Differential Cohesive Type Theory (Extended Abstract)*

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As internal languages of toposes, type theories allow mathematicians to reason *synthetically* about mathematical structures in a concise and natural way. While Homotopy Type Theory provides an internal language for $(\infty, 1)$-toposes, it is also possible to consider type theories that correspond to $(\infty, 1)$-toposes with extra structure of interest in algebraic and differential geometry. In this work we consider type theories for *differential cohesive* and *cohesive* structure, following a line of work begun by Shulman (2015) and Licata and Shulman (2016). In his real-cohesive type theory, the 0-type $\mathbb{S}^1$ for the topological circle is different from the usual homotopy type S^1, but the two are connected by the cohesive operation $\int \mathbb{S}^1 = S^1$, where $\int$ maps a type to its path ∞-groupoid. This allows Shulman to use homotopy theoretic statements about S^1 to prove Brower's fixed-point theorem—that all continuous maps over the topological disk have a fixed point.

This line of work aims to go beyond cohesive type theories by constructing a *differential cohesive* homotopy type theory, which would allow synthetic reasoning about smooth manifolds and their ∞-stack variants, which are of great interest in current pure mathematics. For example, Sati et al. (2012) uses spaces locally modeled on 2- and 6-types. These are supported by and can already be reasoned about in a fragment of differential cohesive homotopy type theory used by Wellen (2017) to develop the basics of Cartan geometry. Furthermore, differential cohesive toposes are used to great extent by Schreiber (2013) to reason about spaces with geometric structures of interest to modern physics, and the differential part of differential cohesion plays an important part in algebraic geometry, as first noted in the form of an adjoint triple by Simpson and Teleman (1997).

The type theory described here supports all operations of differential cohesion arranged in a predictable pattern, but lacks dependent and identity types. Future work will extend it to a true *homotopy* type theory.

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$$\begin{array}{ccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \cup & & \cup & & \\ & & \int & \dashv & \flat & \dashv & \sharp \end{array}$$

Figure 1: On the bottom, the real-cohesion operations $\int$ (the connected components with discrete topology on 1-toposes and the fundamental ∞-groupoid in the $(\infty, 1)$-topos), $\flat$ (the underlying set with discrete topology), and $\sharp$ (the underlying set with codiscrete topology). On the top, the differential operations $\Re$ (the underlying space without infinitesimal directions), $\Im$ (the underlying space where all maps out of it are trivial on tangent spaces), and & (on manifolds, this is just the discrete underlying set; applied to the right stacks, it might be a coefficient object for interesting cohomology theories). The inclusion symbol $F \subset G$ indicates that the image of F is contained in the image of G.

Cohesion in adjoint type theory. Shulman (2015) constructs a variant of homotopy type theory for $(\infty, 1)$-toposes with an additional cohesive structure using a *modal* type theory (Pfenning and Davies, 2001) to describe the categorical structure of cohesive $(\infty, 1)$-toposes. This structure consists of three *modalities* that form the adjoint triple $\int \dashv \flat \dashv \sharp$ described in Figure 1. In the type theory, variables are marked as either *cohesive* or *crisp* depending on how they are used in a term—a typing judgment $\Gamma \mid \Delta \vdash e : \tau$ is continuous on the cohesive variables in Δ, but may be discontinuous on the crisp variables in Γ. The modalities $\flat$ and $\sharp$ are defined as type and term formation and introduction rules, while $\int$ is defined as a higher inductive type.

In particular, Shulman defines $\int$ as a localization with respect to the Dedekind-reals. This approach admits useful constructions of familiar topological spaces using the real numbers, including certain topological spheres or disks, but it is also interesting to look at general cohesion, where $\int$ is not a priori linked to the Dedekind-reals.

In a different approach, Licata et al. (2017) provide a general purpose framework for (non-dependent) modal type theories, called *adjoint type theory*, which we can instantiate with the three cohesive operations independent of the Dedekind-reals. The result is a typing judgment with three sorts of contexts, written $\Gamma \mid \Delta \mid \Xi \vdash e : \tau$, where Γ and Δ still hold crisp and cohesive variables, respectively, and where Ξ contains *shapely* variables, which are constant on the connected components of the topological structure. In particular, crisp variables correspond directly to the modality $\flat$, while shapely variables correspond to the modality $\int$.

Differential cohesion in adjoint type theory. To extend the type theory to differential cohesive toposes, we consider three additional operations, described in the top row of Figure 1, in addition to the cohesive operations. Wellen (2017) extends HoTT axiomatically with the $\Im$ modality (similar to Shulman's approach to $\int$) but the comonadic modalities $\Re$, $\&$, and (in the case of ordinary cohesion) $\flat$ cannot be added axiomatically in the same way. Further, Wellen's construction relies on a particular formulation of the differentially cohesive real line, but it is unknown whether that can be defined internally to the type theory; in typical models of differential cohesion, the Dedekind reals do not correspond to the real line as a smooth space!

Our approach is thus to continue with Licata et al. (2017)'s adjoint type theory framework to describe the differential operations, instead of Shulman (2015)'s more specialized type theory. We extend the typing judgment to differential cohesive variables, where the judgment $\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e : \tau$ uses six different sorts of contexts, as described in Figure 2. We add the modalities $\Re$, $\Im$, and $\&$ to this judgment as inference rules following Licata et al.'s framework. Crucially, because the type theory is derived straightforwardly from this framework, we get some meta-theoretic results—substitution, structural rules, *etc.*—for free!

Γ	crisp	$\flat$
Δ	reduced	$\Re$
Θ	differentially cohesive	
Λ	coreduced	$\Im$
Ξ	shapely	$\int$

Figure 2: Variable usage of the judgment $\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e : \tau$.

Where are the homotopies? Future work will extend this type theory to dependent and identity types, but this is not a straightforward task because of the dependency structure of a differential cohesive type. In Shulman (2015)'s type theory, cohesive variables can depend on crisp variables but not vice versa; in a dependent version of adjoint cohesive type theory, both crisp and cohesive variables will also depend on shapely variables, and vice versa, which causes problems when constructing terms and types. The same dependency problem arises in the differential case. The solution will require a more nuanced view on the dependency graph of a type, but we are confident it is surmountable in future work.

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