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11.8 Power Series

Power Series: $\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1(x-a) + C_2(x-a)^2 + ...$

Integral of Convergence:

$\sum_{n=0}^{\infty} C_n x^n \Rightarrow$ all x such that series is convergent

Radius of convergence: distance between end-points divided by 2

Center: midpoint of interval

Observation: $f(x) = \sum_{n=0}^{\infty} C_n x^n$ is a function with a domain I.O.C.

Fact: I.O.C of $\sum_{n=0}^{\infty} C_n x^n$ can be

a. $\lim_{n\to\infty} \left|\frac{an+1}{an}\right| = \infty \Rightarrow R = 0 \mid \{0\}$
   i. converges at 0
b. $\lim_{n\to\infty} \left|\frac{an+1}{an}\right| = 0 \Rightarrow R = \infty \mid (-\infty, \infty)$
   i. converges for all x-values
c. $\lim_{n\to\infty} \left|\frac{an+1}{an}\right| = \frac{1}{R}|x| < 1 \Rightarrow R > 0$ | (-R,R) , [-R,R] , (-R,R] ,[-R,R)
   i. $|x| < R \to$ converges
   ii. $|x| > R \to$ divergence

   Where R is the radius of convergence (with a center 0)

> Steps:
> 1. Identify series. Which test should you use?
>   a. Ratio/Root
>   b. Geometric
> 2. Determine convergence or divergence
> 3. Find bounds (x values) where the power series converges
> 4. $\to$ will always absolutely converge.
> 5. Test x-values in the series and determine if the series conv/div at that point.
> $\to \{ (-R,R)\ ,\ [-R,R]\ ,\ (-R,R]\ ,[-R,R)\ \}$

Translating the Power Series:

$f(x) = \sum_{n=0}^{\infty} C_n x^n \Rightarrow f(x-a) = \sum_{n=0}^{\infty} C_n (x-a)^n$

a. $\lim_{n\to\infty} \left|\frac{an+1}{an}\right| = \frac{1}{R}|x-c| < 1 \Rightarrow R > 0$ | (-R+c,R+c) , [-R+c,R+c] , (-R+c,R+c] ,[-R+c,R+c)
   i. $|x-c| < R \to$ converges
   ii. $|x-c| > R \to$ divergence

11.10 Taylor and Maclaurin Series

Taylor Series: $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$

> Steps:
> 1. Given the function with a center at a, determine its derivatives and find the value of each derivative at a
> 2. Plug in the values to the equation
> 3. $\frac{f^n(a)}{n!}$ for each term
> 4. List the sequence
> 5. Using the sequence, find the general patent for the series
> 6. Lastly, find the series
> 7. If the ask for radius of convergence, follow the steps in the previous lesson

Maclaurin Series: Taylor Series centered at 0: $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x)^n$

Known Power and Maclaurin Series:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... \Rightarrow R = 1$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \Rightarrow R = \infty$

$sinx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \Rightarrow R = \infty$

$cosx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \Rightarrow R = \infty$

$arctanx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \Rightarrow R = 1$

$ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + ... \Rightarrow R = 1$

$(1+x)^k = \sum_{n=0}^{\infty} (\frac{k}{n}) x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 ... \Rightarrow R = 1$

11.11 Application of Taylor's Polynomials

Taylor's Inequality:

$\left|f^{(n+1)}(x)\right| \le M_n$ for $|x-a| \le d$

$|R(x)| \le \frac{e^d|x|^{n+1}}{n+1!}$ where $(n+1)! \to \infty$ as $n \to \infty$, which means $\frac{1}{(n+1)!} \to 0$ as $n \to \infty$

Therefore, $|R(x)| \le \frac{e^d|x|^{n+1}}{n+1!} \to 0$ as $n \to \infty$