**Proposition:** Let $X$ be a vector space and $L \in \mathcal{L}(X, Y)$. Prove that

$$\dim(X) = \dim(\text{null}L) + \dim(L(X))$$

**Proof:** $\text{null}L$ is a subspace of $X$. As such it has a base, say $V$. If $|V| = \infty$, there is nothing to prove. Similarly, if $\text{span}V = X$ there is nothing to prove.[1] So let $|V| < \infty$ and $\dim(\text{null}L) < \dim(X)$. We can extend $V$ to $\bar{V} \supset V$ a base for $X$. Define the set $W := \{L(v) : v \in \bar{V} \setminus V\}$.

**Claim:** $W$ is base for $L(X)$.

*Subproof.* First we want to show $\text{span}W = L(X)$. The inclusion $\subseteq$ is trivial. For the other inclusion, let $y \in L(X)$. Then there exists $x \in X$ s.t. $L(x) = y$. Being $\bar{V}$ a base for $X$, there exists a linear combination of element of $V$ that equals $x$, i.e. $x = \sum^n \lambda_i v_i$ for some $\{\lambda_i, v_i\}_{i=1}^n \subset \mathbb{R} \times \bar{V}$. Then we have

$$y = L(x) = L\left(\sum^n \lambda_i v_i\right) = L\left(\sum_{v_i \in V} \lambda_i v_i + \sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i\right) = L\left(\sum_{v_i \in V} \lambda_i v_i\right) + L\left(\sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i\right)$$

Notice that $\sum_{v_i \in V} \lambda_i v_i \in \text{null}L$ hence $L(\sum_{v_i \in V} \lambda_i v_i) = 0$. So we are left with

$$y = L\left(\sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i\right) = \sum_{v_i \in \bar{V} \setminus V} \lambda_i L(v_i) = \sum_{w \in W} \lambda_i w_i.$$

Thus, we have found a linear combination of elements of $W$ that generates $y$. Since $y$ was arbitrary, $\text{span}W = L(X)$.

Second, we want to show that $W$ is linearly independent. Suppose not, i.e. suppose you can find a combination with non zero coefficients such that $0 = \sum_{w \in W} \lambda_i w_i$. It follows that,

$$0 = \sum_{w \in W} \lambda_i w_i = \sum_{v_i \in \bar{V} \setminus V} \lambda_i L(v_i) = L\left(\sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i\right)$$

or that $\sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i \in \text{null}L$. Since $V$ spans $\text{null}L$, there exists a combination of elements of $V$ that generates $\sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i$. That is for some $\{\lambda'_i, v'_i\}_{i=1}^n \subset \mathbb{R} \times V$, $\sum_{v'_i \in V} \lambda'_i v'_i = \sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i$. To reach a contradiction, simply notice that we have found a $\{(\lambda'_i, \lambda), (v'_i, v_i)\}_{i=1}^n \subset \mathbb{R} \times \bar{V}$ such that

$$\sum_{v'_i \in V} \lambda'_i v'_i - \sum_{v_i \in \bar{V} \setminus V} \lambda_i v_i = 0.$$

Since at least some of these $\lambda_i$ are non zero (by our contrapositive assumption), we can argue that $\bar{V}$ is not independent. But $\bar{V}$ was a base for $X$, hence it must be linearly independent, a contradiction. $\square$

[1] In the first case, this is because $V \subset X$ implies $\infty = \dim(\text{null}L) \leq \dim(X)$. In the latter, if $\text{span}V = X$, then $X = \text{null}L$ and $L(X) = \{0\}$.