Control:

$$\mathbf{u} = \begin{bmatrix} \tau \\ \theta \end{bmatrix} \tag{1}$$

Cost functional:

$$\begin{aligned} J[u] &= \int_0^{t_f} \alpha\tau^2 \ dt + \gamma t_f + \beta\phi(\dot{x}(t_f), \dot{y}(t_f)) & (2) \\ &= \int_0^{t_f} \alpha\tau^2 \ dt + \gamma t_f + \beta \left(\dot{x}(t_f)^2 + \dot{y}(t_f)^2\right) & (3) \end{aligned}$$

Hamiltonian:

$$\begin{aligned} H &= \mathbf{p} \cdot \mathbf{f} - L & (4) \\ &= \mathbf{p} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \tau\cos(\theta) \\ \tau\sin(\theta) - g \end{bmatrix} - \alpha\tau^2 & (5) \\ &= p_1\dot{x} + p_2\dot{y} + p_3\tau\cos(\theta) + p_4(\tau\sin(\theta) - g) - \alpha\tau^2 & (6) \end{aligned}$$

$$\begin{aligned} \frac{\partial H}{\partial \theta} &= -p_3\tau\sin\theta + p_4\tau\cos\theta = 0 \implies (c_3 - c_1 t)\tan\theta = c_4 - c_2 t & (7) \\ \frac{\partial H}{\partial \tau} &= p_3\cos\theta + p_4\sin\theta - 2\alpha\tau = 0 \implies 2\alpha\tau = p_3\cos\theta + p_4\sin\theta & (8) \end{aligned}$$

Costate equations satisfy:

$$\begin{aligned} \dot{p_1} &= -\frac{\partial H}{\partial x} = 0 \implies p_1 = c_1 & (9) \\ \dot{p_2} &= -\frac{\partial H}{\partial y} = 0 \implies p_2 = c_2 & (10) \\ \dot{p_3} &= -\frac{\partial H}{\partial \dot{x}} = -p_1 \implies p_3 = -c_1 t + c_3 & (11) \\ \dot{p_4} &= -\frac{\partial H}{\partial \dot{y}} = -p_2 \implies p_4 = -c_2 t + c_4 & (12) \end{aligned}$$

State equations satisfy:

$$\begin{aligned} \dot{x} &= \dot{x} & (13) \\ \dot{y} &= \dot{y} & (14) \\ \ddot{x} &= \tau\cos\theta & (15) \\ \ddot{y} &= \tau\sin\theta - g & (16) \end{aligned}$$

Boundary conditions:

$$
\begin{align}
x(0) &= x_0 \tag{17}\\
y(0) &= y_0 \tag{18}\\
\dot{x}(0) &= v_0 \tag{19}\\
\dot{y}(0) &= 0 \tag{20}\\
p_1(t_f) &= -\frac{\partial\phi}{-x(t_f)} = 0 \implies c_1 = 0 \tag{21}\\
p_3(t_f) &= 2\beta\dot{x}(t_f) \tag{22}\\
p_4(t_f) &= 2\beta\dot{y}(t_f) \tag{23}\\
H(t_f) &= p_1\dot{x}(t_f) + p_2\dot{y}(t_f) + p_3\tau(t_f)\cos(\theta(t_f)) + p_4(\tau(t_f)\sin(\theta(t_f)) - g) - \alpha\tau(t_f)^2 \tag{24}\\
&= c_2\dot{y}(t_f) + 2\beta\dot{x}(t_f)\tau(t_f)\cos\theta(t_f) + p_3\tau(t_f)\cos(\theta(t_f)) + p_4\left(\tau(t_f)\sin(\theta(t_f)) - g\right) - \alpha\tau(t_f)^2 \tag{25}\\
&= -\frac{\partial\phi}{\partial t_f} \tag{26}\\
&= -\gamma \tag{27}
\end{align}
$$

Implications:

$$
\begin{align}
p_3(t_f) = p_3(t) &= 2\beta\dot{x}(t_f) = c_3 \tag{28}\\
p_4(t_f) &= 2\beta\dot{y}(t_f) = c_4 - c_2 t_f \tag{29}\\
p_4(t) &= 2\beta\dot{x}(t_f)\tan(\theta(t)) \tag{30}\\
&= \frac{2\alpha\tau}{\sin\theta} - 2\beta\dot{y}\cot\theta(t) \tag{31}\\
\tan\theta(t_f) &= \frac{\dot{y}(t_f)}{\dot{x}(t_f)} \tag{32}
\end{align}
$$

And now we're scrapping all this and starting over with the control $\mathbf{u} = \begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix}$. Something to look forward to :)

New cost:

$$
J[\mathbf{u}] = \int_0^{t_f} \alpha\|u\|^2 \, dt + \gamma t_f + \beta\left(\dot{x}(t_f)^2 + \dot{y}(t_f)^2\right)
$$

New Hamiltonian:

$$
H = p_1\dot{x} + p_2\dot{y} + p_3\ddot{x} + p_4(\ddot{y} - g) - \alpha\left(\ddot{x}^2 + ddoty^2\right)
$$

New costate:

$$
\begin{aligned}
\dot{p}_1 = 0 &\implies p_1 = c_1\\
\dot{p}_2 = 0 &\implies p_2 = c_2\\
\dot{p}_3 = -p_1 &\implies p_3 = c_3 - c_1 t\\
\dot{p}_4 = -p_2 &\implies p_4 = c_4 - c_2 t
\end{aligned}
$$

Optimization condition:

$$\frac{\partial H}{\partial \ddot{x}} = p_3 - 2\alpha\ddot{x} = 0$$
$$\frac{\partial H}{\partial \ddot{y}} = p_4 - 2\alpha\ddot{y} = 0$$

Boundary conditions:

$$\begin{aligned} H(t_f) &= \gamma \\ x(0) &= x_0 \\ y(0) &= y_0 \\ \dot{x}(0) &= v_0 \\ \dot{y}(0) &= 0 \end{aligned}$$