Differential Cohesive Type Theory (Extended Abstract)*

Jacob A. Gross[1], Daniel R. Licata[2], Max S. New[3], Jennifer Paykin[4], Mitchell Riley[2], Michael Shulman[5], and Felix Wellen[6]

[1]University of Pittsburgh, [2]Wesleyan University, [3]Northeastern University, [4]University of Pennsylvania, [5]University of San Diego, [6]Karlsruhe Institute of Technology

As the internal languages of toposes, type theories allow mathematicians to reason *synthetically* about mathematical structures in a concise and natural way. While Homotopy Type Theory provides an internal language for $(\infty, 1)$-toposes, it is also possible to consider type theories that correspond to $(\infty, 1)$-toposes with extra structure of interest in algebraic and differential geometry. In this work we consider type theories for *differential cohesive* and *cohesive* structure, following a line of work begun by Shulman (2015). In his real-cohesive type theory, the type $\mathbb{S}^1$ for the topological circle is different from the usual homotopy type S^1, but the two are connected by the cohesive operation $\int \mathbb{S}^1 = S^1$, where $\int$ collapses the cohesive structure of the circle into its connected components.

This work aims at constructing a differential cohesive homotopy type theory, which would allow synthetic reasoning not just about ordinary smooth manifolds, but also ∞-stack variants which are of great interest in current pure mathematics. For example, Sati et al. (2012) uses spaces locally modeled on 2- and 6-types, which are supported by and may already be reasoned about in a fragment of differential cohesive homotopy type theory used by Wellen (2017) to develop the basics of Cartan geometry.

The most advanced type theory presented in this work supports all operations of differential cohesion arranged in an easy to work with pattern, but lacking dependent types and identity types.

Cohesion in adjoint type theory. In previous work, Shulman (2015) constructs a variant of homotopy type theory for $(\infty, 1)$-toposes with an additional "cohesive" structure. In this setting, derivations in the type theory correspond to continuous maps that respect the cohesive structure of the space, providing a framework for a wide variety of synthetic proofs. For example, Brower's fixed-point theorem, which states that all continuous maps over the topological disk have a fixed point has no synthetic proof in ordinary homotopy type theory, but has a proof in cohesive HoTT (Shulman, 2015).

To capture cohesion in the type theory, Shulman (2015) uses a *modal* type theory (Pfenning and Davies, 2001) to describe the categorical structure of cohesive $(\infty, 1)$-toposes. This structure consists of three *modalities* that form an adjoint triple $\int \dashv \flat \dashv \sharp$; they are described in black in Figure 1. In addition, variables in the type theory are marked as either *cohesive* or *crisp* depending on how they are used in a term—a typing judgment $\Gamma \mid \Delta \vdash e : \tau$ is continuous on the cohesive variables in Δ, but may be discontinuous on the crisp variables in Γ. The modalities $\flat$ and $\sharp$ are defined as type and term formation and introduction rules, while $\int$ is defined as a higher inductive type.

In particular, Shulman defines the $\int$ functor as a localization using the Dedekind-reals. While this approach admits very useful constructions of familiar topological spaces using the real numbers, like the *topological spheres* or disks which have the right homotopy type, it is also interesting to look at general cohesion, where $\int$ is not a priori linked to the Dedekind-reals.

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Differential cohesion in adjoint type theory. In this work we aim to construct a type theory similar to Shulman's cohesive type theory that captures the additional structure of what are called differential cohesive toposes. Differential cohesive toposes, used by Schreiber (2013) to reason about spaces with geometric structures of interest to modern physics, have already been explored from within a type theory by Wellen (2017). There, just a small fragment of differential cohesive type theory is used to define and use notions of differential geometry. The only difference to homotopy type theory is the existence of the $\Im$ modality given by axioms. The comonadic modalities, $\Re$, &, $\flat$ can not be added to a type theory in this way – the rules have to be changed. And it is sometimes better, to also implement the monadic modalities by modifying the rules instead of adding axioms.

$$\begin{array}{ccccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \cup & & \cup & & \\ & & \int & \dashv & \flat & \dashv & \sharp \end{array}$$

Figure 1: On the bottom, the real-cohesion operations $\int$ (the connected components with discrete topology on 1-toposes and the fundamental ∞-groupoid in the $(\infty, 1)$-topos version), $\flat$ (the underlying set with discrete topology), and $\sharp$ (the underlying set with codiscrete topology). On the top, the differential operations $\Re$ (the underlying space without infinitesimal directions), $\Im$ (a space with the same points but all maps from it to any other space induce trivial maps on tangent spaces), and & (on manifolds, this is just the discrete underlying set; applied to the right stacks, it might be a coefficient object for interesting cohomology theories). The inclusion symbol $F \subset G$ indicates that the image of F is contained in the image of G.

A full type theory for differential cohesion would cover not just the three differential modalities (in blue in Figure 1), but also the three cohesive modalities. Unfortunately, Shulman's construction cannot be extended directly from real to differential cohesion—in typical models of differential cohesion the Dedekind reals do not correspond to the real line as a smooth space! Furthermore, it is not known if there is a way to define the differentially cohesive real line internally to the type theory.

Following the 2016 paper on cohesion, Licata et al. (2017) developed a general construction for non-dependent modal type theories, which we instantiate for differential cohesion in this paper. This takes place in two stages.

First, in the absence of dependent types, we cannot utilize Shulman's definition of $\int$ as a higher inductive type characterized by its localization properties. Instead we give introduction and elimination rules for $\int$ internally to the type theory, guided by Licata et al.'s general construction. The result is a typing judgment with three sorts of contexts, written $\Gamma \mid \Delta \mid \Xi \vdash e : \tau$, where Γ and Δ still hold crisp and cohesive variables, respectively, and where Ξ contains *shapely* variables, which are constant on the connected components of the topological structure. In particular, crisp variables correspond directly to the modality $\flat$, while shapely variables correspond to the modality $\int$.

Next, we extend the judgment to differentially cohesive variables. The judgment $\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e : \tau$ uses variables in the following way:

Γ	crisp	$\flat$
Δ	reduced	$\Re$
Θ	differentially cohesive	
Λ	coreduced	$\Im$
Ξ	shapely	$\int$

We add the modalities $\Re$, $\Im$, and & to this judgment as inference rules using Licata et al.'s framework. Because the type theory is derived straightforwardly from this framework, we get some meta-theoretic results—substitution, structural rules, *etc.*—for free.[1]

Future work will extend this type theory to dependent and identity types.

[1]We must still be careful about the proof that our type system corresponds exactly to an instance of Licata et al.'s framework.

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