E.1: Practice finding the derivatives of these functions:

E1

1) $f(x) = \sin(6x-1)$

Chain rule

$t = 6x-1$

$y = \sin(t)$

$\frac{dt}{dx} = 6$ $\quad \frac{dy}{dt} = \cos(t)$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 6 \cdot \cos(t)$

$\boxed{f'(x) = 6 \cos(6x-1)}$

2) $f(x) = x^8 + 30 + \frac{1}{x^4}$ $\quad x^{-4}$

$f'(x) = 8x^7 + -4x^{-5}$

$\boxed{f'(x) = 8x^7 - \frac{4}{x^5}}$

3) $f(x) = e^{\left(\frac{1}{x}\right) + \frac{1}{x^2}}$

$f(x) = e^{x^{-1} + x^{-2}}$

$t = x^{-1} + x^{-2}$

$t' = -x^{-2} - 2x^{-3}$

$f(x) = e^{f(x)}$

$f'(x) = e^{f(x)} \cdot f'(x)$

$\boxed{f'(x) = e^{\left(\frac{1}{x}\right) + \left(\frac{1}{x^2}\right)} \cdot \left(-\frac{1}{x^2} - \frac{2}{x^3}\right)}$

4) $f(x) = \sin^2(6x-1)$

$= (\sin(6x-1))^2$

$f(u) = u^2$ $\quad f'(u) = 6\cos(6x-1)$

$u = \sin(6x-1)$

$u \cdot f'(u) = \boxed{6\sin(6x-1)\cos(6x-1) = f'(x)}$

E.2: Finding when a function is increasing/decreasing and concave up/down. When is the function $f(x) = 2x^3+24x^2-54x$ decreasing? When is it concave up? Plot the function and find your check your answer.

E2

$f(x) = 2x^3 + 24x^2 - 54x$

$f'(x) = 6x^2 + 48x - 54$

$= 6(x^2 + 8x - 9)$

$= 6(x+9)(x-1)$

$(x+9)(x-1) = 0$

$x = -9, 1$

If $f'(x) < 0$ at -4, then decreasing

$f'(x) = 6(-4)^2 + 48(-4) - 54$

$= 96 - 192 - 54$

$= -150$

$\boxed{\text{increasing: } (-\infty, -9),\ (1, \infty) \quad \text{decreasing: } (-9, 1)}$

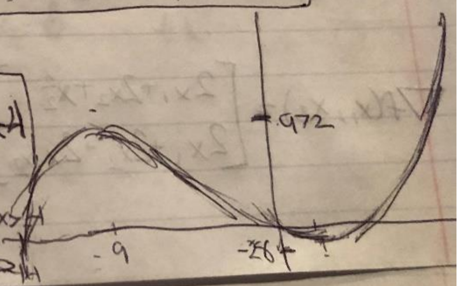
$f''(x) = 12x + 48 = 0$

$12(x+4) = 0$

$x = -4$

$f''(-5) = -60 + 48 = -12$ concave down $x < -4$

$f''(3) = -36 + 48 = 12$ concave up $x > -4$



E.3: Finding critical points, local max/min, global max/min, and inflection points. Find all critical points and inflection points of f(x) = $2x^3+24x^2-54x$. Classify the critical points as local min, local max, or neither. Find the global max and min of this function on [−3,3] and on (−∞,0). Plot the function and find your check your answer?

$f'(c)=0$

or

$f'(c) = \text{undefined}$

E3

$f(x)=2x^3+24x^2-54x$

$f'(x)=6x^2+48x-54=0$

$6(x^2+8x-9)=0$

$(x+9)(x-1)=0$

$\boxed{x=-9,\ 1}$

$f''(x)=12x+48$

~~global~~

local max $f''(-9)=12(-9)+48=-60$ concave down

-108

local min $f''(1)=12(1)+48=60$ concave up

global max $=\infty$

global min $=-\infty$

inflection point: $f''(x)=0$

$12x+48=0$

$x=-4$

E.4:

E4

$f(x,y) = x^2 + y^2$ $\quad \frac{df}{dx} = 2x + y^2 \quad \frac{df}{dy} = x^2 + 2y$

vector

i. $\nabla f(x,y) = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = \begin{bmatrix} 2x+y^2 \\ x^2+2y \end{bmatrix}$

ii. $\nabla f(1,2) = \begin{bmatrix} 2+4 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ $\quad \nabla f(2,1) = \begin{bmatrix} 4+1 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ $\quad \nabla f(0,0) = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

E.5:

E5

$f(x,y) = 2xy + x^2 + y$ $\quad \frac{df}{dx} = 2y + 2x \quad \frac{df}{dy} = 2x + 1$

$\nabla f(x,y) = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} = \begin{bmatrix} 2x+2y \\ 2x+1 \end{bmatrix}$ i $\quad \nabla f(1,1) = \begin{bmatrix} 2+2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\nabla f(0,-1) = \begin{bmatrix} 0-2 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ii

$\nabla f(0,0) = \begin{bmatrix} 0+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

iii $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2 + x_1x_2^2$ $\quad \frac{df}{dx_1} = 2x_1 + 2x_2 + x_2^2$

$\frac{df}{dx_2} = 2x_1 + 2x_2 + 2x_1x_2$

$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 + 2x_2 + x_2^2 \\ 2x_1 + 2x_2 + 2x_1x_2 \end{bmatrix}$

E.6:

EC

i. $y = mx + b$

$y = 3x - 0.5$

ii. $m = \frac{14-8}{6-4} = \frac{6}{2} = 3$

$8 = 4(3) + b$

$b = -4$

$y = 3x - 4$

iii. $m = \frac{1}{5}$

$2 = \frac{1}{5}(3) + b$

$\frac{10}{5} = \frac{3}{5} + b$

$b = \frac{7}{5}$

$y = \frac{1}{5}x + \frac{7}{5}$

iv. $1 = 2m + 3$

$-2 = 2m$

$m = 1$

$y = x + 3$

v. $m = \frac{4-(-1)}{6-1} = \frac{5}{5} = 1$

$4 = 1(6) + b$

$b = -2$

$y = x - 2$

E.7: Find the eigenvalues and eigenvectors of the given matrix by hand and check results by the computer (use Python to check your results).

E.7

1\. $\begin{bmatrix}2&0\\0&5\end{bmatrix}$

$\det\left(\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}2&0\\0&5\end{bmatrix}\right)=0$

$\det\left(\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}-\begin{bmatrix}2&0\\0&5\end{bmatrix}\right)=0$

$\det\left(\begin{bmatrix}\lambda-2&0\\0&\lambda-5\end{bmatrix}\right)=0$

$(\lambda-2)(\lambda-5)-0(0)=0$

$\boxed{\lambda=2,5}$

assuming nonzero eigenvectors

$A\vec{v}=\lambda\vec{v}$

$\vec{0}=\lambda\vec{v}-A\vec{v}$

$=\lambda I_n\vec{v}-A\vec{v}$

$=\vec{v}(\lambda I_n-A)$

for any eigenvalue λ the eigenvectors is the null space of $\lambda I_n - A$

$\lambda=5$ $\quad 5I_n$

$E_5=N\left(\begin{bmatrix}5&0\\0&5\end{bmatrix}-\begin{bmatrix}2&0\\0&5\end{bmatrix}\right)=N\left(\begin{bmatrix}3&0\\0&0\end{bmatrix}\right)$

$\begin{bmatrix}3&0\\0&0\end{bmatrix}\vec{v}=\begin{bmatrix}0\\0\end{bmatrix}$

$3v_1+0v_2=0$ $\quad v_2=t$

$0v_1+0v_2=0$ $\quad v_1$

$3v_1=0v_2$

$v_2=t \quad 3v_1=0t$

$v_1=0t$

$\boxed{t\begin{bmatrix}0\\1\end{bmatrix} \text{ for } \lambda=5}$

ii. $\lambda=2$

$A-2I=0$

$\begin{bmatrix}2&0\\0&5\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}=0$

$\begin{bmatrix}0&0\\0&3\end{bmatrix}=0$

$\begin{bmatrix}0&0\\0&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$0v_1+0v_2=0$

$0v_1+3v_2=0$

$v_2=t$

$0v_1=0t$

$v_1=0$

$0=3t$

$t=0$

$\boxed{t\begin{bmatrix}1\\0\end{bmatrix} \text{ for } \lambda=2}$

7 cont

ii. $\begin{bmatrix}5&1\\4&5\end{bmatrix}$

$\det(A-\lambda I)=0$

$\det\left(\begin{bmatrix}5&1\\4&5\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right)=0$

$\det\left(\begin{bmatrix}5-\lambda&1\\4&5-\lambda\end{bmatrix}\right)=0$

$(5-\lambda)(5-\lambda)-4=0$

$\lambda^2-10\lambda+25-4=0$

$\lambda^2-10\lambda+21=0$

$(\lambda-7)(\lambda-3)=0$

$\boxed{\lambda=7,3}$

$A\vec{v}=\lambda\vec{v}$

$(A-7I)\vec{v}=0$

$\left(\begin{bmatrix}5-7&1\\4&5-7\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix}=0$

$\begin{bmatrix}-2&1\\4&-2\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$-2v_1+v_2=0$ $\quad v_2=t$

$4v_1-2v_2=0$ $\quad -2v_1+t=0$

$v_1=\frac{1}{2}t$

$\boxed{\begin{bmatrix}1/2\\1\end{bmatrix}t \text{ for } \lambda=7}$

$(A-3I)\vec{v}=0$

$\left(\begin{bmatrix}5-3&1\\4&5-3\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix}=0$

$2v_1+v_2=0$ $\quad v_2=t$

$4v_1+2v_2=0$ $\quad 2v_1=-t$

$v_1=-\frac{1}{2}t$

for $\lambda=3$

$\boxed{\begin{bmatrix}-1/2\\1\end{bmatrix}t}$

iii. $\begin{bmatrix}3&5\\3&1\end{bmatrix}$

$\det\left(\begin{bmatrix}3&5\\3&1\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right)=0$

$\det\left(\begin{bmatrix}3-\lambda&5\\3&1-\lambda\end{bmatrix}\right)=0$

$(3-\lambda)(1-\lambda)-15=0$

$\lambda^2-4\lambda-12=0$

$(\lambda-6)(\lambda+2)=0$

$\boxed{\lambda=6,-2}$

$(A-6\lambda)\vec{v}=0$

$\begin{bmatrix}-3&5\\3&-5\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=0$

$-3v_1+5v_2=0$

$3v_1-5v_2=0$

$v_2=t \quad -3v_1=-5t$

$v_1=\frac{5}{3}t$

$\boxed{\begin{bmatrix}\frac{5}{3}\\1\end{bmatrix}t \text{ for } \lambda=6}$

$A+2\lambda=0$

$\begin{bmatrix}5&5\\3&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=0$

$5v_1+5v_2=0$

$3v_1+3v_2=0$

$v_2=t \quad 5v_1=-5t$

$v_1=-1t$

$\boxed{\begin{bmatrix}-1\\1\end{bmatrix}t \text{ for } \lambda=-2}$

E.8:

i. $f(0) = 0$

Conditions

1) continuous ✓
2) ✓
3) ✓
4) passes through origin ✓
5) ✓
6) ✓
7) ✓
8) ✓
9) ✓
10) ✓

E.9: Which of the following sets of vectors are independent? Find the dimension of the vector space spanned by each set. (Verify your answers using Python).

E9

$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

i. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$ $\quad \begin{bmatrix}a_1 + a_2 + a_3\\ 2a_1 + 0a_2 + 2a_3\\ 3a_1 + a_2 + a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$a_1 = -a_3$
$a_2 = -2a_1$
$1 = -1$

$2a_1 + 0a_2 + 2a_3 = 0$
$2a_1 = -2a_3$
$3a_1 + a_2 - a_1 = 0$
$2a_1 = -a_2$
$a_2 = -2a_1$

independent

$a_1 = 1$
$a_2 = -2$
$a_3 = -1$

$\cancel{a_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + a_2 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_3 \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}}$

$\begin{bmatrix}1 & 1 & 1\\ 2 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

check
for inverse
if inverse, then independent
if deter $= 0$, then dependent

$= 1\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix} - 1\begin{bmatrix}2 & 2\\ 3 & 1\end{bmatrix} + 1\begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix}$

$= 1(1-2) - 1(2-6) + 1(2+0)$

$= -1 + 4 + 2 = 5$

ii. $\sin(t)$ $\cos(t)$ $\cos(2t)$

$a_1 \sin(t) + a_2 \cos(t) + a_3 \cos(2t) = 0$

dependent

iii. $1+t$, $1-t$

dependent

vi. $\begin{array}{ccc} x_1 & x_2 & x_3 \\ \begin{bmatrix}1\\2\\2\\1\end{bmatrix}, & \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, & \begin{bmatrix}3\\4\\4\\3\end{bmatrix} \end{array}$

Grammian

$9+16+16+9$
$32+18=50$

$G = \begin{bmatrix}(x_1, x_1) & (x_1, x_2) & (x_1, x_3)\\ (x_2, x_1) & (x_2, x_2) & (x_2, x_3)\\ (x_3, x_1) & (x_3, x_2) & (x_3, x_3)\end{bmatrix}$

$G = \begin{bmatrix}10 & 2 & 22\\ 2 & 2 & 6\\ 22 & 6 & 50\end{bmatrix} = 10\begin{bmatrix}2 & 6\\ 6 & 50\end{bmatrix} - 2\begin{bmatrix}2 & 6\\ 22 & 50\end{bmatrix} + 22\begin{bmatrix}2 & 2\\ 22 & 6\end{bmatrix}$

$= 10(64) - 2(-32) + 22(-32)$

$= 640 + 64 - 704 \neq 0$

independent

E.10: Using the following basis vectors, find an orthogonal set using Gram-Schmidt orthogonalization.

E10

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad y_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = y_2 - \frac{v_1^T y_2}{v_1^T v_1} v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[1\ 0\ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{[1\ 0\ 0]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = y_3 - \frac{v_1^T y_3}{v_1^T v_1} v_1 - \frac{v_2^T y_3}{v_2^T v_2} v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[1\ 0\ 0]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[1\ 0\ 0]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{[0\ 1\ 0]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{[0\ 1\ 0]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

E.11: Expand x = [1 2 2]T in terms of the following basis set.

E11

$x = [1\ 2\ 2]^T$ in terms of basis set $v_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ $v_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix}$ $v_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

Reciprocal Base Vectors

$$B = \begin{bmatrix}-1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 2 & 0\end{bmatrix} \quad \cancel{B^{-1}B = I}$$

$$\cancel{B^T = \begin{bmatrix}-1 & 1 & 0\\ 1 & 1 & 2\\ 1 & 1 & 0\end{bmatrix}}$$

Cofactor:

$$\begin{bmatrix}-1 & -1 & 0\\ 0 & 0 & -2\\ 2 & -2 & -2\end{bmatrix} \times \begin{bmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{bmatrix} = \begin{bmatrix}-1 & 1 & 0\\ 0 & 0 & 2\\ 2 & 2 & -2\end{bmatrix} \quad Adj(B)$$

$$\left[\begin{array}{ccc|ccc}-1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0 & 1 & 0\\ 0 & 2 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow{a_1+a_2} \left[\begin{array}{ccc|ccc}-1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 2 & 1 & 1 & 0\\ 0 & 2 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow{a_2-a_3} \left[\begin{array}{ccc|ccc}-1 & 0 & 2 & 3/2 & 1/2 & 0\\ 0 & 2 & 2 & 1 & 1 & 0\\ 0 & 0 & -2 & -1 & -1 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}-2 & 0 & 0 & 1 & -1/2 & 1\\ 0 & 2 & 0 & 0 & 0 & -1\\ 0 & 0 & -2 & -1 & -1 & 1\end{array}\right] \quad \begin{bmatrix}-1/2 & 1/2 & 0\\ 0 & 0 & -1/2\\ 1/2 & 1/2 & 1/2\end{bmatrix}$$

$$B^{-1} = \begin{bmatrix}-1/2 & 1/2 & 0\\ 0 & 0 & -1/2\\ 1/2 & 1/2 & 1/2\end{bmatrix} \quad r_1 = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix} \quad r_2 = \begin{bmatrix}0\\0\\-1/2\end{bmatrix} \quad r_3 = \begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}$$

$x_1^v = r_1^T x$

$x_1^v = [-1/2\ \ 1/2\ \ 0]\begin{bmatrix}1\\2\\2\end{bmatrix}$

$x_1^v = 1/2$

$x_2^v = r_2^T x$

$x_2^v = [0\ \ 0\ \ -1/2]\begin{bmatrix}1\\2\\2\end{bmatrix}$

$x_2^v = -2/2 = -1$

$x_3^v = r_3^T x$

$x_3^v = [1/2\ \ 1/2\ \ 1/2]\begin{bmatrix}1\\2\\2\end{bmatrix}$

$x_3^v = \frac{5}{2}$

Expansion: $x = x_1^v v_1 + x_2^v v_2 + x_3^v v_3 = \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\-2\end{bmatrix} + \frac{5}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$

$$x^v = B^{-1}x = \begin{bmatrix}-1/2 & 1/2 & 0\\ 0 & 0 & -1/2\\ 1/2 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}1/2\\-1\\5/2\end{bmatrix}$$