Module 1 Review | 7.1 Integration by Parts | 7.4 Rational Functions | 7.2 Trigonometric Integrals | 7.3 Trigonometric Substitutions | 7.5 Strategies of Integration | 7.7 Approximate Integration | 7.8 Improper Integrals | 8.1 Arc Length

7.1 Integration by Parts

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

Deriving the Integration by Parts Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int f'(x)g(x) + f(x)g'(x)dx$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \text{ or } uv - \int vdu$$

Useful when:

1. $g'(x)$ is easy to integrate
2. $f'(x)g(x)dx$ is easy to integrate
3. Lonely integrals $arcsin(u)/tan(u)/sec(u)$ or $ln(u)dx$

How we can think about it:

$f(x) \leftrightarrow g'(x)$ want to integrate product

$\downarrow \quad \downarrow$

$f'(x) \to g(x)$ easy to integrate $\Rightarrow \int f'(x)g(x)dx$

Ex. 1 Integration by Parts

$x \leftrightarrow cos(x)$

$\downarrow \quad \downarrow$

$1 \to sin(x) \quad \Rightarrow \int sin(x)dx$

7.4 Rational Functions

Aim: Find a general method to calculate $\int \frac{p(x)}{q(x)}dx$

Useful Fact: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}arctan(\frac{x}{a}) + C$

Rational Cases:

1. $\int \frac{1}{(ax+b)^j}dx \to u = ax + b \to \frac{A}{a}\int u^{-j}du$
2. $\int \frac{Bx}{(x^2+k^2)^j}dx \to u = x^2 + k^2 \to \frac{B}{2}\int u^{-j}du$
3. $\int \frac{C}{(Cx^2+k^2)^j}dx \to \int \frac{C}{x^2+k^2}dx \to \int \frac{C}{(x^2+k^2)^2}dx \to \int \frac{C}{(x^2+k^2)^3}dx$ IBP

Strategy: Reduce all polynomials into rational integrals and these three cases.

Partial Fraction Function Rules: $\frac{r(x)}{p(x)^j}$

1. Degree of $r(x) < p(x)$
2. $p(x)$ is irreducible $\to$ cannot be factored into lower degree polynomials
3. $j \geq 1$ whole number

Irreducible Polynomial:

1. Linear $\to p(x) = ax + b,\ a \neq 0$
2. Unbreakable quadratic $\to p(x) = ax^2 + bx + c,\ bx - 4ac < 0$ *negative root*

Partial Fraction: $\frac{A}{(ax+b)^j}$ or $\frac{Bx+C}{(ax^2+bx+c)^j}$

Ex. 1 Long Division

$$\frac{x^5+3x^4+7x^3+12x^2+8x+5}{x^4+2x^3+2x^2+2x+1}$$

$= (x+1) + \frac{3x^3+8x^2+5x+4}{x^4+2x^3+2x^2+2x+1}$ degree on $top \geq bottom$

$= (x+1) + \frac{3x^3+8x^2+5x+4}{(x+1)^2(x^2+1)}$ factor the bottom into irreducibles

$$= (x+1) + \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}$$

Ex. 2 Breaking a Partial Function Apart

$$\frac{x^3+2x+1}{(x^2+3x+2)(x+1)(x^2+2)^3} = \frac{x^3+2x+1}{(x+2)(x+1)^2(x^2+2)^3}$$

$$\Rightarrow \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} + \frac{Dx+E}{(x^2+2)} + \frac{Fx+G}{(x^2+2)^2} + \frac{Hx+I}{(x^2+2)^3}$$

Ex. 3 Solving for the Coefficients

$$\frac{2-3x}{x-x^2}$$

$$\frac{2-3x}{x(1-x)} = \frac{A}{x} + \frac{B}{(1-x)}$$

$$(\frac{A}{x} + \frac{B}{(1-x)})(x(1-x))$$

$\frac{A(1-x)+B(x)}{x(1-x)} = \frac{A+(B-A)x}{x-x^2}$ compare new equation to original

$$A = 2\ B = 1$$

$\frac{2}{x} + \frac{1}{(1-x)}$ after all fractions are irreducibles, solve for the integral

7.2 Trigonometric Integrals

Aim: Calculate $\int sin^m(x)cos^n(x)dx$ where m and n are integers

Strategy: Using simple trigonometric identities, make appropriate substitutions to get a rational integral.

Trigonometric Integral Rules:

$\int sin^m(x)cos^n(x)dx$

1. m odd: $u = cos(x)\ sin^2(x) = 1 - cos^2(x)$
2. n odd: $u = sin(x)\ cos^2(x) = 1 - sin^2(x)$
3. m and n odd: either 1 or 2
4. m and n odd: $sin(2x) = 2cos(x)sin(x)$ or $cos^2(x) = \frac{1+cos(2x)}{2}$ or $sin^2(x) = \frac{1-cos(2x)}{2}$
5. $-(m+n) > 0$ and even: $u = tan(x)$ convert to $\int tan^m(x)sec^{-(a+b)}(x)dx$

$\int tan^m(x)sec^n(x)dx$

1. m odd: $u = sec(x)\ tan^2(x) = sec^2(x) - 1$
2. n even: $u = tan(x)\ sec^2(x) = 1 + tan^2(x)$, strip 2 secants out
3. m odd n even: either 1 or 2
4. m even n odd: each integral will be dealt with differently

7.3 Trigonometric Substitutions

Inverse Substitution:

$$x - world \Leftrightarrow \theta - world$$

$$x = h(\theta)$$

$$dx = h'(\theta)d\theta$$

$$\int f(x)dx \to \int f(x)h'(\theta)d\theta \to \int f(h'(\theta)h'(\theta)d\theta$$

$F(u(\theta)) + C = \int f(u(\theta)h'(\theta)d\theta$ solve in $\theta - world$

Facts:

1. $\sqrt{a^2 - x^2} \Rightarrow x = asin(\theta) \Rightarrow sin(\theta) = \frac{x}{a}$
2. $\sqrt{a^2 + x^2} \Rightarrow x = atan(\theta) \Rightarrow tan(\theta) = \frac{x}{a}$
3. $\sqrt{x^2 - a^2}\ dx \Rightarrow x = asec(\theta) \Rightarrow sec(\theta) = \frac{x}{a}$ or $cos(\theta) = \frac{a}{x}$

Ex. 1 Solving from $x - world$ to $\theta - world$ to $u - world$

$$\int \frac{1}{\sqrt{1+x^2}}dx \Rightarrow x = tan(\theta),\ dx = sec^2(\theta)$$

$$= \int \frac{sec^2\theta}{sec(\theta)}d\theta = \int sec(\theta)d\theta = \int sin^0(\theta)cos^{-1}(\theta) \Rightarrow u = sin(\theta),\ d\theta = \frac{dx}{cos(\theta)}$$

$$= \int sin^0(\theta)cos^{-2}(\theta)d\theta = \int \frac{1}{cos^2(\theta)}d\theta$$

$= \int \frac{1}{(1-sin^2(\theta))}du = \int \frac{1}{(1-u^2)}du$ looks like partial fraction

$$(\frac{A}{(1-u)} + \frac{B}{(1+u)})(1-u)(1+u)$$

$$= A + Au + B - Bu = 1$$

$$= u(A - B) + A + B = 1$$

$$A \text{ and } B = 1/2$$

$$\int \frac{\frac{1}{2}}{(1-u)}du + \int \frac{\frac{1}{2}}{(1+u)}du = \frac{ln|1+u|}{2} - \frac{ln|1-u|}{2} + C$$

$$= \frac{ln|1+sin(\theta)|}{2} - \frac{ln|1-sin(\theta)|}{2} + C \Rightarrow tan(\theta) = \frac{x}{1},\ sin(\theta) = \frac{x}{\sqrt{1+x^2}}$$

Ex. 1 Type 1

$$\frac{\ln\left|1+\frac{x}{\sqrt{1+x^2}}\right|}{2} - \frac{\ln\left|1+\frac{x}{\sqrt{1+x^2}}\right|}{2} + C$$

7.5 Strategies of Integration

Strategy: Reduce to power x^n, exponential b^x, trigonometric $sin(x)/cos(x)/etc.$ using sum rule/constant multiple rule/substitution/IBP.

Integration by Substitution:

$$\int f(g(x))g'(x)dx \qquad F(g(x)) + C$$

$$\uparrow \quad u = g(x),\ du = g'(x) \qquad \uparrow$$

$$\int f(g(x))g'(x)\frac{du}{g'(x)} \rightarrow \int f(u)du = F(x) + C$$

Common Substitutions:

$$\int f(x^k)x^{k-1}dx \Rightarrow u = x^k \qquad \int f(cos(x))sin(x)dx \Rightarrow u = sin(x)$$

$$\int f(b^x)b^x dx \Rightarrow u = b^x \qquad \int \frac{f(arctan(x))}{1+x^2}dx \Rightarrow u = arctan(x)$$

$$\int \frac{f(ln|x|)}{x}dx \Rightarrow u = ln|x| \qquad \int \frac{f(arcsin(x))}{\sqrt{1-x^2}}dx \Rightarrow u = arcsin(x)$$

7.7 Approximate Integration

Aim: Approximate $\int_a^b f(x)dx$ if we cannot find an antiderivative.

Approximation Formulas:

1. Left Endpoint: $x_i^* = x_i - 1$
2. Right Endpoint: $x_i^* = x_i$
3. Midpoint: $x_i^* = \frac{x_{i-1}+x_i}{2} = \bar{x}$
   a. concave down $M_n > \int_a^b f(x)dx$ <u>overestimate</u>
   b. concave up $M_n < \int_a^b f(x)dx$ <u>underestimate</u>
4. Trapezoidal Rule: T_n $= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]$
   a. concave down $T_n < \int_a^b f(x)dx$ <u>underestimate</u>
   b. concave up $T_n > \int_a^b f(x)dx$ <u>overestimate</u>
5. Simpson's Rule: $S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + f(x_n)]$

Error bounds:

1. $\left|E_{M_n}\right| \le \frac{K(b-a)^3}{24n^2}$ $\quad \left|f^{(2)}(x)\right| \le K$ for some constant K for all x in $[a, b]$
2. $\left|E_{T_n}\right| \le \frac{K(b-a)^3}{12n^2}$ $\quad \left|f^{(2)}(x)\right| \le K$ for some constant K for all x in $[a, b]$
3. $\left|E_{S_n}\right| \le \frac{K(b-a)^5}{180n^4}$ $\quad \left|f^{(4)}(x)\right| \le K$ for some constant K for all x in $[a, b]$

7.8 Improper Integrals

Type 1 Cases | Improper Integrals:

1. $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$ <u>convergent</u> if the limit exists
2. $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$ <u>convergent</u> if the limit exists
3. $\int_{-\infty}^\infty f(x)dx = \lim_{t\to\infty}\int_0^t f(x) + \lim_{t\to-\infty}\int_t^0 f(x)dx$

   <u>convergent</u> $\Leftrightarrow$ <u>both convergent</u> or <u>divergent</u> $\Leftrightarrow$ <u>both divergent</u>

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx$$

$$= \lim_{t\to\infty} ln|t| - ln|1|$$

$$= \lim_{t\to\infty} ln|t| = \infty \text{ divergent}$$

Type 2 Cases | Vertical Asymptotes:

1. If f has a vertical asymptote at
2. a $\int_a^b f(x)dx = \lim_{t\to a+}\int_t^b f(x)dx$
3. If f has a vertical asymptote at b $\int_a^b f(x)dx = \lim_{t\to b-}\int_a^t f(x)dx$
4. If f has a vertical asymptote at c, $a < c < b$

   $\int_a^b f(x)dx = \lim_{t\to c-}\int_a^t f(x)dx + \lim_{t\to c+}\int_t^b f(x)dx$

Ex. 2 Type 2 Vertical Asymptote

$$\int_{-1}^1 \frac{1}{x^2}dx = \int_{-1}^0 \frac{1}{x^2}dx + \int_0^1 \frac{1}{x^2}dx$$

$$\int_0^1 \frac{1}{x^2}dx = \lim_{t\to0+}\int_t^1 \frac{1}{x^2}dx$$

$$\lim_{t\to0+} \frac{1}{t} - 1$$

$$= \infty \text{ divergent} \Leftrightarrow \text{both divergent}$$

Facts:

1. $\int_1^\infty \frac{1}{x^p}dx$ <u>convergent</u> if $p > 1$ <u>divergent</u> if $p \le 1$
2. $\int_0^1 \frac{1}{x^p}dx$ <u>convergent</u> if $p < 1$ <u>divergent</u> if $p \ge 1$
3. $\int_0^\infty \frac{1}{b^x}dx$ <u>convergent</u> $\Rightarrow$ $\int_{-\infty}^0 b^x$ <u>convergent</u>

Comparison Test: Assume $0 \le g(x) \le f(x)$ or $f(x) \ge g(x)$

1. $\int_a^\infty g(x)dx \mid \int_{-\infty}^b g(x)dx \mid \int_{-\infty}^\infty g(x)dx \mid \int_a^b g(x)dx$ <u>divergent</u> $\Rightarrow$

   $\int_a^\infty f(x)dx \mid \int_{-\infty}^b f(x)dx \mid \int_{-\infty}^\infty f(x)dx \mid \int_a^b f(x)dx$ <u>divergent</u>
2. $\int_a^\infty f(x)dx \mid \int_{-\infty}^b f(x)dx \mid \int_{-\infty}^\infty f(x)dx \mid \int_a^b f(x)dx$ <u>convergent</u> $\Rightarrow$

   $\int_a^\infty g(x)dx \mid \int_{-\infty}^b g(x)dx \mid \int_{-\infty}^\infty g(x)dx \mid \int_a^b g(x)dx$ <u>convergent</u>

Ex. 3 Comparison Test

$$\int_1^\infty e^{-x^2}dx,\ 0 \le e^{-x^2} \le e^{-x} \text{ on } [1, \infty]$$

$$f(x) = \int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx$$

$$\lim_{t\to\infty} e^{-1} - e^{-t}$$

$$\lim_{t\to\infty} e^{-1} - \frac{1}{e^\infty}$$

$$= e^{-1} \text{ convergent}$$

$$e^{-x} \ge e^{-x^2}$$

$$\int_1^\infty e^{-x^2}dx \text{ convergent}$$

8.1 Arc Length

Arc length of $y = f(x)$ over $[a, b] = \lim_{n\to\infty}\sum_{i=1}^n length\ li$

$$= \int_a^b \sqrt{1 + (f'(x))^2 dx}$$