1)

a) $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i)$

$= \sum_{i=1}^{n} y_i - (\bar{y} + r \frac{\sigma_y}{\sigma_x} (x_i - \bar{x}))$

$= \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \bar{y} - \sum_{i}^{n} r \frac{\sigma_y}{\sigma_x} x_i - \sum r \frac{\sigma_y}{\sigma_x} \bar{x}$

$= \sum_{i=1}^{n} y_i - n\bar{y} - r \frac{\sigma_y}{\sigma_x} \sum_{i}^{n} x_i - n r \frac{\sigma_y}{\sigma_x} \bar{x}$

$= \sum_{i=1}^{n} y_i - \frac{1}{n} n \sum_{i=1}^{n} y_i - r \frac{\sigma_y}{\sigma_x} \sum_{i}^{n} x_i - \frac{1}{n} n r \frac{\sigma_y}{\sigma_x} \sum_{i=1}^{n} x_i$

$= 0$

$\therefore \sum_{i=1}^{n} e_i = 0$

b) $\bar{y} = \bar{\hat{y}}$

$\bar{\hat{y}} = \frac{1}{n} \sum_{i=1}^{n} [(\bar{y} + r \frac{\sigma_y}{\sigma_x} (x_i - \bar{x})]$

$= \frac{1}{n} \sum_{i=1}^{n} \bar{y} + \frac{1}{n} r \frac{\sigma_y}{\sigma_x} \sum_{i=1}^{n} x_i + \frac{1}{n} r \frac{\sigma_y}{\sigma_x} \sum_{i=1}^{n} \bar{x}$

$= \bar{y} + r \frac{\sigma_y}{\sigma_x} \bar{x} - r \frac{\sigma_y}{\sigma_x} \bar{x} = \bar{y}$

$\therefore \bar{y} = \bar{\hat{y}}$

c) SLR $\Rightarrow \hat{y} = \bar{y} + r \sigma_y \frac{x - \bar{x}}{\sigma_x}$

$(\bar{x}, \bar{y})$ plug in $\bar{x}$ as x

$\hat{y} = \bar{y} + r \sigma_y \frac{\bar{x} - \bar{x}}{\sigma_x}$

$\hat{y} = \bar{y}$

Plugging in $\bar{x}$ as x into the SLR gives $\bar{y}$

$\therefore (\bar{x}, \bar{y})$ is in the SLR

2)

a) $\sum_{i=1}^{n} e_i = 1^T e = 0$

$$X^T(Y - X\hat{\theta}) = 0$$

$$\hat{\theta} = (X^T X)^{-1} X^T Y$$

$$1^T e = 1^T(Y - \hat{Y})$$

$$= 1^T X^T (I - X\hat{\theta}) Y$$

$$= 1^T X^T (I - X(X^T X)^{-1} X^T) Y$$

$$= e^T (X^T - X^T X (X^T X)^{-1} X^T) Y$$

$$= e^T (X^T - X^T) Y$$

$$= 0$$

b) $X^T e = X^T(Y - \hat{Y})$

$$= X^T Y - X^T X \hat{\theta}$$

$$= (X^T - X^T X (X^T X)^{-1} X^T) Y$$

$$= (X^T - X^T) Y$$

$$= 0$$

c) $\hat{Y}^T e = X\hat{\theta}^T e = X\hat{\theta}^T (Y - \hat{Y})$

$$= X\hat{\theta}^T Y - X\hat{\theta}^T X \hat{\theta}$$

$$= (X(X^T X)^{-1} X^T - X(X^T X)^{-1\,T} X^T X (X^T X)^{-1} X^T) Y$$

$$= 0$$

3)

a) $R(r) = \frac{1}{n} \sum_{i=1}^{n} (y_i - r x_i)^2$

$R'(r) = -\frac{2}{n} \sum_{i=1}^{n} (y_i - r x_i) x_i$

$-\frac{2}{n} \sum_{i=1}^{n} (y_i - r x_i) x_i = 0$

$\sum_{i=1}^{n} (y_i x_i - r x_i^2) = 0$

$\sum_{i=1}^{n} y_i x_i = r x_i^2$

$r = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$

$R''(r) = \frac{d}{dr} -\frac{2}{n} \sum_{i=1}^{n} (y_i - r x_i)^2$

$= \frac{-2}{n} \sum_{i=1}^{n} (-x_i)^2$

$= \frac{2}{n} \sum_{i=1}^{n} (x_i)^2 > 0$

$\therefore$ minimum

4)

a) Does not hold w/out an intercept term because w/out the column of 1's, the dot product btwn 1 & e is not always 0.

b) Holds true even w/out an intercept term because even w/out the column of 1's to create the matrix X, the dot product btwn $\vec{x}$ & e is 0.

c) This holds true even w/out an intercept term because $\hat{y}$ w/out the columns of 1's is the same things as the column vector $\vec{x}$. Therefore, the therefore the dot product between $\hat{y}$ and e will still be 0.

d) This will not hold w/out an intercept term because $(\bar{x}, \bar{y})$ will only be on the regression line if the intercept term exists in the SLR and is equal to $\bar{y}$.