Whirlwind Tour of LA
Part 1: Some Nitty-Gritty Stuff

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Logistics

- PS1/2 deferred to next Weds/Fri
  - Brandon has OH tonight 5-7
  - I will arrange extra OH early next week (TBA)
- Please keep up with the reading!
- Ask me questions!

Big picture: What's a matrix?

An array of numbers, or a representation of

- A tabular data set?
- A graph?
- A linear function between vector spaces?
- A bilinear function on two vectors?
- A pure quadratic function?

It's all the above, plus being an interesting object on its own!

Let's start concrete.

Basics: Constructing matrices and vectors

```
x = [1; 2];           % Column vector
y = [1, 2];           % Row vector
M = [1, 2; 3, 4];     % 2-by-2 matrix
M = [I, A];           % Horizontal matrix concatenation
```

Basics: Constructing matrices and vectors

```
I  = eye(n);     % Build n-by-n identity
Z  = zeros(n);   % n-by-n matrix of zeros
b  = rand(n,1);  % n-by-1 random matrix (uniform)
e  = ones(n,1);  % n-by-1 matrix of ones
D  = diag(e);    % Construct a diagonal matrix
e2 = diag(D);    % Extract matrix diagonal
```

Basics: Transpose, rearrangements

```
% Reshape A to a vector, then back to a matrix
% Note: MATLAB is column-major
avec = reshape(A, prod(size(A)));
A = reshape(avec, n, n);

A = A';  % Conjugate transpose
A = A.'; % Simple transpose

idx = randperm(n); % Random permutation of indices
Ac = A(:,idx);     % Permute columns of A
Ar = A(idx,:);     % Permute rows of A
Ap = A(idx,idx);   % Permute rows and columns
```

Basics: Submatrices, diagonals, triangles

```
A = randn(6,6);     % 6-by-6 random matrix
A(1:3,1:3)          % Leading 3-by-3 submatrix
A(1:2:end,:)        % Rows 1, 3, 5
A(:,3:end)          % Columns 3-6

Ad = diag(A);       % Diagonal of A (as vector)
A1 = diag(A,1);     % First superdiagonal
Au = triu(A);       % Upper triangle
Al = tril(A);       % Lower triangle
```

Basics: Matrix and vector operations

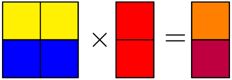
```
y = d.*x;     % Elementwise multiplication of vectors/matrices
y = x./d;     % Elementwise division
z = x + y;    % Add vectors/matrices
z = x + 1;    % Add scalar to every element of a vector/matrix

y = A*x;      % Matrix times vector
y = x'*A;     % Vector times matrix
C = A*B;      % Matrix times matrix

% Don't use inv!
x = A\b;      % Solve Ax = b *or* least squares
y = b/A;      % Solve yA = b or least squares
```

Two basic operations

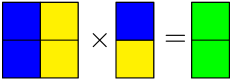
- Matrix-vector product (matvec): $O(n^2)$
- Matrix-matrix product (matmul): $O(n^3)$

Matvec

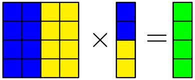
A matvec is a collection of dot products.



A matvec is a sum of scaled columns.



Can also think of *block* rows/columns.



Same (scalar) operations, different order!

Matvec: Diagonal

```
% Bad idea
D = diag(1:n); % O(n^2) setup
y = D*x;       % O(n^2) matvec

% Good idea
d = (1:n)';    % O(n) setup
y = d.*x;      % O(n) matvec
```

- Matrix-vector products are a basic op.
- Can you write the two nested loops?
- Obvious form: `y = Ax`
- Obvious isn't always best!

Matvec: Low rank

```
A = u*v';   % O(n^2)
y = A*x;

a = v'*x;   % O(n)
y = u*a;
```

Don't form low rank matrices explicitly!

Matvec: Low rank

Write an outer-product decomposition for

- A matrix of all ones
- A matrix of ± 1 in a checkerboard
- A matrix of ones and zeros in a checkerboard

Matvec: Sparse

```
% Sparse (O(n) = number nonzeros to form / multiply)
e = ones(n−1,1);
T = speye(n)−spdiags(e,−1,n,n)−spdiags(e,1,n,n);

% Dense (O(n^2))
T = eye(n)−diag(e,−1)−diag(e,1);
```

Will talk about this in more detail – keep it in mind!

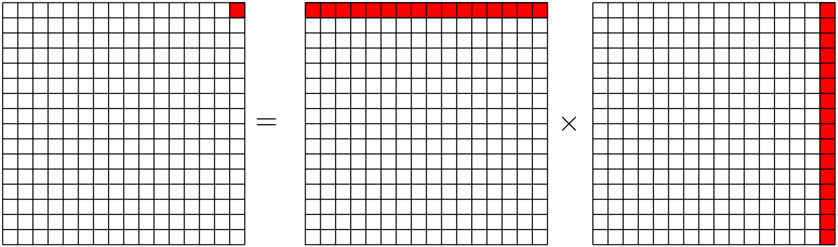
From matvec to matmul

Matrix-vector product is a key kernel in *sparse* NLA.
Matrix-matrix product is a key kernel in *dense* NLA.

Surprisingly tricky to get fast – so let someone else write fast matmul, and use it to accelerate our codes!

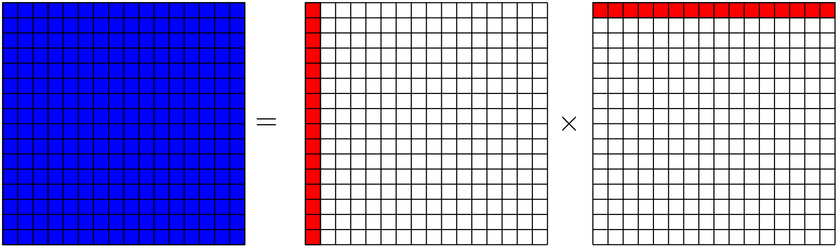
Matmul: Inner product version

An entry in C is a dot product of a row of A and column of B.



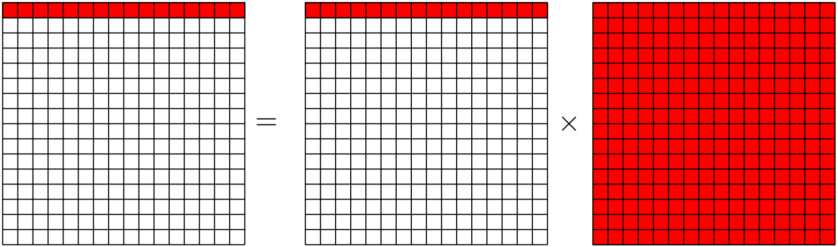
Matmul: Outer product version

C is a sum of outer products of columns of A and rows of B.



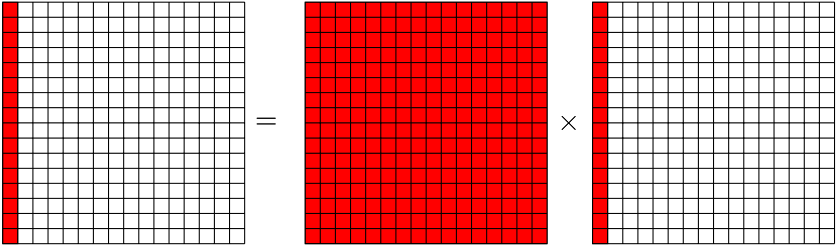
Matmul: Row-by-row

A row in C is a row of A multiplied by B.



Matmul: Col-by-col

A column in C is A multiplied by a column of B.

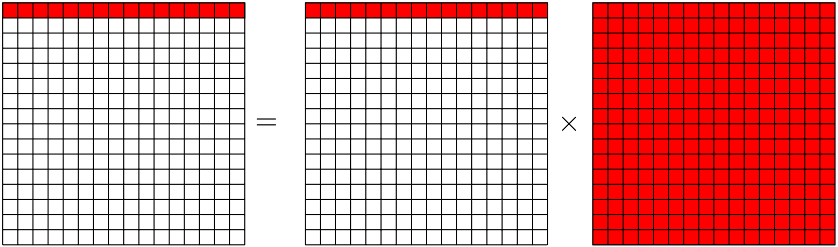


Reality intervenes

These arrangements of matmul are theoretically equivalent.
What about in practice?

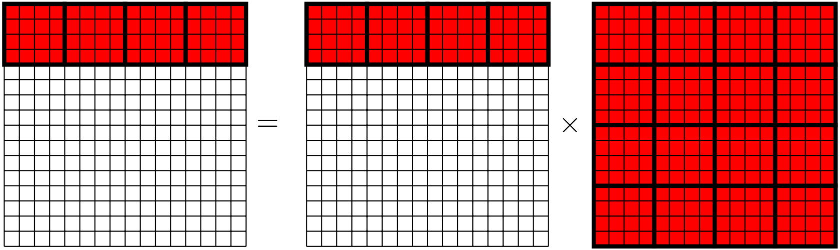
Answer: Big differences due to *memory hierarchy*.

One row in naive matmul



- Access A and C with stride of $8M$ bytes
- Access *all* $8M^2$ bytes of B before first re-use
- Poor *arithmetic intensity*

Engineering strategy: blocking/tiling



Simple model

Consider two types of memory (fast and slow) over which we have complete control.

- m = words read from slow memory
- t_m = slow memory op time
- f = number of flops
- t_f = time per flop
- $q = f/m$ = average flops / slow memory access

Time:

$$ft_f + mt_m = ft_f \left(1 + \frac{t_m/t_f}{q}\right)$$

Larger q means better time.

How big can q be?

1. Dot product: n data, $2n$ flops
2. Matrix-vector multiply: n^2 data, $2n^2$ flops
3. Matrix-matrix multiply: $2n^2$ data, $2n^3$ flops

These are examples of level 1, 2, and 3 routines in *Basic Linear Algebra Subroutines* (BLAS). We like building things on level 3 BLAS routines.

q for naive matrix multiply

$q \approx 2$ (on board)

q for blocked matrix multiply

$q \approx b$ (on board). If M_f words of fast memory, $b \approx \sqrt{M_f/3}$.

Th: (Hong/Kung 1984, Irony/Tishkin/Toledo 2004): Any reorganization of this algorithm that uses only associativity and commutativity of addition is limited to $q = O(\sqrt{M_f})$

Note: Strassen uses distributivity...

Concluding thoughts

- Will not focus on performance *details* here (see CS 5220!)
- Knowing "big picture" issues makes a big difference
  - Order-of-magnitude improvements through blocking ideas
  - Even more possible through appropriate use of structure
- Next time: More theoretical stuff!