Generate Geodesics from Metric!

https://physics.stackexchange.com/questions/137422/geodesic-equation-from-euler-lagrange

Metric

```
clear
syms lambda t(lambda) r(lambda) theta(lambda) phi(lambda)

% G = 1;
% M = 1;
% c = 1;
% r_s = 2*G*M/c^2;
%
% metricName = 'Schwarz'
% metric = [-(1-r_s/r) 0 0 0; ...
%          0 1/(1-r_s/r) 0 0; ...
%          0   0    r^2    0; ...
%          0 0 0 r^2*sin(theta)^2]

G = 1;
M = 1;
c = 1;
J = 2;
r_s = 2*G*M/c^2;
a = J/(M*c);
sig = r^2 + a^2*cos(theta)^2;
del = r^2 - r_s*r + a^2;

metricName = 'Kerr'
```

```
metricName =
'Kerr'
```

```
metric = [-(1-(r_s*r)/sig) 0 0 -r_s*r*a*sin(theta)^2/sig; ...
          0 sig/del 0 0; ...
          0 0 sig 0; ...
          -r_s*r*a*sin(theta)^2/sig 0 0 (r^2+a^2+r_s*r*a^2*sin(theta)^2/
sig)*sin(theta)^2]
```

```
metric(lambda) =
```

$$\begin{pmatrix} \frac{2\,r(\lambda)}{\sigma_2} - 1 & 0 & 0 & \sigma_1 \\ 0 & \frac{\sigma_2}{r(\lambda)^2 - 2\,r(\lambda) + 4} & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 & \sigma_3\,\left(r(\lambda)^2 + \frac{8\,\sigma_3\,r(\lambda)}{\sigma_2} + 4\right) \end{pmatrix}$$

where

$$\sigma_1 = -\frac{4\,\sigma_3\,r(\lambda)}{\sigma_2}$$

$$\sigma_2 = 4\cos(\theta(\lambda))^2 + r(\lambda)^2$$

$$\sigma_3 = \sin(\theta(\lambda))^2$$

Lagrangian:

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

```
syms dt dr dtheta dphi ddt

vars = [t r theta phi];
dvars = diff(vars,lambda);

L = 0.5 * sum(metric .* (dvars.'*dvars),'all')
```

L =

$$\frac{\left(\frac{2\,r(\lambda)}{\sigma_1} - 1\right)\left(\frac{\partial}{\partial\lambda}\,t(\lambda)\right)^2}{2} + \frac{\sigma_1\,\left(\frac{\partial}{\partial\lambda}\,\theta(\lambda)\right)^2}{2} + \frac{\sigma_1\,\left(\frac{\partial}{\partial\lambda}\,r(\lambda)\right)^2}{2\,(r(\lambda)^2 - 2\,r(\lambda) + 4)} + \frac{\sigma_2\,\left(\frac{\partial}{\partial\lambda}\,\phi(\lambda)\right)^2\,\left(r(\lambda)^2 + \frac{8\,\sigma_2\,r(\lambda)}{\sigma_1} + \right.}{2}$$

where

$$\sigma_1 = 4\cos(\theta(\lambda))^2 + r(\lambda)^2$$

$$\sigma_2 = \sin(\theta(\lambda))^2$$

Geodesics

$$\frac{d}{d\lambda}\frac{\partial L}{\partial(dx^\mu/d\lambda)} = \frac{\partial L}{\partial x^\mu}$$

```
geodesics = cell(4);

% tIndex is a function I wrote that just allows me
% to index vectors/matrices of symbolic functions
for i = 1:4
    thisVar = tIndex(vars,':',i);
    thisDVar = tIndex(dvars,':',i);
    eqtn = diff(diff(L,thisDVar),lambda) == ...
        diff(L,thisVar);
    geodesics{i} = symfun(isolate(eqtn,diff(thisDVar,lambda)),lambda);
end
```

Output

```
geodesics{1}
```

ans(lambda) =

$$\frac{\partial^2}{\partial\lambda^2}\, t(\lambda) = -\frac{4\,\sigma_2\, r(\lambda)^3\,\sigma_3 - 8\,\sigma_1 \frac{\partial}{\partial\lambda}\, t(\lambda)\frac{\partial}{\partial\lambda}\, r(\lambda) + 2\, r(\lambda)^2 \frac{\partial}{\partial\lambda}\, t(\lambda)\frac{\partial}{\partial\lambda}\, r(\lambda) + 16\,\sigma_1\,\sigma_2 \frac{\partial}{\partial\lambda}\, r(\lambda)\frac{\partial}{\partial\lambda}\,\phi(\lambda) - 4\,\sigma_2}{}$$

where

$$\sigma_1 = \cos(\theta(\lambda))^2$$

$$\sigma_2 = \sin(\theta(\lambda))^2$$

$$\sigma_3 = \frac{\partial^2}{\partial\lambda^2}\,\phi(\lambda)$$

```
geodesics{2}
```

ans(lambda) =

$$\frac{\partial^2}{\partial \lambda^2}\ r(\lambda) = -\frac{\sigma_2 \left(\frac{\left(\frac{4\,r(\lambda)^2}{{\sigma_1}^2} - \frac{2}{\sigma_1}\right)\ \left(\frac{\partial}{\partial \lambda}\ t(\lambda)\right)^2}{2} - r(\lambda)\ \left(\frac{\partial}{\partial \lambda}\ \theta(\lambda)\right)^2 - \frac{\sigma_3\ \left(\frac{\partial}{\partial \lambda}\ \phi(\lambda)\right)^2\ \left(2\,r(\lambda) + \frac{8\,\sigma_3}{\sigma_1} - \frac{16\,\sigma_3}{\sigma_1}\right.}{2}\right.}{}$$

where

$\sigma_1 = 4\cos(\theta(\lambda))^2 + r(\lambda)^2$

$\sigma_2 = r(\lambda)^2 - 2\,r(\lambda) + 4$

$\sigma_3 = \sin(\theta(\lambda))^2$

$\sigma_4 = 2\,r(\lambda)\frac{\partial}{\partial \lambda}\ r(\lambda)$

$\sigma_5 = \left(\frac{\partial}{\partial \lambda}\ r(\lambda)\right)^2$

```
geodesics{3}
```

```
ans(lambda) =
```

$$\frac{\partial^2}{\partial \lambda^2}\ \theta(\lambda) = -\frac{\left(2\,r(\lambda)\frac{\partial}{\partial \lambda}\ r(\lambda) - 8\cos(\theta(\lambda))\sin(\theta(\lambda))\frac{\partial}{\partial \lambda}\ \theta(\lambda)\right)\frac{\partial}{\partial \lambda}\ \theta(\lambda) + 4\cos(\theta(\lambda))\sin(\theta(\lambda))\ \left(\frac{\partial}{\partial \lambda}\ \theta(\lambda)\right)^2}{}$$

where

$\sigma_1 = 4\cos(\theta(\lambda))^2 + r(\lambda)^2$

$\sigma_2 = \sin(\theta(\lambda))^3$

$\sigma_3 = \sin(\theta(\lambda))^2$

$\sigma_4 = \left(\frac{\partial}{\partial \lambda}\ \phi(\lambda)\right)^2$

```
geodesics{4}
```

```
ans(lambda) =
```

$$\frac{\partial^2}{\partial \lambda^2}\ \phi(\lambda) = -\frac{\sigma_5 \left(\sigma_3 + \frac{8\,\sigma_5 \frac{\partial}{\partial \lambda}\ r(\lambda)}{\sigma_4} - \frac{\sigma_5\, r(\lambda)\, \sigma_1\, 8}{{\sigma_4}^2} + \frac{16\cos(\theta(\lambda))\sin(\theta(\lambda))\, r(\lambda) \frac{\partial}{\partial \lambda}\ \theta(\lambda)}{\sigma_4}\right) \frac{\partial}{\partial \lambda}\ \phi(\lambda) - \frac{4\,\sigma_5\, r(\lambda}{}}{}$$

where

$$\sigma_1 = \sigma_3 - 8\cos(\theta(\lambda))\sin(\theta(\lambda)) \frac{\partial}{\partial \lambda}\ \theta(\lambda)$$

$$\sigma_2 = r(\lambda)^2 + \frac{8\,\sigma_5\, r(\lambda)}{\sigma_4} + 4$$

$$\sigma_3 = 2\, r(\lambda) \frac{\partial}{\partial \lambda}\ r(\lambda)$$

$$\sigma_4 = 4\cos(\theta(\lambda))^2 + r(\lambda)^2$$

$$\sigma_5 = \sin(\theta(\lambda))^2$$

```
filename = ['geodesics_lagrange_' metricName '.mat'];
save(filename, 'geodesics', 'r_s', 'G', 'M', 'c');
```