MICROECONOMICS II.I – PS2 SOLUTIONS

New York University, a.y. 2012-2013

General Comment:

With regard to Exercise 1.b, whenever you are asked to disprove a statement, providing a counterexample is as sound as (and much simpler than) proving that the negation of that statement is true. Some of you represented three-players games in way that is different than Figure 1. That's fine as long as your representation is equivalent to a three-players game with simultaneous moves, i.e. as long as you don't get information sets wrong. Exercise 2 was in general well understood. In Exercise 3, nearly half of the students misunderstood the very last question. The correct answer was that for $n > 3$, the probability the ambulance arrives in a symmetric equilibrium is 0. You don't need to use the limit as n goes to infinity to have zero probability. Only one student reported an answer for the optional exercise on correlated equilibria, even though I know some of you ventured in it without success (a thing that for me is nonetheless notable). *jp*

Exercise 1

Consider the two following definitions.

Definition 1. A two-person game $G = \langle \{1,2\}, (S_i, u_i)_{i\in\{1,2\}} \rangle$ is **constant sum** if for some $c \in \mathbb{R}$,

$$u_1(s) + u_2(s) = c \text{ for each } s \in S_1 \times S_2.$$

Definition 2. Two Nash equilibria of G, say (m_1, m_2) and (m_1', m_2'), are

- (i) **Equivalent**: if $u_i(m_1, m_2) = u_i(m_1', m_2')$ for $i = 1, 2$, and
- (ii) **Interchangeable**: if (m_1, m_2') and (m_1', m_2) are both Nash equilibria of G.

Now answer the following two questions:

- (a) Prove that any two Nash equilibria of a two-person constant-sum game G are equivalent and interchangeable.
- (b) Is the result from part (a) also true for three-person constant-sum games (where $\sum_{i=1}^{3} u_i(s) = c$ for all $s \in S$)?

Solution, Part (a)

Let $m = (m_1, m_2)$ and $m' = (m_1', m_2')$ be Nash equilibria of the game G. We want to show that they are *equivalent*, i.e. $u_i(m) = u_i(m')$ for all $i \in \{1, 2\}$. Fix $i \in \{1, 2\}$, and recall that

February 20, 2013. These proposed solutions may contain minor typos. If you spot any, please e-mail me at jacopo.perego@nyu.edu.

the definition of Nash Equilibrium implies

$$u_i(m_i, m_{-i}) \geq u_i(\tilde{m}_i, m_{-i}) \qquad \forall \tilde{m}_i \in \Delta(S_i)$$

and in particular $u_i(m_i, m_{-i}) \geq u_i(m'_i, m_{-i})$. Now, we take advantage of the fact that G is constant sum and write

$$u_i(m_i, m_{-i}) \geq u_i(m'_i, m_{-i}) = c - u_{-i}(m'_i, m_{-i}) \geq c - u_{-i}(m'_i, m'_{-i}),$$

where the last inequality[1] comes from the fact that, being m' a Nash Equilibrium, it satisfies $u_{-i}(m'_i, m'_{-i}) \geq u_{-i}(m'_i, m_{-i})$. Now we can use the fact that $u_{-i}(m'_i, m'_{-i}) = c - u_i(m'_i, m'_{-i})$ to conclude

$$u_i(m_i, m_{-i}) \geq u_i(m'_i, m'_{-i})$$

Since the choice of m over m' and of $i \in \{1, 2\}$ was arbitrary, we can repeat the argument to get

$$u_i(m'_i, m'_{-i}) \geq u_i(m_i, m_{-i})$$

which gives us $u_i(m_i, m_{-i}) = u_i(m'_i, m'_{-i})$ as desired.

We prove next that m and m' are *interchangeable*. By Part (a), $u_1(m_1, m_2) = u_1(m'_1, m'_2)$. Since m' is a NE, we have $u_1(m_1, m_2) \geq u_1(m_1, m'_2)$. Given that G is constant sum, the previous inequality is equivalent to $u_2(m_1, m'_2) \geq u_2(m_1, m_2)$. But since also m' is a NE, we have $m_2 \in r_2(m_1)$ and the previous inequality implies $m'_2 \in r_2(m_1)$. We are left to show that $m_1 \in r_1(m'_2)$. To do that, we repeat the same argument switching the role of 1 and 2. From Part (a) we get, $u_2(m'_1, m'_2) = u_2(m_1, m_2) \geq u_2(m_1, m'_2)$. Constant-sum of G implies $u_1(m_1, m'_2) \geq u_2(m'_1, m'_2)$. Since m' is a NE we have $m_1 \in r_1(m'_2)$. Hence, (m_1, m'_2) is a NE, as we seek. By switching the role of m and m' we can get to the conclusion that also (m'_1, m_2) is a NE. $\square$

Part (b).

The answer is no, as already motivated in the footnote. Consider the following game as a counterexample.[2] Consider (S, S, B) and (B, B, S), two Nash equilibria in pure strategies. Clearly, these are not equivalent. Moreover, they are neither interchangeable, as, for example, (S, B, S) is not a NE.

Exercise 2

Modify the standard symmetric linear Cournot model with $p = a - bQ$ (where Q is market output) by letting marginal cost for a firm equal c times that firm's output (c is the same for all n firms).

(a) Argue that only pure Nash equilibria can exist.

[1] This is the point where we are using the fact that G is a two-person game. When $-i$ is just *one* player and $u_{-i}(m'_i, m_{-i})$ is a scalar, his unilateral deviation must necessarily make him better off, by definition of Nash equilibrium. However, when $-i$ is a *vector* of opponents, a profitable deviation for one of them can be detrimental for some others, and the above inequality may be false.

[2] For those of you who never saw a normal form game with three players, notice that the following convention is in place: Player 1 chooses the row and gets x in the payoff triple (x, y, z), Player 2 chooses the column and gets y, and Player 3 chooses the matrix and gets z.

	L	R
B	$2,1,0$	$0,0,3$
S	$0,0,3$	$1,2,0$

B

	B	S
B	$1,2,0$	$0,0,3$
S	$0,0,3$	$2,1,0$

S

FIGURE 1.— Counterexample

(b) Argue that all equilibria are symmetric.

(c) Each firm's best reply function shows the firm's optimal output level as a function of what it believes other firms are putting on the market. Compare the best reply function here to that in the standard model with constant marginal cost. Give an intuition based on marginal analysis for the difference.

(c) How much does each firm produce in Nash equilibrium? What happens to output per firm, aggregate output and aggregate profits as n grows large?

Solution

First let's define the game $G = \langle N, (S_i, u_i)_{i \in N} \rangle$ we are working on. We have $N = \{1, 2, \ldots, n\}$, $S_i = \mathbb{R}_+$ and

$$u_i(q_i, q_{-i}) = \Big(a - b \sum_{j \neq i} q_j - (b + c) q_i\Big) q_i \qquad \text{for all } (q_i, q_{-i}) \in S$$

Notice that u_i is continuous in (q_i, q_{-i}) and strictly quasi-concave in q_i. Existence of Nash is ensured[3] and maximum principle tells us that the best response correspondence r_i is single-valued.

Part (a).

Can we have (non-degenerate) mixed strategy Nash equilibria? The answer is negative. To see that, recall that if $(m_i, m_{-i}) \in \Delta(S)$ is a mixed equilibrium, we have $\text{Supp } m_i \subset r_i(m_{-i})$, that is, the strategies that receive strictly positive probability weight according to m_i must be best responses themselves. But we have just argued that r_i is single-valued, hence the support of m_i cannot be non-singleton. Hence, we cannot have non-pure equilibria.

Part (b).

First, let's compute the best reply function from $u_i(q_i, q_{-i})$. We have,

$$q_i = \arg\max u_i(q_i, q_{-i}) = \frac{a - b \sum_{j \neq i} q_j}{2b + c}.$$

Recall that a profile of strategy is a Nash equilibrium if $q_i = r_i(q_{-i})$ for all i. In this case, the linearity of r_i makes the fixed point argument particularly simple. Indeed, the fixed point of

[3] *Quiz* Why is the case that a non-compact strategy space is not a concern here?

r can be found as the solution of than the linear system

$$q_i = \frac{a - b\sum_{j\neq i} q_j}{2b + c} \qquad \forall\, i \in N$$

For non trivial values of a, b and c, this system can have either no solution or one solution. However, we already know that a Nash equilibrium exists, i.e. this system has exactly one solution. Hence, to show that the unique equilibrium is symmetric, we simply have to solve the linear system imposing symmetry and confirm that the solution exists, i.e.

$$q = \frac{a - b(n-1)q}{2(b+c)} \qquad \text{implies} \qquad q = \frac{a}{2c + (n+1)b}$$

Part (c).

The standard Cournot model delivers

$$q_i = \frac{a - c - b\sum_{j\neq i} q_j}{2b}$$

while this modified version,

$$q_i = \frac{a - b\sum_{j\neq i} q_j}{2b + c}$$

In both models, an increase in q_j linearly decrease q_i, however the response is flatter in the modified Cournot, as the slope coefficient is smaller.

Part (d).

As by Part (b), the unique Nash is

$$q_i = \frac{a}{2c + (n+1)b} \qquad \forall\, i \in N$$

The total output is

$$Q = \frac{na}{2c + (n+1)b}$$

Equilibrium price is $p = \frac{a(b+c)}{c+(n+1)b}$ and firm profits

$$u_i(q) = (p - \frac{1}{2}cq)q = \frac{a^2(2b+c)}{(c + (n+1)b)^2}$$

As n grows large, we have $q \to 0$, $Q \to \frac{a}{b}$ and $u_i(q) \to 0$. Each firm becomes non-atomic and price and total output converge to the competitive market outcome.

Exercise 3

There are n people living in separate apartments overlooking an icy road. An old lady (not one of the retired pirates) has fallen on the road and needs to be taken to the hospital quickly. For each $i = 1, ..., n$, the occupant of the i^{th} apartment gains two utils if an ambulance arrives to collect the old lady, but loses one util if he gets up to call 9-1-1 himself. If he were the only person living there, he would make the call for a net gain of one util. But he's not. Treat this as a simultaneous game with each of the n players having two pure strategies.

(a) If $n = 2$, find the symmetric equilibrium of the game. Are there asymmetric equilibria? Explain.

(b) As n increases, how does the old lady's expected welfare change in the symmetric equilibrium?

(c) Suppose instead that the ambulance will be sent only if calls are received from at least *two* different apartments (so it takes at least two "good Samaritans" to get help for the old lady). Again, this is a simultaneous game with two pure strategies for each player.

(i) When $n = 3$, in the symmetric equilibrium (ignoring the one in pure strategies), how likely is the ambulance to arrive?

(ii) Prove that for sufficiently large n, there is *no* chance of the ambulance arriving in any symmetric equilibrium! It may be useful to remember that for x close to 0, the n^{th} power of $(1 - x)$ is well approximated by $(1 - nx)$.

Solution, Part (a).

This strategic situation can be represented through the following normal form game. Clearly,

	C	I
C	$1, 1$	$1, 2$
I	$2, 1$	$0, 0$

Figure 2.— Normal Form representation when $n = 2$.

there are three Nash equilibria, two in pure strategies (C, I) and (I, C) and one in mixed strategies, $((.5, .5), (.5, .5))$. Notice that the unique symmetric equilibrium among the three is the latter, the others being asymmetric.

Part (b).

Now let $n > 2$, we want to find the symmetric equilibrium.[4] Notice that this ought to be in mixed strategies as in this game there cannot be symmetric equilibria in pure strategies. Indeed, neither $(C, \ldots, C)$ nor $(I, \ldots, I)$ are not immune to unilateral deviations, hence are not equilibria. Let $m = ((p, 1 - p), (p, 1 - p))$ be the symmetric mixed equilibrium. Our task

[4]Notice in passing that *every symmetric finite game has a symmetric Nash equilibrium* (Proof?).

is to find p. In order to randomize between action C and I, a generic player i has to be indifferent. Playing C will lead to the following expected payoff

$$\mathbb{E}_m(u_i(C, m_{-i})) = 1$$

Similarly, playing I would yield to an expected payoff of

$$\mathbb{E}_m(u_i(I, m_{-i})) = 2\mathbb{P}\big(\exists\, j \neq i : \ s_j = C\big) + 0\Big(1 - \mathbb{P}\big(\exists\, j \neq i : \ s_j = C\big)\Big)$$

where $\mathbb{P}(\exists\, j \neq i : \ s_j = C)$ is the probability that at least one opponent calls the ambulance. Since, each j calls with probability p the probability that at least one call is

$$\mathbb{P}(\exists\, j \neq i : \ s_j = C) = 1 - (1-p)^{n-1}$$

We have

$$\mathbb{E}_m(u_i(I, m_{-i})) = 2(1 - (1-p)^{n-1}).$$

Setting $\mathbb{E}_m(u_i(I, m_{-i})) = \mathbb{E}_m(u_i(C, m_{-i}))$ to make i indifferent we get in equilibrium

$$p = 1 - .5^{\frac{1}{n-1}}.$$

The probability that under m the ambulance arrives is $1 - .5^{\frac{n}{n-1}}$ which is also the expected utility of the old lady. As n grows large this number decreases to $\frac{1}{2}$. Notice the perverse effect of strategic thinking here. As n grows, player i (correctly) believes that the probability that at least some of his opponents will call the ambulance increased, leaving little scope to him calling the ambulance. Hence, he decreases the probability of calling.

Part (c).

Consider now the game with $n = 3$ and two good samaritans needed for the ambulance to arrive. We can represent the strategic interaction with the following normal form game. We apply the same reasoning of Part (b) being however careful with the fact the we have a

	C	I
C	$1,1,1$	$1,2,1$
I	$2,1,1$	$0,0,-1$

C

	C	I
C	$1,1,2$	$-1,0,0$
I	$0,-1,0$	$0,0,0$

I

FIGURE 3.— Two Good Samaritans

different payoff structure. There are many pure strategy equilibria in this game. In particular there are $\binom{n}{2}$ equilibria in which only two players choose C and the others free ride. We also have a symmetric equilibrium in pure strategies in which no one of the player calls, leaving the old lady to her faith with probability one. We want to see if there is also a non-degenerate mixed equilibrium m. To find it, as we did before, we need to make player i is indifferent between calling and not calling.

$$\mathbb{E}_m(u_i(C,(p,1-p))) = 1\mathbb{P}\big(\exists\, j \neq i : \ s_j = C\big) + (-1)\mathbb{P}\big(\nexists\, j \neq i : \ s_j = C\big)$$

and

$$\mathbb{E}_m(u_i(I,(p,1-p))) = 2\mathbb{P}\big(\exists\, j \text{ and } k \neq i : j \neq k, \ s_j = s_k = C\big)$$

We have as before
$$\mathbb{P}\big(\exists\, j \neq i : \; s_j = C\big) = 1 - (1-p)^{n-1},$$
and
$$\mathbb{P}\big(\exists\, j \text{ and } k \neq i : j \neq k, \; s_j = s_k = C\big) = 1 - (1-p)^{n-1} - (n-1)p(1-p)^{n-2}.$$
That is, we have to solve the $n-1$th-order equation
$$1 - 2(1 - p^{n-1}) = 2(1 - (1-p)^{n-1} - (n-1)p(1-p)^{n-2}),$$
or
$$p(1-p)^{n-2} = \frac{1}{2(n-1)} \tag{1}$$

If we let $n = 3$, we get a unique solution $\big((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\big)$. The total probability that an ambulance will arrive is $1 - (1-p)^3 - 3p(1-p)^2 = \frac{1}{2}$ which is lower than what we found in Part (b).

Now we can try to characterize what happens when n is large enough. We can approximate the equation (1) as
$$p(1-(n-2)p) = \frac{1}{2(n-1)} \qquad \Rightarrow \qquad (n-2)p^2 - p + \frac{1}{2(n-1)} = 0$$
Notice that the Δ of this equation is positive only if $n \leq 3$. That is, when $n > 3$ there are no real solutions that make i indifferent between C and I. The corner solution is the one in which $p = 0$, which is the only symmetric equilibrium of this game. In conclusion, for $n > 3$, in the unique symmetric equilibrium of the game, the ambulance will arrive with probability 0 and the old lady will not survive.

Exercise 4. Optional, not graded

This exercise is optional and covers correlated equilibrium, discussed in the third TA session. Consider the relationship between Nash and correlated equilibrium payoffs:

(a) Can there be a correlated equilibrium where a player gets less than her lowest Nash equilibrium payoff? Explain or give an example. (hint: Is it possible in a two players game?)

(b) (*Harder*) Can there be a correlated equilibrium where every player gets less than her lowest Nash payoff?

Solution, Part (a) and (b).[5]

The answer is affirmative. To show this I will provide two examples. The first one is a three players game and is rather simple. The second is a two-players game and it's more contrived. Since I have not dedicated much time in the last Lab to show you how to find a correlated

	L	R
U	$4,4,4$	$4,4,4$
D	$4,4,4$	$4,4,-4$

A

	L	R
U	$0,0,0$	$-1,-1,-1$
D	$-1,-1,-1$	$0,0,0$

B

	L	R
D	$4,4,-4$	$4,4,4$
D	$4,4,4$	$4,4,4$

C

FIGURE 4.— Three-Players game and Correlated Equilibria

equilibrium, I am going slow on the latter - which is more pedagogical - and I will leave you to check that the proposed equilibrium in the former is indeed a correlated equilibrium.

The proposed canonical correlated equilibrium is $\mu \in \Delta(S)$ s.t. $\mu(U,L,B) = \mu(D,R,B) = \frac{1}{2}$ and zero otherwise. This equilibrium will lead to an expected payoff of 0 which is certainly lower than any payoff reachable in a Nash equilibrium, both pure and mixed. Recall that in order to check whether or not μ is a canonical correlated equilibrium we need to make sure it satisfies the following system:

$$\forall i \in N,\ \forall s_i \in S_i,\ \forall s_i' \in S_i \setminus \{s_i\}, \qquad \sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) \Big[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \Big] \geq 0 \tag{2}$$

This defines a system of 10 inequalities. For example, when $i = 1$, $s_1 = U$ and $s_1' = D$ we have

$$\mu(U,L,A)(4-4)+\mu(U,L,B)(0+1)+\mu(U,L,C)(4-4)+\mu(U,R,A)(4-4)+\mu(U,R,B)(-1-0)+\mu(U,R,C)(4-4) \geq 0$$

As a second example, consider now the following two-players game:

		$P2$		
		A	B	C
$P1$	A	$5,2$	$0,0$	$1,1$
	B	$6,0$	$9,9$	$0,0$
	C	$4,4$	$0,6$	$2,5$

FIGURE 5.— Two-Players game and Correlated Equilibria.

Consider the following randomizing device:

	A	B	C
A	$\frac{1}{3}$	0	$\frac{1}{3}$
B	0	0	0
C	0	0	$\frac{1}{3}$

FIGURE 6.— Joint distribution of the candidate equilibrium $\mu \in \Delta(S)$.

We want to show that μ is indeed a canonical correlated equilibrium. To see this we need to retrieve the system of inequalities in (2). There are two players, each one having three strategies, and two possible deviation for each one of them. We get a total of $2 \cdot 3 \cdot 2 = 12$

[5]In this exercise, I elaborate on two examples provided by last year TA, Joao Ramos.

inequalities. After some boring algebra we get the system:[6]

$$
\begin{cases}
P1: & \\
A \text{ dev to } B & \mu(A,A)(5-6)+\mu(A,B)(0-9)+\mu(A,C)(1-0)\geq 0\\
A \text{ dev to } C & \mu(A,A)(5-4)+\mu(A,B)(0-0)+\mu(A,C)(1-2)\geq 0\\
B \text{ dev to } A & \mu(B,A)(1)+\mu(B,B)(9)+\mu(B,C)(-1)\geq 0\\
B \text{ dev to } C & \mu(B,A)(2)+\mu(B,B)(9)+\mu(B,C)(-2)\geq 0\\
C \text{ dev to } A & \mu(C,A)(-1)+\mu(C,B)(0)+\mu(C,C)(1)\geq 0\\
C \text{ dev to } B & \mu(C,A)(-2)+\mu(C,B)(-9)+\mu(C,C)(1)\geq 0\\
P2: & \\
A \text{ dev to } B & \mu(A,A)(2)+\mu(B,A)(-9)+\mu(C,A)(-2)\geq 0\\
A \text{ dev to } C & \mu(A,A)(-1)+\mu(B,A)(0)+\mu(C,A)(-1)\geq 0\\
B \text{ dev to } B & \mu(A,B)(-2)+\mu(B,B)(9)+\mu(C,B)(2)\geq 0\\
B \text{ dev to } C & \mu(A,B)(-1)+\mu(B,B)(9)+\mu(C,B)(1)\geq 0\\
C \text{ dev to } A & \mu(A,C)(-1)+\mu(B,C)(0)+\mu(C,C)(1)\geq 0\\
C \text{ dev to } B & \mu(A,C)(1)+\mu(B,C)(-9)+\mu(C,C)(-1)\geq 0
\end{cases}
$$

Check that the proposed μ in Figure 6 satisfies all these constraints and it is indeed a canonical correlated equilibrium. This equilibrium gives both players an expected payoff of $\frac{8}{3}$, which is smaller than the payoff they get when they play the unique Nash in pure strategies (B, B) or the mixed equilibirum $((.5, 0, .5), (.5, 0, .5))$.

[6]The first 6 inequalities are relative to Player 1. A dev to B means that I am fixing strategy A and considering the deviation to B.