Math for Economists I - Midterm Exam 2013
Suggested Solutions

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Exercise 1

Let (x_n) and (y_n) be two Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(x_n, y_n)\}$ converges.

Proof. Let $\varepsilon > 0$ and notice that, by Cauchyness of the two sequences, there is a $N \in \mathbb{N}$ such that for all $m, n > N$ we have both $d(x_m, x_n) < \varepsilon/2$ and $d(y_m, y_n) < \varepsilon/2$. Using triangle inequality we have $d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$, that is $d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n)$. Since the role of n and m is interchangeable we also have $d(x_m, y_m) - d(x_n, y_n) \leq d(x_n, x_m) + d(y_m, y_n)$. Putting all together

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_n, x_m) + d(y_m, y_n) < \varepsilon.$$

That is the sequence $\{d(x_m, y_m)\}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence converges. $\square$

Exercise 2

(a) USC iff $\text{ipo}(\varphi)$ is closed in $X \times \mathbb{R}$.

Proof: $\Rightarrow$). We show that $\text{ipo}(\varphi)^c$ is open. Take $(x, y) \notin \text{ipo}(\varphi)$. Then, $\varphi(x) < y$. Define $\varepsilon := \frac{y - \varphi(x)}{2} > 0$. Notice that by definition of USC there is a $\delta > 0$ s.t. for all $z \in X$ s.t. $d(x, z) < \delta$ we have $\varphi(z) < \varphi(x) + \varepsilon$. But $\varphi(x) + \varepsilon < y$ by assumption, hence for all $z \in X$ s.t. $d(x, z) < \delta$ we have $\varphi(z) < y$, i.e. $(z, y) \notin \text{ipo}(\varphi)$. In particular, there is a ball in $X \times \mathbb{R}$ of radius ε entirely contained in $\text{ipo}(\varphi)^c$. Hence, $\text{ipo}(\varphi)^c$ is open.

$\Leftarrow$). Let $\text{ipo}(\varphi)$ be closed and fix a point $x \in X$. Consider a converging sequence $x_m \to x$. Let $y := \limsup \varphi(x_m)$ and notice that $(x, y) \in \text{ipo}(\varphi)$, since $\text{ipo}(\varphi)$ is closed in the product metric and y is the limit point of the greatest convergent subsequence of $\varphi(x_m)$. Hence $\varphi(x) \geq y = \limsup \varphi(x_m)$. By definition of $\limsup$, this implies φ is USC at x. $\square$

(b) Let φ_i be USC for all $i \in \mathbb{N}$. Show that $\varphi := \inf \varphi_i$ is USC.

Proof: Notice that $\text{ipo}(\varphi) = \bigcap \text{ipo}(\varphi_i)$. Indeed, $(x, y) \in \text{ipo}(\varphi)$ iff $\inf \varphi_i(x) \geq y$ iff for all $i \in \mathbb{N}$ we have $\varphi_i(x) \geq y$ iff for all $i \in \mathbb{N}$, $(x, y) \in \text{ipo}(\varphi_i)$ iff $(x, y) \in \bigcap \text{ipo}(\varphi_i)$. By

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assumption and (a), $\text{ipo}(\varphi_i)$ is closed for all i. Hence, $\text{ipo}(\varphi)$ is closed. By (a), again, φ is USC. $\square$

(c) Show $\max\{\varphi_1, \varphi_2\}$ is USC. Is $\sup_i \varphi_i$ USC?

Proof: A similar argument can lead us to show $\text{ipo}(\max\{\varphi_1, \varphi_2\}) = \text{ipo}(\varphi_1) \cup \text{ipo}(\varphi_2)$, from which it is immediato to argue that $\text{ipo}(\max\{\varphi_1, \varphi_2\})$ is closed, hence $\max\{\varphi_1, \varphi_2\}$ USC. Similarly, $\text{ipo}\sup\{\varphi_i\} = \bigcup \text{ipo}(\varphi_i)$, from which it is immediate to argue that the LHS is not necessarily closed, hence $\sup\{\varphi_i\}$ is not necessarily USC. $\square$

EXERCISE 3

Define $\Phi : C[0,1] \to C[0,1]$

$$\Phi(f)(x) := \frac{2}{3}x^2 + \int_0^x t^2 f(t)^2 dt \qquad \forall x \in [0,1]$$

(a) Show Φ is continuous, wrt d_∞.

Proof: Take $f, g \in C[0,1]$ and notice that these are bounded by $M_f, M_g \in \mathbb{R}$. Let $x \in [0,1]$ and notice

$$|\Phi(f)(x) - \Phi(g)(x)| \leq \int_0^x t^2 |f(t)^2 - g(t)^2| dt = \int_0^x t^2 |(f(t) - g(t))(f(t) + g(t))| dt \leq$$

$$\leq \int_0^x t^2 |\sup_{t\in[0,1]} (f(t) - g(t)) \sup_{t\in[0,1]} (f(t) + g(t))| dt = d_\infty(f,g)\frac{(M_f + M_g)}{3}$$

Notice that if $d_\infty(f,g) < \delta$ then $M_g < M_f + \delta$. This is because $d_\infty(f,g) < \delta$ implies $|M_f - M_g| < \delta$. Hence, for any $\varepsilon > 0$, putting $\delta < \frac{3\varepsilon}{1+2M_f}$, we have that for all $g \in C[0,1]$ s.t. $d_\infty(f,g) < \delta$ implies

$$d_\infty(\Phi(f)(x), \Phi(g)(x)) < d_\infty(f,g)\frac{(M_f + M_g)}{3} < \delta\frac{(2M_f + \delta)}{3} < \varepsilon.$$

(b) Show that there exists $f \in C[0,1]$ such that $d_\infty(f, 0) \leq 1$ and $f = \Phi(f)$.

Proof: First notice that $\mathcal{F} := \{f \in C[0,1] : d_\infty(f,0) \leq 1\}$ is a closed subset of $C[0,1]$ hence it is complete. (Take a converging sequence in $\mathcal{F}$ and show its limit is continuous and satisfies $d_\infty(f,0) \leq 1$). Moreover we have that Φ maps elements in $\mathcal{F}$ into elements in $\mathcal{F}$. Indeed, letting $f \in \mathcal{F}$,

$$|\Phi(f)(x)| = \frac{2}{3}x^2 + \int_0^x t^2 f(t)^2 dt \leq \frac{2}{3} + \frac{1}{3} = 1$$

for any $x \in [0,1]$. Now we show Φ is a contraction,

$$|\Phi(f)(x) - \Phi(g)(x)| \leq d_\infty(f,g)\frac{(M_f + M_g)}{3} < \frac{2}{3} d_\infty(f,g)$$

hence, $d_\infty(\Phi(f)(x), \Phi(g)(x)) < \frac{2}{3} d_\infty(f,g)$. Applying Banach concludes the proof. $\square$

EXERCISE 4

Show that

$$K := \{x \in l^2 | |x_i| \leq \frac{1}{i} \text{ for all } i \in \mathbb{N}\}$$

is compact in (l^2, d_2).

Proof: It is enough to show that every sequence has a convergent subsequence. Let $(x^m) \in K^\infty$. Notice that $(x_1^m) \in [-1,1]$. As such (Bolzano-Weierstrass), there is a subsequence $(x_1^{m_k}) \in [-1,1]^\infty$ s.t. $x_1^{m_k} \to x_1 \in [-1,1]$ in the metric $|\cdot|$. Notice that $(x^{m_k}) \in K^\infty$. Hence, $(x_2^{m_k}) \in [-\frac{1}{2}, \frac{1}{2}]^\infty$. Again, being a sequence in a compact set, it has a convergent subsequence, call it $(x_2^{m_{k_l}}) \in [-\frac{1}{2}, \frac{1}{2}]^\infty$, s.t. $x_2^{m_{k_l}} \to x_2 \in [-\frac{1}{2}, \frac{1}{2}]$ in the metric $|\cdot|$. Notice that $\{m_{k_l}\} \subset \{m_k\}$. Proceeding inductively, we can find a subsequence (x^{m_t}) of $(x^m) \in K^\infty$ s.t. $x^{m_t} \to x := (x_1, x_2, \dots)$ pointwise, with $x \in K$ by construction. We are left to show $x^{m_t} \to x$ in d_2, too. Choose ε arbitrary. We want to show $d_2(x^{m_t}, x) < \varepsilon$ for sufficiently large m_t. To this purpose notice that there exists a N s.t.

$$d_2(x^{m_t}, x)^2 = \sum_{i=1}^{\infty} |x_i^{m_t} - x_i|^2 = \sum_{i=1}^{N} |x_i^{m_t} - x_i|^2 + \sum_{i=N+1}^{\infty} |x_i^{m_t} - x_i|^2 < \sum_{i=1}^{N} |x_i^{m_t} - x_i|^2 + \frac{\varepsilon}{2}.$$

Indeed, $\sum_{i=N'+1}^{\infty} |x_i^{m_t} - x_i|^2 < \sum_{i=N'+1}^{\infty} \frac{4}{i^2}$ which goes to zero as N' grows large. Hence, there must be a N such that for all $N' > N$, we have $\sum_{i=N'+1}^{\infty} |x_i^{m_t} - x_i|^2 < \varepsilon/2$. Let's focus now on the latter term of the summation. Recall that, for all $i \in \{1, \dots, N\}$, there is a M_i such that for all $m_t > M_i$ we have $|x_i^{m_t} - x_i|^2 < \frac{\varepsilon}{2N}$. Fix $M := \max_i M_i$. Then

$$\sum_{i=1}^{N} |x_i^{m_t} - x_i|^2 < \sum_{i=1}^{N} \frac{\varepsilon}{2N} = \frac{\varepsilon}{2}.$$

Putting all together, for all $m_t > M$,

$$d_2(x^{m_t}, x)^2 = \sum_{i=1}^{\infty} |x_i^{m_t} - x_i|^2 = \sum_{i=1}^{N} |x_i^{m_t} - x_i|^2 + \sum_{i=N+1}^{\infty} |x_i^{m_t} - x_i|^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since, ε was arbitrary, we are done.[1] □

[1] **Alternative Proof:** Recall that a set is compact iff it is complete and totally bounded. To argue that K is complete we will show it is a closed subset of l^2, a complete metric space. Take $x \notin K$. Then there exists some $n \in \mathbb{N}$ such that $|x_n| > \frac{1}{n}$. Now let $\varepsilon < |x_n| - \frac{1}{n}$ and notice that $x \in B(x, \varepsilon) \cap K = \emptyset$. That is, $B(x, \varepsilon) \subseteq K^c$. Since x was arbitrary, K^c is open. To show K is totally bounded, let > 0. Fix $N > \frac{1}{\varepsilon}$. Notice that the product space $C = \times_{n=1}^{N} [-\frac{1}{n}, \frac{1}{n}]$ is compact, hence totally bounded. Hence, there exists a finite collection of points in C, $\{x_1, \dots, x_K\}$ such that for all $y \in C$ there is a x_k such that $d_2(y, x_k) < \varepsilon$. Now we want to find ε-net for K. For all $k \in [K]$, define $z_m^k := x_m^k \mathbb{1}_{m \leq N}$ and notice that for all $y \in K$, there is a z_k such that $d_2(y, z_k) < \varepsilon$.