The Fine-Structure Constant: Various Representations and Relationships

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1 Introduction to the Fine-Structure Constant

The fine-structure constant (α) is a dimensionless physical constant that plays a fundamental role in quantum electrodynamics. It describes the strength of the electromagnetic interaction between elementary particles. In its most well-known form, the formula is:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137.035999}$$

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2 Explanation of Symbols and Units Used

- α: Fine-structure constant (dimensionless) - e: Elementary charge (unit: Coulomb, C) - ε_0: Electric field constant (unit: Farad per meter, F/m) - μ_0: Magnetic field constant (unit: Newton per Ampere squared, N/A²) - $\hbar$: Reduced Planck's constant (unit: Joule-second, Js) - c: Speed of light (unit: Meter per second, m/s) (unit: $\mathrm{m\,s^{-1}}$) - h: Planck's constant (unit: Joule-second, Js) - m_e: Electron mass (unit: Kilogram, kg) - λ_C: Compton wavelength (unit: Meter, m) - Q: Electric charge (unit: Coulomb, C) - C: Capacitance (unit: Farad, F) - V: Voltage (unit: Volt, V) - E: Energy (unit: Joule, J)

3 Alternative Formulations of the Fine-Structure Constant

3.1 Representation with Permeability

Starting from the standard form, we can replace the electric field constant ε_0 with the magnetic field constant μ_0 by using the relationship $c^2 = \frac{1}{\varepsilon_0 \mu_0}$:

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}$$

$$\alpha = \frac{e^2}{4\pi \left(\frac{1}{\mu_0 c^2}\right) \hbar c}$$

$$= \frac{e^2 \mu_0 c^2}{4\pi \hbar c}$$

$$= \frac{e^2 \mu_0 c}{4\pi \hbar}$$

Using the relationship $\hbar = \frac{h}{2\pi}$, we obtain an alternative form:

$$\alpha = \frac{\mu_0 e^2}{2h}$$

3.2 Formulation with Electron Mass and Compton Wavelength

Planck's constant h can be expressed through other physical quantities:

$$h = \frac{m_e c \lambda_C}{2\pi}$$

where λ_C is the Compton wavelength of the electron:

$$\lambda_C = \frac{h}{m_e c}$$

Substituting this into the fine-structure constant:

$$\begin{aligned}
\alpha &= \frac{\mu_0 e^2}{2h} \\
&= \frac{\mu_0 e^2}{2\frac{m_e c \lambda_C}{2\pi}} \\
&= \frac{\mu_0 e^2 \cdot 2\pi}{2 m_e c \lambda_C} \\
&= \frac{\mu_0 e^2 \pi}{m_e c \lambda_C}
\end{aligned}$$

3.3 Expression with Classical Electron Radius

The classical electron radius is defined as:

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}$$

Using $\varepsilon_0 = \frac{1}{\mu_0 c^2}$, it follows:

$$r_e = \frac{e^2 \mu_0}{4\pi m_e c^2}$$

The fine-structure constant can be written as the ratio of the classical electron radius to the Compton wavelength:

$$\alpha = \frac{r_e}{\lambda_C}$$

This leads to another form:

$$\begin{aligned}
\alpha &= \frac{e^2 \mu_0}{4\pi m_e c^2} \cdot \frac{2\pi m_e c}{h} \\
&= \frac{e^2 \mu_0}{2hc}
\end{aligned}$$

3.4 Formulation with μ_0 and ε_0 as Fundamental Constants

Using the relationship $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$, the fine-structure constant can be expressed as:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \cdot \sqrt{\mu_0\varepsilon_0}$$
$$= \frac{e^2}{4\pi\varepsilon_0\hbar} \cdot \sqrt{\mu_0\varepsilon_0}$$

These different representations allow for various physical interpretations and demonstrate the connections between fundamental natural constants.

4 Derivation of Planck's Constant Through Fundamental Electromagnetic Constants

4.1 Dimensional Interchangeability: A New Perspective

Before beginning the derivation, it is important to explain the fundamental assumption of this work: In a fundamental theory, the dimensions of quantum quantities (such as Planck's constant) and electromagnetic quantities (such as permittivity and permeability) might be interchangeable or reducible to a common basis.

This assumption is based on the observation that nature, at its most fundamental level, might require a unified description in which the seemingly different dimensions of quantum mechanics and electrodynamics appear as different manifestations of the same underlying structure. In such a theory, the established dimensional barriers would not be absolute but rather artifacts of our macroscopic perspective.

This interchangeability of dimensions allows us to establish a direct connection between h and the electromagnetic constants, going beyond conventional physics and possibly hinting at a deeper structure of the universe.

4.2 Relationship Between h, μ_0, and ε_0

First, we consider the fundamental relationship between the speed of light c, permeability μ_0, and permittivity ε_0: (unit: $\mathrm{m\,s^{-1}}$)

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

We also use the fundamental relationship between Planck's constant h and the Compton wavelength λ_C of the electron:

$$h = \frac{m_e c \lambda_C}{2\pi}$$

The Compton wavelength is defined as:

$$\lambda_C = \frac{h}{m_e c}$$

By substituting the speed of light $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, we obtain: (unit: $\mathrm{m\,s^{-1}}$)

$$h = \frac{m_e}{2\pi} \cdot \frac{\lambda_C}{\sqrt{\mu_0 \varepsilon_0}}$$

Now we replace λ_C with its definition:

$$h = \frac{m_e}{2\pi} \cdot \frac{h}{m_e c \sqrt{\mu_0 \varepsilon_0}}$$

This leads to:

$$h^2 = \frac{1}{\mu_0 \varepsilon_0} \cdot \frac{m_e^2 \lambda_C^2}{4\pi^2}$$

Using $\lambda_C = \frac{h}{m_e c}$, it follows:

$$h^2 = \frac{1}{\mu_0 \varepsilon_0} \cdot \frac{m_e^2}{4\pi^2} \cdot \frac{h^2}{m_e^2 c^2}$$

After canceling m_e^2 and substituting $c^2 = \frac{1}{\mu_0 \varepsilon_0}$, we finally obtain:

$$h = \frac{1}{2\pi \sqrt{\mu_0 \varepsilon_0}}$$

This equation shows that, under the assumption of dimensional interchangeability, Planck's constant h can indeed be expressed through the electromagnetic vacuum constants μ_0 and ε_0. This relationship suggests a deeper connection between quantum mechanics and electrodynamics.

For the reduced Planck's constant $\hbar = \frac{h}{2\pi}$, it follows accordingly:

$$\hbar = \frac{h}{2\pi} = \frac{1}{2\pi} \cdot \frac{1}{2\pi \sqrt{\mu_0 \varepsilon_0}} = \frac{1}{4\pi^2 \sqrt{\mu_0 \varepsilon_0}}$$

5 Redefinition of the Fine-Structure Constant

5.1 What Does the Elementary Charge e Mean?

The elementary charge e represents the electric charge of an electron or proton and is approximately $e \approx 1.602 \times 10^{-19}$ C (Coulomb).

5.2 The Fine-Structure Constant Through Electromagnetic Vacuum Constants

The fine-structure constant α is traditionally defined as:

$$\alpha = \frac{e^2}{4\pi \varepsilon_0 \hbar c}$$

By substituting the derivation for $\hbar = \frac{1}{4\pi^2 \sqrt{\mu_0 \varepsilon_0}}$, we get:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \cdot \frac{1}{4\pi^2\sqrt{\mu_0\varepsilon_0}} \cdot c}$$
$$= \frac{e^2}{4\pi\varepsilon_0} \cdot 4\pi^2\sqrt{\mu_0\varepsilon_0} \cdot \frac{1}{c}$$
$$= \frac{e^2 \cdot 4\pi^2 \cdot \sqrt{\mu_0\varepsilon_0}}{4\pi\varepsilon_0 \cdot c}$$
$$= \frac{\pi e^2 \cdot \sqrt{\mu_0\varepsilon_0}}{\varepsilon_0 \cdot c}$$

Using $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$, it follows:

$$\alpha = \frac{\pi e^2 \cdot \sqrt{\mu_0\varepsilon_0}}{\varepsilon_0} \cdot \sqrt{\mu_0\varepsilon_0}$$
$$= \frac{\pi e^2 \cdot \mu_0\varepsilon_0}{\varepsilon_0}$$
$$= \pi e^2 \cdot \mu_0$$

Alternatively, with the other derivation for $h = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}$:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$
$$= \frac{e^2}{4\pi\varepsilon_0 \cdot \frac{h}{2\pi} \cdot c}$$
$$= \frac{e^2 \cdot 2\pi}{4\pi\varepsilon_0 \cdot h \cdot c}$$
$$= \frac{e^2}{2\varepsilon_0 \cdot h \cdot c}$$

Substituting $h = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}$ here:

$$\alpha = \frac{e^2}{2\varepsilon_0 \cdot \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}} \cdot c}$$
$$= \frac{e^2 \cdot 2\pi\sqrt{\mu_0\varepsilon_0}}{2\varepsilon_0 \cdot c}$$
$$= \frac{\pi e^2 \cdot \sqrt{\mu_0\varepsilon_0}}{\varepsilon_0 \cdot c}$$

Using $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ again:

$$\alpha = \pi e^2 \cdot \mu_0$$

This representation shows that the fine-structure constant can be directly derived from the electromagnetic structure of the vacuum without explicitly involving h or $\hbar$.

6 Consequences of a Redefinition of Coulomb

6.1 Is Coulomb Incorrectly Defined if $\alpha = 1$ is Assumed?

The hypothesis is that if the fine-structure constant $\alpha = 1$ were set, the definition of the Coulomb, and thus the elementary charge e, would need to be adjusted.

6.2 New Definition of the Elementary Charge

If we set $\alpha = 1$ and use the original formula $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$, then for the elementary charge e:

$$e^2 = 4\pi\varepsilon_0\hbar c$$

$$e = \sqrt{4\pi\varepsilon_0\hbar c}$$

Alternatively, with the newly derived form $\alpha = \pi e^2 \cdot \mu_0$:

$$1 = \pi e^2 \cdot \mu_0$$

$$e = \sqrt{\frac{1}{\pi\mu_0}}$$

This would mean that the numerical value of e would change because it would then depend directly on $\hbar$, c, and ε_0, or on μ_0.

6.3 Physical Significance

The unit Coulomb (C) is an arbitrary convention in the SI system. If $\alpha = 1$ were chosen instead, the definition of e would change. In natural unit systems (common in high-energy physics), $\alpha = 1$ is often set, meaning that charge is measured in a unit other than Coulomb.

The current value of the fine-structure constant $\alpha \approx \frac{1}{137}$ is not "wrong" but a consequence of our historical definitions of units. The electromagnetic unit system could originally have been defined such that $\alpha = 1$.

7 Effects on Other SI Units

7.1 What Effects Would a Coulomb Adjustment Have on Other Units?

Adjusting the charge unit so that $\alpha = 1$ would have consequences for numerous other physical units:

7.1.1 New Charge Unit

The new elementary charge would be:

$$e = \sqrt{4\pi\varepsilon_0\hbar c} = \sqrt{\frac{1}{\pi\mu_0}}$$

7.1.2 Change in Electric Current (Ampere)

Since 1 A = 1 C/s, the unit of Ampere would also change accordingly.

7.1.3 Changes in Electromagnetic Constants

Since ε_0 and μ_0 are linked to the speed of light: (unit: $\mathrm{m\,s^{-1}}$)

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

perhaps either μ_0 or ε_0 would need to be adjusted.

7.1.4 Effects on Capacitance (Farad)

Capacitance is defined as $C = \frac{Q}{V}$. Since Q (charge) changes, the unit of Farad would also change.

7.1.5 Changes in Voltage Unit (Volt)

Electric voltage is defined as 1 V = 1 J/C. Since Coulomb would have a different magnitude, the size of the Volt would also shift.

7.1.6 Indirect Effects on Mass

In quantum field theory, the fine-structure constant is linked to the rest mass energy of electrons, which could have indirect effects on the definition of mass.

7.1.7 Consistency Check of the Fine-Structure Constant with SI Units

To check whether an adjustment of μ_0 or ε_0 would be necessary, we consider the relationship:

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

with the known values: - Speed of light: $c = 299792458$ m/s - Magnetic field constant: $\mu_0 = 4\pi \times 10^{-7}$ H/m - Electric field constant: $\varepsilon_0 = \frac{1}{\mu_0 c^2}$ F/m

The fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

with: - Planck constant: $\hbar = 1.054571817 \times 10^{-34}$ J·s - Elementary charge: $e = 1.602176634 \times 10^{-19}$ C

Substituting the values yields:

$$\alpha_{\text{calculated}} = 0.007297352569776441$$

The official value is:

$$\alpha_{\text{official}} = 0.0072973525692838015$$

The difference is 4.93×10^{-13}, which is extremely small. Thus, α remains consistent within the expected precision, and an adjustment of μ_0 or ε_0 is not necessary.

The gravitational constant can be expressed through Planck units as:

$$G = \frac{\hbar c}{m_P^2}$$

with: - $\hbar = 1.054571817 \times 10^{-34}$ J·s - $c = 299792458$ m/s - $m_P = 2.176434 \times 10^{-8}$ kg (Planck mass)

Substituting the values:

$$G_{\text{calculated}} = \frac{(1.054571817 \times 10^{-34}) \times (299792458)}{(2.176434 \times 10^{-8})^2}$$

Result:

$$G_{\text{calculated}} = 6.6743021 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$$

The official value:

$$G_{\text{official}} = 6.67430 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$$

The deviation is only 2.1×10^{-17}, which is within measurement accuracy. This means G is correctly defined in the SI system and remains consistent with Planck units.

In an alternative unit system, we set certain constants to 1: - $c = 1$ (speed of light) - $\hbar = 1$ (reduced Planck constant) - $e = 1$ (elementary charge) - $\alpha = 1$ (fine-structure constant)

From this, the electric field constant follows as:

$$\varepsilon_0 = \frac{1}{4\pi}$$

and with the relationship $c^2 = 1/(\mu_0 \varepsilon_0)$, we obtain the magnetic field constant:

$$\mu_0 = 4\pi$$

In this system, units change: - Charge unit: $e = 1$ fixes the unit of charge. - Energy unit: Since $\hbar = 1$, energy is measured directly in frequency units. - Time unit: With $c = 1$, length units are directly linked to time units.

The fine-structure constant remains 1 by definition, but physical quantities are scaled differently in this system.

In our new unit system, the following constants were set to 1: - Speed of light $c = 1$ - Reduced Planck constant $\hbar = 1$ - Elementary charge $e = 1$ - Fine-structure constant $\alpha = 1$

Not set to 1: - Planck energy E_P, which serves as a measure for the energy unit. - Gravitational constant G, as it relates to Planck units.

This means all other physical constants can be derived from these values.

The Planck energy is defined as:

$$E_P = \sqrt{\frac{\hbar c^5}{G}}$$

Substituting the known values:

$$E_P = \sqrt{\frac{(1.054571817 \times 10^{-34})(299792458)^5}{6.67430 \times 10^{-11}}}$$

This yields:

$$E_P \approx 1.956 \times 10^9 \text{ J}$$

Since $E_P = 1$ in our new unit system, it follows:

$$1 \text{ energy unit} = 1.956 \times 10^9 \text{ J}$$

Thus, any energy in our system can be converted to Joules by multiplying by this factor.

Choice of the Fine-Structure Constant in a Natural Unit System

In a natural unit system, fundamental natural constants can be set to 1 to simplify equations. A key question is whether the fine-structure constant α should also be set to 1 or if its measured value $\alpha \approx \frac{1}{137}$ must be retained.

Setting $\alpha = 1$

If $\alpha = 1$ is set, several simplifications arise:

- The electric permittivity becomes $\varepsilon_0 = \frac{1}{4\pi}$, and the magnetic constant $\mu_0 = 4\pi$.
- The Maxwell equations take a particularly simple form.
- The elementary charge e is directly linked to energy units.
- All fundamental constants become dimensionless.
- Simplified quantum electrodynamics (QED), as the coupling strength no longer appears as a perturbation parameter.
- Potential for a direct relationship between gravitation and electrodynamics if G is chosen accordingly.

However, there are also challenges:

- SI values for charge, energy, and electrical constants would need to be rescaled.

Retaining $\alpha \approx 1/137$

Alternatively, α can be left at its measured value. Then, the known conversion factors remain:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$$

This maintains physical consistency with experimental values, but some equations become slightly more complex.

Possible Alternative Formulations

This natural form allows various alternative formulations:

- Directly derivable relationships between electromagnetic and gravitational forces.
- Maxwell equations in a particularly simple form.
- Simplified quantum electrodynamics, as the coupling strength no longer acts as a perturbation parameter.
- Improved comparability of gravitation and electrodynamics through scaled constants.
- Simplification of Planck units and their connection to electromagnetic quantities.

Conclusion

There is no definitive answer as to which choice is better:

- **If simplifying equations is the main goal**, $\alpha = 1$ makes sense.
- **If maintaining physical consistency with measured values is desired**, $\alpha \approx 1/137$ is preferable.

Since $c = 1$, $\hbar = 1$, and $e = 1$ have already been set in this unit system, it would be consistent to also choose $\alpha = 1$ to further simplify the equations.

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