CX 4230 Mini Project 2 Report

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Introduction

This report is for CX 4230 Mini-project 2: Traffic jams. In this report, we'll analyze models for freeway traffic in two different models - using the PDE model, and the CA model. Given mathematical relations from the textbook, we will discuss the implementation of our codes, its result using various plots, and finally an in-depth critique of those results.

#1.1

We first started off by defining the time and spatial domain in a similar manner as we did in class. We defined a discrete time domain such that t is between 0s and 120s, which spans two minutes. The step size is 0.1s, which will allow us to evaluate 1200 time steps in total. We defined a discrete spatial domain such that x is between -5km and 5km, which results in a total of 10 km. The step size is 0.01km, which will allow us to evaluate 1000 spacial steps in total. Then we defined a solution grid, also using similar logic from the class; we've used ghost cells and a helper function that will visualize our plots. We define our impulse function as shown in Figure 2, where p(x,t) is 80 for -5 < x < -0.25, 160 for -0.25 < x < 0, and 0 for x > 0. This impulse function represents the occurrence of shock around x = -0.25 km. Now that we have set up the basic grid and domains, we've defined the contour function as well as the step_upwind function using

$$p_{i,j+1} = p_{i,j} - \frac{h}{s} v_{max}((1 - \frac{p_{i+1,j}}{p_{max}})p_{i+1,j} - (1 - \frac{p_{i,j}}{p_{max}})p_{i,j})$$

Figure 1: the upwind scheme relation

In the step_upwind function, we've used P1 and P2 to indicate $p_{i,j}$ and $p_{i+1,j}$ respectively. We've set our p.max to be 160 cars/km, and v.max to be 120km/h as well. By doing so, we were able to achieve plot_grid and contour plots. We've shown those in #1.2 and discussed its result in #1.3.

#1.2

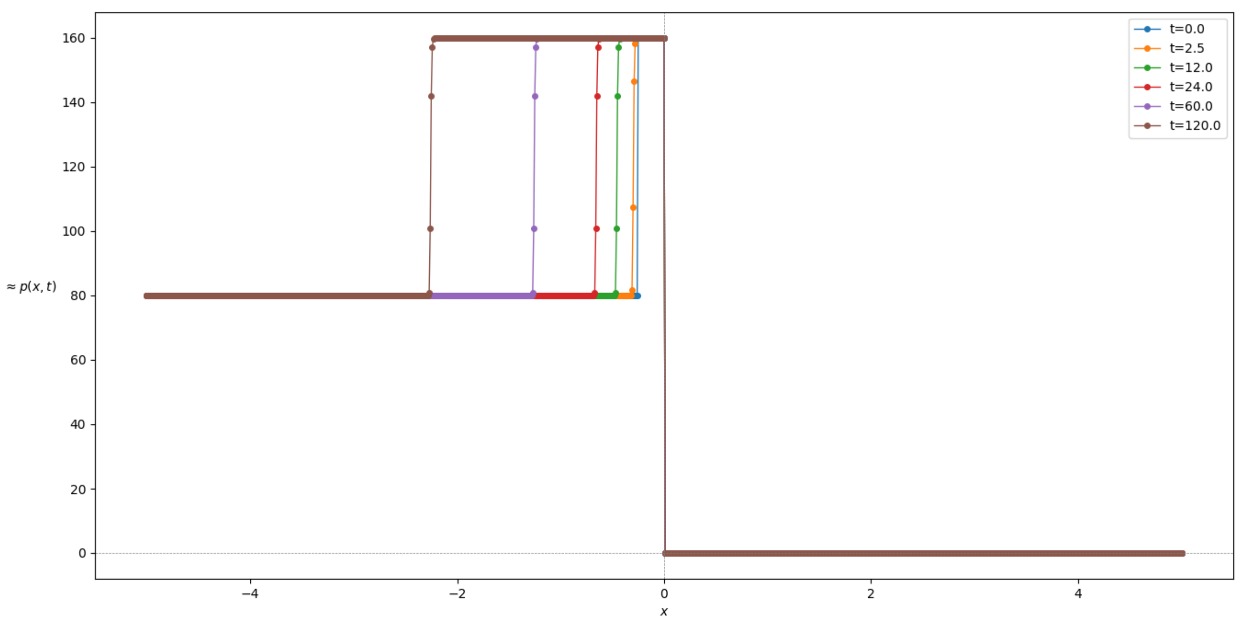


Figure 2: upwind plot at t = 0, t = 2.5, t = 12, t = 24, t = 60, t = 120

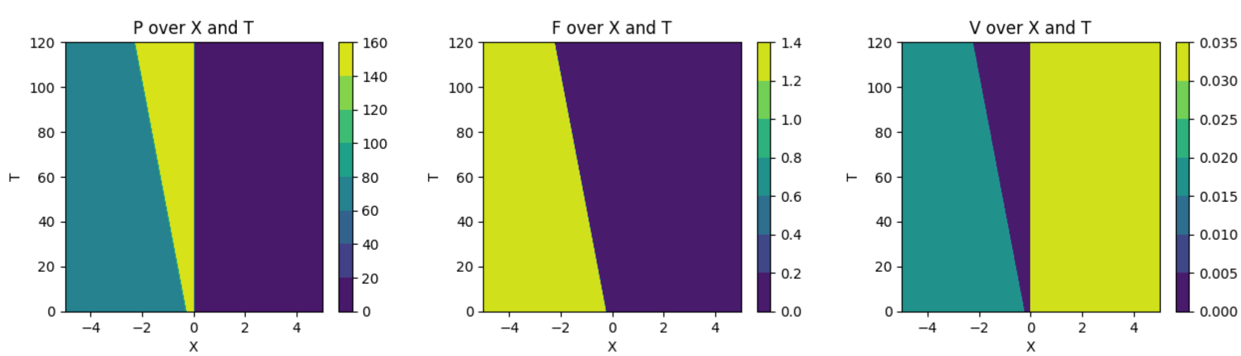


Figure 3: contour plots for P, F, and V

#1.3

The contour plots shown above represent what we've expected. Until the shock occurs at x = -0.25km, we can see that the density remains to be 80 car/km, and suddenly becomes 160 car/km when the shock occurs. This could've happened due to various reasons in real life, such as drivers in front pressing brakes which slows down entire traffic. This brake will propagate to people behind the driver who pressed the brake. This will essentially act as if there was a red light or stop sign on the highway. After the shock, the density remains to be 0 car/km, which indicates according to this mode, all the cars are not moving past x = 0km line. Realistically, this is impossible as if there's an empty road past x = 0, cars would be moving and the traffic should be a little bit looser. Even if there was a stop sign or red light at x = 0km, which could make 0 car/h density possible for that time period, the traffic should be mitigated as time goes by it'll eventually turn to green light. The density will definitely be low in a certain area, but it certainly won't be 0 after a certain time. In the real world, after some time, the shock will go away and happen again periodically, which means cars are still moving. If we had a better model, it would indicate a possibility of a decrease in shock as time goes by, however, in this mode, as shown in Figure 2, the shock still remains even after two minutes - the shock is simply propagated to an earlier stage of the road (as you can see, it's simply shifted to left). We were able to predict this from Figure 1, as when x > 0 (taking one spacial step), $p_{i,j}$ = 0 and $p_{i+1,j}$ = 0, and plugging this into Figure 1, it concludes the density will be 0 no matter how much time goes by. This is the issue of this upwind scheme mode, which is also known as the Euler forward method's instability. It's not a matter of big step size, as our step size could be considered small enough, and we have plotted with even smaller time and spatial step sizes. However, the issue comes from Euler's forward model numerical instability. In other words, the method can become unstable when applied to stiff ODEs due to the rapidly changing components of the ODE. We can see when x = 0, the stiffness is causing the issue for the upwind scheme. In Figure 1, we can see this method really only takes into consideration the difference between flow-density relations at the next spacial step and the current spacial step. As the textbook stated, "Nothing changes as time proceeds if the density at the starting time is held constant along the entire road." Looking at flow, this is also shown as after the shock, the flow is only colored purple, indicating there was no flow at all. Velocity is equal to v_max (which is shown as 0.035 km/s which is around 120 km/h, there is some rounding here), which also makes sense since there will be no car at all, (shown by 0 density as well as 0 flow) so velocity will equal to the speed limit, which we set is as 120km/h earlier. Essentially, this model suggests once the shock occurs, all the cars will remain stopped, not moving at all. This is clearly an incorrect reflection of what happens in real life, so we'll need an improvement. The improvements will be discussed below.

#2.1

To deal with the instability issue we had from the upwind scheme also known as Euler's forward method, we know instead will use the Lax-Friedrichs formula shown below.

$$p_{i,j+1} = \frac{p_{i+1,j} + p_{i-1,j}}{2} - \frac{h}{s}\frac{f_{i+1,j} - f_{i-1,j}}{2}$$

Figure 4: Lax-Friedrichs' formula

This time, our P1 and P2 indicate $p_{i+1,j}$ and $p_{i-1,j}$ respectively. We also introduced new variables F1 and F2 which indicate $f_{i+1,j}$ and $f_{i-1,j}$ respectively. Besides that, we pretty much repeat the same process from #1.1, as we use the same Solution1D function and plot_grid to show the result. Besides those changes, we have adapted the same initial values for everything else. We'll again plot those results in #2.2 and further, analyze those in #2.3

#2.2

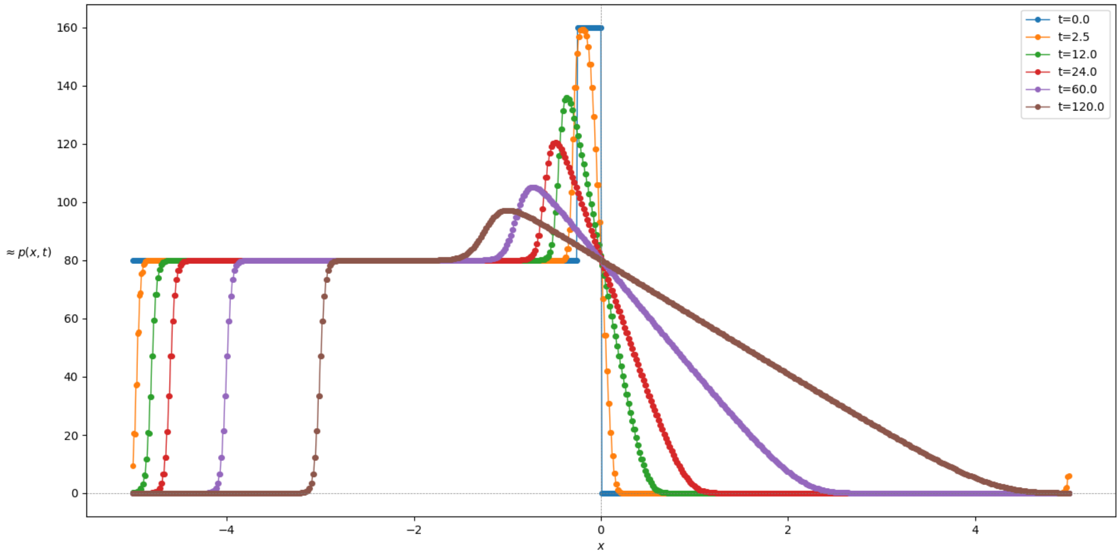


Figure 5: updated Lax-Friedrichs plot at t = 0, t = 2.5, t = 12, t = 24, t = 60, t = 120

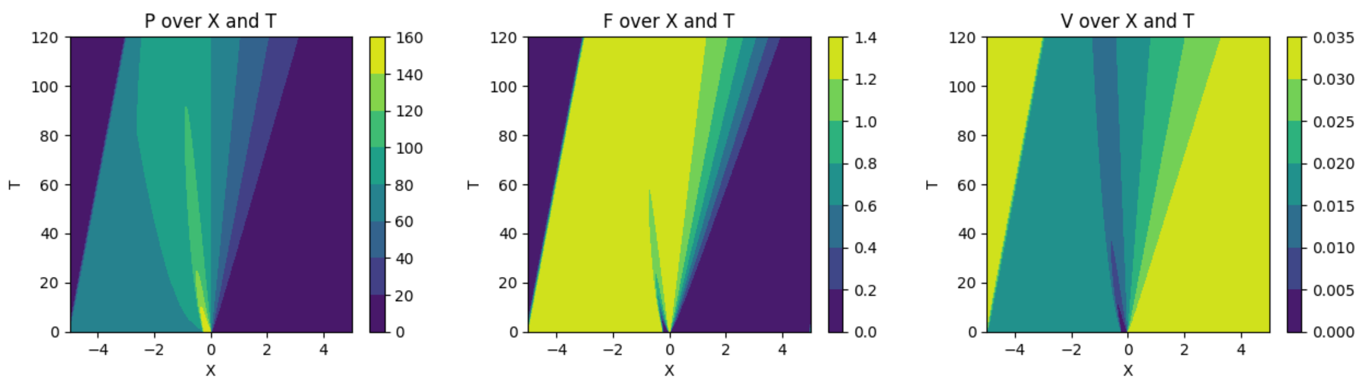


Figure 6: updated contour plots for P, F, V

#2.3

Comparing Figure 4 and Figure1, we can see our updated formula takes $p_{i-1,j}$ and $f_{i-1,j}$ into consideration as well. Instead of simply $p_{i,j}$, we're now using ($p_{i+1,j}$ + $p_{i-1,j}$) / 2, and instead of simply $f_{i+1,j}$ - $f_{i,j}$, we're now using ($f_{i+1,j}$ - $f_{i-1,j}$) / 2. The main differences are coming from now we're also looking at earlier spacial step instead of only current and next. We expected this to deal with Euler's forward method's limitation of numerical instability, and produce a better result. As shown in Figure 5, it has indeed improved and reflects a lot closer to real-life traffic. When t = 0, there's no change. However, as time goes by, we can see how the peak of y values of the graph (which is the density at a given time) decreases as t increases. This means although the shock occurs at x = 0 and density remains 0 for x > 0, as time goes by, eventually, some cars will move. We can see the plots are getting flatter as t increases, and we can expect it to be completely flattened out as the shock goes away at some point and the traffic returns to the equilibrium distribution. In other words, after a few more minutes, assuming no other shock occurs, the density will return to 80 cars/km as long as new cars incoming maintain the pace. This is a huge improvement from the upwind scheme, as the upwind scheme couldn't seem to recover from the shock at all. Looking at contour plots, we can see the density and flow gradually become 0, and velocity increases. This reflects what we've discussed so far; right after the shock, the flow and density gradually decrease as if the road past x = 0 were empty, and those cars in front of the traffic will continue to move.

#3.1

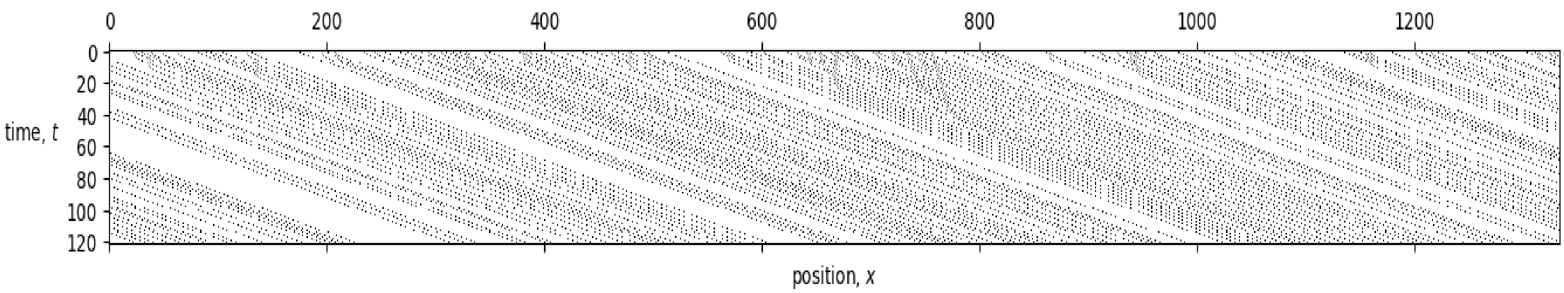
For this part, we modified our domain a little bit - specifically, for our time step, we now have 1 instead of 0.1. This was because the textbook suggested using 1s instead, which roughly corresponds to the average duration of one's reaction time. We've also set v.max to be 5 cells/timestep, and 135 km/h as well. This indicates each additional cell will increase the velocity by 27 km/h each time. We'll be able to represent (0, 27, 54, 81, 108, 135)km/h for each car's velocity on the road. In our impulseCA function, we've set the initial density to be 0.1, which means 10% of the road(=10km) has been occupied as we start. In our SolutionCA function, we've also updated our number of ghost cells to 5 to reflect the maximum number of cell steps that could be taken.

Algorithm 8.1 (The rules of the CA model).
Update for vehicle i:

1. **Accelerate**: $v_i := \min\{v_i + 1, v_{\max}\}$
2. **Decelerate**: $v_i := d(i, i+1)$, if $v_i > d(i, i+1)$
3. **Move**: vehicle i moves v_i cells forward

For our CA_algorithm function, we've used the relations from the textbook, where there are three main states: accelerate, decelerate, and move. These updates are being reflected at every new time step for each individual car. These formulas essentially look ahead at vehicles in front of them and update what actions to take. We've also made sure the cars that exit (in the far right section) come back to x = 0, which acts like a circular road. Finally, we've used simulate function and show_grid function that was used earlier for part 1 and part 2. Below in 3.2 shows the result of our plots.

#3.2



#3.3

Above plot is quite different than what we've seen so far; instead of average flow or density at a certain point, we now look at each cell which indicates a car, and the cell's color which indicates each vehicle's speed. We're using the concept of Cellular Automata as each cell is getting affected by its neighbor cells; although the cell doesn't die or stay alive, it adjusts the color of its cell which determines the speed of the car. As noted in the textbook, we can see the absence of collision and conservation of vehicles in the above plot. Not more than one vehicle can occupy one cell at a given point, and none of the cars disappear. Of course, we're neglecting some important real-life situations such as passing other cars or leaving the interstate at some point. We're simply looking at one lane that has a density of 0.1, and at the beginning the speed and location of each car are randomized. Due to this random distribution at the beginning, we can observe some traffic jams, however, we can see as time progresses, it dissolves relatively quickly. In other words, even if there was some shock happening, it gets fixed relatively quickly, and once it's fixed it won't happen much frequently due to relatively low initial density. This is also not exactly realistic, however, it allows us to view overall traffic view in the perspective of individual vehicles. This is what we've wanted, as microscopic simulation helps us understand the individuals' behavior and interaction with the system. This provides a different way of analyzing the model than Part 1 and Part 2, which is a lot closer to macroscopic simulation.

Conclusion

In this project, we've looked at a basic traffic model - we tried to understand why traffic jam out of nowhere happens, how it dissolves, and how the overall flow, density, and velocity relate to each other in a comprehensive manner. We started from upwind scheme modeling, which used Euler's forward method; we've found that there's an issue when traffic shock happens, as it failed to update any density or flows when the shock happens. We looked at the improved version of the upwind scheme, which is the Lax-Friedrichs model. Lax-Friedrichs modeling was able to avoid the numerical instability of Euler's forward method, by having $p_{i-1,j}$, $f_{i-1,j}$ components. Finally, we've also looked at the Cellular Automata model to rather provide a different view of this analysis, which shifted to microscopic simulation from macroscopic simulation. This allowed us to look at individual vehicles' behavior, which was heavily affected by the vehicle in front of them. We observed that even though there was some traffic in the beginning due to the randomness of car initialization, traffic dissolved quickly, and a steady traffic situation occurs.