

Control:

$$\mathbf{u} = \begin{bmatrix} \tau \\ \theta \end{bmatrix} \quad (1)$$

Cost functional:

$$J[u] = \int_0^{t_f} \alpha \tau^2 dt + \gamma t_f + \beta \phi(\dot{x}(t_f), \dot{y}(t_f)) \quad (2)$$

$$= \int_0^{t_f} \alpha \tau^2 dt + \gamma t_f + \beta (\dot{x}(t_f)^2 + \dot{y}(t_f)^2) \quad (3)$$

Hamiltonian:

$$H = \mathbf{p} \cdot \mathbf{f} - L \quad (4)$$

$$= \mathbf{p} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \tau \cos(\theta) \\ \tau \sin(\theta) - g \end{bmatrix} - \alpha \tau^2 \quad (5)$$

$$= p_1 \dot{x} + p_2 \dot{y} + p_3 \tau \cos(\theta) + p_4 (\tau \sin(\theta) - g) - \alpha \tau^2 \quad (6)$$

$$\frac{\partial H}{\partial \theta} = -p_3 \tau \sin \theta + p_4 \tau \cos \theta = 0 \implies (c_3 - c_1 t) \tan \theta = c_4 - c_2 t \quad (7)$$

$$\frac{\partial H}{\partial \tau} = p_3 \cos \theta + p_4 \sin \theta - 2\alpha \tau = 0 \implies 2\alpha \tau = p_3 \cos \theta + p_4 \sin \theta \quad (8)$$

Costate equations satisfy:

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = 0 \implies p_1 = c_1 \quad (9)$$

$$\dot{p}_2 = -\frac{\partial H}{\partial y} = 0 \implies p_2 = c_2 \quad (10)$$

$$\dot{p}_3 = -\frac{\partial H}{\partial \dot{x}} = -p_1 \implies p_3 = -c_1 t + c_3 \quad (11)$$

$$\dot{p}_4 = -\frac{\partial H}{\partial \dot{y}} = -p_2 \implies p_4 = -c_2 t + c_4 \quad (12)$$

State equations satisfy:

$$\dot{x} = \dot{x} \quad (13)$$

$$\dot{y} = \dot{y} \quad (14)$$

$$\ddot{x} = \tau \cos \theta \quad (15)$$

$$\ddot{y} = \tau \sin \theta - g \quad (16)$$

Boundary conditions:

$$x(0) = x_0 \quad (17)$$

$$y(0) = y_0 \quad (18)$$

$$\dot{x}(0) = v_0 \quad (19)$$

$$\dot{y}(0) = 0 \quad (20)$$

$$p_1(t_f) = -\frac{\partial\phi}{-x(t_f)} = 0 \implies c_1 = 0 \quad (21)$$

$$p_3(t_f) = 2\beta\dot{x}(t_f) \quad (22)$$

$$p_4(t_f) = 2\beta\dot{y}(t_f) \quad (23)$$

$$H(t_f) = p_1\dot{x}(t_f) + p_2\dot{y}(t_f) + p_3\tau(t_f)\cos(\theta(t_f)) + p_4(\tau(t_f)\sin(\theta(t_f)) - g) - \alpha\tau(t_f)^2 \quad (24)$$

$$= c_2\dot{y}(t_f) + 2\beta\dot{x}(t_f)\tau(t_f)\cos\theta(t_f) + p_3\tau(t_f)\cos(\theta(t_f)) + p_4(\tau(t_f)\sin(\theta(t_f)) - g) - \alpha\tau(t_f)^2 \quad (25)$$

$$= -\frac{\partial\phi}{\partial t_f} \quad (26)$$

$$= -\gamma \quad (27)$$

Implications:

$$p_3(t_f) = p_3(t) = 2\beta\dot{x}(t_f) = c_3 \quad (28)$$

$$p_4(t_f) = 2\beta\dot{y}(t_f) = c_4 - c_2t_f \quad (29)$$

$$p_4(t) = 2\beta\dot{x}(t_f)\tan(\theta(t)) \quad (30)$$

$$= \frac{2\alpha\tau}{\sin\theta} - 2\beta\dot{y}\cot\theta(t) \quad (31)$$

$$\tan\theta(t_f) = \frac{\dot{y}(t_f)}{\dot{x}(t_f)} \quad (32)$$