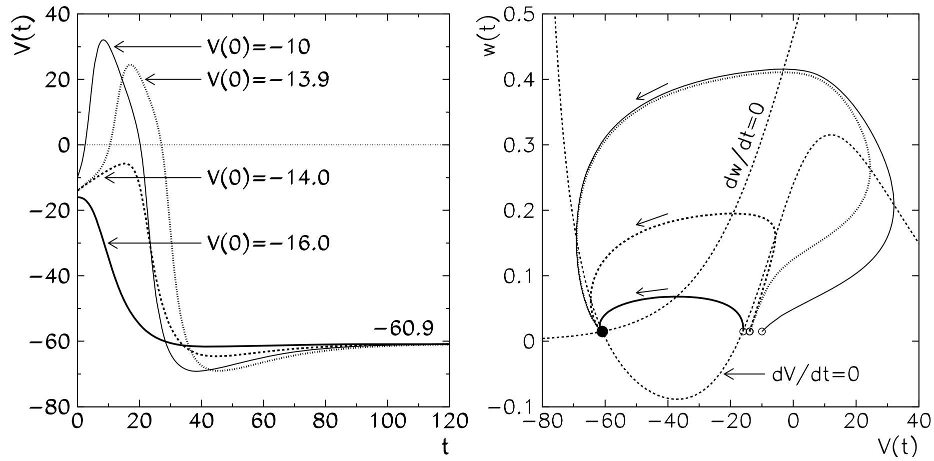

Fig. 7.19 Solutions of the Morris–Lecar model. [*Left*] The membrane potential V as a function of time for different initial conditions. [*Right*] Phase diagram ($w(t)$ versus $V(t)$) for the same initial conditions (*symbols* $\circ$) as in the left figure. Also shown are the stationary-state curves connecting the points with $\mathrm{d}V/\mathrm{d}t = 0$ and $\mathrm{d}w/\mathrm{d}t = 0$. They intersect at $(V, w) \approx (-60.9, 0.015)$ corresponding to the stationary state (*symbol* $\bullet$)

Table 7.1 Initial conditions for V and w, the current density I, the integration interval length T, and the list of tasks for solving the Morris–Lecar model. The curves of the stationary state (s.s.) and the shape of the pulse $s(t)$ are defined in the text

$V(0)$	$w(0)$	I	T	Observe
−16.0	0.014915	0	150	$V(t)$, $w(t)$, $V(w)$, s.s. curves
−14.0	0.014915	0	150	
−13.9	0.014915	0	150	
−10.0	0.014915	0	150	
−10.0	0.014915	0(1)300	800	$V_{\max}(I)$, $V_{\min}(I)$
−26.59	0.129	90	800	$V(t)$ with double pulse $s(t)$

where $H(t)$ is the Heaviside (step) function. In an actual experiment this truly represents a 5 ms long step pulse with the peak current density 30 μA/cm^2 at time 100 ms and another such pulse at time 470 ms.

7.14.6 Restricted Three-Body Problem (Arenstorf Orbits)

A basic problem in astronomy is to find the trajectory of a light object (for example, a satellite) in the presence of two much heavier bodies whose motion is not influenced by the light object. The heavy bodies with a mass ratio $\mu : (1-\mu)$ circle in the (x, y) plane with frequency 1 around their common center of gravity, which is at the origin: the ratio of their orbital radii is then $(1-\mu) : \mu$. In general the light object may move outside the plane defined by the orbits of heavy bodies. The dimensionless equations of motion for the light body are

$$\begin{aligned}
\ddot{x} &= x + 2\dot{y} - \frac{(1-\mu)(x+\mu)}{r^3} - \frac{\mu(x-1+\mu)}{s^3}, \\
\ddot{y} &= y - 2\dot{x} - \frac{(1-\mu)y}{r^3} - \frac{\mu y}{s^3}, \\
\ddot{z} &= -\frac{(1-\mu)z}{r^3} - \frac{\mu z}{s^3},
\end{aligned}$$

where

$$r = \sqrt{(x+\mu)^2 + y^2 + z^2}, \qquad s = \sqrt{(x-1+\mu)^2 + y^2 + z^2}.$$

A detailed analysis of planar solutions of the special case $\mu = 0.5$ (frequently used as a benchmark test for symplectic methods) can be found in [50].

$\odot$ Solve the restricted three-body problem corresponding to the Earth and Moon as the heavy objects, hence $\mu = M_{\mathrm{Moon}}/M_{\mathrm{Earth}} = 0.012277471$. Consider the planar case ($z = 0$) with the initial conditions (example parameters from [28])

$$\begin{aligned}
\big(x(0), y(0)\big) &= (0.994, 0), \\
\big(\dot{x}(0), \dot{y}(0)\big) &= (0, -2.00158510637908252240537862224),
\end{aligned}$$

for which the solution is periodic with the period

$$T = 17.0652165601579625588917206249.$$

Use the Euler explicit method with step $h = T/24000$ and the RK4 method with $h = T/6000$. Plot the dependence of x, $\dot{x}$, y, and $\dot{y}$ on time, as well as the phase diagrams $(x, \dot{x})$ and $(y, \dot{y})$, as in Fig. 7.20 (left). Can you obtain a periodic solution by reducing the step size in either the Euler or RK4 method? (Pretend that the solution is periodic if the deviation of $x(10T)$, $\dot{x}(10T)$, $y(10T)$, and $\dot{y}(10T)$ from the initial conditions after ten returns does not exceed 1 %.)

$\oplus$ Use the Dormand–Prince method 5(4) with adaptive step size. Select a few tolerances on the local error and determine the number of steps needed for one cycle in the phase diagram at this precision (Fig. 7.20 (right)). Enrich the problem by allowing non-planar orbits of the third body ($z \neq 0$).

The problem outlined here is an example of motion of satellites along the "horseshoe" orbits in the gravitational field of the Sun–Earth system. Recently two nearby Earth asteroids have been discovered, 3753 Cruithne [51] and 2002 AA29 [52], whose orbits have similar properties. Even before that, the restricted three-body problem was analyzed theoretically for the Sun–Jupiter system [53]. You can find a rich set of initial conditions for it in [54] or [55].

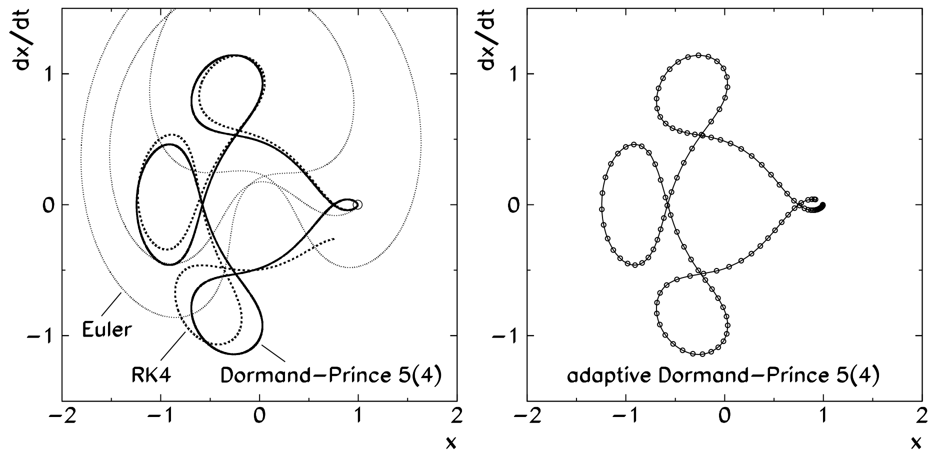

Fig. 7.20 Orbits of the light object in the restricted three-body problem. [*Left*] Solution by the explicit Euler method (7.5) with step $h = T/24000$ and by the RK4 method (7.10) and Dormand–Prince 5(4) method (p. 345) with steps $h = T/6000$. [*Right*] Solution by the Dormand–Prince 5(4) method with adaptive step size. At 0.001 tolerance on the local error a mere 134 steps are needed

7.14.7 Lorenz System

A classical problem of atmospheric physics is the convection of a fluid with kinematic viscosity ν, volume expansion coefficient β, and heat diffusion coefficient D, in a layer of thickness H in which a constant temperature difference $\Delta T_0 = T(0) - T(H) > 0$ persists between the top and bottom. Simpler cases were treated by Lord Rayleigh already in 1916, but in certain regimes of Prandtl ($\sigma = \nu/\beta$) and Rayleigh ($R = g\beta H^3 \Delta T_0 / D\nu$) numbers the convection dynamics (velocity of motion, heat transfer) becomes very complex [56].

The Lorenz system [38]

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y, \\ \dot{Y} &= -XZ + rX - Y, \\ \dot{Z} &= XY - bZ, \end{aligned} \tag{7.62}$$

offers a simplified picture of such convection. The variable X represents the velocity of the convection current. The variable Y is proportional to the temperature difference between the ascending and descending convection currents (warmer fluid goes up, colder goes down), while Z is the deviation of the vertical temperature profile from the linear height dependence (a positive value implies strong gradients in the vicinity of the top or bottom edge). The derivatives in (7.62) are with respect to dimensionless time $\tau = \pi^2(1 + a^2)\kappa t/H^2$, where a is a parameter. The coefficient $r = R/R_c$ is the Rayleigh number in units of the critical value $R_c = \pi^4(1 + a^2)^3/a^2$ (with a minimum at $27\pi^4/4$), while $b = 4/(1 + a^2)$.

⊙ Solve the Lorenz system with parameters $\sigma = 10$, $b = 8/3$, $r = 28$, and initial conditions $X(0) = 0$, $Y(0) = 1$, $Z(0) = 0$, which are slightly "off" the