Far from complete and far from correct. These notes exist because I learn better when I express things in my own words. In some cases I might be assembling material for teaching purposes. Might delete later.

Linear Algebra

Contents of this chapter

Spectral Theorem

Singular Value Decomposition

Generalized Eigenvectors

Affine Transformations

Affine transformations are the combination of a linear map and a translation, which has the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$.

$$f : V \to W \tag{1}$$

Where V and W are vector spaces. Affine transformations can be expresses as matrices by adding an entry with a constant to the vectors that describe a point in space. For example, for $\mathbf{x} \in \mathbb{R}^n$, the affine transform $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ with $A \in \mathbb{R}^{n,n}$ and $x, b \in \mathbb{R}^n$ can be expressed as the product of a rectangular matrix $\mathbf{M}$ and a vector $\mathbf{c}$ as:

$$\mathbf{A}\mathbf{x} + \mathbf{b} = \underbrace{\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]}_{\mathbf{M}} \underbrace{\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}}_{\mathbf{c}} \tag{2}$$

Where $\mathbf{c}^T = [x_1, x_2, x_3, ..., x_n, 1]$ and $\mathbf{M} \in \mathbb{R}^{n,n+1}$.

Multilinear Maps

A multilinear map acts on several vectors in a way that is linear in each of its arguments. A k-linear map acts on k vectors, where $k = 2$ are bilinear maps and $k = 1$ are linear maps.

$$f : V_1 \times V_2 \times ... \times V_n \to W \tag{3}$$

Where $V_1, V_2, ..., V_n$ and W are vector spaces. An example would be the addition or subtraction of two or more vectors.

Multilinear Forms

Multilinear forms are multilinear maps that have a scalar output. An example is the dot product between two vectors, or summing over the elements of one or more vectors.

$$f : V_1 \times V_2 \times ... \times V_n \to K \tag{4}$$

Where $V_1, V_2, ..., V_n$ and K is a scalar field.

Taking Derivatives

$$\frac{d}{d\mathbf{x}}\left(u^T x\right) = \left[\frac{d}{dx_1}\left(\sum_i u_i x_i\right), ..., \frac{d}{dx_n}\left(\sum_i u_i x_i\right)\right] = u^T$$

$$\frac{d}{d\mathbf{x}}\left(x^T u\right) = \left[\frac{d}{dx_1}\left(\sum_i u_i x_i\right), ..., \frac{d}{dx_n}\left(\sum_i u_i x_i\right)\right] = u^T$$

$$\frac{d}{d\mathbf{x}}\left(x^T x\right) = \left[\frac{d}{dx_1}\left(\sum_i x_i^2\right), ..., \frac{d}{dx_n}\left(\sum_i x_i^2\right)\right] = 2x^T$$

$$\frac{d}{d\mathbf{x}}\left(\mathbf{A}x\right) = \begin{bmatrix} \underbrace{\dfrac{d}{dx_1}\left(\displaystyle\sum_i A_{1i}x_i\right)}_{A_{11}} & \cdots & \underbrace{\dfrac{d}{dx_n}\left(\displaystyle\sum_i A_{1i}x_i\right)}_{A_1 n} \\ \vdots & \vdots & \vdots \\ \underbrace{\dfrac{d}{dx_1}\left(\displaystyle\sum_i A_{ni}x_i\right)}_{A_{n1}} & \cdots & \underbrace{\dfrac{d}{dx_n}\left(\displaystyle\sum_i A_{ni}x_i\right)}_{A_{nn}} \end{bmatrix} = \mathbf{A} \tag{5}$$

Combinatorics

Contents of this chapter

The Twentyfold Way

The twentyfold way is a taxonomy of distribution problems developed by Kenneth Bogard in his book *Combinatorics through Guided Discovery*. It divides up the way in which k objects may be assigned to n individuals, subject to whether the objects are distinct or identical, and subject to conditions on how the objects are received.

> *When we are passing out objects to recipients, we may think of the objects as being either identical or distinct. We may also think of the recipients as being either identical (as in the case of putting fruit into plastic bags in the grocery store) or distinct (as in the case of passing fruit out to children). We may restrict the distributions to those that give at least one object to each recipient, or those that give exactly one object to each recipient, or those that give at most one object to each recipient, or we may have no such restrictions. If the objects are distinct, it may be that the order in which the objects are received is relevant (think about putting books onto the shelves in a bookcase) or that the order in which the objects are received is irrelevant (think about dropping a handful of candy into a child?s trick or treat bag). If we ignore the possibility that the order in which objects are received matters, we have created* $2 \times 2 \times 4 = 16$ *distribution problems. In the cases where a recipient can receive more than one distinct object, we also have four more problems when the order objects are received matters. Thus we have 20 possible distribution problems.* - Bogart, "Combinatorics Though Guided Discovery", Chapter 3.

The weakness (in my opinion) is that the language of "objects" and "recipients" is unclear because in practice it's not obvious which is which: if there are k students and n teachers, do the teachers receive students, or do the students receive a teacher?

The best way I can think of to resolve this is to say that an object can have only one recipient, but a recipient might receive more than one object. A more formal path is to think of the act of creating combinations in terms of functions. The elements of the domain are the objects. The elements of the range are the recipients. A function can be many-to-one, but it should not be one-to-many.

Favorite Teachers At a school with k students and n teachers, the students all have a favorite teacher. (They might all like the same one.). How many ways are there for the k students to pick a favorite?

Objects: k students. *Recipients:* n teachers. Many students might have one favorite teacher. There are n^k combinations.

Assembling A Team Out of a choice of n athletes, a coach must assemble a team of k. How many ways are there to form a team?

Objects: n athletes. *Recipients:* team, not on the team. Many athletes can be assigned to one outcome of being on the team or not being on the team. There are $\binom{n}{k}$ combinations for the team, which is the same number as the $\binom{n}{n-k}$ selections for the bench.

Distinct Objects, Without Conditions

Distinct Recipients

The k objects are assigned to n recipients with no conditions as to the number of objects each recipient receives. This is the same as assigning the elements of a k-tuple from a selection of n with replacement.

$$\begin{aligned} & S = \{(i_1, i_2, ..., i_k) | i_j \in A, |A| = n\} \\ & |S| = n^k \end{aligned} \tag{6}$$

Binary Strings of Length k The k distinct positions of a binary string $(i_1, i_2, ..., i_k)$ of length k are assigned to an element of the set $A \in [0, 1]$. The number of possible binary strings of length k is 2^k.

Subsets of a k-Element Set The subsets of a set of k distinct elements are formed by assigning each of its k distinguishable elements to one of the two labels $A \in [\text{included}, \text{excluded}]$. The number of possible subsets, including the empty subset and the full set, is 2^k.

Indistinct Recipients

Distinct Objects, At Most One is Assigned

Distinct Recipients

Exacly one of k distinct objects are assigned to a single one of n recipients.

Linear Regression

Ordinary Linear Regression

R^2 **Value**

Regression Diagnostics

t-Statistics

AIC and BIC

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