PSEUDO-RANDOM SEQUENCES IDENTIFICATION BY PROBABILITY DISTRIBUTION MOMENTS

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ABSTRACT

The paper discusses the method of discovering of the deterministic nature of a pseudo-random process, formed by a nonlinear mapping $[0, 1]$ to $[0, 1]$. The proposed method is based on comparing two-dimensional mixed moments of distributions of values generated by this mapping. The possibility of approximation for the mapping that generates pseudo-random process based on aggregation of two-dimensional mixed distribution moment is considered.

Keywords One-dimensional mapping · Moments of distribution · Non-linear approximation

1 Introduction

Dynamical systems (as a rule, nonlinear), that may, with certain parameters, generate complex non-fading aperiodic trajectories looks like random non-deterministic "noise" processes, have recently attracted particular interest. Such systems and the processes they form are known as “chaotic”. In applied areas, the methods for discovering the deterministic nature of the observed process with purely random features can be used for system identification, non-linear regression models construction , machine learning algorithms, etc. [1].

This paper considers the possibility of identifying deterministic systems with discrete time, given by a recurrent equation of the form

$$x_{n+1} = f(x_n), \tag{1}$$

(also known as mapping a segment to itself), which, under certain parameters, are chaotic [2].

Let we have such system that produce a chaotic trajectory $\{..x_{n-1}, x_n, x_{n+1}, ..\}$, where $x_n = f^{(n)}(x_0) = \underbrace{f(f...(f(x_0)))}_{n}$, x_0 - is a trajectory starting point. The values of the sequence $\{..x_{n-1}, x_n, x_{n+1}, ..\}$ will form the asymptotic distribution with density, described by a smooth function

$$W^*(x) = \lim_{\Delta x \to 0} \lim_{N\to\infty} \frac{1}{\Delta x} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{\Delta x}(f^{(n)}(x_0))$$

(here Δx is the partition interval of the range of $\{x_n\}0$ - segment $[0, 1]$, χ - is the indicator function on Δx).

If the transformation (1) is chaotic, then the ergodicity condition is satisfied for it, that is $W^*(x)$ must be invariant with respect to the transformation f[3]

$$W^*(x) = \frac{d}{dx} \int_{f_{-1}([0,1])} W^*(x')\, dx' \equiv W_1(x).$$

Here $W_1(x)$ - is one-dimensional probability distribution (PDF) of the values $\{x_n\}$.

In this case, the two-dimensional joint probability density is defined as

$$W_2(x,y) = W_1(x)\delta(y - f(x)). \tag{2}$$

For this transformation and the associated PDF function, one can derive one-dimensional moments

$$m_p = \int_0^1 x^p W_1(x)\, dx \tag{3}$$

and two-dimensional mixed moments

$$m_{k,l} = \int_0^1 \int_0^1 x^k y^l W_2(x,y)\, dxdy = \int_0^1 x^k (f(x))^l W_1(x)\, dx. \tag{4}$$

The fulfillment of the ergodicity condition for a chaotic process makes it possible to consider, in the numerical analysis of moments, their estimates obtained by averaging the values of x_n, taken from the same trajectory $\{..x_{n-1}, x_n, x_{n+1}, ..\}$.

2 Comparison of two-dimensional mixed moments for one-dimensional chaotic mapping

Consider the class G of one-dimensional chaotic mappings of the form (1), defined by analytic functions that can be represented as an expansion

$$f(x) = \sum_{m=0}^{\infty} a_m \cdot x^m\ . \tag{5}$$

For this class of mappings, the following proposition is true:

Proposition 1. *If the one-dimensional mapping defined by the analytic function f is chaotic, then for the set of two-dimensional mixed moments (4) for the distribution of the values generated by this mapping, there is at least one value k such that*

$$m_{k,k+1} \neq m_{k+1,k}\ . \tag{6}$$

The analyticity condition in this case is caused by the need to exclude from consideration piecewise smooth and piecewise continuous mappings (mappings with singular points on the interval $[0,1]$), which are the case of special studies.

Proof. Let we represent $m_{k,k+1}$ in the form

$$\begin{aligned} m_{k,k+1} &= \int_0^1 (xf(x))^k (\sum_{m=0}^{\infty} a_m\, x^m)\, W_1(x)\, dx = \\ &= \sum_{m=0}^{\infty} a_m \int_0^1 x^{k+m} f(x)^k W_1(x)\, dx = \sum_{m=0}^{\infty} a_m m_{k+m,k}\ . \end{aligned} \tag{7}$$

Consider the case when the series (5) is bounded and consists of $M+1$ initial terms (that is, $f(x)$ is a polynomial of finite degree). For the $(M+1)$ coefficients of the series, one can form a system of $(M+1)$ equations of the form $\mathbf{DA} = \mathbf{B}$

$$\begin{pmatrix} m_{0,0} & m_{1,0}\cdots & m_{M,0} \\ m_{1,1} & m_{2,1}\cdots & m_{M+1,1} \\ \multicolumn{3}{c}{\dotfill} \\ m_{M,M} & m_{M+1,M}\cdots & m_{2M,M} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_M \end{pmatrix} = \begin{pmatrix} m_{0,1} \\ m_{1,2} \\ \cdots\cdots \\ m_{M,M+1} \end{pmatrix}. \tag{8}$$

The determinant for the matrix $\mathbf{D}$ is not equal to 0 since the solution of the system (8) is unique because of the uniqueness of the decomposition (5). Suppose that $m_{k,k+1} = m_{k+1,k}$ for any value of k.Given this equality, we rewrite (8) as

$$\begin{pmatrix} m_{0,0} & m_{1,0}\cdots & m_{M,0} \\ m_{1,1} & m_{2,1}\cdots & m_{M+1,1} \\ \multicolumn{3}{c}{\dotfill} \\ m_{M,M} & m_{M+1,M}\cdots & m_{2M,M} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_M \end{pmatrix} = \begin{pmatrix} m_{1,0} \\ m_{2,1} \\ \cdots\cdots \\ m_{M+1,M} \end{pmatrix}. \tag{9}$$

In this case, the only solution is $a_0 = 0$, $a_1 = 1$, $a_2 = .. = a_M = 0$, that is f - is the identity transformation, and it is not in G. Any other solutions are admissible only for $m_{k,k+1} \neq m_{k+1,k}$, therefore, **Proposition 1** is true. $\square$

Obviously, it holds for any number of series members in the expansion (5).

This Proposition makes it possible to show the difference between a chaotic sequence formed by a one-dimensional mapping of the class G, from a purely random sequence whose values are independent of each other (a random "white noise" type process). Indeed, along with a chaotic sequence satisfying the conditions of Proposition 1, we consider a purely random sequence $\{..\xi_{s-1}, \xi_s, \xi_{s+1}, ..\}$, such that both of one-dimensional PDFs of their values are identical. The mutual independence of the values $\{\xi_n\}$ implies, in particular, that for two-dimensional mixed moments of their distribution $m_{\xi k,k+1}$, next equality is true:

$$m_{\xi k,k+1} = m_{\xi k+1} m_{\xi k} = m_{\xi k+1,k},$$

which, by virtue of Proposition 1, does not performed for a chaotic sequence formed by a one-dimensional map as an analytic function.

3 Approximation of the one-dimensional mapping function

Let we have a sequence of values $\{..x_{n-1}, x_n, x_{n+1}, ..\}$, produced by some one-dimensional map. On the basis of this sequence, one can obtain estimates of the 1st and 2nd order moments $\hat{m}_p$ $\hat{m}_{k,l}$ and obtain an approximation of the function $\hat{f}$ as the sum of $\hat{f}(x) = \sum\limits_{m=0}^{M} \hat{a}_m \cdot x^m$ at $[0,1]$ where $\hat{a}_m$ are solutions of the system (8).

The approximated function $\hat{f}$ will be represented as a polynomial of finite degree M.

In applied problems, the restoration of the mapping function will be a numerical procedure, and it arises the well-known problem of the accuracy of computer approximation associated with the restriction of the discharge grid [4]. This leads to the fact that in the case of the convergence of the series (5) during the approximation, the limit value m can be equated to some final value M_z, which does not make sense to exceed.

4 Conclusions

It was shown above that a comparison of two-dimensional moments for a sequence with discrete time may suggests that there is a functional connection between adjacent samples, assuming that this functional dependence is one-step: (x_{n+1} depends on x_n).Meanwhile, there are also mappings of the form $x_n = f(x_{n-1}, x_{n-2}....)$, including dependencies on a larger number of steps, also capable of forming trajectories of the type "chaotic attractor" [5]. The question of how the mixed moments for such mappings correlate remains open.

We also have to substantiate the ergodic nature of the process under study. The ergodicity of the mapping leads to the presence of an invariant probability distribution, whose density is equal to the PDF for a separate trajectory $\{..x_{n-1}, x_n, x_{n+1}, ..\}$, which allows for process, use time-averaged (along the trajectory) estimates of distribution points. That allows use time-averaged (along the trajectory) estimates of distribution moments. In practical application, the ergodicity of a process can be estimated from the decay rate of its correlation function [6].

At the same time, the deterministic nature of the process does not automatically mean that the display that forms it is chaotic. The presence of a chaotic attractor assumes instability of the trajectories of the dynamic system and can be justified, for example, by calculating Lyapunov exponent [7].

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