \dots, \vartheta_n$ , we can express  $\dot{x}$  as a linear combination of  $\dot{z}_i$  and  $\dot{\vartheta}_i$  with coefficients which are bounded because they are smooth vector functions of the bounded signals  $z_1, \dots, z_n$  and  $\vartheta_1, \dots, \vartheta_n$ . Hence, the convergence of  $\dot{z}$  and  $\dot{\vartheta}_i$  to zero implies that  $\dot{x}$  converges to zero. Combining this with (3.70) and the regulation of  $x_1$  leads to the desired result.  $\square$

### 3.3.2 Multi-input systems

The adaptive backstepping design procedure of Theorem 3.5 can be easily extended to nonlinear systems which have been transformed into the *multi-input parametric strict-feedback form*

$$\begin{aligned}
\dot{x}_{1,1} &= x_{1,2} + \varphi_{1,1}^T(x_{1,1}, x_{2,1}, \dots, x_{2,\rho_2-\rho_1+1}, \dots, x_{m,1}, \dots, x_{m,\rho_m-\rho_1+1})\theta \\
\dot{x}_{1,2} &= x_{1,3} + \varphi_{1,2}^T(x_{1,1}, x_{1,2}, x_{2,1}, \dots, x_{2,\rho_2-\rho_1+2}, \\
&\quad \dots, x_{m,1}, \dots, x_{m,\rho_m-\rho_1+2})\theta \\
&\vdots \\
\dot{x}_{1,\rho_1-1} &= x_{1,\rho_1} + \varphi_{1,\rho_1-1}^T(x_{1,1}, \dots, x_{1,\rho_1-1}, x_{2,1}, \dots, x_{2,\rho_2-1}, \\
&\quad \dots, x_{m,1}, \dots, x_{m,\rho_m-1})\theta \\
\dot{x}_{1,\rho_1} &= \sum_{j=1}^m \beta_{1,j}(x)u_j + \varphi_{1,\rho_1}^T(x)\theta \\
&\vdots \\
\dot{x}_{i,j} &= x_{i,j+1} + \varphi_{i,j}^T(x_{1,1}, \dots, x_{1,\rho_1-\rho_i+j}, \dots, x_{i,1}, \dots, x_{i,j}, \\
&\quad \dots, x_{m,1}, \dots, x_{m,\rho_m-\rho_i+j})\theta \\
&\vdots \\
\dot{x}_{m,1} &= x_{m,2} + \varphi_{m,1}^T(x_{1,1}, \dots, x_{1,\rho_1-\rho_m+1}, x_{2,1}, \dots, x_{2,\rho_2-\rho_m+1}, \dots, x_{m,1})\theta \\
\dot{x}_{m,2} &= x_{m,3} + \varphi_{m,2}^T(x_{1,1}, \dots, x_{1,\rho_1-\rho_m+2}, x_{2,1}, \dots, x_{2,\rho_2-\rho_m+2}, \\
&\quad \dots, x_{m,1}, x_{m,2})\theta \\
&\vdots \\
\dot{x}_{m,\rho_m-1} &= x_{m,\rho_m} + \varphi_{m,\rho_m-1}^T(x_{1,1}, \dots, x_{1,\rho_1-1}, x_{2,1}, \dots, x_{2,\rho_2-1}, \\
&\quad \dots, x_{m,1}, \dots, x_{m,\rho_m-1})\theta \\
\dot{x}_{m,\rho_m} &= \sum_{j=1}^m \beta_{m,j}(x)u_j + \varphi_{m,\rho_m}^T(x)\theta,
\end{aligned} \tag{3.81}$$

where  $u_1, \dots, u_m$  are the inputs, and the input matrix is nonsingular  $\forall x \in \mathbb{R}^n$  ( $n = \rho_1 + \dots + \rho_m$ ):

$$\det B(x) \neq 0, \quad \forall x \in \mathbb{R}^n, \quad B(x) = \begin{bmatrix} \beta_{1,1}(x) & \dots & \beta_{1,m}(x) \\ \vdots & & \vdots \\ \beta_{m,1}(x) & \dots & \beta_{m,m}(x) \end{bmatrix}. \tag{3.82}$$

The design procedure for this class of systems consists of applying the design procedure of Theorem 3.5 to the first  $\rho_i - 1$  equations of each of the  $m$  subsystems of (3.81), to obtain the system

$$\begin{aligned}
\dot{z}_{i,j} &= -c_{i,j}z_{i,j} - z_{i,j-1} + z_{i,j+1} + w_{i,j}^T(x, \vartheta_1, \dots, \vartheta_{\ell-1})(\theta - \vartheta_\ell) \\
\ell &= \sum_{k=1}^{i-1} (\rho_k - 1) + j, \quad 1 \leq j \leq \rho_i - 1, \quad 1 \leq i \leq m \\
\dot{\vartheta}_\ell &= \Gamma w_{i,j}(x, \vartheta_1, \dots, \vartheta_\ell)z_{i,j}, \quad 1 \leq \ell \leq n - m
\end{aligned} \tag{3.83}$$

$$\frac{d}{dt} \begin{bmatrix} z_{1,\rho_1} \\ \vdots \\ z_{m,\rho_m} \end{bmatrix} = B(x)u + \Phi(x, \vartheta_1, \dots, \vartheta_{n-m}) + W_{n-m+1}^T(x, \vartheta_1, \dots, \vartheta_{n-m})\theta,$$

where the functions  $w_{i,j}$ ,  $\Phi$  and  $W_{n-m+1}$  are defined appropriately. Now let  $\vartheta_{n-m+1}$  be a new estimate of  $\theta$  and define the control  $u$  as

$$\begin{aligned}
u &= B^{-1}(x) \left\{ - \begin{bmatrix} c_{1,\rho_1}z_{1,\rho_1} + z_{1,\rho_1-1} \\ \vdots \\ c_{m,\rho_m}z_{m,\rho_m} + z_{m,\rho_m-1} \end{bmatrix} - \Phi(x, \vartheta_1, \dots, \vartheta_{n-m}) \right. \\
&\quad \left. - W_{n-m+1}^T(x, \vartheta_1, \dots, \vartheta_{n-m})\vartheta_{n-m+1} \right\}, \tag{3.84}
\end{aligned}$$

and the update law for  $\vartheta_{n-m+1}$  as

$$\dot{\vartheta}_{n-m+1} = \Gamma W_{n-m+1}(x, \vartheta_1, \dots, \vartheta_{n-m+1}) \begin{bmatrix} z_{1,\rho_1} \\ \vdots \\ z_{m,\rho_m} \end{bmatrix}. \tag{3.85}$$

The stability properties of the resulting closed-loop system are analogous to those listed in Theorem 3.5, and can be similarly established using the Lyapunov function

$$V(z, \vartheta_1, \dots, \vartheta_{n-m+1}) = \frac{1}{2}z^T z + \frac{1}{2} \sum_{i=1}^{n-m+1} (\theta - \vartheta_i)^T \Gamma^{-1} (\theta - \vartheta_i). \tag{3.86}$$

### 3.3.3 Parametric block-strict-feedback systems

Lemma 3.4 can be applied repeatedly to design adaptive controllers for nonlinear systems which can be transformed, after a change of coordinates, into the *parametric block-strict-feedback form*

$$\begin{aligned}
\dot{\chi}_1 &= \tilde{f}_1(\chi_1) + \tilde{F}_1(\chi_1)\theta + \tilde{g}_1(\chi_1)y_2 \\
y_1 &= h_1(\chi_1) \\
\dot{\chi}_2 &= \tilde{f}_2(\chi_1, \chi_2) + \tilde{F}_2(\chi_1, \chi_2)\theta + \tilde{g}_2(\chi_1, \chi_2)y_3 \\
y_2 &= h_2(\chi_2) \\
&\vdots \\
\dot{\chi}_i &= \tilde{f}_i(\chi_1, \dots, \chi_i) + \tilde{F}_i(\chi_1, \dots, \chi_i)\theta + \tilde{g}_i(\chi_1, \dots, \chi_i)y_{i+1} \\
y_i &= h_i(\chi_i) \\
&\vdots \\
\dot{\chi}_{\rho-1} &= \tilde{f}_{\rho-1}(\chi_1, \dots, \chi_{\rho-1}) + \tilde{F}_{\rho-1}(\chi_1, \dots, \chi_{\rho-1})\theta + \tilde{g}_{\rho-1}(\chi_1, \dots, \chi_{\rho-1})y_\rho \\
y_{\rho-1} &= h_{\rho-1}(\chi_{\rho-1}) \\
\dot{\chi}_\rho &= \tilde{f}_\rho(\chi) + \tilde{F}_\rho(\chi)\theta + \tilde{g}_\rho(\chi)u \\
y_\rho &= h_\rho(\chi_\rho),
\end{aligned} \tag{3.87}$$

where each of the  $\rho$  subsystems with state  $\chi_i \in \mathbb{R}^{n_i}$ , output  $y_i \in \mathbb{R}$ , and input  $y_{i+1}$  (for convenience we denote  $y_{\rho+1} \equiv u$ ) satisfies conditions (BSF-1) and (BSF-2) (see Chapter 2, equation (2.198)), that is, it has relative degree one uniformly in  $\chi_1, \dots, \chi_{i-1}$ , and its zero dynamics subsystem is ISS with respect to  $\chi_1, \dots, \chi_{i-1}, y_i$ .

Using the change of coordinates which transformed (2.198) into (2.201) in Section 2.3.3, we can now transform the system (3.87) into

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, \zeta_1) + \bar{\varphi}_1^T(y_1, \zeta_1)\theta + g_1(y_1, \zeta_1)y_2 \\ \dot{y}_2 &= f_2(y_1, \zeta_1, y_2, \zeta_2) + \bar{\varphi}_2^T(y_1, \zeta_1, y_2, \zeta_2)\theta + g_2(y_1, \zeta_1, y_2, \zeta_2)y_3 \\ &\vdots \\ \dot{y}_{\rho-1} &= f_{\rho-1}(y_1, \zeta_1, \dots, y_{\rho-1}, \zeta_{\rho-1}) + \bar{\varphi}_{\rho-1}^T(y_1, \zeta_1, \dots, y_{\rho-1}, \zeta_{\rho-1})\theta \\ &\quad + g_{\rho-1}(y_1, \zeta_1, \dots, y_{\rho-1}, \zeta_{\rho-1})y_\rho \\ \dot{y}_\rho &= f_\rho(y_1, \zeta_1, \dots, y_\rho, \zeta_\rho) + \bar{\varphi}_\rho^T(y_1, \zeta_1, \dots, y_\rho, \zeta_\rho)\theta + g_\rho(y_1, \zeta_1, \dots, y_\rho, \zeta_\rho)u \\ \dot{\zeta}_1 &= \bar{\Phi}_{1,0}(y_1, \zeta_1) + \bar{\Phi}_1(y_1, \zeta_1)\theta \\ &\vdots \\ \dot{\zeta}_\rho &= \bar{\Phi}_{\rho,0}(y_1, \zeta_1, \dots, y_{\rho-1}, \zeta_{\rho-1}, y_\rho, \zeta_\rho) + \bar{\Phi}_\rho(y_1, \zeta_1, \dots, y_{\rho-1}, \zeta_{\rho-1}, y_\rho, \zeta_\rho)\theta. \end{aligned} \quad (3.88)$$

Then we employ another change of coordinates which replaces  $y_i$  by  $x_i = \psi_i(y_1, \zeta_1, \dots, y_{i-1}, \zeta_{i-1}, y_i)$ , where

$$\begin{aligned} x_1 &= y_1 \triangleq \psi_1(y_1) \\ x_2 &= \frac{\partial \psi_1}{\partial y_1}(f_1 + g_1 y_2) = f_1(y_1, \zeta_1) + g_1(y_1, \zeta_1)y_2 \triangleq \psi_2(y_1, \zeta_1, y_2) \\ x_{i+1} &= \sum_{j=1}^{i-1} \frac{\partial \psi_i}{\partial y_j}(f_j + g_j y_{j+1}) + \sum_{j=1}^{i-1} \frac{\partial \psi_i}{\partial \zeta_j} \bar{\Phi}_{j,0} + g_1 \cdots g_{i-1} f_i + g_1 \cdots g_i y_{i+1} \\ &\triangleq \psi_{i+1}(y_1, \zeta_1, \dots, y_i, \zeta_i, y_{i+1}), \quad i = 2, \dots, \rho-1. \end{aligned} \quad (3.89)$$

Finally, we use the feedback transformation

$$v = \sum_{j=1}^{\rho-1} \frac{\partial \psi_\rho}{\partial y_j}(f_j + g_j y_{j+1}) + \sum_{j=1}^{\rho-1} \frac{\partial \psi_\rho}{\partial \zeta_j} \bar{\Phi}_{j,0} + g_1 \cdots g_{\rho-1} f_\rho + g_1 \cdots g_\rho u. \quad (3.90)$$

Condition (BSF-1) guarantees that  $g_1, \dots, g_\rho \neq 0$  everywhere. Hence, the change of coordinates (3.89) relating  $[y_1, \dots, y_\rho, \zeta_1^T, \dots, \zeta_\rho^T]^T$  to  $[x_1, \dots, x_\rho, \zeta_1^T, \dots, \zeta_\rho^T]^T$  is a global diffeomorphism, and the feedback transformation (3.90) relating  $u$  to  $v$  is nonsingular. It is now straightforward to verify that (3.89) and (3.90) transform (3.88) into a form reminiscent of the

parametric strict-feedback form (3.70):

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1^T(x_1, \zeta_1)\theta \\ \dot{x}_2 &= x_3 + \varphi_2^T(x_1, x_2, \zeta_1, \zeta_2)\theta \\ &\vdots \\ \dot{x}_{\rho-1} &= x_\rho + \varphi_{\rho-1}^T(x_1, \dots, x_{\rho-1}, \zeta_1, \dots, \zeta_{\rho-1})\theta \\ \dot{x}_\rho &= v + \varphi_\rho^T(x, \zeta)\theta \\ \dot{\zeta}_1 &= \Phi_{1,0}(x_1, \zeta_1) + \Phi_1(x_1, \zeta_1)\theta \\ &\vdots \\ \dot{\zeta}_\rho &= \Phi_{\rho,0}(x_1, \dots, x_\rho, \zeta_1, \dots, \zeta_{\rho-1}, \zeta_\rho) + \Phi_\rho(x_1, \dots, x_\rho, \zeta_1, \dots, \zeta_{\rho-1}, \zeta_\rho)\theta \\ y &= x_1. \end{aligned} \quad (3.91)$$

In (3.91) each  $\zeta_i$ -subsystem is ISS with respect to  $x_1, \dots, x_i, \zeta_1, \dots, \zeta_{i-1}$  as its inputs, and  $\varphi_i, \Phi_{i,0}, \Phi_i, i = 1, \dots, \rho$  are defined as

$$\begin{aligned} \varphi_i^T(x_1, \dots, x_i, \zeta_1, \dots, \zeta_i) &\triangleq \sum_{j=1}^i \frac{\partial \psi_i}{\partial y_j}(y_1, \zeta_1, \dots, y_{i-1}, \zeta_{i-1}, y_i) \\ &\quad \bar{\varphi}_j^T(y_1, \zeta_1, \dots, y_j, \zeta_j) \\ &\quad + \sum_{j=1}^{i-1} \frac{\partial \psi_i}{\partial \zeta_j}(y_1, \zeta_1, \dots, y_{i-1}, \zeta_{i-1}, y_i) \\ &\quad \bar{\Phi}_j(y_1, \zeta_1, \dots, y_j, \zeta_j) \end{aligned} \quad (3.92)$$

$$\Phi_{i,0}(x_1, \dots, x_i, \zeta_1, \dots, \zeta_i) \triangleq \bar{\Phi}_{i,0}(y_1, \zeta_1, \dots, y_i, \zeta_i) \quad (3.93)$$

$$\Phi_i(x_1, \dots, x_i, \zeta_1, \dots, \zeta_i) \triangleq \bar{\Phi}_i(y_1, \zeta_1, \dots, y_i, \zeta_i). \quad (3.94)$$

It is now clear that the class of parametric block-strict-feedback nonlinear systems strictly contains the class of parametric strict-feedback nonlinear systems, since (3.70) can be obtained by setting  $n_i = 1, \rho = n$ , and  $v = \beta(x)u$  in (3.91).

We now state and prove the generalization of Theorem 3.5 to block-strict-feedback systems of the form (3.91).

**Theorem 3.6 (Parametric Block-Strict-Feedback Systems)** *For the system (3.91), consider the adaptive controller*

$$v = \alpha_\rho(x_1, \dots, x_\rho, \zeta_1, \dots, \zeta_\rho, \vartheta_1, \dots, \vartheta_\rho) \quad (3.95)$$

$$\dot{\vartheta}_i = -\Gamma \left[ \varphi_i^T - \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j^T + \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_j \right) \right]^T z_i, \quad i = 1, \dots, \rho, \quad (3.96)$$

where  $\vartheta_i \in \mathbb{R}^p$  are multiple estimates of  $\theta$ ,  $\Gamma = \Gamma^T > 0$  is the adaptation gain matrix, and the variables  $z_i$  and the stabilizing functions  $\alpha_i$ ,  $i = 1, \dots, \rho$ , are defined by the following recursive expressions (with  $c_i > 0$  being design constants, and  $z_0 \equiv \alpha_0 \equiv 0$  used for notational convenience):

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_{i-1}, \zeta_1, \dots, \zeta_{i-1}, \vartheta_1, \dots, \vartheta_{i-1}) \quad (3.97)$$

$$\begin{aligned} \alpha_i = & -c_i z_i - z_{i-1} - \left[ \varphi_i^T - \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j^T + \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_j \right) \right] \vartheta_i + \sum_{j=1}^{i-1} \left\{ \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} \right. \\ & \left. + \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_{j,0} - \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left[ \varphi_j^T - \sum_{k=1}^{j-1} \left( \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k^T + \frac{\partial \alpha_{j-1}}{\partial \zeta_k} \Phi_k \right) \right]^T z_j \right\}. \end{aligned} \quad (3.98)$$

This overparametrized adaptive controller guarantees global boundedness of  $x_1, \dots, x_\rho$ ,  $\zeta_1, \dots, \zeta_\rho$ ,  $\vartheta_1, \dots, \vartheta_\rho$  and regulation of  $y = x_1$ .

**Proof.** As one would expect from the similarities between the systems (3.91) and (3.70), the expressions (3.95)–(3.98) are similar to (3.71)–(3.74). Using the same arguments as in the proof of Theorem 3.5, we write the derivative of the error variable  $z_i$ ,  $i = 1, \dots, \rho$ , as

$$\begin{aligned} \dot{z}_i = & \dot{x}_i - \dot{\alpha}_{i-1}(x_1, \dots, x_{i-1}, \zeta_1, \dots, \zeta_{i-1}, \vartheta_1, \dots, \vartheta_{i-1}) \\ = & x_{i+1} + \varphi_i^T \theta - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \underbrace{(x_{j+1} + \varphi_j^T \theta)}_{\dot{x}_j} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \underbrace{(\Phi_{j,0} + \Phi_j \theta)}_{\dot{\zeta}_j} \\ & + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \underbrace{\left[ \varphi_j^T - \sum_{k=1}^{j-1} \left( \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k^T + \frac{\partial \alpha_{j-1}}{\partial \zeta_k} \Phi_k \right) \right]^T}_{-\dot{\vartheta}_j} z_j \\ = & \alpha_i + z_{i+1} + \left[ \varphi_i^T - \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j^T + \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_j \right) \right] \theta - \sum_{j=1}^{i-1} \left\{ \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} \right. \\ & \left. + \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_{j,0} - \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left[ \varphi_j^T - \sum_{k=1}^{j-1} \left( \frac{\partial \alpha_{j-1}}{\partial x_k} \varphi_k^T + \frac{\partial \alpha_{j-1}}{\partial \zeta_k} \Phi_k \right) \right]^T z_j \right\} \\ = & -c_i z_i - z_{i-1} + z_{i+1} + \left[ \varphi_i^T - \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j^T + \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_j \right) \right]^T (\theta - \vartheta_i). \end{aligned} \quad (3.99)$$

The closed-loop error system can thus be expressed in the matrix form

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{\rho-1} \\ z_\rho \end{bmatrix} = & \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & -c_{\rho-1} & 1 \\ 0 & \cdots & 0 & -1 & -c_\rho \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{\rho-1} \\ z_\rho \end{bmatrix} \\ & + \begin{bmatrix} \bar{w}_1 & 0 & \cdots & 0 \\ 0 & \bar{w}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{w}_\rho \end{bmatrix}^T \begin{bmatrix} \theta - \vartheta_1 \\ \theta - \vartheta_2 \\ \vdots \\ \theta - \vartheta_\rho \end{bmatrix} \\ \frac{d}{dt} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_\rho \end{bmatrix} = & \begin{bmatrix} \Gamma & 0 & \cdots & 0 \\ 0 & \Gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma \end{bmatrix} \begin{bmatrix} \bar{w}_1 & 0 & \cdots & 0 \\ 0 & \bar{w}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{w}_\rho \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\rho \end{bmatrix}, \end{aligned} \quad (3.100)$$

with the notation

$$\bar{w}_1 = \varphi_1, \quad \bar{w}_i = \varphi_i - \sum_{j=1}^{i-1} \left[ \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j + \left( \frac{\partial \alpha_{i-1}}{\partial \zeta_j} \Phi_j \right)^T \right], \quad i = 2, \dots, \rho. \quad (3.101)$$

The structural properties of this error system are once again apparent. The derivative of the nonnegative function

$$V_n(z_1, \dots, z_\rho, \vartheta_1, \dots, \vartheta_\rho) = \frac{1}{2} \sum_{i=1}^{\rho} [z_i^2 + (\theta - \vartheta_i)^T \Gamma^{-1} (\theta - \vartheta_i)] \quad (3.102)$$

along the solutions of (3.100) is

$$\dot{V}_n = - \sum_{i=1}^{\rho} c_i z_i^2. \quad (3.103)$$

The LaSalle-Yoshizawa theorem (Theorem 2.1) now guarantees the global uniform boundedness of  $z(t)$ ,  $\vartheta_1(t), \dots, \vartheta_\rho(t)$ , as well as the regulation of  $z(t)$ . Since  $z_1 = x_1$ , we see that  $y = x_1$  is also bounded and regulated. The boundedness of  $x_2, \dots, x_\rho$ ,  $\zeta_1, \dots, \zeta_\rho$  and of the transformed control variable  $v$  is then established via an induction argument for  $i = 2, \dots, \rho + 1$  (with  $x_{\rho+1} \equiv v$ ): If  $x_1, \dots, x_{i-1}$  and  $\zeta_1, \dots, \zeta_{i-2}$  are bounded, the ISS property of the  $\zeta_{i-1}$ -subsystem guarantees that  $\zeta_{i-1}$  is bounded. Hence,  $\alpha_{i-1}(x_1, \dots, x_{i-1}, \zeta_1, \dots, \zeta_{i-1}, \vartheta_1, \dots, \vartheta_{i-1})$  is bounded, which implies that  $x_i = z_i + \alpha_{i-1}$  is also bounded.  $\square$

We should note that the adaptive controller (3.95)–(3.96), when applied to the system (3.87) using the expressions (3.89) and (3.90), guarantees the global uniform boundedness of  $\chi$  and  $u$ , as well as the regulation of  $y = h_1(\chi_1)$ . This follows from the fact that  $\frac{\partial h_i}{\partial \chi_j} g_j \neq 0$ , which guarantees that the transformation relating  $(y_1, \dots, y_n, \zeta_1, \dots, \zeta_n)$  to  $(x_1, \dots, x_n, \zeta_1, \dots, \zeta_n)$  is a global diffeomorphism, and the feedback transformation (3.90) from  $u$  to  $v$  is nonsingular.

### 3.4 Extended Matching Design

The increase in the number of parameter estimates caused by overparametrization can be an undesirable feature, since it rapidly increases the dynamic order of the resulting adaptive controller. In Chapter 4, the overparametrization will be eliminated by the tuning functions method. As preliminary to this development, we now show how the overparametrization can be avoided in the case of extended matching, that is, when the uncertain parameters are only one integrator away from the control.

#### 3.4.1 Reducing the overparametrization

We consider again the nonlinear system (3.29),

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta\varphi(x_1) \\ \dot{x}_2 &= u,\end{aligned}$$

and modify its two-step design.

**Step 1.** With  $z_1 = x_1$  and  $x_2$  viewed as the virtual control in the  $\dot{z}_1$ -equation, we define the first stabilizing function  $\alpha_1$  as in (3.34):

$$\alpha_1 = -c_1 z_1 - \hat{\theta}\varphi. \quad (3.104)$$

Comparing (3.104) with (3.34), we see that the parameter estimate  $\vartheta_1$  has been replaced by the parameter estimate  $\hat{\theta}$ . The difference in notation indicates that in this design procedure only one estimate  $\hat{\theta}$  of the unknown parameter will be used.

The first Lyapunov function is now chosen as

$$V_1(z_1, \hat{\theta}) = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{\theta}^2, \quad (3.105)$$

where  $\tilde{\theta} = \theta - \hat{\theta}$  is the parameter error, and  $\gamma > 0$  is the adaptation gain. With  $z_2 = x_2 - \alpha_1$ , the derivative of  $V_1$  is

$$\dot{V}_1 = z_1 z_2 - c_1 z_1^2 + \tilde{\theta} \left( \varphi z_1 - \frac{1}{\gamma} \dot{\tilde{\theta}} \right). \quad (3.106)$$

We postpone the choice of update law for  $\hat{\theta}$  until the next step. The first error subsystem becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + \tilde{\theta}\varphi. \quad (3.107)$$

**Step 2.** The derivative of  $z_2 = x_2 - \alpha_1$  is

$$\begin{aligned}\dot{z}_2 &= u - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \theta\varphi) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &= u - \frac{\partial \alpha_1}{\partial x_1}x_2 - \hat{\theta}\frac{\partial \alpha_1}{\partial x_1}\varphi - \tilde{\theta}\frac{\partial \alpha_1}{\partial x_1}\varphi - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}.\end{aligned} \quad (3.108)$$

To design the control  $u$ , we consider the augmented Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2. \quad (3.109)$$

The only difference between (3.109) and (3.42) is the absence of the new parameter error  $(\theta - \vartheta_2)$  in (3.109). In view of (3.106) and (3.108), the derivative of  $V_2$  is

$$\begin{aligned}\dot{V}_2 &= z_1 z_2 - c_1 z_1^2 + \tilde{\theta} \left( \varphi z_1 - \frac{1}{\gamma} \dot{\tilde{\theta}} \right) \\ &\quad + z_2 \left[ u - \frac{\partial \alpha_1}{\partial x_1}x_2 - \hat{\theta}\frac{\partial \alpha_1}{\partial x_1}\varphi - \tilde{\theta}\frac{\partial \alpha_1}{\partial x_1}\varphi - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \right] \\ &= -c_1 z_1^2 + \tilde{\theta} \left[ \varphi z_1 - z_2 \frac{\partial \alpha_1}{\partial x_1}\varphi - \frac{1}{\gamma} \dot{\tilde{\theta}} \right] \\ &\quad + z_2 \left[ z_1 + u - \frac{\partial \alpha_1}{\partial x_1}x_2 - \hat{\theta}\frac{\partial \alpha_1}{\partial x_1}\varphi - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \right].\end{aligned} \quad (3.110)$$

In the last equation, all the terms containing  $\tilde{\theta}$  have been grouped together. To eliminate them, the update law is chosen as

$$\dot{\hat{\theta}} = \gamma \left( \varphi z_1 - \frac{\partial \alpha_1}{\partial x_1}\varphi z_2 \right). \quad (3.111)$$

Then, the last bracketed term in (3.110) will be rendered equal to  $-c_2 z_2^2$  with the control

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1}x_2 + \hat{\theta}\frac{\partial \alpha_1}{\partial x_1}\varphi + \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}, \quad (3.112)$$

where for  $\dot{\hat{\theta}}$  we use the analytical expression of the update law (3.111). Substituting the expressions (3.111) and (3.112) into (3.110) we obtain

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 \leq 0, \quad (3.113)$$

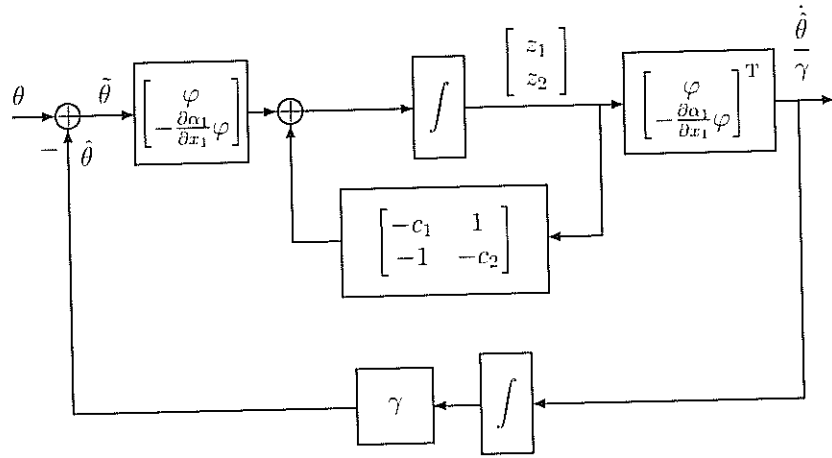


Figure 3.6: The closed-loop adaptive system (3.114).

and the error system becomes (see block diagram in Figure 3.6)

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \tilde{\theta} \\ \dot{\tilde{\theta}} &= \gamma \begin{bmatrix} \varphi & -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \end{aligned} \quad (3.114)$$

Comparing (3.114) with (3.47), we see that the system matrix in (3.114) has preserved the important structural properties it had in (3.47): Its diagonal terms are negative and its off-diagonal terms are skew-symmetric. Furthermore, we see that, as in (3.47), the matrix that multiplies the parameter error  $\tilde{\theta}$  in the  $\dot{z}$ -equation is used (in its transposed form) in the update law for the parameter estimates. It is also instructive to compare the expressions for the parameter update laws in (3.114) and (3.47): Even though the update law for  $\tilde{\theta}$  appears in the form of the sum of the update laws for  $\vartheta_1$  and  $\vartheta_2$ , the expressions (3.34) and (3.104) for  $\alpha_1$  depend on different parameter estimates ( $\vartheta_1$  and  $\tilde{\theta}$ , respectively), and thus  $z_2$  and the partial derivative  $\frac{\partial \alpha_1}{\partial x_1}$  will have different values in (3.44) and (3.111).

Due to the structure of the error system (3.114), its stability and convergence properties are derived in a manner almost identical to those of (3.47) and are therefore omitted here.

In the extended matching case we avoided the overparametrization by postponing the choice of the update law until the second step. When  $\tilde{\theta}$  appeared in the second step, it was replaced by its known analytical expression. Beyond the extended matching case we need more than two steps, so that  $\tilde{\theta}$  and higher derivatives of  $\tilde{\theta}$  appear. Instead of the simple idea of postponing the choice

of the update law, we need the more intricate *tuning functions method*, to be developed in Chapter 4.

### 3.4.2 Example: biochemical process

Extended matching design is also applicable to pure-feedback systems, introduced in Section 2.3.2, provided that the unknown parameters appear linearly. While the general case of these *parametric pure-feedback systems* is presented in Section 4.5.3, the extended matching design will be illustrated on a simplified model of a biotechnological process which goes as far back as Monod [135]. In spite of its simplicity and somewhat unrealistic assumptions, this example is representative of several successful applications of adaptive nonlinear control to more complex processes described by Bastin [7]. In a model of a fed-batch process,  $S$  is the concentration of the growth limiting substrate,  $X$  is the concentration of the growing microbial population,  $k$  is the yield constant,  $D$  is the dilution rate, and the control  $u$  is the substrate feed rate. In a batch process, that is, when both  $D = 0$  and  $u = 0$ , the rate of microbial growth  $\dot{X}$  is modeled as  $\dot{X} = \mu(S)X$ , where  $\mu(S)$  is the “specific growth rate.” The nonlinear function  $\mu(S)$  is usually poorly known, and for our illustrative purpose we parametrize it using unknown parameters:

$$\mu(S) = \varphi_0(S) + \theta_1 \varphi_1(S) + \theta_2 \varphi_2(S). \quad (3.115)$$

Note that  $X$ ,  $S$ , and  $\mu(S)$  are nonnegative quantities. With this parametrization the fed-batch process operating at constant temperature is modeled by the following two mass-balance equations:

$$\dot{X} = [\varphi_0(S) + \theta_1 \varphi_1(S) + \theta_2 \varphi_2(S)] X - DX \quad (3.116a)$$

$$\dot{S} = -k[\varphi_0(S) + \theta_1 \varphi_1(S) + \theta_2 \varphi_2(S)] X - DS + u. \quad (3.116b)$$

The control objective is regulation of  $X$  to the set point  $X_r$ . To further simplify the system (3.116), we use the change of coordinates  $x_1 = \ln X$ ,  $x_2 = S$ , which is well-defined and invertible since  $X > 0$ . Then (3.116) becomes

$$\dot{x}_1 = \varphi_0(x_2) + \theta_1 \varphi_1(x_2) + \theta_2 \varphi_2(x_2) - D \quad (3.117a)$$

$$\dot{x}_2 = -k[\varphi_0(x_2) + \theta_1 \varphi_1(x_2) + \theta_2 \varphi_2(x_2)] e^{x_1} - Dx_2 + u. \quad (3.117b)$$

This system is clearly not in the parametric strict-feedback form (3.70), since the nonlinearities in (3.117a) depend on the second state variable  $x_2$ . However, we can still apply the design procedure illustrated in Section 3.4.1 to this system. In Section 2.3 we saw that our recursive design procedures can be applied not only to strict-feedback systems (2.165), but also to pure-feedback systems (2.180), whose nonlinearities are allowed to depend on one more state variable. The price to be paid is that the stability properties are no longer

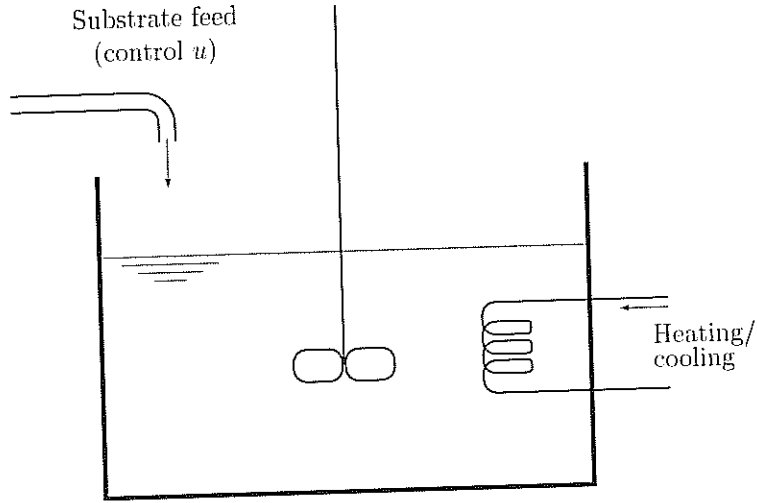


Figure 3.7: Fed-batch stirred tank reactor.

global, but regional: They are guaranteed only for a compact set of initial conditions. The same is possible for our adaptive designs. For the system (3.117), our design proceeds by choosing  $\varphi_0(x_2)$  as the virtual control variable in (3.117a) and designing for it the stabilizing function

$$\alpha_1 = -c_1 z_1 - \hat{\theta}_1 \varphi_1 - \hat{\theta}_2 \varphi_2 + D, \quad (3.118)$$

where  $z_1 = x_1 - \ln X_r$ . With  $z_2 = \varphi_0(x_2) - \alpha_1$ , the error system becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \hat{\theta}_1) \varphi_1 + (\theta - \hat{\theta}_2) \varphi_2 \quad (3.119)$$

$$\begin{aligned} \dot{z}_2 = & \left( \frac{\partial \varphi_0}{\partial x_2} + \hat{\theta}_1 \frac{\partial \varphi_1}{\partial x_2} + \hat{\theta}_2 \frac{\partial \varphi_2}{\partial x_2} \right) \left\{ -k [\varphi_0 + \theta_1 \varphi_1 + \theta_2 \varphi_2] e^{x_1} - D x_2 + u \right\} \\ & + c_1 [-c_1 z_1 + z_2 + (\theta - \hat{\theta}_1) \varphi_1 + (\theta - \hat{\theta}_2) \varphi_2] + \dot{\hat{\theta}}_1 \varphi_1 + \dot{\hat{\theta}}_2 \varphi_2. \end{aligned} \quad (3.120)$$

Following the development of Section 3.4.1, we choose the update law

$$\dot{\hat{\theta}} = \Gamma \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \left\{ z_1 + \left[ c_1 - k \left( \frac{\partial \varphi_0}{\partial x_2} + \hat{\theta}_1 \frac{\partial \varphi_1}{\partial x_2} + \hat{\theta}_2 \frac{\partial \varphi_2}{\partial x_2} \right) \right] z_2 \right\}, \quad (3.121)$$

where  $\theta^T = [\theta_1, \theta_2]$ . The corresponding control law

$$\begin{aligned} u = & \frac{1}{\left( \frac{\partial \varphi_0}{\partial x_2} + \hat{\theta}_1 \frac{\partial \varphi_1}{\partial x_2} + \hat{\theta}_2 \frac{\partial \varphi_2}{\partial x_2} \right)} \left\{ -c_2 z_2 - z_1 - c_1 [-c_1 z_1 + z_2 - \hat{\theta}_1 \varphi_1 - \hat{\theta}_2 \varphi_2] \right. \\ & \left. - [\varphi_1, \varphi_2] \Gamma \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \left\{ z_1 + \left[ c_1 - k \left( \frac{\partial \varphi_0}{\partial x_2} + \hat{\theta}_1 \frac{\partial \varphi_1}{\partial x_2} + \hat{\theta}_2 \frac{\partial \varphi_2}{\partial x_2} \right) \right] z_2 \right\} \right\} \\ & + k [\varphi_0 + \hat{\theta}_1 \varphi_1 + \hat{\theta}_2 \varphi_2] e^{x_1} + D x_2, \end{aligned} \quad (3.122)$$

is feasible only in the region in which  $\frac{\partial \varphi_0}{\partial x_2} + \hat{\theta}_1 \frac{\partial \varphi_1}{\partial x_2} + \hat{\theta}_2 \frac{\partial \varphi_2}{\partial x_2} \neq 0$ . With these choices, the derivative of the Lyapunov function

$$V = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} (\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}) \quad (3.123)$$

is nonpositive:

$$\dot{V} = -c_1 z_1^2 - c_2 z_2^2. \quad (3.124)$$

As we will see in Section 4.5.3, stability is guaranteed for all initial conditions inside the largest level set of the Lyapunov function 3.123 contained in the feasibility region.

### 3.4.3 Transient performance improvement

The nonlinear damping with  $\kappa$ -terms introduced in Section 2.5 can easily be incorporated into the adaptive design procedures we have discussed so far. The resulting adaptive controllers guarantee boundedness even when the adaptation is switched off, and their transient performance can be improved in a systematic way through *trajectory initialization* and the choice of design parameters.

To illustrate the design with  $\kappa$ -terms and the process of trajectory initialization, we consider again the system (3.29) with the output  $y = x_1$ :

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta \varphi(x_1) \\ \dot{x}_2 &= u \\ y &= x_1. \end{aligned} \quad (3.125)$$

The control objective is to asymptotically track a reference output  $y_r(t)$  with the output  $y$  of the system (3.125). We assume that not only  $y_r$ , but also its first two derivatives  $\dot{y}_r$ ,  $\ddot{y}_r$  are known and uniformly bounded, and, in addition,  $\ddot{y}_r$  is piecewise continuous.

**Step 1.** The first error variable is now the *tracking error*

$$z_1 = y - y_r = x_1 - y_r, \quad (3.126)$$

whose derivative is

$$\dot{z}_1 = x_2 + \theta^T \varphi_1(x_1) - \dot{y}_r. \quad (3.127)$$

Viewing  $x_2$  as the virtual control we define the stabilizing function

$$\alpha_1 = -c_1 z_1 - \kappa_1 z_1 \varphi^2 - \hat{\theta} \varphi + \dot{y}_r. \quad (3.128)$$

Comparing (3.128) with (3.104) we note two new terms in (3.128). The term  $\dot{y}_r$ , which is intended to cancel the corresponding term in (3.127), is due to the tracking objective. The nonlinear damping term  $-\kappa_1 z_1 \varphi^2$  is motivated

by Lemma 2.26. It contains the square of the term  $(\varphi)$  which multiplies the parametric uncertainty in the error equation obtained by substituting  $z_2 = x_2 - \alpha_1$  and (3.128) into (3.127):

$$\dot{z}_1 = -c_1 z_1 - \kappa_1 z_1 \varphi^2 + z_2 + \tilde{\theta} \varphi. \quad (3.129)$$

The derivative of the Lyapunov function  $V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\gamma} \tilde{\theta}^2$  becomes

$$\dot{V}_1 = z_1 z_2 - c_1 z_1^2 - \kappa_1 z_1^2 \varphi^2 + \tilde{\theta} \left( \varphi z_1 - \frac{1}{\gamma} \dot{\tilde{\theta}} \right). \quad (3.130)$$

**Step 2.** As in (3.108), the derivative of  $z_2 = x_2 - \alpha_1$  is

$$\begin{aligned} \dot{z}_2 &= u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \theta \varphi) - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \frac{\partial \alpha_1}{\partial \ddot{y}_r} \ddot{y}_r - \frac{\partial \alpha_1}{\partial \dot{\theta}} \dot{\theta} \\ &= u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \tilde{\theta} \frac{\partial \alpha_1}{\partial x_1} \varphi - \tilde{\theta} \frac{\partial \alpha_1}{\partial x_1} \varphi - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \ddot{y}_r - \frac{\partial \alpha_1}{\partial \dot{\theta}} \dot{\theta}, \end{aligned} \quad (3.131)$$

where in the last equality we have used the identity  $\frac{\partial \alpha_1}{\partial y_r} = 1$ . Using (3.130) and (3.131), the derivative of the Lyapunov function

$$V_2 = V_1 + \frac{1}{2} z_2^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma} \tilde{\theta}^2 \quad (3.132)$$

is expressed as

$$\begin{aligned} \dot{V}_2 &= z_1 z_2 - c_1 z_1^2 - \kappa_1 z_1^2 \varphi^2 + \tilde{\theta} \left( \varphi z_1 - \frac{1}{\gamma} \dot{\tilde{\theta}} \right) \\ &\quad + z_2 \left[ u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \tilde{\theta} \frac{\partial \alpha_1}{\partial x_1} \varphi - \tilde{\theta} \frac{\partial \alpha_1}{\partial x_1} \varphi - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \ddot{y}_r - \frac{\partial \alpha_1}{\partial \dot{\theta}} \dot{\theta} \right] \\ &= -c_1 z_1^2 - \kappa_1 z_1^2 \varphi^2 + \tilde{\theta} \left[ \varphi z_1 - z_2 \frac{\partial \alpha_1}{\partial x_1} \varphi - \frac{1}{\gamma} \dot{\tilde{\theta}} \right] \\ &\quad + z_2 \left[ z_1 + u - \frac{\partial \alpha_1}{\partial x_1} x_2 - \tilde{\theta} \frac{\partial \alpha_1}{\partial x_1} \varphi - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \ddot{y}_r - \frac{\partial \alpha_1}{\partial \dot{\theta}} \dot{\theta} \right]. \end{aligned} \quad (3.133)$$

As in (3.110), all the terms containing  $\tilde{\theta}$  have been grouped together. To eliminate them, the update law is chosen as

$$\dot{\tilde{\theta}} = \gamma \left( \varphi z_1 - \frac{\partial \alpha_1}{\partial x_1} \varphi z_2 \right). \quad (3.134)$$

The control law is now chosen to render the last bracketed term in (3.133) equal to  $-c_2 z_2^2 - \kappa_2 z_2^2 \left( \frac{\partial \alpha_1}{\partial x_1} \varphi \right)^2$ , instead of just equal to  $-c_2 z_2^2$  as in (3.112):

$$u = -z_1 - c_2 z_2 - \kappa_2 z_2 \left( \frac{\partial \alpha_1}{\partial x_1} \varphi \right)^2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \tilde{\theta} \frac{\partial \alpha_1}{\partial x_1} \varphi + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r + \ddot{y}_r + \frac{\partial \alpha_1}{\partial \dot{\theta}} \dot{\theta}. \quad (3.135)$$

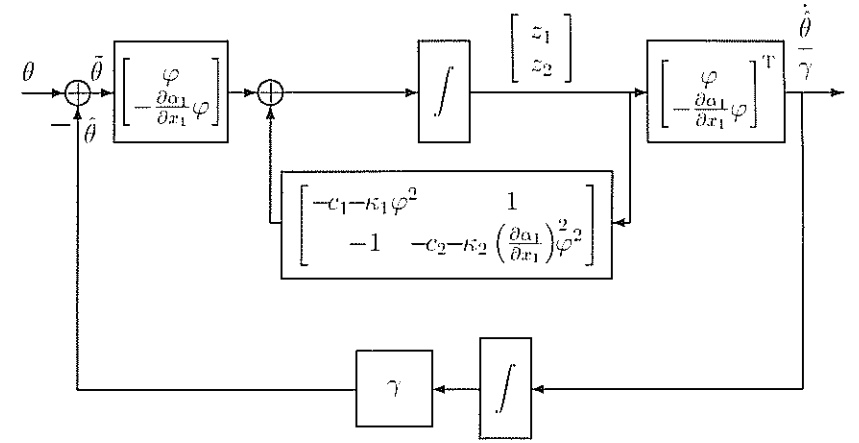


Figure 3.8: The closed-loop adaptive system (3.137).

To implement this control law, we will replace  $\dot{\tilde{\theta}}$  with the analytical expression of the update law (3.134).

Substituting the expressions (3.134) and (3.135) into (3.133) we obtain

$$\dot{V}_2 = -c_1 z_1^2 - \kappa_1 z_1^2 \varphi^2 - c_2 z_2^2 - \kappa_2 z_2 \left( \frac{\partial \alpha_1}{\partial x_1} \varphi \right)^2 \leq 0, \quad (3.136)$$

while the complete error system becomes (see block diagram in Figure 3.8)

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -c_1 - \kappa_1 \varphi^2 & 1 \\ -1 & -c_2 - \kappa_2 \left( \frac{\partial \alpha_1}{\partial x_1} \varphi \right)^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \tilde{\theta} \\ \dot{\tilde{\theta}} &= \gamma \begin{bmatrix} \varphi & -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \end{aligned} \quad (3.137)$$

Comparing (3.137) with (3.114), we see that the system matrix in (3.137) is not constant: Its diagonal terms have been “fortified” with additional nonlinear damping terms. These terms contain the squares of the elements of the vector that multiplies the parameter error  $\tilde{\theta}$ .

Let us now study the properties of the error system (3.137):

**Global stability and asymptotic tracking.** Using (3.132) and (3.136) we conclude that the  $(z, \tilde{\theta})$ -system has a globally uniformly stable equilibrium at the origin, and

$$\lim_{t \rightarrow \infty} z(t) = 0. \quad (3.138)$$

In particular, this implies that the state of the system (3.125) is globally uniformly bounded (since  $y_r$ ,  $\dot{y}_r$ , and  $\ddot{y}_r$  are bounded) and that the tracking error  $z_1 = y - y_r$  converges to zero asymptotically.

**Boundedness without adaptation.** It is also straightforward to see that the designed controller guarantees global uniform boundedness even when the adaptation is turned off, that is, even with  $\gamma = 0$ . In that case, the closed-loop system (3.137) becomes

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 - \kappa_1 \varphi^2 & 1 \\ -1 & -c_2 - \kappa_2 \left( \frac{\partial \alpha_1}{\partial x_1} \varphi \right)^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \tilde{\theta}. \quad (3.139)$$

A candidate Lyapunov function for this system is given by

$$V(z) = \frac{1}{2}|z|^2 = \frac{1}{2}(z_1^2 + z_2^2). \quad (3.140)$$

Its derivative along the solutions of (3.139) satisfies

$$\begin{aligned} \dot{V}_{(3.139)} &= -c_1 z_1^2 - c_2 z_2^2 - \kappa_1 z_1^2 \varphi^2 - \kappa_2 z_2^2 \left( \frac{\partial \alpha_1}{\partial x_1} \varphi \right)^2 + z_1 \varphi \tilde{\theta} - z_2 \frac{\partial \alpha_1}{\partial x_1} \varphi \tilde{\theta} \\ &\leq -c_1 z_1^2 - c_2 z_2^2 - \kappa_1 \left( z_1 \varphi - \frac{\tilde{\theta}}{2\kappa_1} \right)^2 + \frac{\tilde{\theta}^2}{4\kappa_1} - \kappa_2 \left( z_2 \frac{\partial \alpha_1}{\partial x_1} \varphi + \frac{\tilde{\theta}}{2\kappa_2} \right)^2 + \frac{\tilde{\theta}^2}{4\kappa_2} \\ &\leq -c_1 z_1^2 - c_2 z_2^2 + \frac{\tilde{\theta}^2}{4\kappa_1} + \frac{\tilde{\theta}^2}{4\kappa_2} \\ &\leq -c_0 |z|^2 + \frac{\tilde{\theta}^2}{4\kappa_0}, \end{aligned} \quad (3.141)$$

where the constants  $c_0$  and  $\kappa_0$  are defined as

$$c_0 = \min\{c_1, c_2\}, \quad \frac{1}{\kappa_0} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2}. \quad (3.142)$$

It is clear from (3.141) that, for any positive values of  $c_0$  and  $\kappa_0$ , the state of the error system (and hence the state of the plant) is uniformly bounded, since  $\dot{V} < 0$  whenever  $|z|^2 > |\tilde{\theta}|^2 / 4\kappa_0 c_0$ , where  $\tilde{\theta} = \theta - \tilde{\theta}(0)$  is constant since adaptation is turned off.

#### Transient performance improvement with trajectory initialization.

Let us now investigate the transient performance of the adaptive closed-loop system (3.137). The derivative of the nonnegative function  $V(z)$  defined in (3.140) along the solutions of (3.137) satisfies the same inequality as in (3.141):

$$\frac{d}{dt} \left( \frac{1}{2}|z|^2 \right) \leq -c_0 |z|^2 + \frac{\tilde{\theta}^2}{4\kappa_0}. \quad (3.143)$$

Since the boundedness of  $\tilde{\theta}$  has already been established from (3.132) and (3.136), we can strengthen the inequality in (3.143) by replacing  $\tilde{\theta}^2$  with its

bound  $\|\tilde{\theta}\|_\infty^2$ . This bound is estimated from (3.132) using the fact that  $V_2$  is nonincreasing:

$$\begin{aligned} \frac{1}{2\gamma} |\tilde{\theta}(t)|^2 &\leq \frac{1}{2} |z(t)|^2 + \frac{1}{2\gamma} \tilde{\theta}(t)^2 = V_2(t) \\ &\leq V_2(0) = \frac{1}{2} |z(0)|^2 + \frac{1}{2\gamma} \tilde{\theta}(0)^2. \end{aligned} \quad (3.144)$$

This implies

$$\|\tilde{\theta}\|_\infty^2 \leq \gamma |z(0)|^2 + \tilde{\theta}(0)^2. \quad (3.145)$$

Combining (3.143) and (3.145) we obtain

$$\frac{d}{dt} (|z|^2) \leq -2c_0 |z|^2 + \frac{1}{2\kappa_0} [\gamma |z(0)|^2 + \tilde{\theta}(0)^2]. \quad (3.146)$$

Multiplying both sides of (3.146) by  $e^{2c_0 t}$  and integrating over the interval  $[0, t]$  results in

$$|z(t)|^2 \leq |z(0)|^2 e^{-2c_0 t} + \frac{1}{4\kappa_0 c_0} [\gamma |z(0)|^2 + \tilde{\theta}(0)^2]. \quad (3.147)$$

The bound (3.147) suggests that the transient behavior of the error system can be influenced through the choice of design constants  $c_0$ ,  $\kappa_0$  and  $\gamma$ . What is not clear, however, is that an increase of  $\kappa_0 c_0$  alone may not reduce the maximum value of  $|z(t)|$  and will certainly not reduce the computable  $\mathcal{L}_\infty$ -bound of  $z$ . In fact, it may even *increase* this bound by increasing the initial value  $|z(0)|$ . To clarify this point, let us recall the definitions of  $z_1$  and  $z_2$ :

$$\begin{aligned} z_1 &= x_1 - y_r \\ z_2 &= x_2 - \alpha_1 = x_2 + c_1 z_1 + \kappa_1 z_1 \varphi^2 + \tilde{\theta} \varphi - \dot{y}_r. \end{aligned}$$

Suppose now that  $z_1(0)$  is different than zero. In that case, an increase of  $c_1$  and  $\kappa_1$  may increase the value of  $z_2(0)$  and thus also the value of  $|z(0)|$ . Moreover, this increase may more than offset the decreasing effect of the term  $1/4\kappa_0 c_0$  in (3.147), since  $|z(0)|^2$  will increase in proportion to  $c_1^2$  and  $\kappa_1^2$ .

It would seem that the dependence of  $z(0)$  on the design constants  $c_1, c_2, \kappa_1, \kappa_2$  eliminates any possibility of systematically improving the transient performance of the error system through the choice of  $c_0$  and  $\kappa_0$ . Fortunately, it is not so. The remedy for this problem is to use *trajectory initialization* to render  $z(0) = 0$  *independently* of the choice of these design constants. The initialization procedure, presented for the general case in Section 4.3.2, is straightforward and is dictated by the definitions of the  $z$ -variables:

- Starting with  $z_1$ , set  $z_1(0) = 0$  by choosing

$$y_r(0) = x_1(0). \quad (3.148)$$

• Since  $z_1(0) = 0$ , (3.128) shows that

$$\alpha_1(0) = \dot{y}_r(0) - \hat{\theta}(0)\varphi(0), \quad (3.149)$$

where we use the notation  $\varphi(0) = \varphi(x_1(0))$ . From (3.149) it is clear that we can set  $z_2(0) = 0$  with the choice

$$\dot{y}_r(0) = x_2(0) + \hat{\theta}(0)\varphi(0). \quad (3.150)$$

With the trajectory initialization defined by (3.148) and (3.150), we have set  $z(0) = 0$ . In the case of model reference control, this is achieved by adjusting the initial conditions of the reference model. If, on the other hand, the reference trajectory is given as a precomputed function of time, then it can be initialized through the addition of exponentially decaying terms which define the *reference transients*.

We note that (3.148) and (3.150) are independent of the design constants  $c_1, c_2, \kappa_1, \kappa_2$ . This means that different choices of  $c_0$  and  $\kappa_0$  will still result in  $z(0) = 0$  with the same values of  $y_r(0)$  and  $\dot{y}_r(0)$ . Returning to (3.147), we substitute  $z(0) = 0$  to obtain

$$|z(t)|^2 \leq \frac{1}{4\kappa_0 c_0} \bar{\theta}(0)^2, \quad (3.151)$$

which implies

$$\|z\|_\infty \leq \frac{1}{2\sqrt{\kappa_0 c_0}} |\bar{\theta}(0)|. \quad (3.152)$$

Hence, the  $\mathcal{L}_\infty$ -bound on the transient performance of the error system is directly proportional to the initial parametric uncertainty and can be reduced arbitrarily by increasing the values of  $c_0$  and  $\kappa_0$ . In particular, this implies that the transients of the tracking error  $z_1 = y - y_r$  are directly influenced by the design constants  $c_i$  and  $\kappa_i$ . This possibility of arbitrary reduction may seem peculiar, since it can be achieved for all initial conditions. We must remember, however, that this error is defined with respect to the reference signals which have in turn been initialized to set  $z(0) = 0$ . Hence, the effect of the plant initial conditions has been “absorbed” into the reference transients.

To provide some further insight into the process of trajectory initialization, let us return to the Lyapunov function (3.132). When  $z(0) = 0$ , the initial value of this function is reduced to the initial value of the parametric uncertainty. If we interpret the value of this function as a distance between the actual system trajectory and the reference trajectory, we see that *trajectory initialization places the initial point of the reference trajectory as close as possible to the initial point of the system trajectory*. If the parametric uncertainty were zero, trajectory initialization would have placed the reference output and its derivatives at the true values of the plant output and its derivatives. This

is easily seen if  $\hat{\theta}(0)$  is replaced by  $\theta$  in (3.148) and (3.150):

$$\begin{aligned} y_r(0) &= x_1(0) = y(0) \\ \dot{y}_r(0) &= x_2(0) + \theta\varphi(0) = \dot{y}(0). \end{aligned}$$

However, since the parameter  $\theta$  is unknown, trajectory initialization placed only the reference output at the true value of the plant output, while its derivatives were placed at the *estimated* values of the plant output derivatives. Through this process, the initial value of the Lyapunov function is

$$V_2(0) = \frac{1}{2\gamma} \bar{\theta}(0)^2, \quad (3.153)$$

which, in the presence of parametric uncertainty, is its smallest possible value.

## Notes and References

Adaptive backstepping (Kanellakopoulos, Kokotović, and Morse [69]), first presented in a Grainger lecture [87], was a culmination of an intensive effort of several groups of authors. The path to adaptive backstepping was not as direct as it may appear from this chapter. It led through the matched case in Taylor, Kokotović, Marino, and Kanellakopoulos [186], and then the case of extended matching in Kanellakopoulos, Kokotović, and Marino [65], and Bastin and Campion [8]. Even though nonadaptive backstepping was available from Saberi, Kokotović, and Sussmann [163], the steps beyond the extended matching case were delayed and alternative approaches were explored. The focus was on estimation-based designs summarized in Praly, Bastin, Pomet, and Jiang [157].

The class of parametric strict-feedback systems was characterized via coordinate-free geometric conditions by Kanellakopoulos *et al.* [63, 69], and the class of parametric pure-feedback systems by Akhrif and Blankenship [1]. The multi-input design for parametric pure-feedback systems was presented in [87].

Teel [188] increased the feasibility region for parametric pure-feedback systems by casting the scheme [69] in an observer-based setting. Several extensions of [69] were proposed by Seto, Annaswamy, and Baillieul [3, 167].