MATH FOR ECON - PROBLEM SET 2 SOLUTIONS

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Exercise 1

Let $X = (0,1]$ with the usual metric $d(x,y) = |x-y|$. Show that (X,d) is not complete. Let $\tilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ for all $x, y \in X$. Show that $\tilde{d}$ is a metric on X that is equivalent to d, and that $(X, \tilde{d})$ is complete.

Proof: Clearly, (X,d) is not complete (use the sequence $x_m = 1/m$). To see that $\tilde{d}$ is a metric, notice that symmetry and reflexivity holds trivially. For the triangle inequality, pick $x, y, z \in X$. Define $a = x^{-1}$, $b = y^{-1}$ and $c = z^{-1}$. Notice that $\tilde{d}(x,y) = |a-b| \le |a-c|+|c-b| = \tilde{d}(x,z) + \tilde{d}(z,y)$.

Next we show equivalence. In particular, we are going to show that for all $x \in X$ and for all ε, there exists a $\tilde{\varepsilon}$ such that $B(x,\varepsilon) = \tilde{B}(x,\tilde{\varepsilon})$, and vice versa, that for all $\tilde{\varepsilon}$ there exists a ε such that $B(x,\varepsilon) = \tilde{B}(x,\tilde{\varepsilon})$. Fix ε and $x \in X$. Define $\tilde{\varepsilon} = \frac{\varepsilon}{x(x+\varepsilon)}$. Notice that $(x+\varepsilon) < y$. We have that $y \in \tilde{B}(x,\tilde{\varepsilon})$ iff $\tilde{d}(x,y) = |\frac{1}{x} - \frac{1}{y}| < \tilde{\varepsilon} = \frac{\varepsilon}{x(x+\varepsilon)}$ iff $\frac{|x-y|}{xy} < \frac{\varepsilon}{x(x+\varepsilon)} < \frac{\varepsilon}{xy}$ iff $|x-y| < \varepsilon$ iff $d(x,y) < \varepsilon$. To see the other direction, just notice $\tilde{d}(x,y) \ge d(x,y)$. Hence, for any $\tilde{\varepsilon}$ just let $\varepsilon := \tilde{\varepsilon}$. Then $\tilde{d}(x,y) < \tilde{\varepsilon}$ implies $d(x,y) < \varepsilon$ as we seek.

Finally we show completeness. Let (x_m) be cauchy in $(X, \tilde{d})$. By definition, for all ε, there is a N such that for all $m, n > N$, $|\frac{1}{x_m} - \frac{1}{x_n}| < \varepsilon$. This means that $(\frac{1}{x_m})$ is Cauchy in d. But given that $x_m \in (0,1]$ this implies $\frac{1}{x_m} \in [1,\infty)$, a closed subset of a d-complete space. Hence $(\frac{1}{x_m})$ converges in d to some y. Define $z := y^{-1} \in X$. Then for all ε there is an N such that for all $m > N$, $|\frac{1}{x_m} - \frac{1}{z}| < \varepsilon$, which means $x_m \to z$ in $\tilde{d}$, ergo $(X,\tilde{d})$ is complete. □

Exercise 2

Let (X,d) be complete and $\Phi : X \to X$. Show that $d(\Phi(x), \Phi(y)) < d(x,y)$ for all $x, y \in X$, $x \neq y$, is insufficient for the existence of a fixed point of Φ. (An example is enough)

Proof: Let $X = [1,\infty)$ and $\Phi(x) := \frac{1+x^2}{x}$. To check that $d(\Phi(x), \Phi(y)) < d(x,y)$, fix $x, y \in [1,\infty)$, wlog $x < y$. There exists a $z \in (x,y)$ such that $|\Phi(x) - \Phi(y)| < |\Phi'(z)||x-y|$ (Rolle's Theorem) and $|\Phi'(z)| < 1$ everywhere in the domain. Clearly $[1,\infty)$ is complete and clearly there is no x solving $\frac{1+x^2}{x} = x$ in $[1,\infty)$. □

Exercise 3

Let $f : \mathbb{R} \to \mathbb{R}$ be increasing. Show that f can be discontinuous at only countably many points.

Proof: Let $D := \{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$. Then,

$$a(x) := \lim_{y_n \nearrow x} f(y_n) \neq \lim_{y \searrow x} f(y) =: b(x).$$

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Notice $a(x)$ and $b(x)$ exists. Indeed, for any $y_n \nearrow x$, $f(y_n)$ is an increasing bounded real sequence hence has a limit. A similar argument holds for $b(x)$. Moreover, since f is increasing and discontinuous at x we must have $a(x) < b(x)$. By the Archimedean principle, there exists a rational number $q(x) \in (a(x), b(x))$. Also notice that for all $x, x' \in D$ with $x < x'$, $q(x) < q(x')$. Contrapositive argument: suppose D is uncountable. Then, there are uncountably many distinct rational numbers $\{q(x) : x \in D\}$, a contradiction on the cardinality of $\mathbb{Q}$. An equivalent way to get the point home is: interpret q as a function, i.e. let $q : D \to \mathbb{Q}$. Notice is that q is injective. By definition, this means that $\mathbb{Q}$ is cardinally larger then D, again a contradiction. $\square$

Exercise 4

Let $f : X \to X$ be a continuous function and X is a compact metric space. Also, $d(f(x), f(y)) = d(x, y)$. Prove that f is onto.

Proof: Suppose not, that is suppose $f(X) =: Y \subsetneq X$. Let $\bar{x} \in X \setminus Y$. Notice Y is compact (continuity and X being compact). Hence, $X \setminus Y$ is open. So there exists a $\varepsilon > 0$ sufficiently small s.t. $B(\bar{x}, \varepsilon) \cap Y = \emptyset$. Let $\mathcal{T}$ the collections of sets $T \subset X$ with finitely many elements such that $\bigcup_{x \in T} B(x, \varepsilon/2) \supset X$. Since X is compact, we also have that it is totally bounded. Hence, $\mathcal{T} \neq \emptyset$. Moreover, $\underline{N} := \min\{|T| : T \in \mathcal{T}\}$ exists by the Well Ordering principle.[1] Let $\{x_1, \dots, x_{\underline{N}}\}$ (one of) the (possibly multiple) collection(s) of points associated with $\underline{N}$, i.e. $\bigcup_{i=1}^{\underline{N}} B(x_i, \varepsilon/2) \supset X$. Since $\bar{x} \in X$, there exists one j such that $\bar{x} \in B(x_j, \varepsilon/2)$. By assumption, $B(x_j, \varepsilon/2) \subset X \setminus Y$. Therefore, $Y \subset \bigcup_{i \neq j} B(x_i, \varepsilon/2)$, that is Y can be covered by with identical balls centered in $\underline{N} - 1$-many points. Notice that

$$X = f^{-1}(Y) \subset f^{-1}(\bigcup_{i \neq j} B(x_i, \varepsilon/2)) = \bigcup_{i \neq j} f^{-1}(B(x_i, \varepsilon/2)).$$

That is, $\{f^{-1}(B(x_i, \varepsilon/2))_{i=1}^{\underline{N}}$ is a cover for X with $\underline{N} - 1$ many sets. To conclude the proof, we need to show that each one of these sets is (included in) a ball of radius smaller than $\varepsilon/2$. Here is where we use the isometry assumption on f. Indeed, pick $z_i \in f^{-1}(B(x_i, \varepsilon/2))$. Notice that for a generic $z \in X$, $d(z_i, z) < \varepsilon/2$ iff $d(x_i, f(z)) < \varepsilon/2$ (isometry), iff $d(x_i, f(z)) < \varepsilon/2$ iff $z \in f^{-1}(x_i)$. That is, each $f^{-1}(B(x_i, \varepsilon/2))$ is a ball of radius $\varepsilon/2$ centered in z_i. But this means that we have found $\underline{N} - 1$-many balls of radius $\varepsilon/2$ covering X, a contradiction on the minimality assumption of $\underline{N}$. $\square$

A Better Proof:[2] Suppose not, that is suppose $f(X) =: Y \subsetneq X$. Let $\bar{x} \in X \setminus Y$. Since $f(X)$ is compact, hence closed, $\bar{x}$ lives in an open set, hence we can find an $\varepsilon > 0$ s.t. $d(\bar{x}, x) > \varepsilon$ for all $x \in X$. Define the sequence $(y_m) \in Y^\infty$ as follow: $z_1 := f(\bar{x})$ and $z_n = f(z_{n-1})$ for all $n > 1$. Pick $m, n \in \mathbb{N}$ and wlog let $m > n$. Then we have $d(y_m, y_n) = d(f^m(\bar{x}), f^n(\bar{x})) = d(f(f^{m-1}(\bar{x})), f(f^{n-1}(\bar{x}))) = d(f^{m-1}(\bar{x}), f^{n-1}(\bar{x}))$ where the last equality comes from the isometry assumption. Proceeding inductively we can show that $d(y_m, y_n) = d(f^{m-n-1}(\bar{x}), \bar{x}) > \varepsilon$, where the last inequality comes from the fact that $f^{m-n-1}(\bar{x}) \in Y$. Hence, no two members of (y_m) is closer than ε to each other. Ergo, there is no convergent subsequence of (y_m). That is, Y is not compact. Contradiction. $\square$

Exercise 5

Continuous function on compact domains are uniformly continuous.

[1] Every subset of the natural numbers has a minimal element.

[2] I am borrowing this smart idea from you. Unfortunately, I cannot give credit to anyone in particular, since many of you used this argument.

Proof: Let $f : X \to Y$ continuous and X compact. Fix $\varepsilon > 0$ arbitrarily and $\bar{x} \in X$. By continuity of f, for all $x \in X$ there exists a $\delta_x > 0$ such that for all $y \in B(x, \delta_x)$, we have $d_Y(f(x), f(y)) < \frac{\varepsilon}{2}$. Notice that the collection of balls $\{B(x, \delta_x/2) : x \in X\}$ is an open cover of X. By compactness there exists a finite collection of points $\{x_1, \ldots, x_n\} \subset X$ such that $\{B(x_i, \delta_x/2) : i \in \{1, \ldots, n\}\}$ covers X as well. Let $\delta := \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}/2$ and notice this number does not depend on $\bar{x}$.[3] Consider all $y \in X$ such that $d(\bar{x}, y) < \delta$. Clearly, there is some $i \in \{1, \ldots, n\}$ such that $\bar{x} \in B(x_i, \delta_{x_i}/2)$. By definition of δ_{x_i}, this implies that $d_Y(f(\bar{x}), f(x_i)) < \frac{\varepsilon}{2}$. Moreover,

$$d(x_i, y) \leq d(x_i, \bar{x}) + d(\bar{x}, y) < \frac{\delta_{x_i}}{2} + \delta < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} < \delta_{x_i}.$$

Again, this implies that $d_Y(f(x_i), f(y)) < \frac{\varepsilon}{2}$. Summing things up, we can argue that for the ε chosen above, we were able to find a δ such that for all $y \in X$ with $d(\bar{x}, y) < \delta$, we have

$$d_Y(f(\bar{x}), f(y)) \leq d_Y(f(\bar{x}), f(x_i)) + d_Y(f(x_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon,$$

for some x_i, this being true for all $\varepsilon > 0$ and all $\bar{x} \in X$. We conclude that f is uniformly continuous. □

[3] It clearly depends on what finite subcover we choose, but not on $\bar{x}$.