A Linear/Producer/Consumer model of Classical Linear Logic

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***Abstract*—This paper defines a new proof- and category-theoretic framework for *classical linear logic* that separates reasoning into one linear regime and two persistent regimes corresponding to ! and ?. The resulting linear/producer/consumer (LPC) logic puts the three classes of propositions on the same semantic footing, following Benton's linear/non-linear formulation of intuitionistic linear logic. Semantically, LPC corresponds to a system of three categories connected by adjunctions that reflect the linear/producer/consumer structure. The paper's metatheoretic results include admissibility theorems for the cut and duality rules, and a translation of the LPC logic into the category theory. The work also presents several concrete instances of the LPC model.**

I. Introduction

Since its introduction by Girard in 1987, *linear logic* has been found to have a range of applications in logic, proof theory, and programming languages. Its notion of "resource consciousness" sheds light on topics as diverse as proof search [1], memory management [2], alias control [3], computational complexity [4], and security [5], among many others.

Linear logic's power stems from its ability to carefully manage resource usage: it makes a crucial distinction between *linear* (used exactly once) and *persistent* (unrestricted use) hypotheses, internalizing the latter via the ! connective. From a semantic point of view, the literature has converged (following Benton [6]) on an interpretation of ! as a comonad given by $! = \mathrm{F} \circ \mathrm{G}$ where $\mathrm{F} \dashv \mathrm{G}$ is a symmetric monoidal adjunction between categories $\mathcal{L}$ and $\mathcal{P}$ arranged as shown below:

$$\begin{tikzcd} \mathcal{P} \arrow[r, bend right, "\mathrm{F}"'] \arrow[r, phantom, "\top"] & \mathcal{L} \arrow[l, bend right, "\mathrm{G}"'] \end{tikzcd}$$

Here, $\mathcal{L}$ (for "linear") is a symmetric monoidal closed category and $\mathcal{P}$ (for "persistent") is a cartesian category. This is, by now, a standard way of interpreting *intuitionistic* linear logic (for details, see the discussion in Melliès' article [7]).

If, in addition, the category $\mathcal{L}$ is $*$-autonomous, then the structure above is sufficient to interpret *classical* linear logic, where the monad ? is determined by $? = (\mathrm{F}^{op}\,(\mathrm{G}^{op}\,(-^{\perp})))^{\perp}$. While sound, this situation is not entirely satisfactory because it essentially commits to a particular implementation of ? in terms of $\mathcal{P}^{op}$, which, as we show, is not necessary.

With that motivation, this paper defines a proof- and category-theoretic framework for full *classical linear logic* that uses *two* persistent categories: one corresponding to ! and one to ?. The resulting categorical structure is shown in Figure 1, where $\mathcal{P}$ now takes the place of the "producing", category, in duality with $\mathcal{C}$ as the "consuming" category. This terminology comes from the observations that:

$$\begin{array}{lll} !A \vdash 1_{\mathsf{L}} & !A \vdash A & !A \vdash !A \otimes !A \\ \perp_{\mathsf{L}} \vdash ?A & A \vdash ?A & ?A \parr ?A \vdash ?A \end{array}$$

Intuitively, the top row means that $!A$ is sufficient to *produce* any number of copies of A and, dually, the bottom rows says that $?A$ can *consume* any number of copies of A.

Contributions. In Section II we define a linear/producer/consumer (LPC) presentation of classical linear logic that, like Benton's linear/non-linear logic [6], syntactically exhibits the decomposition of ! and ? into their constituent functors. We prove that cut and duality rules are admissible in the logic, and that LPC is consistent.

Section III develops the categorical model for LPC, relates it to other models from the literature, and considers how to interpret judgments of the LPC logic as morphisms in the appropriate categories. Section IV presents several concrete example instances of the LPC categorical framework, and, in particular, gives an example in which $\mathcal{C}$ is not just $\mathcal{P}^{op}$.

We conclude the paper with a discussion of related and future work.

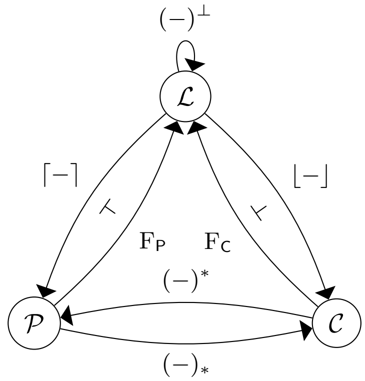

Fig. 1. Categorical model with linear, producing and consuming categories.

II. LPC Logic

The syntax of the LPC logic is made up of three syntactic forms for propositions: linear propositions A, producer propositions P, and consumer propositions C.

$$
\begin{array}{rclclclcl}
A & ::= & 0 & \mid & A_1 \oplus A_2 & \mid & \top & \mid & A_1 \,\&\, A_2 \\
 & \mid & 1_{\mathsf{L}} & \mid & A_1 \otimes A_2 & \mid & \perp_{\mathsf{L}} & \mid & A_1 \parr A_2 \\
 & \mid & F_!\, P & \mid & F_?\, C & & & & \\
P & ::= & 1_{\mathsf{P}} & \mid & P_1 \otimes P_2 & \mid & \lceil A \rceil & & \\
C & ::= & \perp_{\mathsf{C}} & \mid & C_1 \parr C_2 & \mid & \lfloor A \rfloor & &
\end{array}
$$

The syntactic form of a proposition is called its *mode*—one of linear L, producing P or consuming C. The meta-variable X ranges over propositions of any mode, and the tagged meta-variable X^{m} ranges over propositions of mode m. The term *persistent* refers to propositions that are either producers or consumers.

These propositions exclude the usual exponentials ! and ? and instead replace them with two pairs of connectives: $F_!$ and $\lceil - \rceil$ for !, and $F_?$ and $\lfloor - \rfloor$ for ?. If A is a linear proposition, $\lceil A \rceil$ is a producer and $\lfloor A \rfloor$ is a consumer. On the other hand, a producer proposition P may be "frozen" into a linear proposition $F_!\, P$, effectively discarding its persistent characteristics. Similarly for a consumer C, $F_?\, C$ is linear. The linear propositions $!A$ and $?A$ are encoded in this system as $F_!\,(\lceil A \rceil)$ and $F_?\,(\lfloor A \rfloor)$ respectively.

The inference rules of logic are defined in Figures 2 through 5. There are two distinct sequent relations: the linear sequent $\Gamma \vdash \Delta$ and the persistent sequent $\Gamma \Vdash \Delta$. In the linear sequent, the (unordered) contexts Γ and Δ may be made up of propositions of any mode; in the persistent sequent however, the contexts may contain only persistent propositions. Γ^{P} indicates that Γ contains only producer propositions, and Δ^{C} indicates that Δ contains only consumer propositions.

$$\frac{}{X \vdash X}\ \mathrm{Ax}^{\vdash}$$

$$\frac{}{\Gamma \vdash \Delta, \top}\ \top^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{}{\Gamma, 0 \vdash \Delta}\ 0^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L1} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R1}$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L2} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R2}$$

$$\frac{\begin{array}{c}\Gamma \vdash \Delta, A_1 \\ \Gamma \vdash \Delta, A_2\end{array}}{\Gamma \vdash \Delta, A_1 \,\&\, A_2}\ \&^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{\begin{array}{c}\Gamma, A_1 \vdash \Delta \\ \Gamma, A_2 \vdash \Delta\end{array}}{\Gamma, A_1 \oplus A_2 \vdash \Delta}\ \oplus^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\begin{array}{c}\Gamma \vdash \Delta \\ \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}\end{array}}{\Gamma, 1_{\mathsf{m}} \vdash \Delta}\ 1^{\vdash}\text{-L} \qquad \frac{\begin{array}{c}\Gamma \vdash \Delta \\ \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}\end{array}}{\Gamma \vdash \Delta, \perp_{\mathsf{m}}}\ \perp^{\vdash}\text{-R}$$

$$\frac{\mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\cdot \vdash 1_{\mathsf{m}}}\ 1^{\vdash}\text{-R} \qquad \frac{\mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\perp_{\mathsf{m}} \vdash \cdot}\ \perp^{\vdash}\text{-L}$$

$$\frac{\begin{array}{c}\Gamma, {X_1}^{\mathsf{m}}, {X_2}^{\mathsf{m}} \vdash \Delta \\ \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}\end{array}}{\Gamma, {X_1}^{\mathsf{m}} \otimes {X_2}^{\mathsf{m}} \vdash \Delta}\ \otimes^{\vdash}\text{-L} \qquad \frac{\begin{array}{c}\Gamma \vdash \Delta, {X_1}^{\mathsf{m}}, {X_2}^{\mathsf{m}} \\ \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}\end{array}}{\Gamma \vdash \Delta, {X_1}^{\mathsf{m}} \parr {X_2}^{\mathsf{m}}}\ \parr^{\vdash}\text{-R}$$

$$\frac{\Gamma_1 \vdash \Delta_1, {X_1}^{\mathsf{m}} \quad \Gamma_2 \vdash \Delta_2, {X_2}^{\mathsf{m}} \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, {X_1}^{\mathsf{m}} \otimes {X_2}^{\mathsf{m}}}\ \otimes^{\vdash}\text{-R}$$

$$\frac{\Gamma_1, {X_1}^{\mathsf{m}} \vdash \Delta_1 \quad \Gamma_2, {X_2}^{\mathsf{m}} \vdash \Delta_2 \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma_1, \Gamma_2, {X_1}^{\mathsf{m}} \parr {X_2}^{\mathsf{m}} \vdash \Delta_1, \Delta_2}\ \parr^{\vdash}\text{-L}$$

Fig. 2. Linear Inference Rules for Linear Sequent

The linear inference rules in Figures 2 and 3 encompass rules for the units and the linear operators $\oplus$, $\&$, $\otimes$ and $\parr$. It is worth noting that the multiplicative product $\otimes$ is defined only on linear and producer propositions, while the multiplicative sum $\parr$ is defined only on linear and consumer propositions.

The structural inference rules are given in Figure 4. Weakening and contraction apply only for producers on the left and consumers on the right. The rules for the operators $F_!$, $F_?$, $\lceil - \rceil$ and $\lfloor - \rfloor$ are more interesting, as they must be able to encode dereliction and promotion. On the left, the $F_!$ and $\lceil - \rceil$ rules can be applied freely to transform linear propositions into producers and vice versa. These rules emulate the dereliction rule for linear logic by passing through the adjunction:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \text{versus} \qquad \frac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}}{\Gamma, F_!\, \lceil A \rceil \vdash \Delta}$$

On the right however, these rules can only be applied when the contexts are persistent and of the correct form,

$$\frac{}{P \Vdash P}\ \mathrm{Ax}^{\Vdash}_{\mathsf{P}} \qquad \frac{}{C \Vdash C}\ \mathrm{Ax}^{\Vdash}_{\mathsf{C}}$$

$$\frac{}{\cdot \Vdash 1_{\mathsf{P}}}\ 1^{\Vdash}_{\mathsf{P}}\text{-R} \qquad \frac{}{\bot_{\mathsf{C}} \Vdash \cdot}\ \bot^{\Vdash}_{\mathsf{C}}\text{-L}$$

$$\frac{\Gamma \Vdash \Delta}{\Gamma, 1_{\mathsf{P}} \Vdash \Delta}\ 1^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, \bot_{\mathsf{C}}}\ \bot^{\Vdash}_{\mathsf{C}}\text{-R}$$

$$\frac{\Gamma, P_1, P_2 \Vdash \Delta}{\Gamma, P_1 \otimes P_2 \Vdash \Delta}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C_1, C_2}{\Gamma \Vdash \Delta, C_1 \parr C_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-R}$$

$$\frac{\Gamma_1 \Vdash \Delta_1, P_1 \quad \Gamma_2 \Vdash \Delta_2, P_2}{\Gamma_1, \Gamma_2 \Vdash \Delta_1, \Delta_2, P_1 \otimes P_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\frac{\Gamma_1, C_1 \Vdash \Delta_1 \quad \Gamma_2, C_2 \Vdash \Delta_2}{\Gamma_1, \Gamma_2, C_1 \parr C_2 \Vdash \Delta_1, \Delta_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-L}$$

Fig. 3. Linear Inference Rules for Persistent Sequent

$$\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{W}^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, C}\ \mathrm{W}^{\vdash}\text{-R}$$

$$\frac{\Gamma \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{W}^{\Vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, C}\ \mathrm{W}^{\Vdash}\text{-R}$$

$$\frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{C}^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta, C, C}{\Gamma \vdash \Delta, C}\ \mathrm{C}^{\vdash}\text{-R}$$

$$\frac{\Gamma, P, P \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{C}^{\Vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C, C}{\Gamma \Vdash \Delta, C}\ \mathrm{C}^{\Vdash}\text{-R}$$

$$\frac{\Gamma, P \vdash \Delta}{\Gamma, F_{!}\, P \vdash \Delta}\ F_{!}\text{-L} \qquad \frac{\Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, F_{?}\, C}\ F_{?}\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_{!}\, P}\ F_{!}\text{-R} \qquad \frac{\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, F_{?}\, C \vdash \Delta^{\mathsf{C}}}\ F_{?}\text{-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash A, \lfloor A \rfloor}\ \lfloor - \rfloor\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R} \qquad \frac{\Gamma^{\mathsf{P}}, A \vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, \lfloor A \rfloor \Vdash \Delta^{\mathsf{C}}}\ \lfloor - \rfloor\text{-L}$$

Fig. 4. Structural Inference Rules

as in the ! introduction rule in linear logic:

$$\frac{\Gamma^{!} \vdash \Delta^{?}, A}{\Gamma^{!} \vdash \Delta^{?}, !A} \qquad \text{versus} \qquad \frac{\dfrac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_{!}\, \lceil A \rceil}$$

Thus, the $F_{!}$-R and $\lceil - \rceil$-R rules move the sequents between the linear and persistent regimes. The $F_{?}$ and $\lfloor - \rfloor$ rules are dual to those of $F_{!}$ and $\lceil - \rceil$.

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \mathrm{Cut}^{\vdash}_{\mathsf{L}}$$

$$\frac{\Gamma^{\mathsf{P}}_1 \Vdash \Delta^{\mathsf{C}}_1, P \quad P, \Gamma_2 \vdash \Delta_2}{\Gamma^{\mathsf{P}}_1, \Gamma_2 \vdash \Delta^{\mathsf{C}}_1, \Delta_2}\ \mathrm{Cut}^{\vdash}_{\mathsf{P}}$$

$$\frac{\Gamma^{\mathsf{P}}_1 \Vdash \Delta^{\mathsf{C}}_1, P \quad P, \Gamma_2 \Vdash \Delta_2}{\Gamma^{\mathsf{P}}_1, \Gamma_2 \Vdash \Delta^{\mathsf{C}}_1, \Delta_2}\ \mathrm{Cut}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, C \quad C, \Gamma^{\mathsf{P}}_2 \Vdash \Delta^{\mathsf{C}}_2}{\Gamma_1, \Gamma^{\mathsf{P}}_2 \vdash \Delta_1, \Delta^{\mathsf{C}}_2}\ \mathrm{Cut}^{\vdash}_{\mathsf{C}}$$

$$\frac{\Gamma_1 \Vdash \Delta_1, C \quad C, \Gamma^{\mathsf{P}}_2 \Vdash \Delta^{\mathsf{C}}_2}{\Gamma_1, \Gamma^{\mathsf{P}}_2 \Vdash \Delta_1, \Delta^{\mathsf{C}}_2}\ \mathrm{Cut}^{\Vdash}_{\mathsf{C}}$$

Fig. 5. CUT Inference Rules

A. Displacement

The commas on the left-hand-side of both the linear and persistent sequents intuitively correspond to the multiplicative product $\otimes$, and the commas on the right correspond to the multiplicative sum $\parr$. This correspondence motivates the context restriction of the rules that move between the linear and persistent regimes. The restriction ensures that almost all of the propositions have the "natural" mode—producers on the left and consumers on the right. We say "almost" because the principle formula in each of these rules defies this classification. We call such propositions *displaced*.

Definition 1. *In a derivation of* $\Gamma \Vdash \Delta$*, a producer* P *is displaced if it appears in* Δ*. A consumer* C *is displaced if it appears in* Γ*.*

Lemma 2 (Displacement)**.** *For any* LPC *derivation* $\mathcal{D}$ *of* $\Gamma \Vdash \Delta$*,* $\mathcal{D}$ *contains exactly one displaced proposition.*

The proof is by straightforward induction on $\mathcal{D}$.

B. Cut Rules

This section presents the cut rules of Figure 5 and proves they are admissible in LPC. The intuition behind this nontraditional formulation of the cut rules is that whenever a cut term is displaced in a subderivation, that derivation must be persistent and satisfy the restrictions of Lemma 2. Lacking this restriction, simpler formulations of the cut rules disallow cut admissibility. For a thorough discussion of the choice of these cut rules, see the companion paper [8].

To show admissibility of the CUT rules, it is sufficient to show admissibility of an equivalent set of rules called CUT+. For linear cut terms, the CUT+ rule is identical to the corresponding CUT rule. For persistent cut terms, CUT+ uses the observation that when a persistent proposition is *not* displaced in a sequent, it can be replicated any number of times. That is, for any n the linear derivations

$$\frac{\Gamma,(P)_n \vdash \Delta}{\Gamma, P \vdash \Delta} \quad \text{and} \quad \frac{\Gamma \vdash \Delta,(C)_n}{\Gamma \vdash \Delta, C}$$

are admissible in LPC, and similarly for the persistent derivation. The CUT+ rules incorporate this observation as follows (and dually for consumers):

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \text{CUT+}^{\vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \text{CUT+}^{\Vdash}_{\mathsf{P}}$$

It is easy to see that the CUT and CUT+ rules are equivalent in strength.

Lemma 3 (CUT+ Admissibility)**.** *The* CUT+ *rules are admissible in* LPC.

Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the hypotheses of one of the CUT+ rules. We proceed by induction on the cut term primarily and the sum of the depths of $\mathcal{D}_1$ and $\mathcal{D}_2$ secondly. The cases here are representative of the full proof in the companion paper.

Suppose $\mathcal{D}_1$ or $\mathcal{D}_2$ ends in a weakening rule on the cut term. In the case when the cut term is a producer and $\mathcal{D}_2$ is a linear judgment, we have

$$\frac{\mathcal{D}_1}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P} \quad \text{and} \quad \mathcal{D}_2 = \frac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P)_n \vdash \Delta_2}}{\Gamma_2, (P)_{n+1} \vdash \Delta_2}\ \text{W-L}$$

By the inductive hypothesis on P, $\mathcal{D}_1$ and $\mathcal{D}_2'$, there exists a cut-free derivation of $\Delta_1^{\mathsf{P}}, \Delta_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2$. The persistent judgment and consumer cases are similar.

If $\mathcal{D}_1$ or $\mathcal{D}_2$ ends in a contraction rule on the cut term, the reasoning is similar.

Suppose the cut term is the principle formula in both $\mathcal{D}_1$ and $\mathcal{D}_2$. It suffices to exclude weakening and contraction rules, as these have already been covered.

1) Suppose

$$\mathcal{D}_1 = \frac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11} \vdash \Delta_{11}, A_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_{12} \vdash \Delta_{12}, A_2}}{\Gamma_{11}, \Gamma_{12} \vdash \Delta_{11}, \Delta_{12}, A_1 \otimes A_2}\ \otimes^{\vdash}_{\mathsf{L}}\text{-R}$$

$$\text{and} \quad \mathcal{D}_2 = \frac{\dfrac{\mathcal{D}_2'}{\Gamma_2, A_1, A_2 \vdash \Delta_2}}{\Gamma_2, A_1 \otimes A_2 \vdash \Delta_2}\ \otimes^{\vdash}_{\mathsf{L}}\text{-L}$$

By the inductive hypothesis on A_2, $\mathcal{D}_{12}$ and $\mathcal{D}_2'$, there exists a derivation $\mathcal{E}$ of $\Gamma_{12}, \Gamma_2, A_1 \vdash \Delta_{12}, \Delta_2$. By the inductive hypothesis on A_1, $\mathcal{D}_{11}$ and $\mathcal{E}$ we can then obtain the desired derivation of $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \vdash \Delta_{11}, \Delta_{12}, \Delta_2$.

2) Suppose

$$\mathcal{D}_1 = \frac{\dfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, P_1 \otimes P_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\text{and} \quad \mathcal{D}_2 = \frac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P_1 \otimes P_2)_n, P_1, P_2 \vdash \Delta_2}}{\Gamma_2, (P_1 \otimes P_2)_{n+1} \vdash \Delta_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-L}$$

First of all, the inductive hypothesis on $P_1 \otimes P_2$, $\mathcal{D}_1$ itself and $\mathcal{D}_2'$ gives us a derivation

$$\frac{\mathcal{E}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}$$

By the inductive hypothesis on P_2, $\mathcal{D}_{12}$ and $\mathcal{E}$, there exists

$$\frac{\mathcal{E}'}{\Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1 \vdash \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}$$

and the inductive hypothesis on P_1, $\mathcal{D}_{11}$ and $\mathcal{E}'$ gives a derivation

$$\frac{\mathcal{E}''}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}$$

Because the replicated contexts are made up exclusively of non-displaced propositions, it is possible to apply contraction multiple times to $\mathcal{E}''$ to obtain the desired sequent.

Suppose that the cut term is not principle in $\mathcal{D}_1$.

1) If

$$\mathcal{D}_1 = \frac{\dfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, (C)_n, P}}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, (C)_n, F_!\, P}\ F_!\text{-R}$$

then by the structure of the cut rule it must be the case that $\mathcal{D}_2$ has the form $\Gamma_2^{\mathsf{P}}, C \Vdash \Delta_2^{\mathsf{C}}$. By the inductive hypothesis on C, $\mathcal{D}_1'$ and $\mathcal{D}_2$, there exists a derivation $\mathcal{E}$ of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P, \Delta_2^{\mathsf{C}}$. From this we construct the following:

$$\frac{\dfrac{\mathcal{E}}{\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P, \Delta_2^{\mathsf{C}}}}{\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, F_!\, P, \Delta_2^{\mathsf{C}}}\ F_!\text{-R}$$

2) If the cut term is a producer, then $\mathcal{D}_1$ is a persistent judgment and so it cannot be the case that the last rule of $\mathcal{D}_1$ is an $F_!$ rule or a $\lceil - \rceil$-L or $\lfloor - \rfloor$-R rule. But it also cannot be the case that the last rule in $\mathcal{D}_1$ is a in a $\lceil - \rceil$-R or $\lfloor - \rfloor$-L rule because there is a non-principle formula—namely, the cut formula—in a displaced position.

□

Corollary 4 (CUT Admissibility)**.** *The* CUT *rules in Figure 5 are admissible in* LPC.

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}\text{-L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ (-)^{\perp}\text{-R}$$

$$\frac{\Gamma \vdash \Delta, P}{\Gamma, P^{*} \vdash \Delta}\ (-)^{*\vdash}\text{-L} \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \Delta, P^{*}}\ (-)^{*\vdash}\text{-L-R}$$

$$\frac{\Gamma \Vdash \Delta, P}{\Gamma, P^{*} \Vdash \Delta}\ (-)^{*\Vdash}\text{-L} \qquad \frac{\Gamma, P \Vdash \Delta}{\Gamma \Vdash \Delta, P^{*}}\ (-)^{*\Vdash}\text{-R}$$

$$\frac{\Gamma \vdash \Delta, C}{\Gamma, C_{*} \vdash \Delta}\ (-)_{*}{}^{\vdash}\text{-L} \qquad \frac{\Gamma, C \vdash \Delta}{\Gamma \vdash \Delta, C_{*}}\ (-)_{*}{}^{\vdash}\text{-R}$$

$$\frac{\Gamma \Vdash \Delta, C}{\Gamma, C_{*} \Vdash \Delta}\ (-)_{*}{}^{\Vdash}\text{-L} \qquad \frac{\Gamma, C \Vdash \Delta}{\Gamma \Vdash \Delta, C_{*}}\ (-)_{*}{}^{\Vdash}\text{-R}$$

Fig. 6. Duality inference rules

C. *Duality*

Looking again at the LPC inference rules, it is easy to see that every rule has a dual. We take advantage of this implicit duality in our proofs to cut down the number of cases we have to consider, but we have not yet made the notion formal. Unlike standard presentations of classical linear logic, LPC does not contain an explicit duality operator $(-)^{\perp}$, nor a linear implication $\multimap$ with which to encode duality. Instead, we define $(-)^{\perp}$ to be a meta-operation on propositions and prove that the following duality rules are admissible in LPC:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}\text{-L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ (-)^{\perp}\text{-R}$$

In fact, there are three versions of this duality operation: $(-)^{\perp}$ for linear propositions, $(-)^{*}$ for producers and $(-)_{*}$ for consumers. These operators have the property that for a linear proposition A, $A^{\perp}$ is linear, but for a producer P, P^{*} is a consumer, and for a consumer C, C_{*} is a producer. We define these duality operations as follows:

$$\begin{array}{rlrl}
\top^{\perp} & := 0 & (F_{!}\, P)^{\perp} & := F_{?}\, P^{*} \\
(A \,\&\, B)^{\perp} & := A^{\perp} \oplus B^{\perp} & (F_{?}\, C)^{\perp} & := F_{!}\, C_{*} \\
0^{\perp} & := \top & 1_{\mathsf{P}}^{*} & := \bot_{\mathsf{C}} \\
(A \oplus B)^{\perp} & := A^{\perp} \,\&\, B^{\perp} & (P \otimes Q)^{*} & := P^{*} \parr Q^{*} \\
1_{\mathsf{L}}^{\perp} & := \bot_{\mathsf{L}} & \lceil A \rceil^{*} & := \lfloor A^{\perp} \rfloor \\
(A \otimes B)^{\perp} & := A^{\perp} \parr B^{\perp} & \bot_{\mathsf{C}*} & := 1_{\mathsf{P}} \\
\bot_{\mathsf{L}}^{\perp} & := 1_{\mathsf{L}} & (C \parr D)_{*} & := C_{*} \otimes D_{*} \\
(A \parr B)^{\perp} & := A^{\perp} \otimes B^{\perp} & \lfloor A \rfloor_{*} & := \lceil A^{\perp} \rceil
\end{array}$$

The duality rules are given in Figure 6.

Lemma 5. *The following axioms hold in* LPC*:*

$$\overline{A, A^{\perp} \vdash \cdot} \qquad \overline{P, P^{*} \Vdash \cdot} \qquad \overline{C, C_{*} \Vdash \cdot}$$

The proof is by mutual induction on the structures of A, P, and C. Similarly we can prove $\cdot \vdash A, A^{\perp}$, $\cdot \Vdash P, P^{*}$ and $\cdot \Vdash C, C_{*}$.

The above lemma does *not* cover the case of persistent propositions in a linear sequent. We can prove that $P, P^{*} \vdash \cdot$ as follows:

$$\frac{\overline{P^{*} \vdash P^{*}} \qquad \overline{P, P^{*} \Vdash \cdot}}{P, P^{*} \vdash \cdot}\ \textsc{Cut}_{\mathsf{C}}^{\vdash}$$

and similarly for $\cdot \vdash P, P^{*}$.

Theorem 6 (Duality Admissibility)**.** *The duality inference rules given in Figure 6 are admissible in* LPC.

Proof. The linear rules can be encoded directly using cut and the lemmas above. For example:

$$\frac{\Gamma \vdash \Delta, A \qquad \overline{A, A^{\perp} \vdash \cdot}}{\Gamma, A^{\perp} \vdash \Delta}\ \textsc{Cut}_{\mathsf{L}}^{\vdash}$$

Consider the producer duality rule for persistent sequents. The right duality rule can be implemented by means of a cut rule, but when we attempt to do the same for the left duality rule, we are restricted by the cut rule's form, as follows:

$$\frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \overline{P, P^{*} \Vdash \cdot}}{\Gamma^{\mathsf{P}}, P^{*} \Vdash \Delta^{\mathsf{C}}}\ \textsc{Cut}_{\mathsf{P}}^{\Vdash}$$

By the displacement theorem, this restriction of the contexts to non-displaced propositions is actually redundant, as P is the only displaced proposition in any derivation of $\Gamma \Vdash \Delta, P$.

The same restrictions on the cut rules lead us to the following left duality rule for producers in linear sequents, which is *not* equivalent to the one in Figure 6:

$$\frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \overline{P, P^{*} \vdash \cdot}}{\Gamma^{\mathsf{P}}, P^{*} \vdash \Delta^{\mathsf{C}}}\ \textsc{Cut}_{\mathsf{P}}^{\vdash}$$

Not only is the hypothesis a persistent sequent when the conclusion is a linear sequent, but the formulation excludes the occurrence of linear terms in the contexts. Instead we must prove the more general form the rule directly. For any derivation $\mathcal{D}$ of $\Gamma \vdash \Delta, P$, there is a derivation of $\Gamma, P^{*} \vdash \Delta$. The proof is by straightforward induction on $\mathcal{D}$. $\square$

D. *Consistency*

Using the cut and duality rules we can prove that LPC is consistent. Define the negation of a linear proposition to be $\neg A := A^{\perp} \parr 0$.

Lemma 7 (Consistency)**.** *There is no proposition A such that A and $\neg A$ are both provable in* LPC.

Proof. Suppose there were such an A, along with derivations $\mathcal{D}_1$ of $\cdot \vdash A$ and $\mathcal{D}_2$ of $\cdot \vdash A^\perp ⅋ 0$. Then we construct a derivation of $\cdot \vdash 0$ as follows:

$$\dfrac{\dfrac{\mathcal{D}_2}{\cdot \vdash A^\perp, 0} \quad \dfrac{\dfrac{\mathcal{D}_1}{\cdot \vdash A}}{A^\perp \vdash \cdot}\,(-)^\perp\text{-L}}{\cdot \vdash 0}\,\textsc{Cut}^{\vdash}_{\mathsf{L}}$$

However, there is no cut-free proof of $\cdot \vdash 0$ in LPC, which contradicts cut admissibility. $\square$

III. Categorical Model

In this section we describe a categorical model of LPC based on the three-category picture in Figure 1. Some details of the categorical definitions have been left out for brevity, but these can be found in the companion paper [8].

A. Linear Category

The linear category should be able to interpret the inference rules for Figure 2 as well as the linear duality. Traditionally the multiplicative fragment of linear logic is modeled by a *-autonomous category. For LPC, we use an equivalent notion that puts the tensor $\otimes$ and cotensor $⅋$ on equal footing, by modeling the category $\mathcal{L}$ as a symmetric linearly distributive category with negation.

Monoidal Structures: We start with basic definitions about symmetric monoidal structures.

Definition 8. *A* symmetric monoidal category *is a category $\mathcal{C}$ equipped with a bifunctor $\otimes$, an object 1, and the following natural isomorphisms:*

$$\begin{aligned}\alpha_{A_1,A_2,A_3} &: A_1 \otimes (A_2 \otimes A_3) \to (A_1 \otimes A_2) \otimes A_3\\ \lambda_A &: 1 \otimes A \to A\\ \rho_A &: A \otimes 1 \to A\\ \sigma_{A,B} &: A \otimes B \to B \otimes A\end{aligned}$$

satisfying certain coherence conditions.

Definition 9. *Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \sigma)$ and $(\mathcal{C}', \otimes', 1', \alpha', \lambda', \rho', \sigma')$ be symmetric monoidal categories. A functor $\mathrm{F} : \mathcal{C} \Rightarrow \mathcal{C}'$ is* symmetric monoidal *if there exists a map and natural transformation*

$$m_1^{\mathrm{F}} : 1' \to \mathrm{F}\,1 \quad \textit{and} \quad m_{A,B}^{\mathrm{F}} : \mathrm{F}\,(A) \otimes' \mathrm{F}\,(B) \to \mathrm{F}\,(A \otimes B)$$

which commute with α, λ, ρ and σ. F *is* symmetric comonoidal *if there is a map and natural transformation*

$$n_1^{\mathrm{F}} : \mathrm{F}\,1 \to 1' \quad \textit{and} \quad n_{A,B}^{\mathrm{F}} : \mathrm{F}\,(A \otimes B) \to \mathrm{F}\,(A) \otimes' \mathrm{F}\,(B)$$

which commute appropriately.

Definition 10. *Let* F *and* G *be symmetric monoidal (resp comonoidal) functors $\mathrm{F}, \mathrm{G} : \mathcal{C} \Rightarrow \mathcal{C}'$. A natural transformation $\tau : \mathrm{F} \to \mathrm{G}$ is* (co-)monoidal *if it commutes with the monoidal components m^{F} and m^{G} (comonoidal components n^{F} and n^{G}).*

Definition 11. *A* symmetric monoidal *(resp* comonoidal*)* adjunction *is an adjunction $\mathrm{F} \dashv \mathrm{G}$ between symmetric (co-)monoidal functors* F *and* G *where the unit and counit of the adjunction are symmetric (co-)monoidal natural transformations.*

Linearly Distributive Categories: The LPC logic treats $\otimes$ and $⅋$ as independent operations on propositions, and the categorical treatment mirrors this structure. The linear category therefore is taken to be linearly distributive, defined by Cockett and Seely [9], where the two monoidal structures are the primitive components.

Definition 12. *Let $\mathcal{L}$ be a category with two symmetric monoidal structures: $(\otimes, 1_{\mathsf{L}}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \sigma^{\otimes})$ and $(⅋, \perp_{\mathsf{L}}, \alpha^{⅋}, \lambda^{⅋}, \rho^{⅋}, \sigma^{⅋})$. Let*

$$\delta_{A_1,A_2,A_3} : A_1 \otimes (A_2 ⅋ A_3) \to (A_1 \otimes A_2) ⅋ A_3$$

be a natural transformation in $\mathcal{L}$. Then $\mathcal{L}$ is a symmetric linearly distributive category *if δ satisfies a number of coherence conditions [9].*

Definition 13. *A symmetric linear distributive category $\mathcal{L}$ is said to* have negation *if there exists a map $(-)^\perp$ on objects of $\mathcal{L}$, and families of maps*

$$\gamma_A^\perp : A^\perp \otimes A \to \perp_{\mathsf{L}} \qquad\qquad \gamma_A^1 : 1_{\mathsf{L}} \to A ⅋ A^\perp$$

commuting with δ in certain ways.

Lemma 14 (Cockett and Seely)**.** *Symmetric linearly distributive categories with negation correspond to *-autonomous categories.*

Additives: To encode the additives, we require that the linear category has finite products, $\&$, with unit $\top$, and finite coproducts, $\oplus$, with unit 0.

B. Persistent Categories

Definition 15. *Two symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, ⅋)$ are in duality with each other if there exist contravariant functors $(-)^* : \mathcal{P} \Rightarrow \mathcal{C}$ and $(-)_* : \mathcal{C} \Rightarrow \mathcal{P}$ where $(-)^*$ is monoidal and $(-)_*$ is comonoidal, and natural isomorphisms*

$$\epsilon^*_{*\,C} : (C_*)^* \to C \quad \textit{and} \quad \eta^*_{*\,P} : P \to (P^*)_*$$

Definition 16. *Let $(\mathcal{P}, \otimes, 1_{\mathsf{P}})$ be a symmetric monoidal category. A commutative comonoid in $\mathcal{P}$ is an object P along with two morphisms $e^{\otimes} : P \to 1_{\mathsf{P}}$ and $d^{\otimes} : P \to P \otimes P$ such that the following three commuting diagrams are satisfied:*

$$
\begin{tikzcd}
 & P \arrow[dl, "d^{\otimes}"'] \arrow[dr, "d^{\otimes}"] & \\
P \otimes P \arrow[rr, "\sigma_{P,P}"'] & & P \otimes P
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "d^{\otimes}"] \arrow[dr, "{\lambda_P}^{-1}"'] & P \otimes P \arrow[d, "e^{\otimes} \otimes id_P"] \\
 & 1_{\mathsf{P}} \otimes P
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & P \otimes P \arrow[r, "id_P \otimes d^{\otimes}"] & P \otimes (P \otimes P) \arrow[d, "\alpha_{P,P,P}"] \\
P \arrow[ur, "d^{\otimes}"] \arrow[dr, "d^{\otimes}"'] & & \\
 & P \otimes P \arrow[r, "d^{\otimes} \otimes id_P"'] & (P \otimes P) \otimes P
\end{tikzcd}
$$

Dually, a commutative monoid in a symmetric monoidal category $(\mathcal{C}, \parr, \bot_{\mathsf{C}})$ is an object C along with morphisms $e^{\parr} : \bot_{\mathsf{C}} \to C$ and $d^{\parr} : C \parr C \to C$ such that the appropriate (dual) diagrams commute.

C. The LPC *model*

Definition 17. *A linear/producing/consuming (*LPC*) model consists of the following components:*

1) *A symmetric linearly distributive category $(\mathcal{L}, \otimes, \parr)$ with negation $(-)^{\perp}$, finite products $\&$ and finite coproducts $\oplus$.*
2) *Symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, \parr)$ in duality by means of contravariant functors*

$$(-)^* : \mathcal{P} \Rightarrow \mathcal{C} \qquad\qquad (-)_* : \mathcal{C} \Rightarrow \mathcal{P}$$

3) *Monoidal natural transformations*

$$e^{\otimes}_P : P \to 1_{\mathsf{P}} \qquad \textit{and} \qquad d^{\otimes}_P : P \to P \otimes P$$

in $\mathcal{P}$ and comonoidal natural transformations

$$e^{\parr}_C : \bot_{\mathsf{C}} \to C \qquad \textit{and} \qquad d^{\parr}_C : C \parr C \to C$$

in $\mathcal{C}$, interchanged under duality, such that: (a) for every P, $(P, d^{\otimes}_P, e^{\otimes}_P)$ forms a commutative comonoid in $\mathcal{P}$; and (b) for every C, $(C, d^{\parr}_C, e^{\parr}_C)$ forms a commutative monoid in $\mathcal{C}$.

4) *Symmetric monoidal functors*

$$\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P} \qquad F_! : \mathcal{P} \Rightarrow \mathcal{L}$$

and symmetric comonoidal functors

$$\lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C} \qquad F_? : \mathcal{C} \Rightarrow \mathcal{L}$$

which respect the dualities in that

$$(F_!\, P)^{\perp} \simeq F_?\,(P^*) \qquad \textit{and} \qquad \lceil A \rceil \simeq \lfloor A^{\perp} \rfloor$$

and that form monoidal/comonoidal adjunctions

$$\lceil - \rceil \dashv F_! \qquad \textit{and} \qquad F_? \dashv \lfloor - \rfloor$$

The monoidal components $m^{F_!}$ of the $F_!$ functor are necessarily isomorphisms, using the adjunction and monoidal $m^{\lceil - \rceil}$. Similarly for $n^{F_?}$.

D. LPC *and other linear logic models*

As LPC is inspired by Benton's linear/non-linear paradigm, we would like to formalize the relationship between LPC and LNL.

Definition 18 (Melliès [10])**.** *A linear/non-linear (*LNL*) model consists of: (1) a symmetric monoidal closed category $\mathcal{L}$; (2) a cartesian category $\mathcal{P}$; and (3) functors* $\mathrm{G} : \mathcal{L} \Rightarrow \mathcal{P}$ *and* $\mathrm{F} : \mathcal{P} \Rightarrow \mathcal{L}$ *that form a symmetric monoidal adjunction* $\mathrm{F} \dashv \mathrm{G}$.

The LNL model given by Benton [6] has the added condition that the cartesian category be cartesian closed. In LPC, the fact that every object P forms a commutative comonoid in $\mathcal{P}$ means that $\mathcal{P}$ is cartesian. Therefore:

Lemma 19. *Every* LPC *model is an* LNL *model.*

In addition, a *-autonomous category in a linear/non-linear model induces the consumer category in LPC:

Lemma 20. *If the category $\mathcal{L}$ in an* LNL *model is *-autonomous, then the categories $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{P}^{op}$ form an* LPC *model.*

Next we prove that every LPC model contains a classical linear category as defined by Schalk [11].

Definition 21 (Schalk [11])**.** *A category $\mathcal{L}$ is a model for classical linear logic if and only if it: (1) is *-autonomous; (2) has finite products $\&$ and thus finite coproducts $\oplus$; and (3) has a linear exponential comonad* $!$ *and thus a linear exponential monad* $?$.

This definition is just the logical extension of Benton et al's linear category [12] to classical linear logic.

Lemma 22. *The category $\mathcal{L}$ from the* LPC *model is a model for classical linear logic.*

Lemma 23. *Every model for classical linear logic forms an* LPC *category.*

The producer category can be derived from either the Kleisli or Eilenberg-Moore constructions, as demonstrated by Benton [6]. The consumer category is defined dually. Clearly, these constructions are not unique.

E. Interpretation of the Logic

We define an interpretation of the LPC logic that maps propositions to objects in the categories, and derivations to morphisms. For objects, the $[\![-]\!]_{\mathsf{L}}$ interpretation function is defined on all propositions, but $[\![-]\!]_{\mathsf{P}}$ and $[\![-]\!]_{\mathsf{C}}$ are defined only on persistent propositions. On linear units and combinators, the interpretations act as expected. On the adjoint functors, the behavior is as follows:

$$
\begin{aligned}
\llbracket F_! \, P \rrbracket_{\mathsf{L}} &= F_! \, \llbracket P \rrbracket_{\mathsf{P}} & \llbracket F_? \, C \rrbracket_{\mathsf{L}} &= F_? \, \llbracket C \rrbracket_{\mathsf{C}} \\
\llbracket P \rrbracket_{\mathsf{L}} &= F_! \, \llbracket P \rrbracket_{\mathsf{P}} & \llbracket C \rrbracket_{\mathsf{L}} &= F_? \, \llbracket C \rrbracket_{\mathsf{C}} \\
\llbracket \lceil A \rceil \rrbracket_{\mathsf{P}} &= \lceil \llbracket A \rrbracket_{\mathsf{L}} \rceil & \llbracket \lfloor A \rfloor \rrbracket_{\mathsf{C}} &= \lfloor \llbracket A \rrbracket_{\mathsf{L}} \rfloor
\end{aligned}
$$

Finally, for persistent propositions of the opposite mode, duality is used to interpret the propositions:

$$
\llbracket C \rrbracket_{\mathsf{P}} = (\llbracket C \rrbracket_{\mathsf{C}})_* \qquad \llbracket P \rrbracket_{\mathsf{C}} = (\llbracket P \rrbracket_{\mathsf{P}})^*
$$

Contexts can be interpreted with the comma as either the tensor or cotensor in the linear category; in the producer category there is no cotensor and vice versa for the consumer category.

$$
\begin{aligned}
\llbracket \cdot \rrbracket_{\mathsf{L}}^{\otimes} &= 1_{\mathsf{L}} & \llbracket X, \Gamma \rrbracket_{\mathsf{L}}^{\otimes} &= \llbracket X \rrbracket_{\mathsf{L}} \otimes \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \\
\llbracket \cdot \rrbracket_{\mathsf{L}}^{\parr} &= \bot_{\mathsf{L}} & \llbracket X, \Gamma \rrbracket_{\mathsf{L}}^{\parr} &= \llbracket X \rrbracket_{\mathsf{L}} \parr \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\parr} \\
\llbracket \cdot \rrbracket_{\mathsf{P}} &= 1_{\mathsf{P}} & \llbracket X, \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{P}} &= \llbracket X \rrbracket_{\mathsf{P}} \otimes \llbracket \Gamma \rrbracket_{\mathsf{P}} \\
\llbracket \cdot \rrbracket_{\mathsf{C}} &= \bot_{\mathsf{C}} & \llbracket X, \Gamma \rrbracket_{\mathsf{C}} &= \llbracket X \rrbracket_{\mathsf{C}} \parr \llbracket \Gamma \rrbracket_{\mathsf{C}}
\end{aligned}
$$

A linear derivation $\mathcal{D}$ of the form $\Gamma \vdash \Delta$ is interpreted as a morphism $\llbracket \mathcal{D} \rrbracket_{\mathsf{L}} : \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \to \llbracket \Delta \rrbracket_{\mathsf{L}}^{\parr}$, but this will not suffice for persistent derivations $\Gamma \Vdash \Delta$. When mapped into $\mathcal{P}$, the codomain cannot be interpreted as a $\parr$-separated list. We proved in Section II-A that every persistent derivation $\mathcal{D}$ contains exactly one displaced proposition. This means that $\mathcal{D}$ is either of the form $\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P$ or $\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}$. In the category $\mathcal{P}$, this derivation is interpreted as a morphism

$$
\begin{aligned}
&\llbracket \mathcal{D} \rrbracket_{\mathsf{P}} : \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{P}} \otimes \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{P}} \to \llbracket P \rrbracket_{\mathsf{P}} \quad \text{or} \\
&\llbracket \mathcal{D} \rrbracket_{\mathsf{P}} : \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{P}} \otimes \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{P}} \to \llbracket C \rrbracket_{\mathsf{P}},
\end{aligned}
$$

respectively. Similarly in $\mathcal{C}$, $\mathcal{D}$ is interpreted as

$$
\begin{aligned}
&\llbracket \mathcal{D} \rrbracket_{\mathsf{C}} : \llbracket P \rrbracket_{\mathsf{C}} \to \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{C}} \parr \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{C}} \qquad \text{or} \\
&\llbracket \mathcal{D} \rrbracket_{\mathsf{C}} : \llbracket C \rrbracket_{\mathsf{C}} \to \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{C}} \parr \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{C}}.
\end{aligned}
$$

This interpretation is defined by mutual induction. Following are a few representative cases from the definition.

Interpreting weakening and contraction in the persistent sequent is straightforward using the monoid in $\mathcal{C}$ and comonoid in $\mathcal{P}$. For weakening in the linear sequent, suppose we have the following derivation:

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Gamma \vdash \Delta}}{\Gamma, P \vdash \Delta} \; \mathrm{W}^{\vdash}\text{-L}
$$

The interpretation of $\mathcal{D}$ inserts the comonoid in $\mathcal{P}$ into the linear category:

$$
\begin{aligned}
\llbracket \mathcal{D} \rrbracket_{\mathsf{L}} : \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \otimes F_! \, \llbracket P \rrbracket_{\mathsf{P}} &\xrightarrow{\llbracket \mathcal{D}' \rrbracket_{\mathsf{L}} \otimes F_! \, e^{\otimes}} \llbracket \Delta \rrbracket_{\mathsf{L}}^{\parr} \otimes F_! \, 1_{\mathsf{P}} \\
&\xrightarrow{\mathrm{id} \otimes (m^{F_!})^{-1}} \llbracket \Delta \rrbracket_{\mathsf{L}}^{\parr} \otimes 1_{\mathsf{L}} \xrightarrow{\rho^{\otimes}} \llbracket \Delta \rrbracket_{\mathsf{L}}^{\parr}
\end{aligned}
$$

The case for contraction is similar.

The interpretation of the $F_!$-L and $F_?$-R rules are straightforward from the inductive hypothesis, but the $F_!$-R and $F_?$-L rules are more involved. Suppose

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! \, P} \; F_!\text{-R}
$$

The inductive hypothesis provides a morphism $\llbracket \mathcal{D}' \rrbracket_{\mathsf{P}} : \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{P}} \otimes \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{P}} \to \llbracket P \rrbracket_{\mathsf{P}}$. We need to undo this duality transformation for interpretation in the linear category, however. Consider

$$
\begin{aligned}
\llbracket \mathcal{D} \rrbracket_{\mathsf{L}} : \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{L}}^{\otimes} &\xrightarrow{\rho^{\otimes}} \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{L}}^{\otimes} \otimes 1_{\mathsf{L}} \\
&\xrightarrow{\mathrm{id} \otimes \lambda} \llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{L}}^{\otimes} \otimes ((\llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr})^{\bot} \parr \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr}) \\
&\xrightarrow{\delta^{\mathrm{L},\mathrm{L}}} (\llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{L}}^{\otimes} \otimes (\llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr})^{\bot}) \parr \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr} \\
&\xrightarrow{(\mathrm{id} \otimes (m^{(-)_*})^{-1}) \parr \mathrm{id}} (\llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{L}}^{\otimes} \otimes \llbracket (\Delta^{\mathsf{C}})^* \rrbracket_{\mathsf{L}}^{\otimes}) \parr \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr} \\
&\xrightarrow{\text{multiple } m^{F_!}} F_! \, (\llbracket \Gamma^{\mathsf{P}} \rrbracket_{\mathsf{P}} \otimes \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{P}}) \parr \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr} \\
&\xrightarrow{F_! \, \llbracket \mathcal{D}' \rrbracket_{\mathsf{P}} \parr \mathrm{id}} F_! \, \llbracket P \rrbracket_{\mathsf{P}} \parr \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr} \xrightarrow{\sigma^{\parr}} \llbracket \Delta^{\mathsf{C}} \rrbracket_{\mathsf{L}}^{\parr} \parr \llbracket P \rrbracket_{\mathsf{L}}
\end{aligned}
$$

The $\lceil - \rceil$-L rule uses the unit of the adjunction in its interpretation. Suppose

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}'}{\Gamma, A \vdash \Delta}}{\Gamma, \lceil A \rceil \vdash \Delta} \; \lceil - \rceil\text{-L}
$$

Its interpretation is defined to be

$$
\llbracket \mathcal{D} \rrbracket_{\mathsf{L}} : \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \otimes F_! \, (\lceil \llbracket A \rrbracket_{\mathsf{L}} \rceil) \xrightarrow{\mathrm{id} \otimes \epsilon} \llbracket \Gamma \rrbracket_{\mathsf{L}}^{\otimes} \otimes \llbracket A \rrbracket_{\mathsf{L}} \xrightarrow{\llbracket \mathcal{D}' \rrbracket_{\mathsf{L}}} \llbracket \Delta \rrbracket_{\mathsf{L}}^{\parr}
$$

Admissible Rules: The admissible duality and cut rules from the LPC logic should correspond with the expected notions of duality and cut in the categorical model. In the simplest case, suppose a derivation has the form

$$
\mathcal{D} = \cfrac{\cfrac{\mathcal{D}_1}{\Gamma \vdash A} \qquad \cfrac{\mathcal{D}_2}{A \vdash \Delta}}{\Gamma \vdash \Delta} \; \textsc{Cut}_{\mathsf{L}}^{\vdash}
$$

Lemma 24. $\llbracket \mathcal{D} \rrbracket_{\mathsf{L}} = \llbracket \mathcal{D}_2 \rrbracket_{\mathsf{L}} \circ \llbracket \mathcal{D}_1 \rrbracket_{\mathsf{L}}$.

We will sketch some cases from the cut admissibility proof to demonstrate that this property holds.

Suppose

$$
\mathcal{D}_1 = \cfrac{}{\cdot \vdash 1_{\mathsf{L}}} \; 1_{\mathsf{L}}^{\vdash}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \cfrac{\cfrac{\mathcal{D}_2'}{\cdot \vdash \Delta}}{1_{\mathsf{L}} \vdash \Delta} \; 1_{\mathsf{L}}^{\vdash}\text{-L}
$$

By the definition of cut, we know $\mathcal{D} = \mathcal{D}_2'$. So our goal is to show that $\llbracket \mathcal{D}_2' \rrbracket_{\mathsf{L}} = \llbracket \mathcal{D}_2 \rrbracket_{\mathsf{L}} \circ \llbracket \mathcal{D}_1 \rrbracket_{\mathsf{L}}$ which holds because $\llbracket \mathcal{D}_2 \rrbracket_{\mathsf{L}} = \llbracket \mathcal{D}_2' \rrbracket_{\mathsf{L}}$ and $\llbracket \mathcal{D}_1 \rrbracket_{\mathsf{L}} = \mathrm{id}_{1_{\mathsf{L}}}$.

Suppose A has the form $A_1 \mathbin{\&} A_2$ and that

$$
\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_{11}}{\Gamma \vdash A_1} \qquad \cfrac{\mathcal{D}_{12}}{\Gamma \vdash A_2}}{\Gamma \vdash A_1 \mathbin{\&} A_2} \; \&_{\mathsf{L}}^{\vdash}\text{-R} \qquad \mathcal{D}_2 = \cfrac{\cfrac{\mathcal{D}_2'}{A_1 \vdash \Delta}}{A_1 \mathbin{\&} A_2 \vdash \Delta} \; \&_{\mathsf{L}}^{\vdash}\text{-L}
$$

By the definition of cut, $\mathcal{D}$ is equal to the derivation

$$\mathcal{D} = \cfrac{\cfrac{\mathcal{D}_{11}}{\Gamma \vdash A_1} \qquad \cfrac{\mathcal{D}'_2}{A_1 \vdash \Delta}}{\Gamma \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}$$

Meanwhile,

$$[\![\mathcal{D}_1]\!]_{\mathsf{L}} : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}_{11}]\!]_{\mathsf{L}} \& [\![\mathcal{D}_{12}]\!]_{\mathsf{L}}} [\![A_1 \,\&\, A_2]\!]_{\mathsf{L}}$$

$$[\![\mathcal{D}_2]\!]_{\mathsf{L}} : [\![A_1 \,\&\, A_2]\!]_{\mathsf{L}} \xrightarrow{\pi_1} [\![A_1]\!]_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}'_2]\!]_{\mathsf{L}}} [\![\Delta]\!]^{\parr}_{\mathsf{L}}$$

Therefore, $[\![\mathcal{D}_2]\!]_{\mathsf{L}} \circ [\![\mathcal{D}_1]\!]_{\mathsf{L}} = [\![\mathcal{D}_1]\!]_{\mathsf{L}} \circ [\![\mathcal{D}'_2]\!]_{\mathsf{L}} = [\![\mathcal{D}]\!]_{\mathsf{L}}$.

Finally, consider the case where A is not the principle formula in $\mathcal{D}_1$, and instead

$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}'_1}{\Gamma, B \vdash A}}{\Gamma, \lceil B \rceil \vdash A}\ \lceil - \rceil\text{-L}$$

Working through the interpretation function, we know

$$[\![\mathcal{D}_1]\!]_{\mathsf{L}} : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes F_! \lceil [\![B]\!]_{\mathsf{L}} \rceil \xrightarrow{\mathrm{id} \otimes \epsilon} [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes [\![B]\!]_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}'_1]\!]_{\mathsf{L}}} [\![A]\!]_{\mathsf{L}}$$

In addition, since

$$\mathcal{D} = \cfrac{\cfrac{\cfrac{\mathcal{D}'_1}{\Gamma, B \vdash A} \qquad \cfrac{\mathcal{D}_2}{A \vdash \Delta}}{\Gamma, B \vdash \Delta}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}}{\Gamma, \lceil B \rceil \vdash \Delta}\ \lceil - \rceil\text{-L}$$

by the definition of cut we have

$$[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes F_! \lceil [\![B]\!]_{\mathsf{L}} \rceil \xrightarrow{\mathrm{id} \otimes \epsilon} [\![\Gamma]\!]^{\otimes}_{\mathsf{L}} \otimes [\![B]\!]_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}'_1]\!]_{\mathsf{L}}} [\![A]\!]_{\mathsf{L}} \xrightarrow{[\![\mathcal{D}_2]\!]_{\mathsf{L}}} [\![\Delta]\!]^{\parr}_{\mathsf{L}}$$

Therefore it is easy to see that $[\![\mathcal{D}]\!]_{\mathsf{L}} = [\![\mathcal{D}_2]\!]_{\mathsf{L}} \circ [\![\mathcal{D}_1]\!]_{\mathsf{L}}$.

IV. Examples

This section provides some concrete instances of the LPC model. The following chart summarizes the three examples and their LPC categories.

	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{C}$
Vectors	FinVect	FinSet	FinSet^{op}
Relations	Rel	Set	Set^{op}
Bool. Alg.	FinBoolAlg	FinPoset	FinLat

Vector Spaces: The $\otimes$, $(-)^{\perp}$ and $\&$ of linear logic are easily interpreted as the tensor product, duality and direct product. The exponentials $!$ and $?$ are not induced from the usual structures of linear algebra, however.

The LPC model takes $\mathcal{L}$ to be finite-dimensional vector spaces over a finite field, $\mathcal{P}$ to be the category of finite sets and functions, and $\mathcal{C}$ to be FinSet^{op}. The functors $\lceil - \rceil$ and $\lfloor - \rfloor$ are the forgetful functor from vector spaces to sets, and the functors $F_!$ and $F_?$ take a set X to the free vector space generated by X.

Other related interpretations are found in the literature. Ehrhard [13] presents finiteness spaces, where the objects are spaces of vectors with finite support. In his model, the $!$ operator sends a space A to the space supported by finite multisets over A; it takes some effort to show that this comonad respects the finiteness conditions. Pratt [14] proves that finite dimensional vector spaces over a field of characteristic 2 is a Chu space and thus a model of linear logic. Valiron and Zdancewic [15] show that the LPC model of FinVect is a sound and complete semantic model for an algebraic λ-calculus.

Relations: The category Rel of sets and relations, along with the categories Set and Set^{op}, form an LPC model. The functor $F_!$ ($F_?$) is the forgetful functor from functions (inverse functions) to relations, and $\lceil - \rceil$ ($\lfloor - \rfloor$) takes a set to its powerset.

Melliés [10] discusses a non-model of linear logic based on Rel, where the exponential takes a set X to the finite subsets of X. That "model" fails because the comonad unit $\epsilon_A : !A \to A$ is not natural. In the LPC formulation, ϵ is derived from the adjunction.

Boolean Algebras: Let $\mathcal{L}$ be the category of finite boolean algebras, $\mathcal{P}$ be FinPoset of finite partially ordered sets, and $\mathcal{C}$ be FinLat of finite lattices. This trio forms an LPC category where the duality is given by Birkhoff's representation theorem [16]. The functors $\lceil - \rceil$ and $\lfloor - \rfloor$ are forgetful functors, and $F_!$ and $F_?$ take the base set X of their respective structures to the boolean lattice $\mathcal{P}(X)$, where the algebraic structure corresponds to union, intersection and complementation. Unlike the examples where $\mathcal{C} = \mathcal{P}^{op}$, this example utilizes a non-trivial duality between the persistent categories. Although FinLat is both cartesian and cocartesian, FinPoset is only cartesian.

V. Related Work

Girard [17] first introduced linear logic to mix the constructivity of intuitionistic propositional logic with the duality of classical logic. Partly because of this constructivity, there has been great interest in the semantics of linear logic in both the classical and intuitionistic fragments. Consequently, there exist several categorical frameworks for its semantic models.

One influential framework is Benton *et al.*'s *linear category* [12], consisting of a symmetric monoidal closed category with products and a linear exponential comonad $!$. Other characterizations include the Seely category [18], based on a distribution between $!A \otimes !B$ and $!(A \,\&\, B)$, and which was later proved unsound by Wadler [19]. Bierman [20] defined a new Seely category by adding a symmetric monoidal adjunction between the category and its co-Kleisli category.

Except for Seely's original formulation, these works deal with the intuitionistic fragment of linear logic. The multiplicative fragment (with just $\otimes$ and $⅋$) of classical linear logic is usually modeled by *-autonomous categories, introduced by Barr [21]. Schalk [11] adapted linear category to the classical case by requiring that the symmetric monoidal closed category be *-autonomous. The coproduct $⅋$ and coexponential $?$ are then induced from the duality.

Cockett and Seely [9], seeking to study $\otimes$ and $⅋$ as independent structures unobscured by duality, introduced linearly distributive categories, which make up the linear category in the LPC model. The authors extended this motivation to the exponentials by modeling $!$ and $?$ as linear functors [22], meaning that $?$ is not derived from $!$ and $(-)^{\perp}$. The LPC model reflects that work by allowing $!$ and $?$ to have different adjoint decompositions.

Other variations of classical linear logic, notably Girard's Logic of Unity [23], distinguish linear propositions from persistent ones. The sequent $\Gamma; \Gamma' \vdash \Delta'; \Delta$ is meant to be seen as a derivation where Γ' and Δ' are persistent and admit weakening and contraction. In Γ and Δ every proposition is purely linear.

Ramifying LU's separation (in the intuitionistic case), Benton [6] developed the linear/non-linear logic and categorical model described in Section III-D. Barber used this model as the semantics for a term calculus called DILL [24]. In Lemma 19 we prove that every LPC model is an LNL model. A Lafont category [25] is a canonical instance of an LNL model where $!A$ is the free commutative comonoid generated by A. This construction automatically admits an adjunction between automatically forms an adjunction between a linear category $\mathcal{L}$ and the category of commutative comonoids over $\mathcal{L}$. However, the LNL and LPC models have an advantage over Lafont categories by allowing a much greater range of interpretations for the exponential. Lafont's construction excludes traditional models of linear logic like coherence spaces and the category REL.

VI. Conclusion

This work presents a version of classical linear logic and a corresponding semantic model with the goal of shedding light on the fundamentally dual linear structure. We present a logic reminiscent of Benton [6] and prove cut admissibility, duality admissibility, and consistency. We then define a categorical model, prove its compatibility with other models from the literature, and translate derivations from the logic into the categories. Finally we present three new examples of concrete models for linear logic through the scope of the LPC formulation.

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