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11.1 Sequences

Sequence: An infinite list $\{a_1, a_2, a_3, \ldots\} = \{a_n\}_{n=1}^{*} = \{a_n\}$

Limits of Sequences: $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{x\to\infty} f(x) = L$

1. Informal: a_n approaches L as n grows positively without bound
2. Precise: given $\varepsilon > 0$, there exists $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$

11.2 Series

Fact: Monotone Convergence Theorem
$\{a_n\}$ bounded and eventually increasing or decreasing $\Rightarrow$ $\lim_{n\to\infty} = L$ for some number L

Partial Sum: $S_n \longrightarrow S_1 = a_1 + a_2 + a_3 + \ldots$

Series: A sum of a sequence $S_\infty = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

Geometric Series:

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ $\quad |r| < 1 \Rightarrow$ converges

$\sum_{n=1}^{\infty} ar^{n-1}$ $\quad |r| > 1 \Rightarrow$ diverges

P-series Test:

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad p > 1 \Rightarrow$ converges

$p < 1 \Rightarrow$ diverges

Divergence Test:

If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

11.3 The Integral Test and Estimates of Sums

Remainder Estimate for the Integral Test:

$$\int_{N+1}^{\infty} f(x)dx \leq R_N \leq \int_{N}^{\infty} f(x)dx$$

$$\int_{N}^{\infty} f(x)dx \leq \#\ error\ given$$

$$\lim_{N\to\infty} \int_{N}^{\infty} f(x)dx \leq \#\ error\ given$$

Then solve for N

Integral Test:

If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Conditions
1. Continuous
2. $f(x) > 0$
3. $f'(x) < 0$

11.4 The Comparison Tests

Direct Comparison Test:

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges $\quad a_n < b_n$

If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges $\quad a_n > b_n$

Limit Comparison Test:

$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge or diverges

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

a_n and b_n must be positive

11.5 Alternating Series

Alternating Series Test:

If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies the conditions, then it converges

Conditions
1. b_n decreasing
2. $\lim_{n\to\infty} b_n = 0$

11.6 Absolute Convergence and the Ratio and Root Tests

Ratio Test:

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ absolutely converges

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$ Inconclusive

Root Test:

If $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ absolutely converges

If $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1$ or diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \Rightarrow$ Inconclusive