MICROECONOMICS II.I – PS3 SOLUTIONS

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General Comment:

In Exercise 1.*a*, the question was really to *explain* why you can or cannot apply backward induction. This should have led you to discuss *ties* and the possibility of not being able to impute a unique action to some player at some node. The answer "Yes you can because this is a finite EFG" was correct yet not dramatically exciting. In 1.*b*.*i*, a good portion of the class drew the whole tree for each possible Condorcet cycles. This is fine, but very time consuming[1] and sometime even not possible. A more synthetic answer is given below. In Exercise 2.*b*, notice that you can achieve all payoff in the continuum $[.25, .5]$, and not only $\{.25, .33, .5\}$. Finally, Exercise 3 was problematic for most of you. My advice is to spend some time reading its solution below. *jp*

Exercise 1

Consider a roll-call vote with three players 1, 2 and 3 and three alternatives A, B and C. Player 1 casts a vote for some alternative, 2 votes after observing 1's vote and 3 votes after seeing the votes of 1 and 2. An alternative wins if it receives two or more votes. If there is a tie, the alternative preferred by Player 1 wins. Assume no player is indifferent over any two alternatives.

(*a*). Can this game be solved by backward induction? Explain.

(*b*). Now assume there is a Condorcet cycle (that is, no two players have the same alternative ranked first, no two players have the same alternative ranked second and no two players have the same alternative ranked third)

i. Show that voter 1 always gets her second-ranked alternative.

ii. Can player 2 ever get his worst alternative? Can player 3 ever get his worst alternative?

Solution. Part (a)

This is a dynamic finite game with perfect information. Hence, by Khun existence theorem, it has at least a SPE in mixed strategies. Backward induction is an algorithm that, when applicable, computes the *unique* SPE of such games.[2] This algorithm is applicable whenever there is no ambiguity on which action a rational player $i \in N$ will choose at a certain history $h \in H$ at which he is active.

February 17, 2014. These proposed solutions may contain minor typos. If you spot any, please e-mail me at jacopo.perego@nyu.edu.

[1] This is certainly not a good strategy when you are time constrained, as you will be during the exam.

[2] I believe this is more a convention than a definition. As I will explain shortly, BI can be generalized to find a full set of SPEs.

There are several different ways to formalize this idea. The simplest (and most restrictive) is the following: Backward induction selects a unique SPE in *generic* extensive form games with perfect information. A perfect information game is **generic** if no player is indifferent between any two terminal histories, that is if for all $i \in N$ and for all terminal histories $z', z'' \in Z$, with $z' \neq z''$, we have $u_i(z') \neq u_i(z'')$. A larger class of games for which BI yields a unique prediction is the class of games with *no relevant ties.* A perfect information game has **no relevant ties** if for all terminal histories $z', z'' \in Z$, with $z' \neq z''$, the player who is *decisive* for z' *vs* z'' is not indifferent among the two. Clearly generic games have no relevant ties, but the contrary is false. If the *no relevant ties* condition is not met, the procedure of backward induction doesn't work unambiguously. As theorists, we cannot make a positive statement concerning the decision of a rational player when he is indifferent among a set of alternatives.

Nonetheless, backward induction is still a powerful tool, as it allows us, if not to compute the unique SPE, to characterize the entire set of SPEs of a game. How? Simply by assigning an arbitrary decision whenever a player is indifferent and then proceed backwards. To each particular arbitrary decision we fixed, it will correspond a different SPE. Another common way to avoid the problem of relevant ties and apply BI to find a particular SPE is to impose global **conventions** on the behavior of players. You have already done this when analyzing the pirates game. As you certainly recall, each pirate was so cranky that whenever he was indifferent about voting for or against a proposal, he was voting against. This eliminated all relevant ties and allowed you to use backward induction smoothly. Another typical setting where we apply behavioral conventions is Bargaining. In bargaining problems, it is usually assumed that when the respondent is indifferent between saying Yes or No to some proposed allocation, he says Yes.

The same can be done here. Notice that the convention we imposed (whenever there is a tie, the alternative preferred by Player 1 wins) does not resolve all relevant ties in the game. For example, consider the following situation. If player 1 and 2 votes for the same candidate, then Player 3's vote is ineffective. Hence, he is indifferent. To avoid that, we could impose a further convention, namely that whenever indifferent a player chooses among A, B and C in alphabetical order.

Part (b).i

Notice that, in an equilibrium, the outcome chosen at the initial stage by $P1$ is the outcome that will be eventually implemented. To see this consider an arbitrary preference ranking for $P1$. If he choose his least preferred outcome, say x, one among P2 or P3 will want to do the same, being x his most preferred outcome. Hence x is implemented, and P1 gets his worst outcome. If P1 chooses his most preferred outcome, say x, there are two cases. First, it is ranked 2nd and 3rd by Player 2 and 3, respectively. In this case, P2 can implement his first best, which is not x. Second, it is ranked 3rd and 2nd by Player 2 and 3. In this case, 2 and 3 can agree to implement Player 3 most preferred outcome, which is not x. Hence, P1 will never choose x, in the first place, since he can only get his second or third most preferred option. Finally, if P1 chooses his second option, say x, then it is certainly implemented, being the most preferred outcome of someone else. Hence, P1 will always choose his second best option and it will always be implemented.

Part (b).ii

Consider the following payoff structure:

	A wins	B wins	C wins
Player 1	2	0	1
Player 2	1	2	0
Player 3	0	1	2

We have $A \succ_1 C \succ_1 B$, $B \succ_2 A \succ_2 C$ and $C \succ_3 B \succ_3 A$. I show that $P2$ gets his worst alternative. Let's start from player 3. He can be active at 9 different nodes.

Node	Action he chooses	Outcome	Payoff
AA	$\{A, B, C\}$	A	0
AB	B	B	1
AC	C	C	2
BA	$\{B, C\}$	B	1
BB	$\{A, B, C\}$	B	1
BC	C	C	2
CA	C	C	2
CB	C	C	2
CC	$\{A, B, C\}$	C	2

Player 2, is active a 3 nodes. He is a rational player and believes in the rationality of Player 3. Hence he can predict what he will do after his move.

Node	Action he chooses	Outcome	Payoff
A	B	B	2
B	$\{A, B\}$	B	2
C	$\{A, B, C\}$	C	0

Finally Player 1, who is rational, believes in the rationality of 2 and 3 and believes that Players 3 believes that Player 3 is rational faces the following decision problem,

Node	Final outcome	Payoff
A	B	0
B	B	0
C	C	1

This three tables fully characterizes the set of SPE of this game. Notice that $|S_1| = 3$, $|S_2| = 27$ and $|S_3| = 3^9 = 19'683$. That is, there is a total of $1'594'323$ different way to play this game. Among them (I count) 342 different SPE in pure strategies. Without imposing a convention on the behaviour of indifferent players we cannot select a unique SPE using backward induction. If we impose the **alphabetic convention** discussed above we are going to select a unique SPE. Clearly, Player 1 choose C. Player 2 is indifferent among $\{A, B, C\}$ and chooses A. Finally, Player 3, choose C. (C, A, C) is not an equilibrium, but it is the final history induced by this particular equilibrium. The equilibrium is

$$\Big(C,\ (B,B,A),\ (A,B,C,B,A,C,C,C,A)\Big)$$

Without this convention, we can only say that Player 1 always chooses C and get 1. Player 2 always gets 0. Player 3 always gets 2. The bill passing the vote is always C.

Now an example where Player 2 get his best and Player 3 his worse. I show that $P3$ gets his worst alternative. We simply have to better "align" the preferences of 1 and 2.

	A wins	B wins	C wins
Player 1	2	1	0
Player 2	0	2	1
Player 3	1	0	2

Let's start from player 3. He can be active at 9 different nodes.

Node	Action he chooses	Outcome	Payoff
AA	$\{A,B,C\}$	A	1
AB	$\{A,C\}$	A	1
AC	C	C	2
BA	A	A	1
BB	$\{A,B,C\}$	B	0
BC	C	C	2
CA	$\{B,C\}$	C	2
CB	$\{A,C\}$	C	2
CC	$\{A,B,C\}$	C	2

Player 2, is active a 3 nodes.

Node	Action he chooses	Outcome	Payoff
A	C	C	1
B	B	B	2
C	$\{A,B,C\}$	C	1

Finally Player 1 faces the following decision problem,

Node	Final outcome	Payoff
A	C	0
B	B	1
C	C	0

Clearly, Player 1 choose B. Player 2 chooses B and bill B passes. Player 3 is indifferent among $\{A,B,C\}$. Player 3 gets his worse outcome. Player 2 his best outcome.

Exercise 2: Hotelling on a Circle

A unit mass of consumers are spread evenly around a circle of circumference 1. There are three sellers that simultaneously choose locations on the circle (one point on which is labeled zero, for your convenience). Consumers buy from the closest seller; if two sellers locate in the same place, they split evenly the demand generated by that location.

(a). Find at least 2 Nash equilibria in pure strategies.

(b). What can we say about seller 1's payoff in different pure strategy NE?

Solution, Part (a).

In this game, there is a continuum of equilibria where the three firms spread evenly in the circle, i.e. the distance between each one of them is the same. Let's see why this is an equilibrium. Let the circle be of length 60. Suppose P1 is based at 0, P2 is at 20 and P3 is at 40. WLOG, we study only unilateral deviation of Player 1. If he moves towards 2, he gains some costumers from 2 and he looses the same amount of them from 3. So deviating leaves P1 indifferent. The same holds for unileteral deviation of 2 and 3. Hence, this is a NE.

There are other equilibria similar to this one e.g. $(0, 22.5, 37.5)$. In general, if the players are spread across the circle, and there is a pair of players such that their distance is higher than 20 then there are incentives to deviate. The converse is not true. There is a simple way to characterize the best reply of a player given the position (x, y) of his opponents. Indeed, the player will always choose an arbitrary position in the longest arc between $|x - y|$ and $60 - |x - y|$.

In light of this, notice that (x, x, x) for any $x \in [0, 2\pi]$ is not an equilibrium. On the other hand, $(0, \pi, \pi)$ is another equilibrium in pure strategies.

Part (b).

For what concerns the payoff of Player 1, in the evenly shared circle he gets $\frac{1}{3}$. In the other equilibria, his payoff depends on whether he has a dominant position, or if he suffers the dominant position of someone else. In the first case he can get payoffs which ranges from $\frac{1}{3}$ to $\frac{1}{2}$. In the other, he gets payoffs from $\frac{1}{4}$ to $\frac{1}{3}$. Hence $u_1(z) \in [1/4, 1/2]$ for all $z \in Z$ that correspond to a NE.

Exercise 3

Consider two oligopolistic firms with identical constant marginal costs $c > 0$, facing a market inverse demand function $p = a - q_1 - q_2$, with $a > c$. The firms play the following three-stage game Γ. In stage one, Firm 2 costlessly chooses $K \in \mathbb{R}_+$, that we interpret as an upper bound on Firm 2's capacity to produce. In stage two, Firm 1, having observed 2's choice of K, chooses output $q_1 \geq 0$. In stage three, having observed q_1, Firm 2 chooses $q_2 \in [0, K]$. Firm i realizes profits $u_i = (a - q_1 - q_2 - c)q_i$. To summarize, stages two and three comprise a standard Stackelberg model, except that Firm 2 must respect the capacity constraint it chose earlier. We are interested in the subgame perfect solutions of Γ.

(*a*). Will Firm 2 do at least as well as in a standard Stackelberg game (one without stage one)? Will Firm 2 choose K large enough that it is not a binding constraint in equilibrium?

(*b*). How do Firm 1's equilibrium output and profits compare to those in the standard Stackelberg model? Explain carefully.

Solution, Part (*a*) *and* (*b*).

Let's first solve the monopolistic problem. The monopolist solves the following program

$$\max_q (a - c - q)q$$

which yields a unique dominant action $q_M = \frac{a-c}{2}$, with payoff equal to $\frac{(a-c)^2}{4}$.

Now, consider $|N| = 2$ and forget for the moment about the first stage. That is, consider the standard Stackelberg problem. Even before computing the equilibrium, we can guess that the fact that 1 is moving first may give him a *first mover advantage.* Let's see if this actually the case. We can use the backward induction procedure. Let's start from player 2. He solves his problem, knowing already the realization of q_1, let's call it $\bar{q}_1$. That is, he solves

$$r_2(\bar{q}_1) = \arg\max_{q_2} (a - c - \bar{q}_1 - q_2)q_2$$

whose FOC leads to $q_2 = \frac{a-c-\bar{q}_1}{2}$. Player 1, anticipating Player 2's behavior, solves

$$\max_{q_1} (a - c - q_1 - \mathbb{E}(q_2))q_1 = \max_{q_1} (a - c - q_1 - \frac{a - c - \bar{q}_1}{2})q_1$$

which gives $q_1^\star = \frac{a-c}{2}$. We are now able to compute the SPE of this game, which is,

$$(q_1^\star, q_2^\star) = \Big(\frac{a-c}{2}, \frac{a-c}{4}\Big)$$

with profits being respectively

$$(u_1(q_1^\star, q_2^\star), u_2(q_1^\star, q_2^\star)) = \Big(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{16}\Big)$$

Notice that, Player 1 produces the monopolist quantity, and Player 2 is obliged to produce much less than that. The behavior of P1, makes the total quantity produced being higher than the monopolist quantity, prices and aggregate profits being lower. Compare this result with the standard Cournot, two firms simultaneous decisions: the equilibrium is $(\frac{a-c}{3}, \frac{a-c}{3})$. The aggregate production there is $\frac{2}{3}(a-c)$, while here is $\frac{3}{4}(a-c)$, that is the *collusion is higher in the Cournot than in Stackelberg.*

Let's come back to our problem. Consider now the game with all the three stages. If 2 can **credibly commit himself** to a certain level of capacity K at stage 1, is there **more or less scope for collusion**? Since the algebra in the derivation of the equilibrium is boring I'll let

$$\boxed{a - c := 1}$$

to avoid carrying out the term $(a - c)$. First, let's put ourselves in the shoes of Player 2 at stage 3 (again, we are using BI). He will have to maximize the following program:

$$r_2(\bar{q}_1) = \arg \max_{q_2 \in [0,K]} (1 - \bar{q}_1 - q_2)q_2,$$

the same as before, but with a smaller set of feasible actions. It gives us the following best reply function

$$r_2(\bar{q}_1, K) = \min \left\{ K, \frac{1 - \bar{q}_1}{2} \right\}$$

That is,[3].

$$r_2(\bar{q}_1, K) = \begin{cases} \frac{1-\bar{q}_1}{2} & \text{if } \bar{q}_1 > 1 - 2K \\ K & \text{if } \bar{q}_1 \leq 1 - 2K \end{cases}$$

Now, let's consider the behavior of Player 1 at stage 2. He solves the following problem.

$$r_1(K) = \arg \max_{q_1} (1 - q_1 - r_2(q_1, K))q_1 \tag{1}$$

Case 1: $\boxed{q_1 > 1 - 2K}$ Then $r_2(q_1, K) = \frac{1-q_1}{2}$, that is the solution to (1) is

$$r_1(K \mid q_1 > 1 - 2K) = \frac{1}{2}. \tag{2}$$

Does this value match the constraint that we are imposing (i.e. $q_1 > 1 - 2K$)? Clearly, it depends on K. Then, we can say that the constraint is met if

$$\frac{1}{2} > 1 - 2K \quad \Rightarrow \quad \boxed{K > \frac{1}{4}}.$$

[3]Notice I am imposing a convention to break ties (See Exercise 1)

Case 2: $\boxed{q_1 \leq 1 - 2K}$ Then $r_2(q_1, K) = K$, that is the solution to (1) is

$$r_1(K|q_1 \leq 1 - 2K) = \frac{1-K}{2} \tag{3}$$

Does this value match the constraint that we are imposing (i.e. $q_1 \leq 1 - 2K$)? As before, we solve

$$\frac{1-K}{2} \leq 1 - 2K \quad \Rightarrow \quad \boxed{K \leq \frac{1}{3}}.$$

It's easy to see that the interesting part in terms of behavior of Player 1 is in the interval $K \in [\frac{1}{4}, \frac{1}{3}]$ since outside this interval his best reply is unambiguous. If $K > \frac{1}{3}$, then Player 1's optimum is to play his Stackelberg strategy, i.e. $\frac{1}{2}$. On the other hand If $K \leq \frac{1}{4}$, Player 1 understand he can be better off by playing $\frac{1-K}{2}$. That is,

$$r_1(K) = \begin{cases} \frac{1}{2} & \text{if } K > \frac{1}{3} \\ ?? & \text{if } K \in [\frac{1}{4}, \frac{1}{3}] \\ \frac{1-K}{2} & \text{if } K < \frac{1}{4} \end{cases}$$

We have to understand what is the best reply to a $K \in [\frac{1}{4}, \frac{1}{3}]$. To do so, let $K \in [\frac{1}{4}, \frac{1}{3}]$. Since, given this interval, the Stackelberg equilibrium strategy is available to Player 2, when Player 1 plays $\frac{1}{2}$ we know that Player 2 will reply with $\frac{1}{4}$. What is the Player 2's best reply to $\frac{1-K}{2}$? We have

$$r_2\Big(\frac{1-K}{2}\Big|K \in \Big[\frac{1}{4}, \frac{1}{3}\Big]\Big) = \min\{K, \frac{1 - \frac{1-K}{2}}{2}\} = K$$

Now, we can compare the payoffs of player 1. If he plays $\frac{1}{2}$,

$$u_1\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{8}$$

else,

$$u_1\left(\frac{1-K}{2}, K\right) = \frac{(1-K)^2}{4}$$

Compare these two payoff:

$$\frac{1}{8} \gtreqless \frac{(1-K)^2}{4} \qquad \Longleftrightarrow \qquad \frac{\sqrt{2}-1}{\sqrt{2}} \gtreqless K$$

which yields to:

$$r_1(K) = \begin{cases} \frac{1}{2} & \text{if } K > \frac{\sqrt{2}-1}{\sqrt{2}} \\ \frac{1-K}{2} & \text{if } K \le \frac{\sqrt{2}-1}{\sqrt{2}} \end{cases}$$

Now that we have the best reply function of player 1, we can go to stage 1, to solve for K. Player 2 solves

$$\max_K u_2\big(r_1(K), r_2(r_1(K), K)\big)$$

Let's solve this in two steps.

Case 1: $\boxed{K \le \dfrac{\sqrt{2}-1}{\sqrt{2}}}$. Thanks to the work done so far we can say that $r_1(K) = \frac{1-K}{2}$ and $r_2(r_1(K), K) = K$, hence

$$u_2\left(\frac{1-K}{2}, K\right) = \frac{1-K}{2} \cdot K$$

The first derivative in $K \le \frac{\sqrt{2}-1}{\sqrt{2}}$ is

$$\frac{1-K}{2} - \frac{K}{2} > 0$$

that is the utility is increasing in K. Hence, in this step, Player 2 hits the constraint and sets $K^\star = \frac{\sqrt{2}-1}{\sqrt{2}}$.

Case 2: $\boxed{K > \dfrac{\sqrt{2}-1}{\sqrt{2}}}$. Then $r_1(K) = \frac{1}{2}$ and $r_2(r_1(K), K) = \frac{1}{4}$, that is we are in the Stackelberg case.

Let's compare the payoffs of Player 2, in the cases 1 and 2. In the first one,

$$u_2\Big(\frac{1-K}{2}, K\Big) = \frac{1-K}{2}K = \frac{1-\frac{\sqrt{2}-1}{\sqrt{2}}}{2}\frac{\sqrt{2}-1}{\sqrt{2}} = \frac{\sqrt{2}-1}{4}$$

In the second (Stackelberg):

$$u_2\Big(\frac{1}{2}, \frac{1}{4}\Big) = \frac{1}{16}$$

Then,

$$\frac{\sqrt{2}-1}{4} = \frac{4(\sqrt{2}-1)}{16} > \frac{1}{16},$$

since, $4(\sqrt{2}-1) > 1$. That is choosing $K = K^\star$ makes Player 2 strictly better off. Let's double check if Player 1 has actually no incentive to deviate to his Stackelberg equilibrium. Sticking to the strategy outlined above he will get:

$$u_1\Big(\frac{1-K}{2}, K\Big) = \frac{1-K}{2}\frac{1-K'}{2} = \frac{(1-K)^2}{4}$$

The payoff in the Stackelberg is $\frac{1}{8}$. Hence, $K = \frac{\sqrt{2}-1}{\sqrt{2}}$ must be such that:

$$\frac{(1-K)^2}{4} \geq \frac{1}{8} \quad \Longleftrightarrow \quad 2(1-K)^2 \geq 1 \quad \Longleftrightarrow \quad k \leq \frac{\sqrt{2}-1}{\sqrt{2}},$$

which is clearly satisfied. Hence, Player 1 is indifferent and Player 2 is better off. Finally, let's answer the initial question: is there more scope for collusion when players can sign binding agreements? The answer is affirmative.[4] Prices increase since

$$p_{\text{Stackelberg}} = \frac{1}{4} < \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}} = p_{\text{Commit}}$$

Prices are higher in equilibrium. The rent is extracted by Player 2 from the consumers.

[4]For this reason, binding agreements of this type are unlawful. Unfortunately (for antitrust lawyers), life is not always that simple. Indeed, as you might see next year, when Firm 1 and 2 repeatedly interact with each other on the same market, we can get collusion also when binding agreements are unavailable, even in environments with imperfect information. A problem that sometime can make antitrust cases discretionary and controversial.