A DUAL PERSPECTIVE ON INFORMATION DESIGN

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ABSTRACT

The problem of optimally designing information for multiple agents who interact in a game can be formulated as a linear program. We explore its *dual* representation and show that it provides a novel perspective and new economic insights into the information-design problem. Through the lens of the dual, we identify general properties that hold for all information-design problems. Duality also offers a portable, general, method for computing solutions. We illustrate this approach in the context of simple investment games.

Keywords: information design, linear programming, dual.

JEL Codes: C61, C72, D02, D70, D82, D83.

1 Introduction

Information design studies how to provide information to a group of players so as to affect the outcome of their strategic interaction in some desired way. This agenda stems from the idea that, in many situations, information provision may be the most natural available tool a third party can use to influence behavior. This framework has been applied to several areas, including the design of rating systems, stress tests, and political campaigns.

The design of information gives rise to specific constraints and challenges relative to, for instance, the design of monetary incentives. To address them, different solution methods have been proposed. In this paper we focus on one, which is based on the notion of Bayes-correlated equilibrium (Bergemann and Morris (2016)). This method allows us to cast the information-design problem in terms of a linear program. Our contribution is to derive and interpret the *dual* of this program for general settings with multiple players and arbitrary payoff functions. This exercise is useful in three distinct ways.

First, the dual offers a novel economic interpretation of information design. Recall that an information structure specifies the probability that each state of the world gives rise to a signal; these probabilities determine the signal's ultimate meaning for the player observing it. Through the lens of the dual, designing an optimal structure can be viewed as a problem of *minimizing* the marginal value of an extra 'unit' of probability to be assigned to the signals. In fact, we can think of designing a signal as 'buying' units of probability from each state viewed as a 'seller,' whose stock of probability is the prior of that state. The solution of this dual problem requires to find (shadow) prices—one for each state—that minimize the value of one extra unit of supply of probability.

These prices capture the key trade-offs of the information-design problem. The higher the designer's payoff from the behavior induced by a signal, the higher the price she has to pay to the states selling their probability to that signal. However, the designer gets a 'discount' from a state-seller if a signal it generates leads some player to choose an action he would find suboptimal ex post, that is, had he learned the state and the action of others. A 'penalty' is instead added to the state-price if the induced action is strictly optimal based on the same ex-post information. Thus, inducing a player to choose a specific action can trigger a discount for one state and a penalty for another, or even discounts and penalties for the same state depending on the opponents' induced actions. This interdependence between state-prices is what ultimately determines which behavior the designer optimally induces in each state.

The price discounts create the basic incentive for the designer to use *partially* informative signals. Indeed, a player can be induced to choose an ex-post suboptimal action only if he is kept uncertain about the state and others' behavior. The dual also characterizes the optimal way of designing information to achieve this.

The second way our dual approach is useful is that it identifies general properties of optimal information structures. For instance, an optimal use of price discounts and penalties requires that the signals inducing a player to take ex-post suboptimal actions should render that player ex-ante *indifferent* between the chosen action or the ex-post optimal one. When this is not the case, the designer provides either too much or too little information to that player. This property generalizes, to the realm of multi-player games, the insight that Kamenica and Gentzkow (2011) have identified for the case of a single receiver of information. Another optimality property implied by the dual is that it can be optimal to induce multiple action profiles in one state only if profiles giving the designer a higher payoff also allow her to reap more discounts on the price of that state. Again, this is a property of the solution to Kamenica and Gentzkow (2011) leading example, which we find to hold true more generally, even for games.

To the best of our knowledge, these properties have not been derived before for multi-player information-design problems. This highlights the third advantage of the dual approach. Is has been noted before that the concavification method of Kamenica and Gentzkow (2011), though conceptually powerful, may encouter some limitations in some applications. Linear programming has been suggested as a possible alternative (Bergemann and Morris (2018), Kolotilin (2018)). Within this alternative, the dual method seems particularly promising as a portable, general, way to solve information-design problems. It delivers a well-defined recipe to first derive the prices of each state and from them the optimal information structure. For instance, the first step can immediately identify the action profiles that can never arise in a state in any optimal solution. More generally, the dual method here is proposed not as a way to check the optimality of some candidate information structure, but to find solutions directly.

To illustrate these techniques, we consider an application of our method to simple investment games. The example is borrowed from (and compared with) Bergemann and Morris (2018). We show that for those games the dual allows one to derive the optimal information structures in few straightforward steps.

Literature. Information design is the focus of a large and growing literature, which has been recently reviewed by Bergemann and Morris (2018). From the seminal contribution of Brocas and Carrillo (2007), Rayo and Segal (2010), and Kamenica and Gentzkow (2011), the literature has focused on different aspects of the interaction between a designer (or sender) who commits to an information structure in order to persuade a single player (or receiver).[1] Information design generalizes this basic interaction to contexts where the designer persuades *multiple* agents who play a game. The present paper builds on a key result by Bergemann and Morris (2016) which characterizes the players' behaviors that the designer can induce in terms of private recommendations that the players are willing to obediently follow.[2]

Closest to our work are two papers that focus exclusively on the single-receiver case. Dworczak and Martini (2019) offer an insightful interpretation that expresses the persuasion problem as a Walrasian exchange economy with one producer and one consumer of probabilities, the latter being the information designer. This interpretation differs from the one proposed in the present paper in several ways, even beyond the fact that it is restricted to sender-receiver settings. For instance, prices in the persuasion economy of Dworczak and Martini (2019) are assigned to the posterior *beliefs* induced by signals. By contrast, in our interpretation prices are assigned to the states, each viewed as a seller of probabilities.

The second paper, Kolotilin (2018), adopts a linear-programming approach for studying a specific class of problems where the receiver's utility satisfies a strong form of single-crossing. Using duality, Kolotilin (2018) shows when full and no information revelation are optimal. The dual of Kolotilin (2018) is similar to the one we derive in this paper. We go beyond this special class of sender-receiver interactions and consider arbitrary games in full generality. Moreover, we offer an economic interpretation of the dual that sheds new light on the information-design problem and use this approach to derive general optimality properties.

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[1] See, e.g., Alonso and Camara (2018), Alonso and Camara (2018), Gentzkow and Kamenica (2016b), Gentzkow and Kamenica (2016a), Mathevet and Lipnowski (2018a), Mathevet and Lipnowski (2018b), Galperti (2018).

[2] Taneva (2015) also formulates the information-design problem as the selection of the best Bayes-correlated equilibrium. By contrast, Mathevet, Perego, and Taneva (2017) extend the idea of concavification to information-design problems and allow for a rich set of selection criteria and solution concepts. In the context of a global game with adversarial selection, Inostroza and Pavan (2017) studies optimal stress-tests for banks seeking funding from heterogeneously informed investors. The problem of the optimal dynamic information design has been analyzed by Ely (2017), Doval and Ely (2016), Makris and Renou (2018), Ely and Szydlowski (2019).

2 The Information-Design Problem

This section introduces the information-design problem and casts it in terms of the optimal choice of a Bayes-Correlated Equilibrium (Bergemann and Morris (2016)). For simplicity of exposition, we will focus on the case where players do not have exogenous information about the payoff-state.

Primitives. There is a finite set of players $N = \{1, \dots, n\}$ and a finite set of payoff-states Ω. Players have a common prior belief $\mu \in \Delta(\Omega)$. With no loss of generality, we let prior μ to have full support on Ω. An information structure, denoted by (S, π), consists of a finite set of signals S_i for each player $i \in N$ and a function $\pi : \Omega \to \Delta(S)$, where $S = S_1 \times \cdots \times S_n$. Without loss of generality, assume that for all $s \in S$ there exists $\omega \in \Omega$ such that $\pi(s|\omega) > 0$. We sometimes abuse notation by writing π for the information structure (S, π). Let Π be the set of all information structures.

Game and Equilibrium Concept. After *privately* observing the signal realizations generated by (S, π), players interact in a game. Each player i has a finite set of actions A_i and preferences described by the utility function $u_i : A \times \Omega \to \mathbb{R}$, where $A = A_1 \times \cdots \times A_n$. We denote $G = (\Omega, (A_i, u_i)_{i \in N}, \mu)$ the *basic* game. A basic game when paired with any information structure π induces an incomplete-information game, denoted (G, π). We focus on Bayes-Nash equilibria (BNE) of such games. A (behavioral) strategy of player i in (G, π) is a map $\sigma_i : S_i \to \Delta(A_i)$. A strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a BNE of (G, π) if for each i, $s_i \in S_i$, and $a_i \in A_i$ with $\sigma_i(a_i|s_i) > 0$,

$$\sum_{a_{-i} \in A_{-i}, s_{-i} \in S_{-i}, \omega \in \Omega} \left[u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)\right] \sigma(a_i, a_{-i}|s_i, s_{-i}) \pi(s_i, s_{-i}|\omega) \mu(\omega) \geq 0$$

for all $a_i' \in A_i$, where $\sigma(a_i, a_{-i}|s_i, s_{-i}) := \prod_{j=1}^{N} \sigma_j(a_j|s_j)$. Let $\text{BNE}(G, \pi)$ be the set of all BNE's of game (G, π).

Information-Design Problem. The information designer is a third-party with preferences on the outcome of the game described by the payoff-function $v : A \times \Omega \to \mathbb{R}$. The designer chooses the information structure π, hence determines which game (G, π) that is played by players. We assume that she can costlessly commit to any π in Π. Her prior belief is also μ as for the players. For every $\pi \in \Pi$, we define the value of π as

$$V(\pi) = \max_{\sigma \in \text{BNE}(G, \pi)} \sum_{a \in A, s \in S, \omega \in \Omega} v(a, \omega) \sigma(a|s) \pi(s|\omega) \mu(\omega).$$

Note that the value is computed under the assumption that the designer-preferred equilibrium is selected. This selection criterion is *with* loss of generality, but its nice properties have made it especially common in most of the information-design literature (see Bergemann and Morris (2018) for a discussion). The *information-design problem* faced by the designer consists of choosing π to maximize $V(\pi)$. Its value function is $V^* := \max_{\pi \in \Pi} V(\pi)$. In the next section, we will show that this value is in fact always attained by some information structure π, hence the use of the max in our definition.

3 The Primal of Information Design

This section characterizes the set of all outcomes—namely, joint distributions of actions a and states ω—that can arise in some equilibrium of the game for some information structure that the designer chooses. This characterization, which is due to Bergemann and Morris (2016), offers a convenient linear structure to the set of feasible outcomes and will be used in the next section to derive the dual of information design.

Consider an information structure π and recall that each player acts on her private signal by choosing an action, possibly at random. Thus, we can view each information structure as inducing, for each ω, a distribution over pure-action profiles.

Definition 1 (Outcome Function)**.** An *outcome function* is a mapping $x : \Omega \to \Delta(A)$.

Not all outcome functions are compatible with equilibrium behavior. We say that an outcome function is *feasible* if it arises for *some* information structure π and *some* equilibrium of the game (G, π). More formally:

Definition 2 (Feasible Outcomes)**.** An outcome function x is *feasible* for G if there exists $\pi \in \Pi$ and $\sigma \in \text{BNE}(G, \pi)$ such that, for every $\omega \in \Omega$ and $a \in A$,

$$x(a_1, \ldots, a_n|\omega) = \sum_{s \in S} \pi(s|\omega) \prod_{i \in N} \sigma_i(a_i|s_i), \tag{1}$$

Denote $X(G)$ the set of feasible outcome functions for G.

An outcome function can be interpreted either as a probabilistic description of what happens in a game or as an explicit recommendation on how players should behave. Clearly, for such recommendations to coincide with actual behavior, it must be incentive compatible to follow them.

Definition 3 (Obedience)**.** The outcome function x is *obedient* for G if, for each $i \in N$ and $a_i \in A_i$,

$$\sum_{\omega \in \Omega, a_{-i} \in A_{-i}} [u_i(a_i, a_{-i}; \omega) - u_i(a_i', a_{-i}; \omega)] x(a_i, a_{-i}|\omega) \mu(\omega) \geq 0, \quad a_i' \in A_i. \tag{2}$$

As the next result demonstrates, obedience fully characterizes the set of feasible outcomes.

Theorem 1 (Feasibility—Bergemann and Morris (2016)). *The outcome function x is feasible for G if and only if it is obedient for G.*

As observed by Bergemann and Morris (2016), the combination of the linear structure of the obedience conditions and the linearity of the designer's *expected* payoff gives rise to a linear program. In the *primal* of this program, the designer chooses an obedient x to *maximize* his expected payoff. We denote this primal problem by $\mathcal{P}$. Our objective is to derive its *dual* formulation, which we shall denote by $\mathcal{P}^*$. To this end, it is more convenient to work in terms of the *joint* distribution over actions and states induced by the pair of outcome function x and prior μ. That is, given every $x : \Omega \to \Delta(A)$, let

$$\chi(a, \omega) := x(a|\omega)\mu(\omega), \quad a \in A, \omega \in \Omega.$$

Note that χ is a vector in $\mathbb{R}^{A\times\Omega}$. Given this, $\mathcal{P}$ can be expressed as follows:

$$\begin{aligned}
\mathcal{P} \quad = \quad & \max_{\chi \in \mathbb{R}^{A\times\Omega}} \quad \sum_{\omega\in\Omega, a\in A} v(a,\omega)\chi(a,\omega), \\
& \text{sub to,} \quad \forall i \in N, \omega \in \Omega, a \in A, a_i' \in A_i : \\
& (O) \quad \textstyle\sum_{\omega\in\Omega, a_{-i}\in A_{-i}} [u_i(a_i, a_{-i}; \omega) - u_i(a_i', a_{-i}; \omega)]\chi(a_i, a_{-i}, \omega) \geq 0, \\
& (PC) \quad \textstyle\sum_{a\in A} \chi(a,\omega) = \mu(\omega), \\
& (P) \quad \chi(a,\omega) \geq 0.
\end{aligned}$$

The first constraint is obedience (O). The second constraint (*prior consistency, PC*) and the third constraint (*positivity, P*) ensure that χ is a probability distribution. Given any such χ, it is straightforward to derive the corresponding outcome function x. To see this, fix a χ and a profile (a, ω). Define $x(a|\omega) := \frac{\chi(a,\omega)}{\mu(\omega)}$ and notice that $x(a|\omega) \geq 0$, because of non-negativity of χ, and $\sum_a x(a|\omega) = 1$, because of prior consistency.

We can now address the issue of existence of a solution. One can easily check that the subset of $\mathbb{R}^{A\times\Omega}$ defined by the constraints of $\mathcal{P}$ is compact and non-empty. It is non-empty because the basic game G must have at least one equilibrium, which is a feasible outcome under a fully uninformative π and hence corresponds to an obedient x by Theorem 1. Also, the objective of $\mathcal{P}$ is continuous. Therefore, by standard arguments $\mathcal{P}$ has a solution. As anticipated in Section 2, this implies that we can write $V^* = \max_{\pi\in\Pi} V(\pi)$.

4 The Dual of Information Design

In this section, we derive the dual representation of the information-design problem. We discuss its economic interpretation and the new insights it offers. We also characterize some general properties of its solutions.

4.1 Derivation of the Dual

Instead of attacking the information-design problem directly through $\mathcal{P}$, we will look at its dual. The dual problem $\mathcal{P}^*$ involves new control variables, which we divide into two vectors, p and λ. The first vector belongs to $\mathbb{R}^\Omega$ and corresponds to the prior-consistency constraints: To each constraint $\sum_{a\in A}\chi(a,\omega)=\mu(\omega)$ there corresponds one variable $p(\omega)$. The second vector λ is obtained by stacking on top of each other the player-specific vectors $\lambda_i\in\mathbb{R}^{A_i\times A_i}$. That is, each entry of λ corresponds to some player i and some pair (a_i,a_i') of his actions. The entry $\lambda_i(a_i,a_i')$ corresponds to the obedience constraint requiring that player i should prefer to follow his recommendation a_i to deviating to a_i'.

Proposition 1. *The dual information-design problem $\mathcal{P}^*$ consists of solving*

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\begin{aligned}
\mathcal{P}^* \quad = \quad & \min_{\substack{p\in\mathbb{R}^\Omega \\ \lambda_1\in\mathbb{R}^{A_1\times A_1} \\ \vdots \\ \lambda_n\in\mathbb{R}^{A_n\times A_n}}} \quad \sum_{\omega\in\Omega}\mu(\omega)p(\omega), \\
& \textit{sub to,} \qquad \forall i\in N,\omega\in\Omega,a\in A,a_i'\in A_i: \\
& (\star) \qquad p(\omega)\geq v(a,\omega)+\sum_{i\in N}\sum_{a_i'\in A_i}[u_i(a_i,a_{-i},\omega)-u_i(a_i',a_{-i},\omega)]\lambda_i(a_i,a_i'), \\
& (P_\lambda) \qquad \lambda_i(a_i,a_i')\geq 0.
\end{aligned}
$$

The dual $\mathcal{P}^*$ is just a new mathematical problem derived from $\mathcal{P}$. It is of special interest, however, because by Strong Duality, if $\mathcal{P}$ and $\mathcal{P}^*$ have an optimal solutions, their optimal values are equal, i.e. $\mathcal{P}=\mathcal{P}^*$. In our case, the dual objective takes a remarkably simple form. It depends only on the prior, which means that it is unaffected by any change in any other aspect of the basic game G. These aspects are all conveniently collected in the constraint $(\star)$. The dual also conveys a very simple goal for the designer: Choose each $p(\omega)$ as small as possible—subject, of course, to condition $(\star)$. The other dual variables λ can be chosen with the sole objective of relaxing $(\star)$. Since these conditions have to hold across all as and λs, they impose non-trivial lower bounds on p.

Another reason for being interested in $\mathcal{P}^*$ is that its solutions (p,λ) and the solutions χ of $\mathcal{P}$ are tightly related to each other. This relationship is summarized by a set of complementary-slackness conditions (CS), as the next result shows. The proof is standard and therefore omitted.

Proposition 2. *Let χ satisfy the constraints of $\mathcal{P}$ and (p,λ) satisfy the constraints of $\mathcal{P}^*$. Then, χ and (p,λ) are optimal solutions for the two respective problems if and only if,* *for all* $i \in N$, $\omega \in \Omega$, $a \in A$, $a'_i \in A_i$,

CS1. $\lambda_i(a_i, a'_i)\left\{\sum_{\omega' \in \Omega, a_{-i} \in A_{-i}} [u_i(a_i, a_{-i}, \omega') - u_i(a'_i, a_{-i}, \omega')]\chi(a_i, a_{-i}, \omega')\right\} = 0,$

CS2. $p(\omega)\left\{\sum_{a' \in A} \chi(a', \omega) - \mu(\omega)\right\} = 0,$

CS3. $\chi(a, \omega)\left\{p(\omega) - v(a, \omega) - \sum_{i \in N} \sum_{a'_i \in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(a'_i, a_{-i}, \omega)]\lambda_i(a_i, a'_i)\right\} = 0.$

In short, the combination of the dual and the complementary slackness conditions offers an alternative way to find optimal feasible outcomes and to characterize their properties. In this sense, an immediate corollary of Proposition 1 and 2 is the following.

Corollary 1. *Fix* $\omega \in \Omega$ *and* $a', \hat{a} \in A$. *If for all* λ

$$\begin{aligned} & v(a', \omega) - \sum_{i \in N} \sum_{a''_i \in A_i} [u_i(a'_i, a'_{-i}, \omega) - u_i(a''_i, a'_{-i}, \omega)]\lambda_i(a'_i, a''_i) \\ > \ & v(\hat{a}, \omega) - \sum_{i \in N} \sum_{a''_i \in A_i} [u_i(\hat{a}_i, \hat{a}_{-i}, \omega) - u_i(a''_i, \hat{a}_{-i}, \omega)]\lambda_i(\hat{a}_i, a''_i), \end{aligned}$$

then every optimal χ *satisfies* $\chi(\hat{a}, \omega) = 0$.

This corollary can be used to quickly prune the support of optimal χ's, as we illustrate in the examples of Section 5.

4.2 The Economics of the Dual

The dual of the information-design problem provides a novel perspective on the economics of information design. Conditions ($\star$) and complementary slackness convey a great deal of information on both how to optimally choose $p(\omega)$ and how to interpret $p(\omega)$. To gain intuition, let's first consider the hypothetical case in which the designer sets $\lambda = 0$. When this is the case, our dual constraint ($\star$) implies that it is optimal to set $p(\omega) = \max_a v(a, \omega)$. Hence, the resulting value of choosing such pair of dual variables (p, λ) is $\sum_\omega \mu(\omega) \max_a v(a, \omega)$. This upper bound represents the first-best the designer can achieve in this information-design problem: in fact, it implements the designer's most favorite action at each different state. Of course, this recommendation entirely ignores players' incentives and, therefore, it may not be even feasible in the primal, except in very special situations. This translates into the dual problem: the designer may be able to do better, i.e. find prices $p(\omega)$ that lead to a smaller value of the dual, by choosing λ that are not necessarily all equal to zero. To illustrate how, suppose that the pair (a, ω)

has the following property: For some player i, action a_i is sub-optimal given ω and a_{-i}. More formally, for some i and (a, ω) with $\chi(a, \omega) > 0$, there is an a_i', such that

$$u_i(a_i, a_{-i}, \omega) < u_i(a_i', a_{-i}, \omega).$$

In this case, by letting $\lambda_i(a_i, a_i') > 0$, the designer can relax constraint $(\star)$ and lower $p(\omega)$ below $v(a, \omega)$. In other words, the designer benefits when she manages to induce a player to choose an action which that player would regret choosing ex post, namely, under complete information about (a_{-i}, ω). In order to induce such ex-post "mistakes," the designer *must* withhold some information from player i, either about the state or about the action of others. These observations explain why the designer may prefer partial to full information disclosure. Ultimately, the value of $p(\omega)$ results from a trade-off between the direct payoff $v(a, \omega)$ and her ability to benefit by inducing players to choose ex-post suboptimal actions.[3]

There is, of course, a limit to the extent with which the designer can induce such ex-post suboptimal actions for the players. Indeed, as captured by condition (CS1), she can let $\lambda_i(a_i, a_i') > 0$ if and only if

$$\sum_{\omega \in \Omega, a_{-i} \in A_{-i}} [u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)]\chi(a_i, a_{-i}, \omega) = 0,$$

that is, if and only if player i is *indifferent* between a_i and a_i' given the information that he learns from his recommendation a_i about (a_{-i}, ω). aThus, this condition precisely pins down how the designer has to design her information structure to achieve her goal. If indifference does not hold, she is either providing too much or too little information to the players.

These observations suggest an economic interpretation for each $p(\omega)$ and the dual objective. We can view the design of an information structure as doing the following. To form a signal profile s and determine its meaning for the players, the designer 'buys' the probability that s occurs in right proportions from each state ω, which can be viewed as a 'seller.' Each seller ω has a stock $\mu(\omega)$ of probability to sell to all signal profiles. Thus, as usual, $p(\omega)$ captures the shadow value of that probability stock $\mu(\omega)$. Given this, the dual objective can be interpreted as minimizing the value of having 'one extra unit' of total supply of probability. Intuitively, if this value were not minimized, then the current stock of probability could be used more effectively. Equivalently, a better information structure could be designed.

Through the lens of this interpretation, we can view $p(\omega)$ as the unit price the designer pays to seller ω. To see this, fix an action profile a. If the designer buys probability from

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[3] Note that if π is fully reveals to every player ω *and* the actions of others, we must have $p(\omega) = v(a, \omega)$.

seller ω to generate a (i.e., $\chi(a,\omega) > 0$), she has to pay

$$p(\omega) = v(a,\omega) + \sum_{i\in N} \sum_{a'_i \in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(a'_i, a_{-i}, \omega)]\lambda_i(a_i, a'_i).$$

This quoted price has three components. First, a *baseline* price, given by $v(a,\omega)$. This suggests that, everything else equal, the more the designer gains by inducing an outcome a in state ω, the higher the price seller ω will charge. Second, the seller applies a *discount* to the baseline price for each player that the designer induces to make an ex-post mistake, that is, to choose an action that is ex-post strictly sub-optimal. Third, a *penalty* is instead applied to the baseline price for each player that the designer induces to choose an ex-post strictly optimal action. Reaping these discounts while avoiding the penalties gives rise to trade-offs that uncover the essence of the information-design problem.

Triggering a discount requires setting the corresponding λ_i strictly positive. However, λ_i does not depend on a_{-i} nor ω and so cannot be tailored to only the situations where a_i is ex-post suboptimal for player i. Thus, even if $u_i(a_i, a_{-i}, \omega) > u_i(a'_i, a_{-i}, \omega)$, the designer may still set $\lambda_i(a_i, a'_i) > 0$. This happens if the recommendation a_i arises in two (or more) states, say, ω and ω', or together with two (or more) recommendations for the other players, say, a_{-i} and a'_{-i}. When player i's ex-post preference between a_i and a'_i depends on ω (holding a_{-i} fixed), a trade-off arises between states for the behavior of a single player. This trade-off has global effects on ps of different states and its solution depends on their likelihood under μ. By contrast, when player i's ex-post preference between a_i and a'_i depends on a_{-i} (holding ω fixed), a trade-off arises within state ω and between players' behaviors for that state. This has only local effects on $p(\omega)$ and is managed through the choice of a.[4]

4.3 Properties of Optimal Design

The previous discussion reveals some general properties of optimal information design. To state them concisely, let

$$Q(a,\omega|\lambda) = \sum_{i=1}^{N} \sum_{a'_i \in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(a'_i, a_{-i}, \omega)]\lambda_i(a_i, a'_i).$$

We can interpret $Q(a,\omega|\lambda)$ as a measure of persuasion, namely, the extent to which the designer manages to *exploit information* to her advantage when inducing a in state ω.

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[4] Note that this within-state trade-off can arise only in problems with multiple players. However, the between-state trade-off also arises in Bayesian persuasion with a single receiver.

We will say that information is successfully exploited at (a, ω) if $Q(a, \omega|\lambda) < 0$ and hence $p(\omega) < v(a, \omega)$.

Proposition 3. *Let χ be an optimal solution of $\mathcal{P}$ and let (a, ω) have positive probability under χ. Then, information is successfully exploited at (a, ω) if, for some player i and some deviation $a'_i \in A_i$, we have that*

1. *ex post, player i strictly prefers a'_i to a_i,*
2. *after learning only recommendation a_i, player i is indifferent between a_i and a'_i.*

This result follows from the previous observations and so the proof is omitted.

One can view this property as an analog and a generalization to multi-player contexts of properties of the optimal information structures in the single-receiver setting of Kamenica and Gentzkow (2011) (hereafter, KG). By Proposition 3, the designer can improve her objective ––in the sense of lowering $p(\omega)$––by inducing some player to choose actions that are strictly dominated ex-post. In KG's judge example, the prosecutor improves her payoff by inducing the judge to convict the plaintiff with positive probability when he is actually innocent, which is strictly suboptimal for the judge. More generally, a key principle of Bayesian persuasion is that the designer benefits by pooling "bad" and "good" states in the same signal: As long as the receiver remains sufficiently confident about the good states, such pooling increases the chances of desirable outcomes for the designer.

This being said, the single- and multi-players problems differ in an important way. In the former, the designer only faces a trade-off *between* states. For example, she can recommend an action to the receiver which happens to be ex-post suboptimal in a state but strictly optimal in an other.[5] In games, instead, the designer faces an additional trade-off, not just *between* different states, but also *within* a given state. For example, she can recommend an action to some player that is ex-post suboptimal or not depending on the actions that other players take in that same state. This interdependence of players' incentives complicates the search for optimal information structures.

A second aspect that is highlighted by the result above is the optimal way to induce ex-post suboptimal actions. Information has to be designed so as to render the involved

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[5] In KG's main example, the recommendation to convict the plaintiff arises in both states. It is strictly suboptimal to follow such recommendation when the plaintiff is innocent, which allows the prosecutor to lower $p(\mathit{innocent})$ below $v(\mathit{convict}, \mathit{innocent})$. But it is strictly dominant in the guilty state, which constrains $p(\mathit{guilty})$ above $v(\mathit{convict}, \mathit{guilty})$. It is optimal for the designer-prosecutor to solve the trade-off in this way because, as we know, in that example $\mu(\mathit{guilty}) > \mu(\mathit{innocent})$.

players exactly indifferent between the recommended action and a possible alternative. Again, by analogy, in KG's main example, the optimal information structure makes the judge exactly indifferent between following the "recommendation" to convict and the deviation to acquit. More generally, Proposition 5 in KG shows that whenever an optimal signal induces the receiver to have an interior belief (a necessary condition to lead to ex-post mistakes), he must be indifferent between the recommended action and some alternative. When studying information design in games, besides the fact that there are multiple players who can be induced to choose ex-post suboptimal actions, the difference is that, for each player i, the relevant "state" is (a_{-i}, ω). Therefore, the designer can also manipulate player i's uncertainty about his opponents' behavior to induce a suboptimal a_i's.

Another general property that we can derive from the dual is the following.

Proposition 4. *Let χ be an optimal solution of $\mathcal{P}$ and let (a', ω) and (a'', ω) have positive probability under χ. If $v(a', \omega) > v(a'', \omega)$, then information is exploited more at (a', ω) than at (a'', ω), that is, $Q(a', \omega|\lambda) < Q(a'', \omega|\lambda)$.*

Proposition 4 can also be seen as generalizing properties of Bayesian persuasion. Again, in KG's main example, in the state where the plaintiff is innocent, the optimal information structure induces two recommendations (i.e., acquit and convict) and the prosecutor strictly prefers to convict. Moreover, the prosecutor induces the judge to choose a dominated action *only* for the recommendation she likes better. Proposition 4 significantly generalizes the insights from KG's example by showing that this property of the solution holds in general problems, where multiple players interact strategically and the designer's payoff may also depend on the state.

5 Illustrative Examples: Investment Games

5.1 The Problem

This example is borrowed from Bergemann and Morris (2018). Its goal is to illustrate the dual approach to solving the information-design problem and compare it with the more direct approach followed by Bergemann and Morris (2018).

The basic game is as follows. Two firms have to choose whether to invest (y) in a project or not (n). That is, $A_i = \{y, n\}$, $i \in N$. The project can be either good or bad: $\Omega = \{g, b\}$. Firm i's payoffs $u_i(a_1, a_2, \omega)$ are described in Table 1. The designer (the "government") wants to foster investment, irrespective of the quality of the project. Its payoff is given by

$$v(a_1, a_2) = \mathbb{I}\{a_1 = y\} + \mathbb{I}\{a_2 = y\},$$

where $\mathbb{I}\{\cdot\}$ is the indicator function.

		Firm 2	
		y	n
Firm 1	y	$\varepsilon - 1, \varepsilon - 1$	$-1, 0$
	n	$0, -1$	$0, 0$

$\omega = b$

		Firm 2	
		y	n
Firm 1	y	$\varepsilon + q, \varepsilon + q$	$q, 0$
	n	$0, q$	$0, 0$

$\omega = g$

Table 1: Firms' payoffs in G

We make the following assumptions on parameters. The prior $\mu(g) = \mu(b) = \frac{1}{2}$ and $0 < q < 1$. Thus, without additional information neither firm wants to invest. This parametrization is convenient because, if $\varepsilon > 0$, the game features strategic complementarities; if $\varepsilon < 0$, instead, the game features strategic substitutabilities. Finally, we assume that ε is small so that y is dominant in state g and n is dominant in state b—specifically, $|\varepsilon| \leq q - \frac{1}{2}$.

5.2 Independent Decisions: the Single-Receiver Case

Consider first the case of $\varepsilon = 0$. In this case, there are no payoff-externalities, so the firms' decisions are independent of each other. Thus, each firm's problem can be solved independently and, after some relabeling, it becomes equivalent to the leading example in KG. We will therefore solve this case as if there were only one firm and omitting the index i. Let

$$w(a, \omega, \lambda) = v(a, \omega) + \sum_{a' \in A} [u(a, \omega) - u_i(a', \omega)]\lambda(a, a').$$

Given this, the dual becomes

$$\begin{aligned} \mathcal{P}^* \quad = \quad & \min_{p \in \mathbb{R}^{\Omega},\, \lambda \in \mathbb{R}^{A \times A}} \quad \frac{1}{2}p(b) + \frac{1}{2}p(g), \\ & \text{sub to,} \qquad \forall \omega \in \Omega, a \in A, a' \in A : \\ & (\star) \qquad p(\omega) \geq w(a, \omega, \lambda), \\ & (P_\lambda) \qquad \lambda(a, a') \geq 0. \end{aligned}$$

Fix an arbitrary $\lambda \geq 0$ and consider constraint ($\star$). When the state if "favorable" for the designer, i.e. $\omega = g$, the function w takes the following values:

$$w(y, g, \lambda) = 1 + q\lambda(y, n),$$
$$w(n, g, \lambda) = -q\lambda(n, y).$$

Since $w(y, g, \lambda) > w(n, g, \lambda)$ for every λ, constraint ($\star$) implies that

$$p(g) = 1 + q\lambda(y, n).$$

Similarly, when the state is "unfavorable," i.e. $\omega = b$, we have that

$$w(y, b, \lambda) = 1 - \lambda(y, n),$$
$$w(n, b, \lambda) = \lambda(n, y).$$

Thus, in this case the maximum of right-hand side of ($\star$) and so the tighter constraint on $p(b)$ depends on the choice of λ.

Using these steps, we can re-write the dual problem as

$$\mathcal{P}^* = \min_{\lambda(y,n),\lambda(n,y)} \Big(\frac{1}{2}\big(1 + q\lambda(y, n)\big) + \frac{1}{2}\max\{1 - \lambda(y, n), \lambda(n, y)\}\Big)$$

It is easy to see that it is optimal to set $\lambda(n, y) = 0$: It is not possible to set it lower than zero, and any higher level renders it harder to minimize the whole function. Given this and that $0 < q < 1$, it is optimal to set $\lambda(y, n)$ as small as possible while satisfying $1 - \lambda(y, n) \leq 0 = w(n, b, \lambda)$. Therefore, $\lambda(y, n) = 1$. In sum, this yields $p(g) = 1 + q$, $p(b) = 0$, and a value $\mathcal{P}^* = \frac{1+q}{2}$.

While the solution to $\mathcal{P}^*$ tells us the optimal payoff of the designer, it reveals little about the optimal information structure. It is immediate to see that her payoff is $\frac{1}{2}$ from a fully informative structure and 0 from a fully uninformative structure. Therefore, we can at least conclude that neither of these structures is optimal.

To calculate the optimal x, we can use the complementary slackness conditions, as pointed out in Proposition 2. First, by (CS3), since $w(y, g, \lambda) > w(n, g, \lambda)$ for all λ, we must have $\chi(n, g) = 0$ and so $x(y|g) = 1$. Second, since $\lambda(y, n) > 0$, by (CS1) the firm must be indifferent between the two actions conditional on being recommended y. This requires that

$$-x(y|b) + qx(y|g) = 0.$$

Therefore, $x(n|b) = q$. Note that under such x, conditional on being recommended n, the firm knows that the state is b and strictly prefers n to y, which is consistent with $\lambda(n, y) = 0$ and (CS1). Also, both $\chi(n, b) > 0$ and $\chi(y, b) > 0$ and

$$0 = p(b) = w(y, b, \lambda) = w(n, b, \lambda)$$

for the optimal λ, which satisfies (CS3). Finally, note that

$$Q(n, b|\lambda) = 0 > -1 = -\lambda(y, n) = Q(y, b|\lambda),$$

which is an instance of the general property in Proposition 4: The designer "exploits information" more when inducing the firm to invest in the unfavorable state b, which the designer prefers to not invest.

5.3 Strategic Decisions: the Multi-Player Case

We now consider the case of $\varepsilon \neq 0$. In this case, firms interact strategically in the context of a "proper" game. Relative to the previous case, the objective of the dual is unchanged. The dual constraint $(\star)$, instead, becomes the following: For all $i = 1, 2$, $\omega \in \Omega$, $a \in A$, and $a_i \in A_i$,

$$(\star)\ p(\omega) \geq w(a, \omega, \lambda) := v(a, \omega) + \sum_{i=1,2} \sum_{a_i' \in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)]\lambda_i(a_i, a_i')$$

and $\lambda_i(a_i, a_i') \geq 0$. As argued by Bergemann and Morris (2018), it is without loss of generality to consider symmetric solutions: This is because the problem is symmetric and the feasible set $X(G)$ is convex. For the dual, this means that $\chi(y, n, \omega) = \chi(n, y, \omega)$ for all ω and $\lambda_1 = \lambda_2$.

Given this, we can easily compute the function $w(a, \omega, \lambda)$. When $\omega = b$, we have

$$w(a_1, a_2, b, \lambda) = \begin{cases} 2 - 2[1-\varepsilon]\lambda(y, n) & \text{if } (a_1, a_2) = (y, y) \\ 1 - \lambda(y, n) + [1-\varepsilon]\lambda(n, y) & \text{if } (a_1, a_2) = (y, n) \\ 1 + [1-\varepsilon]\lambda(n, y) - \lambda(y, n) & \text{if } (a_1, a_2) = (n, y) \\ 2\lambda(n, y) & \text{if } (a_1, a_2) = (n, n). \end{cases}$$

Similarly, when $\omega = g$, we have

$$w(a_1, a_2, g, \lambda) = \begin{cases} 2 + 2[q+\varepsilon]\lambda(y, n) & \text{if } (a_1, a_2) = (y, y) \\ 1 + q\lambda(y, n) - [q+\varepsilon]\lambda(n, y) & \text{if } (a_1, a_2) = (y, n) \\ 1 - [q+\varepsilon]\lambda(n, y) + q\lambda(y, n) & \text{if } (a_1, a_2) = (n, y) \\ -2q\lambda(n, y) & \text{if } (a_1, a_2) = (n, n). \end{cases}$$

Using this, we can quickly solve $\mathcal{P}^*$ and derive the optimal information structure in few steps:

Step 1: Since $0 < q < 1$ and $|\varepsilon| \leq q - \frac{1}{2}$ imply that $|\varepsilon| \leq \frac{q}{2}$, we have that for all λ

$$w(y, y, g, \lambda) > w(y, n, g, \lambda) = w(n, y, g, \lambda) > w(n, n, g, \lambda).$$

Therefore, $p(g) = 2 + 2[q + \varepsilon]\lambda(y, n)$ and $x(y, y|g) = 1$ by (CS3). That is, upon inspection of the dual, we can immediately conclude that the designer will never recommend any profile other than $a = (y, y)$ conditional on state $\omega = g$.

Step 2: Since $x(y, y|g) = 1$, we must have $x(n, n|b) > 0$ which implies by (CS3) that

$$p(b) = 2\lambda(n, y) = 0.$$

Otherwise, at least one firm always receives the recommendation y, which renders x uninformative for that firm about the state; hence, that firm would never choose y, contradicting $x(y, y|g) > 0$. (Recall that an optimal solution must, first of all, be feasible, which in our case means obedient.) Since the recommendation n reveals that the state is b, the firm cannot be indifferent and so $\lambda(n, y) = 0$ by (CS1).

Step 3: Two cases remain to be considered:

Case 1. $0 = p(b) = w(y, y, b, \lambda) > w(y, n, b, \lambda) = w(n, y, b, \lambda)$ if and only if $\lambda(y, n) = \frac{1}{1-\varepsilon}$ and $\varepsilon > 0$. In this case, $x(y, n|b) = x(n, y|b) = 0$ by (CS3). Also, since the candidate $\lambda(y, n) > 0$, each firm must be indifferent by (CS1) after receiving recommendation y. This holds provided that

$$\begin{aligned} 0 =& \mu(g) \left[u_i(y, y, g)x(y, y|g) + u_i(y, n, g)x(y, n|g)\right] \\ &+ \mu(b) \left[u_i(y, y, b)x(y, y|b) + u_i(y, n, b)x(y, n|b)\right] \end{aligned}$$

or equivalently

$$(q + \varepsilon)x(y, y|g) + qx(y, n|g) = (1 - \varepsilon)x(y, y|b) + x(y, n|b),$$

which boils down to

$$(q + \varepsilon) = (1 - \varepsilon)x(y, y|b) + x(y, n|b).$$

Therefore, we conclude that $x(y, y|b) = \frac{q+\varepsilon}{1-\varepsilon}$. The value of $\mathcal{P}^*$ in this case is

$$\mathcal{P}^* = \frac{1}{2}p(g) = 1 + \frac{q + \varepsilon}{1 - \varepsilon}.$$

		Firm 2	
		y	n
Firm 1	y	$\frac{q+\varepsilon}{1-\varepsilon}$	0
	n	0	$1 - \frac{q+\varepsilon}{1-\varepsilon}$

$x(a_1, a_2|\omega = b)$

		Firm 2	
		y	n
Firm 1	y	1	0
	n	0	0

$x(a_1, a_2|\omega = g)$

Table 2: Optimal x for strategic complements (i.e., $\varepsilon > 0$)

		Firm 2	
		y	n
Firm 1	y	0	$q+\varepsilon$
	n	$q+\varepsilon$	$1-2q-2\varepsilon$

$x(a_1, a_2|\omega = b)$

		Firm 2	
		y	n
Firm 1	y	1	0
	n	0	0

$x(a_1, a_2|\omega = g)$

Table 3: Optimal x for strategic substitutes (i.e., $\varepsilon < 0$)

Case 2. $0 = p(b) = w(y, n, b, \lambda) = w(n, y, b, \lambda) > w(y, y, b, \lambda)$ if and only if $\lambda(y, n) = 1$ and $\varepsilon < 0$. In this case, $x(y, y|b) = 0$ by (CS3). Again, since the candidate $\lambda(y, n) > 0$, each firm must be indifferent by (CS1) after receiving recommendation y. This again requires that

$$(q + \varepsilon) = (1 - \varepsilon)x(y, y|b) + x(y, n|b).$$

Therefore, $x(y, n|b) = q + \varepsilon$. The value of $\mathcal{P}^*$ in this case is

$$\mathcal{P}^* = \frac{1}{2}p(g) = 1 + q + \varepsilon.$$

Tables 2 and 3 report the optimal information structures. As already noted by Bergemann and Morris (2018), in the case of strategic complementarities, the optimal x features public information. Inutitively, this is because with strategic complementarities each firm is more willing to invest when recommended y knowing that, if the state is b, also the other firm will invest, which reduces the loss of the bad decision. This higher willingness to invest allows the designer to pool more the unfavorable state b with the favorable g in the recommendation y, thereby increasing the overall chances of investment. In the case of strategic substitutabilities, instead, the optimal x features private information. Intuitively, this is because with strategic substitutabilities instead each firm is more willing to invest when recommended y knowing that, if the state is b, the other firm will not invest.

A Proofs

Proof of Proposition 1. It is convenient to first write $\mathcal{P}$ in matrix form. Fix an arbitrary total ordering of the set $A \times \Omega$. We denote by $v \in \mathbb{R}^{A\times\Omega}$ the vector whose entry corresponding to (a, ω) is $v(\omega, a)$. For every player i, let $U_i \in \mathbb{R}^{(A_i\times A_i)\times(A\times\Omega)}$ be a matrix thus defined: For each row $(a_i', a_i'') \in A_i \times A_i$ and column $(a, \omega) \in A \times \Omega$, denote

$$U_i((a_i', a_i''), (a, \omega)) = \begin{cases} u_i(a_i', a_{-i}, \omega) - u_i(a_i'', a_{-i}, \omega) & \text{if } a_i' = a_i \\ 0 & \text{else.} \end{cases}$$

Define the matrix U by stacking all the matrices $\{U_i\}_{i\in N}$ on top each other. Finally, define the indicator matrix $I \in \{0, 1\}^{\Omega\times(A\times\Omega)}$ such that, for each row ω' and column (a, ω'),

$$I(\omega', (a, \omega)) := \begin{cases} 1 & \text{if } \omega' = \omega \\ 0 & \text{else.} \end{cases}$$

With this notation, $\mathcal{P}$ can be written as follows:

$$\begin{aligned} \mathcal{P} \quad = \quad \max_{\chi} \quad & v'\chi \\ \text{sub to:} \quad U\chi \quad & \geq \quad \mathbf{0}, \\ I\chi \quad & = \quad \mu, \\ \chi \quad & \geq \quad 0. \end{aligned} \tag{$\mathcal{P}$}$$

Given this, by standard linear-programming arguments[6] the dual $\mathcal{P}^*$ can be written as

$$\min_{\lambda,\zeta} -\lambda'\mathbf{0} - \zeta'\mu$$

subject to for all $i = 1, \ldots, N$ and $a_i, a_i' \in A_i$,

$$\lambda_i(a_i, a_i') \geq 0,$$

for all $\omega \in \Omega$, $\zeta(\omega) \in \mathbb{R}$ (i.e., it is unconstrained), and for all $(a, \omega) \in A \times \Omega$

$$v(a, \omega) \leq -\zeta(\omega) - \sum_{i=1}^{N} \left\{ \sum_{a_i'\in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)]\lambda_i(a_i, a_i') \right\}.$$

Letting $p = -\zeta$, the objective simplifies to

$$\min_{\lambda,p} \sum_{\omega\in\Omega} p(\omega)\mu(\omega).$$

The second set of constraints can be written as

$$p(\omega) \geq v(a, \omega) + \sum_{i=1}^{N} \sum_{a_i'\in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)]\lambda_i(a_i, a_i').$$

□

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[6] See Bertsimas and Tsitsiklis (1997) for a reference.

Proof of Proposition 4. By condition (CS3), it must be that

$$\begin{aligned} 0 &< v(a', \omega) - v(a'', \omega) \\ &= \sum_{i=1}^{N} \sum_{\hat{a}_i \in A_i} [u_i(a''_i, a''_{-i}, \omega) - u_i(\hat{a}_i, a''_{-i}, \omega)] \lambda_i(a''_i, \hat{a}_i) \\ &\quad - \sum_{i=1}^{N} \sum_{\hat{a}_i \in A_i} [u_i(a_i, a_{-i}, \omega) - u_i(\hat{a}_i, a_{-i}, \omega)] \lambda_i(a_i, \hat{a}_i) \end{aligned}$$

and therefore $Q(a', \omega|\lambda) < Q(a'', \omega|\lambda)$. □

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