Math for Econ I: Solutions

Problem Set 3*

Exercise 1

Let (Y, d) be a metric space.[1]

(a) True or False: If Y is bounded, $(2^Y \setminus \{\emptyset\}, d_H)$ is a semimetric space.

(b) Give an example of Y s.t. $(2^Y \setminus \{\emptyset\}, d_H)$ is not a metric space.

(c) Let $\mathbf{c}(Y)$ be the class of all nonempty compact subset of Y. Show that $(\mathbf{c}(Y), d_H)$ is a metric space.

Proof: (*Notation*: let $\mathcal{P}(Y) = 2^Y \setminus \{\emptyset\}$). (**a**). We need to show d_H is a semi-metric.[2] Let (Y, d) be a bounded metric space. For any $A, B \in \mathcal{P}(Y)$, A, B are bounded, hence $w(A, B) = \sup_{a \in A} d(a, B) < \mathrm{diam}(Y) < \infty$. That is, w is (positive) real valued. It follows that $d_H : \mathcal{P}(Y)^2 \to \mathbb{R}_+$. To prove (i), take $A \in \mathcal{P}(Y)$. Then, $d_H(A, A) = w(A, A) = \sup_{a \in A} d(a, A) = \sup_{a \in A} \inf_{a' \in A} d(a, a') = 0$. To prove (ii), simply notice that $d_H(A, B) = \max\{w(A, B), w(B, A)\} = \max\{w(B, A), w(A, B)\} =: d_H(B, A)$. To prove (iii), consider any $A, B, C \in \mathcal{P}(Y)$ and $(a, b, c) \in A \times B \times C$. Since d is a metric, we have

$$d(a, b) \leq d(a, c) + d(c, b)$$

Taking $\inf_{b \in B}$ on both sides we get

$$d(a, B) \leq d(a, c) + d(c, B) \leq d(a, c) + \sup_{c \in C} d(c, B) = d(a, c) + w(C, B).$$

Taking $\inf_{c \in C}$ on both sides we get

$$d(a, B) \leq d(a, C) + w(C, B).$$

*Please mail typos and comment to jp2841@nyu.edu.

[1] For all subset $A, B \subset Y$, define $w(A, B) := \sup\{d(z, B) : z \in A\}$ and the function $d_H : 2^Y \setminus \{\emptyset\} \times 2^Y \setminus \{\emptyset\} \to \bar{\mathbb{R}}$ with $d_H(A, B) := \max\{w(A, B), w(B, A)\}$.

[2] A semi-metric on X is a function $d : X^2 \to \mathbb{R}_+$ s.t. for all $x, y, z \in X$, (i) $d(x, x) = 0$, (ii) $d(x, y) = d(y, x)$ and (iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Finally, taking the $\sup_a \in A$ on both sides we get

$$w(A,B) \leq w(A,C) + w(C,B).$$

Now, without loss of generality, put $w(A,B) \geq w(B,A)$. Thanks to the inequality we have just established we have,

$$d_H(A,B) = w(A,B) \leq w(A,C) + w(C,B) \leq$$

$$\leq \max\{w(A,C), w(C,A)\} + \max\{w(C,B), w(B,C)\} = d_H(A,C) + d_H(C,B).$$

We have thus proven (i), (ii) and (iii) and we can conclude d_H is a semi-metric. Notice that the only use of boundedness here was to establish that the function d_H was real valued. □

($\mathbf{b}$). Let (Y,d) be s.t. there is a $A \subset Y$ s.t. $\mathrm{cl}A \neq A$. Since, $d(\mathrm{cl}A, A) = 0$, we can conclude that the d_H is not a metric. □

($\mathbf{c}$). Not let $\mathbf{c}(Y)$ be the class of all nonempty compact subset of Y. Suppose that for some $A, B \in \mathbf{c}(Y)$, $d(A,B) = 0$. Then, $0 = w(A,B) = \sup_{a \in A} d(a,B) = \max_{a \in A} d(a,B)$, that is for all $a \in A$, $d(a,B) = 0$. This can be true iff $A \subset \mathrm{cl}(B) = B$. A symmetric argument leads us to conclude that $B \subset A$, hence $A = B$, as we seek. □

EXERCISE 2

Prove that if Y is complete, then so is $(\mathbf{c}(Y), d_H)$. (Hint. Use Cantor Intersection Theorem)

Proof: Let $(A_m) \subset \mathbf{c}(Y)$ be cauchy. We want to show that $A_m \to A \in \mathbf{c}(Y)$ in d_H. For all $m \in \mathbb{N}$, define $B_m := \mathrm{cl}(A_m \cup A_{m+1} \cup \ldots)$. We want to show that B_1 is compact. Being a closed subset of a complete space, it is **complete**. Hence, we only need to show it is **totally bounded**. Suppose not. That is, suppose that there exists a $\varepsilon > 0$ such that B_1 cannot be covered by balls of radius ε centered in finitely many point of B_1. For any $m \in \mathbb{N}$, thanks to compactness, we can find finitely many points in $A_1 \cup \ldots \cup A_m$ s.t. the union of ε-balls centered at these points cover the whole set $A_1 \cup \ldots \cup A_m$. Hence, it must be the case that no B_m is totally bounded. However (A_m) is Cauchy. That is, there exists a $N \in \mathbb{N}$, s.t. $d(A_m, A_n) < \frac{\varepsilon}{2}$ for all $m, n \geq N$. That is, $d(A_N, A_n) < \frac{\varepsilon}{2}$ for all $n \geq N$. Compactness of A_N implies there exists a finite collections of $T_N \subset A_N$, s.t. $\bigcup_{t \in T_N} B(t, \varepsilon) \supset A_N$. Cauchyness implies, $\bigcup_{t \in T_N} B(t, \varepsilon) \supset A_n$ for all $n \geq N$. A contradiction on the fact that B_N is not totally bounded. Hence, B_1 is totally bounded and complete, that is, **compact**.

Since (B_m) is a nested sequence of non-empty closed sets, each one of them is compact. Moreover, all B_m are bounded, or $\text{diam}B_m < \infty$. In particular, $\text{diam}(B_m)$ is a monotonic sequence bounded above by 0. Hence it converges to some real number r. We want to show that $\bigcap B_i \neq \emptyset$. Suppose $r > 0$ and $\bigcap B_m = \emptyset$. Since each B_m is non-empty and (B_m) is decreasing, it must be the case that for some m large enough $\text{diam}B_m < r$, a contradiction. Suppose $r = 0$, then we can just invoke Cantor Frechet Intersection Theorem to argue that $\bigcap^\infty B_m \neq \emptyset$. Notice that $\bigcap^\infty B_m$ is closed (arbitrary intersection of closed sets). Hence, it is compact (closed subset of compact set).

We are left to show $A_m \to \bigcap_m B_m$. That is, for any ε, there is a $m \in \mathbb{N}$ such that $d_H(A_m, \bigcap^\infty B_n) < \varepsilon$. Notice two things. Clearly, for appropriately large m, $d_H(B_m, \bigcap^\infty B_n) < \frac{\varepsilon}{2}$. Moreover, by Cauchyness of (A_m), for appropriately large m', $d_H(A_{m'}, B_{m'}) < \frac{\varepsilon}{2}$. Thus

$$d_H(A_m, \bigcap^\infty B_n) < d_H(A_m, B_m) + d_H(B_m, \bigcap^\infty B_n) < \varepsilon,$$

which conclude the proof. □

Exercise 3

Let X and Y be two metric spaces. Prove that the correspondence $\Gamma : X \rightrightarrows Y$ satisfies the closed graph property if and only if $\text{Gr}(\Gamma)$ is closed in the product space $X \times Y$.

Proof: $\Rightarrow$. Suppose not, that is, suppose $\text{Gr}(\Gamma)$ is not closed. Hence there must exists some sequence in $\text{Gr}(\Gamma)$ converging outside $\text{Gr}(\Gamma)$. Let $(x_m, \Gamma(x_m)) \in \text{Gr}(\Gamma)^\infty$ be such a sequence. We have $x_m \to x$ and $\Gamma(x_m) \to y \notin \Gamma(x)$. But this implies Γ is not closed at x, a contradiction.

$\Leftarrow$. Let $\text{Gr}(\Gamma)$ be closed and choose $x \in X$ arbitrarily. Consider convergent sequences $x_m \in X^\infty$ and $y_m \in Y^\infty$ such that for all $m \in \mathbb{N}$ $y_m \in Gamma(x_m)$. Then $(x_m, y_m) \in \text{Gr}(\Gamma)$ is convergent in the product space. But $\text{Gr}(\Gamma)$ is closed, hence $y \in \Gamma(x)$, that is Γ is closed at x. □

Exercise 4

Let T be a bounded metric space and $\mathcal{F} \subset C(T)$. Define $\Gamma : T \rightrightarrows \mathbb{R}$ by $\Gamma(t) := \cup\{f(t) : f \in \mathcal{F}\}$. Prove that if $\mathcal{F}$ is compact in $(C(T), d_\infty)$, then Γ is upper hemicontinuous and compact valued.

Proof: First I will show Γ is compact valued. Pick arbitrary $t^\star \in T$. Consider a sequence $(y_m) \in \Gamma(t^\star)^\infty$. By definition of Γ, there exists an associated sequence $(f_m) \in \mathcal{F}^\infty$ such that for any $m \in \mathbb{N}$, $y_m = f_m(t^\star)$. Since $\mathcal{F}$ is compact, there is a converging subsequence $f_{m_k} \to f \in \mathbb{F}$ in d_∞, that implies $f_{m_k}(t^\star) \to f(t^\star)$, or $y_{m_k} \to f(t^\star)$. Finally, notice that, since $f \in \mathcal{F}$, $f(t^\star) \in \Gamma(t^\star)$. Hence, $\Gamma(t^\star)$ is sequentially compact.

Second, I will show Γ is UHC. Consider converging sequences $t_m \to t$ and $y_m \to y$, with $y_m \in \Gamma(t_m)$. As before, there exists an associated sequence of functions $f_m \in \mathcal{F}^\infty$, such that for any $m \in \mathbb{N}$, $y_m = f_m(t_m)$. Since $\mathcal{F}$ is compact, there is a converging subsequence $f_{m_k} \to f \in \mathcal{F}$ in d_∞, that implies $f_{m_k}(t_{m_k}) \to f(t_{m_k})$. Moreover, since f is continuous, $f(t_{m_k}) \to f(t)$. Hence, for m_k large enough,

$$|y_{m_k} - f(t)| = |f_{m_k}(t_{m_k}) - f(t)| \leq |f_{m_k}(t_{m_k}) - f(t_{m_k})| + |f(t_{m_k}) - f(t)| < \varepsilon,$$

that is, $y_{m_k} \to f(t) \in \Gamma(t)$. Hence, Γ is UHC. $\square$

Exercise 5

Prove that a subset S of a linear space X is a basis for X iff S is linearly independent and $X = \text{span}(S)$.

Proof: $\Rightarrow$. Let S be a basis and suppose the implication is false. That is let S span X but not linearly independent. There exists a vector $\lambda \neq 0$ s.t. one linear combination $\sum^n \lambda_i s_i = 0$. Wlog let $\lambda_1 \neq 0$. Thus, $s_1 = -\frac{1}{\lambda_1}\sum_2^n \lambda_i s_i$. That is, $\text{span}(S) = \text{span}(S \setminus \{s_1\}) = X$. Since $S \setminus \{s_1\} \subsetneq S$, we have reached a contradiction on the fact that S is a basis (if this is still not clear, go and check the definition of a basis).

$\Leftarrow$. Let S span X and be linearly independent. We only need to check that for no $T \subsetneq S$, we have $\text{span}(T) = X$. It is enough to show[3] that for any $s \in S$, $\text{span}(S \setminus \{s\}) \subsetneq X$. Suppose not, then, since $s \in X$, we have $s = \sum_{i=1}^n \lambda_i s_i$ for finitely many $s_i \in S \setminus \{s\}$. That is, $s - \sum_{i=1}^n \lambda_i s_i = 0$ which amounts to say that S is not linearly independent, a contradiction. $\square$

Exercise 6

Let X and Y be linear spaces. Show that $\dim(X \times Y) = \dim(X) + \dim(Y)$.[4]

[3] Since the operator $\text{span} : 2^X \to 2^X$ is $\supseteq$-order preserving.

[4] Clearly, $(X \times Y, +, \cdot)$ is a linear space where the operation of sum and scalar multiplication are inherited from those in $(X, \oplus_X, \odot_X)$ and $(Y, , \oplus_Y, \odot_Y)$. E.g. $(x, y) + (x', y') := (x \oplus_X x', y \oplus_Y y')$, and $\lambda \cdot (x, y) := (\lambda \odot_X x, \lambda \odot_Y y)$.

Proof:[5] Define $L_X : X \to X \times Y$ and $L_Y : Y \to X \times Y$ s.t. $L_X(x) := (\mathrm{id}_X(x), 0)$ and $L_Y(y) := (0, \mathrm{id}_Y(y))$.[6] Notice this two functions are linear and injective. Let V_X and V_Y be bases for X and Y respectively. By the previous Exercise they are linearly independent. We know that linear injections preserve linear independence. Thus, $(L_X(v_x))_{v_x \in X}$ and $(L_Y(v_y))_{v_y \in Y}$ are linearly independent sets in $X \times Y$. Now the question is whether the union $(L_X(v_x))_{v_x \in X} \cup (L_Y(v_y))_{v_y \in Y}$ is linearly independent in $X \times Y$. Suppose not, that is suppose there exists a linear combination of elements in this set s.t.

$$0 = \sum_{v_x \in V_X} \lambda_{v_x} L_X(v_x) + \sum_{v_y \in V_Y} \lambda_{v_y} L_Y(v_y) =,$$

$$= L_X(\sum_{v_x \in V_X} \lambda_{v_x} v_x) + L_Y(\sum_{v_y \in V_Y} \lambda_{v_y} v_y) = \Big(\sum_{v_x \in V_X} \lambda_{v_x} v_x, \sum_{v_y \in V_Y} \lambda_{v_y} v_y\Big),$$

which can be true iff $\sum_{v_x \in V_X} \lambda_{v_x} v_x = 0$ and $\sum_{v_y \in V_Y} \lambda_{v_y} v_y = 0$, clearly a contradiction on the fact that V_X and V_Y are linearly independent. Hence, $(L_X(v_x))_{v_x \in X} \cup (L_Y(v_y))_{v_y \in Y}$ is linearly independent in $X \times Y$. Moreover, it clearly spans $X \times Y$. By previous Exercise, we get that $(L_X(v_x))_{v_x \in X} \cup (L_Y(v_y))_{v_y \in Y}$ is base for $X \times Y$, a fact that proves the statement. $\square$

EXERCISE 7

Let S be a non-empty subset of a linear space X. Show that

$$\mathrm{aff}(S) = \bigcap \{Y \subset X \mid Y \text{ is an affine manifold of } X \text{ and } S \subseteq Y\}$$

Proof: $\supseteq$. Trivial since $\mathrm{aff}(S) \in \{Y \subset X \mid Y \text{ is an a.m. of } X \text{ and } S \subseteq Y\}$.

$\subseteq$. Let $Y \supset S$ be an affine manifold in X. We want to show $\mathrm{aff}(S) \subset Y$. Let $x \in \mathrm{aff}(S)$. Then there exists appropriate (s_i, λ_i) with $\sum_i \lambda_i = 1$ s.t. $x = \sum_i \lambda_i s_i$. But $S \subset Y$, and Y is an affine manifold. Hence, it is closed wrt affine combinations.[7] But, $x = \sum_i \lambda_i s_i$ is an affine combination of element in S, hence it must belong to Y. Thus, $x \in Y$ and we are done. $\square$

[5] The very elegant idea of using linear injections is borrowed from the solutions of some of you.

[6] Remember, $\mathrm{id}_X : X \to X$ is the identity function, $\mathrm{id}_X(x) = x$.

[7] In class we proved that if Y is an affine manifold iff it is closed to affine combinations of doubletons, that is iff $\lambda y + (1 - \lambda) y' \in Y$, for all λ and $y, y' \in Y$. Here, I am just extending this characterization to combinations of arbitrary finite subsets of Y, that is for any $y_1, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n$ with $\sum_i \lambda_i = 1$.

EXERCISE 8

Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear and onto, then the image of any open set is open.

Proof: First notice that it is enough to show $f(B_{\mathbb{R}^n}(0,1))$ is open in $\mathbb{R}^m$.[8]

Claim: $f(B_{\mathbb{R}^n}(0,1))$ is open in $\mathbb{R}^m$.

Sub-Proof: Suppose not. This means there exists some $y \in f(B_{\mathbb{R}^n}(0,1))$ such that, for all $\varepsilon > 0$, $B_{\mathbb{R}^m}(y,\varepsilon) \setminus f(B_{\mathbb{R}^n}(0,1)) \neq \emptyset$. That is, for all $\varepsilon > 0$, there is some $y' \in B_{\mathbb{R}^m}(y,\varepsilon)$, s.t. $f(x') = y'$ implies $\|x'\| \geq 1$.

Now pick ε arbitrarily and let $y' \in B_{\mathbb{R}^m}(y,\varepsilon) \setminus f(B_{\mathbb{R}^n}(0,1))$. Since $B_{\mathbb{R}^m}(y,\varepsilon) = y + \varepsilon B_{\mathbb{R}^m}(0,1)$, there is a $y'' \in B_{\mathbb{R}^m}(0,1)$ such that $y' = y + \varepsilon y''$. Moreover, since f is onto, there is a $x \in B_{\mathbb{R}^n}(0,1)$ s.t. $f(x) = y$ and $x'' \in \mathbb{R}^n$ s.t. $f(x'') = y''$. Let $y' \in B_{\mathbb{R}^m}(y,\varepsilon)$ be s.t. for all $\varepsilon > 0$, there is some $y' \in B_{\mathbb{R}^m}(y,\varepsilon)$, s.t. $f(x') = y'$ implies $\|x'\| \geq 1$. By linearity, $f(x') := f(x + \varepsilon x'') = y + \varepsilon y'' = y'$. Since $y' \in B_{\mathbb{R}^m}(y,\varepsilon) \setminus f(B_{\mathbb{R}^n}(0,1))$ and $f(x') = y'$, we have $1 \leq \|x'\| = \|x + \varepsilon x''\|$.

Since ε was arbitrary, we have established that $\|x + \varepsilon x''\| \geq 1$, for all $\varepsilon > 0$. By continuity of $\|\cdot\|$, this implies $\|x\| \geq 1$, or $x \notin B_{\mathbb{R}^n}(0,1)$, or $y = f(x) \notin f(B_{\mathbb{R}^n}(0,1))$, a contradiction. □

[8] Indeed, for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we have $B_{\mathbb{R}^n}(x,\varepsilon) = x + \varepsilon B_{\mathbb{R}^n}(0,1)$. Moreover by linearity, $f(B_{\mathbb{R}^n}(x,\varepsilon)) = f(x) + \varepsilon f(B_{\mathbb{R}^n}(0,1))$.