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# The Newtonian potential and the demagnetizing factors of the general ellipsoid

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The objective of this paper is to present a modern and concise new derivation for the explicit expression of the interior and exterior Newtonian potential generated by homogeneous ellipsoidal domains in $\mathbb{R}^N$ (with $N \geq 3$). The very short argument is essentially based on the application of Reynold's transport theorem in connection with the Green–Stokes integral representation formula for smooth functions on bounded domains of $\mathbb{R}^N$, which permits to reduce the $N$-dimensional problem to a one-dimensional one. Owing to its physical relevance, a separate section is devoted to the the derivation of the *demagnetizing factors* of the general ellipsoid which are one of the most fundamental quantities in ferromagnetism.

## 1. Historical introduction, motivations

The computation of the gravitational potential induced by a homogeneous ellipsoid was one of the most important problems in mathematics for more than two centuries after Newton enunciated his universal law of gravitation [1–8]. Once an ellipsoidal domain $\Omega$ of $\mathbb{R}^3$ is fixed, the problem consists in finding an *explicit* expression for the Newtonian potential induced by a constant mass/charge density on $\Omega$ [9]

$$\mathcal{N}_\Omega[1_\Omega](x) := \frac{1}{4\pi} \int_\Omega \frac{1}{|x-y|}\,\mathrm{d}y, \tag{1.1}$$

in the internal points of $\Omega$ (interior problem) and in the exterior points of $\Omega$ (exterior problem).

In the case of a homogeneous *spherically symmetric* region, Newton (in 1687) proved what nowadays is known as Newton's shell theorem [6,10]: *if $\Omega$ is a homogeneous spherical region centred at the origin, then for all $t > 1$, $\mathcal{N}_{t\Omega\backslash\Omega}[1_{t\Omega\backslash\Omega}]$ is constant in $\Omega$, i.e. $t\Omega\backslash\Omega$* (the so-called *hollow ball*) *induces no gravitational force inside $\Omega$.*



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*Moreover, a spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its centre.*

For what concerns an ellipsoidal domain $\Omega$ of $\mathbb{R}^3$, the *ellipsoidal interior* problem was for the first time solved by Gauss (in 1813) by means of what is nowadays known as the Gauss divergence theorem [11]. Later, in 1839, Dirichlet proposed a solution of the interior problem based on the theory of Fourier integrals [12].

The results of Gauss and Dirichlet can be summarized by saying that if $\Omega$ is an ellipsoidal domain centred at the origin, then the gravitational potential induced by a homogeneous ellipsoid in its internal points is a second-order polynomial. In other terms

$$\mathcal{N}_\Omega[1_\Omega](x) = c - Px \cdot x, \quad \forall x \in \Omega, \tag{1.2}$$

for some constant $c \in \mathbb{R}$ and some matrix $P \in \mathbb{R}^{3\times 3}$ whose values can be expressed in terms of elliptic integrals [2,5,13,14].

**Remark 1.1.** For the sake of completeness we recall that the converse statement (the *inverse homogeneous ellipsoid problem*) is also true [9,15–17], namely *if $\Omega$ is a bounded domain of $\mathbb{R}^N$ such that $\mathbb{R}^N \backslash \Omega$ is connected and* (1.2) *holds, then $\Omega$ is an ellipsoid*. Historically speaking, the inverse homogeneous ellipsoid problem was for the first time solved by Dive [18] in 1931 for $N = 3$ and in 1932 by Hölder [19] for $N = 2$. A modern proof of this result can be found in DiBenedetto & Friedman [15], who, in 1985, extended it to all $N \geq 2$. In 1994, Karp [17], by the means of certain topological methods, obtained an alternative proof of the inverse homogeneous ellipsoid problem.

Despite the well-knowness of (1.2) in the mathematical and physical community, and its importance in theoretical and applied studies [20–29], rigorous proofs of that result are not readily available in the literature: to the best knowledge of the author, relative modern treatments of the interior problem can be found in [13,14], and more recently in [30,31] where also the the exterior problem is investigated. However, in all the cited references, the solution of the problem is always based on the use of ellipsoidal coordinates which tends to focus the attention on the computational details of the question rather than on its geometric counterpart. Modern proofs of Newton's theorem and relation (1.2), as well as far reaching beautiful generalizations, can be found in [32], where nevertheless the problem of finding an analytic expression for the coefficients $c$ and $P$ is not touched.

The aim of this paper is to give a modern and concise derivation for the expression of the *interior* and *exterior* Newtonian potential (induced by homogeneous ellipsoids). The very short argument is essentially based on the application of Reynold's transport theorem in connection with the Green–Stokes integral representation formula for smooth functions on bounded domains of $\mathbb{R}^N$. This approach permits to reduce the $N$-dimensional problem to a one-dimensional one, providing (in particular) at once a proof of (1.2) together with an explicit expression of the coefficients $P$ and $c$ in terms of one-dimensional integrals. More precisely, the paper is organized as follows.

Section 2 is devoted to the main result of the paper. We give a concise proof of the homogeneous ellipsoid problem. For completeness, we then derive Newton's shell theorem as a corollary.

In §3, we focus attention to the three-dimensional case. An expression in terms of the elliptic integrals is given for the coefficients of $P$. Owing to its physical relevance, particular attention is paid to the eigenvalues of $P$. Indeed, when $N = 3$, the matrix $P$ and its eigenvalues, known in the theory of ferromagnetism, respectively, as *the demagnetization tensor* and *the demagnetizing factors*, are one of the most important and well-studied quantities of ferromagnetism [20,21,23,28,29,33]. In fact, the following magnetostatic counterpart of the homogeneous ellipsoid problem holds: *given a uniformly magnetized ellipsoid, the induced magnetic field is also uniform inside the ellipsoid*. This result was for the first time showed by Poisson [34], while an explicit expression for the demagnetizing factors was for the first time obtained by Maxwell [35]. Their importance is in that they encapsulate the self-interaction of magnetized bodies: their knowledge being equivalent to the one of the corresponding demagnetizing (stray) fields [21]. Moreover, a similar result also occurs in linearized elasticity, where the Eshelby solution asserts that a uniform eigenstress on an ellipsoidal inclusion in an infinite elastic medium induces uniform strain inside the ellipsoid [36,37].

## 2. The interior and exterior potential of a homogeneous ellipsoid

In what follows, we denote by $\Omega$ the ellipsoidal domain of $\mathbb{R}^N$ ($N \geq 3$) having $a_1, a_2, \ldots, a_N$ as semi-axes lengths. We then denote by $(\Omega_t)_{t\in[0,+\infty)}$ the family of ellipsoidal domains of $\mathbb{R}^N$, given by the inverse image $\phi_t^{-1}(B_N)$ of the unit ball of $\mathbb{R}^N$ under the one parameter family of diffeomorphisms $\phi_t : x \in \mathbb{R}^N \mapsto \sqrt{A_t}x \in \mathbb{R}^N$, where

$$\sqrt{A_t} := \operatorname{diag}\left[\frac{1}{\sqrt{a_1^2+t}}, \frac{1}{\sqrt{a_2^2+t}}, \ldots, \frac{1}{\sqrt{a_N^2+t}}\right]. \tag{2.1}$$

Note that each diffeomorphism $\phi_t^{-1}$ maps the unit ball of $\mathbb{R}^N$ onto the ellipsoidal domain of $\mathbb{R}^N$ defined by the position $\Omega_t = \{x \in \mathbb{R}^N : |\phi_t(x)|^2 \leq 1\}$. In particular, $\partial\Omega_t = \{x \in \mathbb{R}^N : |\phi_t(x)|^2 = 1\}$ and $\Omega \equiv \Omega_0$. Finally, we denote by $\mathcal{N}_{\Omega_t}[1_{\Omega_t}]$ the Newtonian potential generated by the uniform space density of masses or charges on $\Omega_t$:

$$\mathcal{N}_{\Omega_t}[1_{\Omega_t}](x) = c_N \int_{\Omega_t} \frac{1}{|x-y|^{N-2}}\, \mathrm{d}y, \tag{2.2}$$

with $c_N := [(N-2)\omega_N]^{-1}$ and $\omega_N$ the surface measure of the unit sphere in $\mathbb{R}^N$ (cf. [9,38]).

The main result of the paper is stated in the following.

**Theorem 2.1.** *Let $\Omega = \{x \in \mathbb{R}^N : |\phi_0(x)|^2 \leq 1\}$ be the ellipsoidal domain of $\mathbb{R}^N$ having $(a_1, a_2, \ldots, a_N) \in \mathbb{R}_+^N$ as semi-axes lengths. For every $x \in \mathbb{R}^N$*

$$\mathcal{N}_\Omega[1_\Omega](x) = \frac{1}{4}\int_{\tau(x)}^{+\infty} \gamma_t(1 - A_t x \cdot x)\, \mathrm{d}t, \quad \gamma_t := \prod_{i=1}^{N} \frac{a_i}{\sqrt{a_i^2+t}}, \tag{2.3}$$

*where we have denoted by $\tau$ the non-negative real-valued function*

$$\tau : x \in \mathbb{R}^N \mapsto \begin{cases} 0 & \text{if } x \in \bar{\Omega} \\ \lambda & \text{if } x \in \Omega^c \cap \partial\Omega_\lambda. \end{cases} \tag{2.4}$$

**Remark 2.2.** The function $\tau$ defined by (2.4) can be computed by solving, for every $x \in \Omega^c$, the equation in the $\lambda$ variable given by $|\phi_\lambda(x)|^2 = 1$. Indeed, for every $x \in \Omega^c$ there exists a unique $\lambda \in \mathbb{R}^+$ such that $|\phi_\lambda(x)|^2 = 1$ (cf. figure 1). In particular, when $\Omega$ is a spherical region of radius $a$ (centred around the origin), the function $\tau$ reduces to the function $\tau(x) := (|x|^2 - a^2)1_{\Omega^c}(x)$ and the integral in (2.3) can be readily computed. For general ellipsoidal domains the integral in (2.3) can be evaluated by the means of the theory of elliptic integrals (cf. [39,40]).

*Proof.* For every $t \in \mathbb{R}^+$, the function $|\phi_t|^2 - 1$ is the unique solution of the homogeneous Dirichlet problem for the Poisson equation $\Delta u = 2\,\mathrm{tr}(A_t)$ in $\Omega_t$, $u \equiv 0$ on $\partial\Omega_t$. Thus, according to the Green–Stokes representation formula [9], for every $t \in \mathbb{R}^+$ we have

$$(|\phi_t|^2 - 1)1_{\Omega_t} = -2\,\mathrm{tr}(A_t)\mathcal{N}_{\Omega_t}[1_{\Omega_t}] + \mathcal{S}_{\partial\Omega_t}[\partial_{\boldsymbol{n}}|\phi_t|^2] \quad \text{in } \mathbb{R}^N \backslash \partial\Omega_t, \tag{2.5}$$

where we have denoted by

$$\mathcal{S}_{\partial\Omega_t}[\partial_{\boldsymbol{n}}|\phi_t|^2](x) := c_N \int_{\partial\Omega_t} \frac{\partial_{\boldsymbol{n}(y)}|\phi_t(y)|^2}{|x-y|^{N-2}}\, \mathrm{d}\sigma(y) \tag{2.6}$$

the simple layer potential generated by the space density of masses or charges $\partial_{\boldsymbol{n}}|\phi_t|^2$ concentrated on $\partial\Omega_t$ (cf. [9,38]). Next, we observe that the $N$-dimensional Newtonian kernel

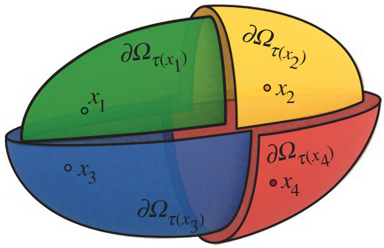


**Figure 1.** The family of confocal ellipsoidal surfaces $(\partial\Omega_t)_{t\in[0,+\infty)}$ induces a partition of the exterior domain $\Omega^c$. Indeed, for every $x \in \Omega^c$ there exists a unique positive real number $\tau(x)$ such that $x \in \partial\Omega_{\tau(x)}$. Here we have $0 < \tau(x_i) < \tau(x_{i+1})$, for every $i \in \mathbb{N}_3$, and $x_i \in \partial\Omega_{\tau(x_i)}$ for every $i \in \mathbb{N}_4$. (Online version in colour.)

$|x|^{2-N}$ is in $W^{1,1}_{\rm loc}(\mathbb{R}^N)$; therefore, due to Reynold's transport theorem,

$$\partial_t\mathcal{N}_{\Omega_t}[1_{\Omega_t}] = c_N \int_{\partial\Omega_t} \frac{v_t(y)\cdot \boldsymbol{n}(y)}{|x-y|^{N-2}}\,\mathrm{d}\sigma(y) = \frac{1}{4}\mathcal{S}_{\partial\Omega_t}[\partial_{\boldsymbol{n}}|\phi_t|^2], \tag{2.7}$$

where we have denoted by $v_t := \partial_t[\phi_t^{-1}]\circ\phi_t$ the eulerian velocity field associated with the motion $\phi_t^{-1}$, for which one has

$$v_t(y) := \partial_t[\phi_t^{-1}]\circ\phi_t(y) = \tfrac{1}{2}A_t y = \tfrac{1}{4}\nabla|\phi_t(y)|^2, \quad \forall (t,y)\in\mathbb{R}^+\times\partial\Omega_t. \tag{2.8}$$

Hence, substituting (2.7) into (2.5) we get

$$\tfrac{1}{4}(|\phi_t(x)|^2-1)1_{\Omega_t}(x) = -\tfrac{1}{2}\,\mathrm{tr}(A_t)\mathcal{N}_{\Omega_t}[1_{\Omega_t}](x) + \partial_t\mathcal{N}_{\Omega_t}[1_{\Omega_t}](x), \quad \forall x\in\mathbb{R}^N\backslash\partial\Omega_t. \tag{2.9}$$

Moreover, by the assignment

$$\gamma_t := \exp\left(-\frac{1}{2}\int_0^t \mathrm{tr}(A_s)\,\mathrm{d}s\right) = \prod_{i=1}^N \frac{a_i}{\sqrt{a_i^2+t}}, \quad \gamma_0 = 1, \tag{2.10}$$

the equality (2.9) reads as

$$\partial_t(\gamma_t\mathcal{N}_{\Omega_t}[1_{\Omega_t}]) = \tfrac{1}{4}\gamma_t(|\phi_t|^2-1)1_{\Omega_t}, \quad \forall x\in\mathbb{R}^N\backslash\partial\Omega_t. \tag{2.11}$$

Thus, once introduced the non-negative real function defined by (2.4), we have for every $t\in\mathbb{R}^+$ and for every $x\in\mathbb{R}^N$

$$\partial_t(\gamma_t\mathcal{N}_{\Omega_t}[1_{\Omega_t}])(x) = \tfrac{1}{4}\gamma_t(|\phi_t(x)|^2-1)1_{[\tau(x),+\infty)}(t). \tag{2.12}$$

Integrating both members of (2.12) on $[0,+\infty)$; taking into account that due to the well-known decay at infinity of the Newtonian potential [41] one has $\lim_{t\to+\infty}\gamma_t\mathcal{N}_{\Omega_t}[1_{\Omega_t}] = 0$; we finish with (2.3). ∎

**Corollary 2.3 (Newton's shell theorem).** *Let $\Omega\subseteq\mathbb{R}^3$ be a homogeneous spherical region (centred around the origin) of radius a and of total mass M. For every $x\in\mathbb{R}^3\backslash\Omega$ the induced gravitational potential is the same as though all of its mass were concentrated at a point at its centre. Moreover, for all $t>1$, $\mathcal{N}_{t\Omega\backslash\Omega}[1_{t\Omega\backslash\Omega}]$ is constant in $\Omega$, i.e. the hollow ball induces no gravitational force inside $\Omega$.*

*Proof.* We denote by $\rho 1_\Omega(x)$, $(\rho := M/|\Omega|)$ the uniform density of mass in $\Omega$. The gravitational potential induced by $\rho$ in $\mathbb{R}^3$ is then given by $u_\rho = 4\pi G\mathcal{N}_\Omega[\rho] = 4\pi G\mathcal{N}_\Omega[1_\Omega]$, where we have denoted by $G$ the *gravitational constant*. In this geometrical setting the function $\tau$ defined by (2.4) reduces to the function $\tau(x) := (|x|^2 - a^2)1_{\mathbb{R}^N\backslash\Omega}(x)$ and the integral in (2.3) immediately gives

$$u_\rho(x) = \frac{2}{3}\pi G\rho\left(\frac{a^2}{2} - |x|^2\right) \quad \text{if } x \in \bar{\Omega} \quad \text{and} \quad u_\rho(x) = \frac{4}{3}\pi a^3 \frac{G\rho}{|x|} = \frac{GM}{|x|} \quad \text{if } x \in \mathbb{R}^3\backslash\Omega. \tag{2.13}$$

Thus, for every $x \in \mathbb{R}^3\backslash\Omega$ the induced gravitational potential is equal to the one induced by a Dirac mass concentrated in the centre of $\Omega$. The gravitational field is given by $\boldsymbol{g} := -\nabla u_\rho$ and the fact that the hollow ball $t\Omega\backslash\Omega$, $t > 1$, induces no gravitational force inside $\Omega$ can be immediately seen by splitting the uniform density of mass in $t\Omega\backslash\Omega$ in the form $\rho = (M/|t\Omega\backslash\Omega|)1_{t\Omega} - (M/|t\Omega\backslash\Omega|)1_\Omega$. Indeed by linearity we get that $u_\rho$ is constant in $\Omega$ and therefore $\boldsymbol{g} = \boldsymbol{0}$. ■

# 3. The demagnetizing factors of the general ellipsoid

We know focus on the three-dimensional framework ($N = 3$) and, in particular, on the so-called demagnetizing factors of the general ellipsoid [29]. To this end, we recall that the demagnetizing (stray) field $\boldsymbol{h}[\boldsymbol{m}]$ associated with a magnetization $\boldsymbol{m} \in C^\infty(\bar{\Omega})$ can be expressed as the gradient field of a suitable magnetostatic potential $\varphi_{\boldsymbol{m}}$ [42,43]. Precisely, $\varphi_{\boldsymbol{m}} := -\mathrm{div}\mathcal{N}_\Omega[\boldsymbol{m}]$ and $\boldsymbol{h}[\boldsymbol{m}] := -\nabla\varphi_{\boldsymbol{m}}$ in $\mathbb{R}^3$. In particular, if $\boldsymbol{m}$ is constant in $\Omega$, then $\varphi_{\boldsymbol{m}} = -\boldsymbol{m}\cdot\nabla\mathcal{N}_\Omega[1_\Omega]$. Thus, from (2.3)

$$\varphi_{\boldsymbol{m}}(x) = Px\cdot\boldsymbol{m} \quad \text{and} \quad \boldsymbol{h}[\boldsymbol{m}] = -P\boldsymbol{m}, \ \forall x \in \Omega, \tag{3.1}$$

where we denote by $P := \nabla\mathcal{N}_\Omega[1_\Omega]$ the diagonal matrix, known in the literature as the *demagnetizing tensor*, whose diagonal *i*-entry (the *i*th *demagnetizing factor*), by virtue of (2.3), is given by

$$P_i := \frac{1}{2}\int_0^{+\infty} \frac{1}{(a_i^2 + t)} \prod_{j=1}^{3} \frac{a_j}{\sqrt{a_j^2 + t}}\,\mathrm{d}t, \quad \forall i \in \mathbb{N}_3. \tag{3.2}$$

**Proposition 3.1.** *We have $P_i \geq 0$ for every $i \in \mathbb{N}_3$ and if $a_1 \geq a_2 \geq a_3$ then $P_1 \leq P_2 \leq P_3$. The trace of $P$ satisfies the relation* $\mathrm{tr}(P) = 1$.

*Proof.* The first statement is obvious. The relation $\mathrm{tr}(P) = 1$ can of course be verified by a direct evaluation of the integrals in (3.2), but it is also possible to observe that as the Newtonian potential $\mathcal{N}_\Omega[1_\Omega]$ satisfies the Poisson equation $\Delta\mathcal{N}_\Omega[1_\Omega] = -1_\Omega$, one has $\mathrm{tr}(P) = \mathrm{div}(Px) = -\Delta\mathcal{N}_\Omega[1_\Omega] = 1_\Omega$. ■

Assuming $a_1 \geq a_2 \geq a_3$, from (3.2) and the theory of elliptic integrals, we get

$$P_1 = 1 - P_2 - P_3, \tag{3.3}$$

$$P_2 = -\frac{a_3}{a_2^2 - a_3^2}\left[a_3 - \frac{a_1a_2}{(a_1^2 - a_2^2)^{1/2}}E\left(\arccos\left(\frac{a_2}{a_1}\right)\middle|\frac{a_1^2 - a_3^2}{a_1^2 - a_2^2}\right)\right] \tag{3.4}$$

and

$$P_3 = +\frac{a_2}{a_2^2 - a_3^2}\left[a_2 - \frac{a_1a_3}{(a_2^2 - a_3^2)^{1/2}}E\left(\arccos\left(\frac{a_3}{a_1}\right)\middle|\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}\right)\right], \tag{3.5}$$

where, for every $y \in \mathbb{R}$ and every $0 < p < 1$, we have denoted by

$$E(y|p) := \int_0^y (1 - p\sin^2\theta)^{1/2}\,\mathrm{d}\theta, \tag{3.6}$$

the incomplete elliptic integral of the second kind expressed in *parameter form* [39,40]. In particular, in the case of a *prolate spheroid* ($a_1 \geq a_2 = a_3$), we get

$$P_1 = -\frac{a_3^2}{(a_1^2 - a_3^2)^{3/2}}\left[(a_1^2 - a_3^2)^{1/2} + a_1\,\mathrm{arccoth}\left(\frac{a_1}{(a_1^2 - a_3^2)^{1/2}}\right)\right] \tag{3.7}$$

and

$$P_2 = P_3 = \frac{1 - P_1}{2}, \tag{3.8}$$

while in the case of an *oblate spheroid* ($a_1 = a_2 \geq a_3$)

$$P_1 = P_2 = \frac{1 - P_3}{2} \tag{3.9}$$

and

$$P_3 = \frac{a_1^2}{(a_1^2 - a_3^2)^{3/2}} \left[ (a_1^2 - a_3^2)^{1/2} + a_3 \arctan\left( \frac{a_3}{(a_1^2 - a_3^2)^{1/2}} \right) - a_3 \frac{\pi}{2} \right]. \tag{3.10}$$

Finally, in the case of a *sphere* ($a_1 = a_2 = a_3$), one finish with $P_1 = P_2 = P_3 = \frac{1}{3}$.

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## References

1. Beltrami E. 1880 *Sulla teoria dell'attrazione degli ellissoidi: memoria del Prof. Eugenio Beltrami*. Bologna, Italy: Tip. Gamberini e Parmeggiani.
2. Chandrasekhar S. 1969 *Ellipsoidal figures of equilibrium*. New Haven, CT: Yale University Press.
3. Daniele E. 1911 Sul problema dell'induzione magnetica di un ellissoide a tre assi. *Il Nuovo Cimento (1911–1923)* **1**, 421–430. (doi:10.1007/BF02958242)
4. Daniele E. 1911 Sull'impiego delle funzioni ellissoidali armoniche nei problemi relativi ad un involucro ellissoidico. *Il Nuovo Cimento (1911–1923)* **2**, 445–452. (doi:10.1007/BF02958516)
5. Lebovitz NR. 1998 The mathematical development of the classical ellipsoids. *Int. J. Eng. Sci.* **36**, 1407–1420. (doi:10.1016/S0020-7225(98)00040-8)
6. Newton SI. 1687 *Philosophiae naturalis principia mathematica*. London, UK: Jussu Societatis Regiae ac Typis Josephi Streater. Prostat apud plures Bibliopolas.
7. Rahman M. 2001 On the newtonian potentials of heterogeneous ellipsoids and elliptical discs. *Proc. R. Soc. Lond. A* **457**, 2227–2250. (doi:10.1098/rspa.2001.0822)
8. Shahgholian H. 1991 On the newtonian potential of a heterogeneous ellipsoid. *SIAM J. Math. Anal.* **22**, 1246–1255. (doi:10.1137/0522080)
9. DiBenedetto. E. 2010 *Partial differential equations*. Boston, MA: Birkhäuser.
10. Arens R. 1990 Newton's observations about the field of a uniform thin spherical shell. *Note Mat.* **10**, 39–45. (doi:10.1285/i15900932v10supn1p39)
11. Gauss CF. 1813 Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractate. In *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores*.
12. Kroneker L. 1969 *Dirichlet's werke*. New York, NY: Chelsea.
13. Kellogg OD. 2010 *Foundations of potential theory*, vol. 31. New York, NY: Dover Publications.
14. Stratton JA. 2007 *Electromagnetic theory*, vol. 33. New York, NY: Wiley-IEEE Press.
15. DiBenedetto E, Friedman A. 1986 Bubble growth in porous media. *Indiana Univ. Math. J.* **35**, 573–606. (doi:10.1512/iumj.1986.35.35030)
16. Howison SD. 1986 Bubble growth in porous media and Hele–Shaw cells. *Proc. R. Soc. Edinb. A* **102**, 141–148. (doi:10.1017/S0308210500014554)
17. Karp L. 1994 On the newtonian potential of ellipsoids. *Complex Var. Elliptic Equ.* **25**, 367–371. (doi:10.1080/17476939408814757)
18. Dive P. 1931 Attraction des ellipsoides homogènes et réciproques d'un théorème de Newton. *Bull. Soc. Math. France* **59**, 128–140.
19. Hölder E. 1932 Über eine potentialtheoretische eigenschaft der ellipse. *Math. Z.* **35**, 632–643. (doi:10.1007/BF01186582)
20. Alouges F, Beauchard K. 2009 Magnetization switching on small ferromagnetic ellipsoidal samples. *ESAIM* **15**, 676–711. (doi:10.1051/cocv:2008047)
21. Beleggia M, De Graef M. 2003 On the computation of the demagnetization tensor field for an arbitrary particle shape using a fourier space approach. *J. Magn. Magn. Mater.* **263**, L1–L9. (doi:10.1016/S0304-8853(03)00238-5)
22. Beleggia M, De Graef M, Millev YT. 2009 Magnetostatics of the uniformly polarized torus. *Proc. R. Soc. A* **465**, 3581–3604. (doi:10.1098/rspa.2009.0355)

23. Brown WF. 1962 *Magnetostatic principles in ferromagnetism*. Amsterdam, The Netherlands: North-Holland.
24. d'Aquino M, Di Fratta G, Serpico C, Bertotti G, Bonin R, Mayergoyz ID. 2011 Current-driven chaotic magnetization dynamics in microwave assisted switching of spin-valve elements. *J. Appl. Phys.* **109**, 07D349. (doi:10.1063/1.3556701)
25. DeSimone A. 1993 Energy minimizers for large ferromagnetic bodies. *Arch. Rat. Mech. Anal.* **125**, 99–143. (doi:10.1007/BF00376811)
26. DeSimone A. 1995 Hysteresis and imperfections sensitivity in small ferromagnetic particles. *Mecca* **30**, 591–603. (doi:10.1007/BF01557087)
27. Di Fratta G, Serpico C, d'Aquino M. 2008 A rigorous treatment of nucleation modes spectrum in micromagnetics. *Phys. B* **403**, 346–349. (doi:10.1016/j.physb.2007.08.046)
28. Di Fratta G, Serpico C, d'Aquino M. 2008 A generalization of the fundamental theorem of brown for fine ferromagnetic particles. *Phys. B* **403**, 346–349. (doi:10.1016/j.physb.2007.08.046)
29. Osborn JA. 1945 Demagnetizing factors of the general ellipsoid. *Phys. Rev.* **67**, 351–357. (doi:10.1103/PhysRev.67.351)
30. Miloh T. 1999 A note on the potential of a homogeneous ellipsoid in ellipsoidal coordinates. *J. Phys. A* **23**, 581. (doi:10.1088/0305-4470/23/4/027)
31. Pohánka V. 2011 Gravitational field of the homogeneous rotational ellipsoidal body: a simple derivation and applications. *Contrib. Geophys. Geod.* **41**, 117–157. (doi:10.2478/v10126-011-0005-0)
32. Khavinson D, Lundberg E. 2013 A tale of ellipsoids in potential theory. *Not. AMS* **61**, 148–156. (doi:10.1090/noti1082)
33. Alouges F, Di Fratta G, Merlet B. 2015 Liouville type results for local minimizers of the micromagnetic energy. *Calc. Var. Partial Differ. Equ.* **53**, 525–560. (doi:10.1007/s00526-014-0757-2)
34. Poisson SD. 1825 Second mémoire sur la théorie du magnétisme. *Mém. Acad. Sci. France* **5**, 488–533.
35. Maxwell. JC. 1873 *A treatise on electricity and magnetism*. Oxford, UK: Clarendon Press.
36. Eshelby JD. 1957 The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. R. Soc. Lond. A* **241**, 376–396. (doi:10.1098/rspa.1957.0133)
37. Liu LP. 2008 Solutions to the Eshelby conjectures. *Proc. R. Soc. A* **464**, 573–594. (doi:10.1098/rspa.2007.0219)
38. Dautray R, Lions JL. 2000 *Mathematical analysis and numerical methods for science and technology: physical origins and classical methods*, vol. 1. Berlin, Germany: Springer.
39. Abramowitz M, Stegun IA. 1964 *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. New York, NY: Dover Publications.
40. Prasolov V, Solovyev Y. 1997 *Elliptic functions and elliptic integrals*, vol. 170. Providence, RI: American Mathematical Society.
41. Salsa S. 2010 *Partial differential equations in action: from modelling to theory*. Berlin, Germany: Springer.
42. Friedman MJ. 1980 Mathematical study of the nonlinear singular integral magnetic field equation. I. *SIAM J. Appl. Math.* **39**, 14–20. (doi:10.1137/0139003)
43. Praetorius D. 2004 Analysis of the operator $\Delta^{-1}$div arising in magnetic models. *J. Anal. Appl.* **23**, 589–605. (doi:10.4171/zaa/1212)