IMSE 780: Methods of Operations Research
Final Exam
Fall 2023

- Test Period: 48 hours.
- Format: Take home, open book, open note. Use any resources you want, so long as you DO NOT DISCUSS THE EXAM MATERIAL WITH ANYONE else except the professor.
- Python portions should be turned in as a Colab or Jupyter notebook, preferably only one notebook per submission (make clear which cells correspond to which problem). You may either upload the .ipynb file or give editor permissions to the professor and include the link in your submission.
- Submit your work via Canvas in any combination of documents (typed, scanned, or photographed), notebook files, and links. Using multiple formats may require multiple submissions on Canvas.
- You MUST sign this sheet and submit a scanned/photographed copy to receive any credit.
- Good luck!

I ______________________ affirm that I have not given or received any unauthorized aid on this exam, and acknowledge that sharing/discussing exam material with anyone besides the professor during the exam window constitutes a violation of the university honor policy.

Signature: ______________________

Date: ______________________

1. (4pts) For each of the following optimization problems, identify which algorithm from class you would use to find an optimal solution. Please explain your reasoning. (Note: I only want you to identify the algorithm to use, you do **not** need to solve the problem)

   (a)
$$\begin{array}{ll} \max & 4x_1 + 2x_2 - x_1^4 - x_2^2 \\ \text{s.t.} & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

   (b)
$$\begin{array}{ll} \max & -3x^2 - x^4 \\ \text{s.t.} & 2x^2 \leq 7 \end{array}$$

   (c)
$$\max \quad 3x - 4x^6$$

   (d)
$$\begin{array}{ll} \max & 2x_1 + 3x_2 - x_1^2 - x_2^2 \\ \text{s.t.} & 6x_1 + 5x_2 \leq 5 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

2. (4pts) Suppose you are using the modified simplex algorithm to optimize a quadratic programming problem. At some point in the algorithm, your simplex system looks like this (where the variables follow the naming conventions we used in class):

$$\begin{bmatrix} 1 & 0 & -2 & -1 & 1 & 0 & 4 & 0 & 5 \\ 0 & 0 & 4 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ u_1 \\ y_1 \\ y_2 \\ v_1 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} ?? \\ 10 \\ 15 \\ 20 \end{bmatrix}$$

   (a) Which variables are in the current basis?
   (b) What should be the value of Z (replacing the ?? on the right-hand side) at the current basic solution?
   (c) Which of the non-basic variables are eligible to enter the basis in the next iteration?
   (d) Of the eligible non-basic variables you identified in part (c), choose one to enter the basis. Given that selection, which variable will need to leave the basis?

3. (4pts) Consider the optimization problem

$$\begin{array}{ll} \max & 25x_1 + 20x_2 + 2x_1x_2 - 3x_1^2 - x_1^4 - 5x_2^2 - x_2^4 \\ \text{s.t.} & 3x_1 + 5x_2 \leq 16 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

   Starting from the initial trial solution $(x_1, x_2) = (0, 0)$, apply two iterations of the Frank-Wolfe algorithm. Make sure to show your work, and clearly identify the new trial solution in each iteration.

4. (6pts) Your company makes hats (one size fits all) in four different colors: blue, red, green, and orange. You have been contacted by 10 stores that would like to buy your hats in bulk for resale. Each store desires a different number of hats and a different mix of colors. Your company has determined that your costs can be reduced significantly if hats are not produced and packaged individually but instead in "packs" of six hats. The manufacturing process can be set up to produce three different pack types, where each pack type can have a different distribution of colors.

   Your task is to determine the three best pack types to produce, based on how closely you can match the orders from the 10 stores. One of your colleagues began working on the problem, and decided it could be modeled as an integer program with some quadratic constraints. Unfortunately, your colleague became very ill before finishing, and left behind only this unfinished notebook:

   https://colab.research.google.com/drive/1cJol72edXkglyp3wp9gNLdb3GKyIefhd?usp=sharing

   Complete your colleague's model to determine the three pack types that can be used to most closely match the store orders, according to the criterion outlined in the notebook.

5. (6pts) Suppose your house has both a front door and a back door, and you own three pairs of shoes. Every time you leave your house, you will exit out the front door with probability $\frac{2}{3}$, and will return to the same door you exited from with probability $\frac{2}{3}$. Each time you leave, if there is at least one pair of shoes at the door you are leaving from, you will choose one of them to put on before you leave (otherwise you will go barefoot). When you return, if you are wearing shoes, you will leave them at the door you returned to. In what percentage of trips will you leave barefoot? (Hint: model this process as a Markov chain with the states encoding how many shoes are at each door.)

6. (6pts) A delivery-only pizza shop receives a steady flow of orders at all hours of the day, which can be modeled via a Poisson process with a rate of 20 orders per hour. Seven employees are working at any given time, each paid \$15 per hour. Whenever an order comes in, the next available employee will be assigned to it, at which point they are tasked with making the pizza and then taking it to the customer's address before returning to the store. The time for this process is exponentially distributed with a mean time of 15 minutes.

   All orders come in via a digital platform that lets customers track their orders, with status updates when the order has been received, when an employee starts working on the order, and when the pizza is on the way. Management noticed that if work has not begun for some time after the order is received, that customers would often cancel the order (cancellations are not allowed after an employee has started work). To combat this, they offer customers a \$10 discount any time 5 minutes elapse between an order being received and work beginning. Since implementing the discount, canceled orders are no longer a problem.

   (a) What is the probability that any given customer gets the \$10 discount?

   (b) How many dollars per day can the shop expect to give out in discounts?

   (c) Management is considering having an extra employee working at all times to help them keep up with orders. Would it make financial sense to do so?

7. (6pts) Every weekend, you and a group of friends meet up for one of three activities. You will either play trivia at a local restaurant (costing you an average of \$10), go to a movie (\$20) or go to a concert (\$45). The activity is chosen by a vote each week. Whichever activity was most recently chosen has a probability $\frac{1}{5}$ of being chosen again for the following week, while the other two activities have probability $\frac{2}{5}$ of being chosen.

   You enjoy all the activities equally, but you are also trying to save some money, and thus prefer trivia over the other options. You know that two of your friends could be bribed into voting in your favor for a cost of \$5 each. If you bribe one friend, then the probability of playing trivia next week will increase by $\frac{2}{5}$ over the baseline, with the probability of the other two activities falling by $\frac{1}{5}$ each. If you bribe both friends and trivia was the most recent activity, trivia will be chosen again the following week with probability $\frac{8}{10}$ while the other activities will have probability $\frac{1}{10}$ of being chosen. Otherwise, if you bribe both friends and the most recent activity was not trivia, then trivia is guaranteed to be chosen the following week.

Formulate this problem as an MDP, and identify:

- The stationary, deterministic policy that minimizes your long-run average cost (bribes plus activity costs) per week.
- The cost attained by the above policy.