Math for Econ I: Solutions

Problem Set 4*

Exercise 1

Let A and B be any two sets in $\mathbb{R}^n$. Show that:

(a) $\text{co}(A) + \text{co}(B) = \text{co}(A + B)$.

Assume now and for the rest of the exercise that A and B are convex.

(b) True or false: If $A \cap B \neq \emptyset$, $\text{aff}(A \cap B) = \text{aff}(A) \cap \text{aff}(B)$.

(c) If $\text{ri}(A) \cap \text{ri}(B) \neq \emptyset$, then $\text{ri}(A) \cap \text{ri}(B) = \text{ri}(A \cap B)$.

(d) If $\text{ri}(A) \cap \text{ri}(B) = \emptyset$, then A and B can be properly separated.

Proof:

(a) Let $x \in \text{co}(A) + \text{co}(B)$. Thus, $x = \sum_{i=1}^{n+1} \lambda_i^a x_i^a + \sum_{i=1}^{n+1} \lambda_i^b x_i^b$, with $\sum_{i=1}^{n+1} \lambda_i^a = \sum_{i=1}^{n+1} \lambda_i^b = 1$. Hence,

$$x = \sum_{i=1}^{n+1} \lambda_i^a x_i^a \sum_{j=1}^{n+1} \lambda_j^b + \sum_{j=1}^{n+1} \lambda_j^b x_j^b \sum_{i=1}^{n+1} \lambda_i^a = \sum_{i,j=1}^{n+1} \lambda_i^a \lambda_j^b x_i^a + \sum_{i,j=1}^{n+1} \lambda_i^a \lambda_j^b x_j^b = \sum_{i,j=1}^{n+1} \lambda_i^a \lambda_j^b (x_i^a + x_j^b).$$

which implies $x \in \text{co}(A + B)$. Conversely, let $x \in \text{co}(A + B)$. Then $x = \sum_{i=1}^{n+1} \lambda_i (x_i^a + x_i^b)$.

(b) False. Consider the following counter example. Let X be the real line and $A = [0, 1]$ and $B = A + 1$. Thus, $\text{aff}(A \cap B) = \text{aff}(\{1\}) = \{1\} \neq \mathbb{R} = \text{aff}(A) \cap \text{aff}(B)$.

*Please email typos and comment to jp2841@nyu.edu.

(c) Let $x \in \text{ri}(A) \cap \text{ri}(B)$. Then there exists a $\varepsilon > 0$ s.t. $B_{\text{aff}(A)}(x, \varepsilon) \subseteq A$ and $B_{\text{aff}(B)}(x, \varepsilon) \subseteq B$. Since, $\text{aff}(A \cap B) \subseteq \text{aff}(A)$ and $\text{aff}(A \cap B) \subseteq \text{aff}(B)$,[1] we also have $B_{\text{aff}(A\cap B)}(x, \varepsilon) \subseteq B_{\text{aff}(A)}(x, \varepsilon)$ and $B_{\text{aff}(A\cap B)}(x, \varepsilon) \subseteq B_{\text{aff}(B)}(x, \varepsilon)$. Hence, $B_{\text{aff}(A\cap B)}(x, \varepsilon) \subseteq A \cap B$ which gives $x \in \text{ri}(A \cap B)$.

Conversely, notice that $\text{ri}(A \cap B) \subseteq \text{ri}(A)$ and $\text{ri}(A \cap B) \subseteq \text{ri}(B)$. This proves $\text{ri}(A \cap B) \subseteq \text{ri}(A) \cap \text{ri}(B)$.

(d) This is just an application of Minkowsky Separation Theorem. Alternatively, you can prove that $\text{ri}(A) - \text{ri}(B) = \text{ri}(A - B)$.[2] This gives you that $\text{ri}(A - B)$ is not only a convex set but also does not contain 0. Thus, by the point-set separation theorem you have proved in class, we can properly separate 0 and $A - B$, which amount to say we can properly separate A and B.

EXERCISE 2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map (that is, $T(x) = A(x) + b$ for some linear function $A : \mathbb{R}^n \to \mathbb{R}^m$ and vector $b \in \mathbb{R}^m$). Show that if $C \subset \mathbb{R}^n$ is convex, then $T(C) \subset \mathbb{R}^m$ is also convex.

Proof: By affinity of T, there is some linear map L s.t. $T = L + b$. Let $y, y' \in T(C)$ and fix $\gamma \in (0, 1)$. Wts $\gamma y + (1 - \gamma)y' \in T(C)$. We have that there exists $x, x' \in C$ s.t. $y = T(x)$ and $y' = T(x')$. Thus, $\gamma y + (1 - \gamma)y' = \gamma T(x) + (1 - \gamma)T(x') = \gamma(L(x) + b) + (1 - \gamma)(L(x') + b) = \gamma L(x) + (1 - \gamma)L(x') + b = L(\gamma x + (1 - \gamma)x') + b$. Since C is convex, we are done.

EXERCISE 3

Let $M = T + z$ be a linear manifold in $\mathbb{R}^n$. We want to construct the orthogonal projection map $P_M : \mathbb{R}^n \to M$, that is a function such that $P_M(y) = x$ if $d(y, x) = d(y, M)$. As we showed in class, there exist $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$ such that $M = \{x \in \mathbb{R}^n | Ax = b\}$.

(a) Show that $T = \text{null}(A)$ and that $T^\perp = \{A^T\lambda \mid \lambda \in \mathbb{R}^m\}$.

(b) Show that for any $y \in \mathbb{R}^n$, $x^\star = P_M(y)$ if and only if $x^\star \in M$ and $\langle y - x^\star, x - x^\star\rangle = 0$ for all $x \in M$. Conclude that $x^\star = P_M(y)$ if and only if $x^\star \in M$ and $y - x^\star \in T^\perp$.

[1] It comes straight from the characterziation of $\text{aff}(A)$ as the smallest affine manifold containing A.

[2] Corollary 6.6.2. in Rockafeller's *Convex Analysis* textbook.

(c) Assume that $m < n$ and that A has rank m (so A is full rank). Show that there exists a matrix $B \in \mathbb{R}^{n\times n}$ and a vector $d \in \mathbb{R}^n$ such that $P_M(y) = By + d$ for all $y \in \mathbb{R}^n$. Explicitly construct B and d.

EXERCISE 4

Leontief Production: Each industry produces a single consumption good using as inputs the goods produced by other industries and raw materials. There are n consumption (intermediary) goods and m raw materials. The economy is endowed with $\omega_k > 0$ units of raw material k, $k = 1, \dots, m$. The production of one unit of (consumption) good j requires a_{ij} units of good i ($i = 1, \dots, n$) and b_{kj} units of raw material k ($k = 1, \dots, m$). If for each $i = 1, \dots, n$, x_i and c_i denote, respectively, the total amount of good i that is produced and consumed (by consumers), then

$$x = Ax + c.$$

The production schedule x is feasible if $x \geq 0$, $(I - A)x \geq 0$ and $Bx \leq \omega$. The feasible schedule x is efficient if there is no feasible schedule y such that $(I - A)y \geq (I - A)x$ and $(I - A)y \neq (I - A)x$. Show that if x is efficient then there exist price vectors $p \in \mathbb{R}^n_+$ and $q \in \mathbb{R}^m_+$ such that $(p^T, q^T) \neq (0, 0)$,

$$p^T(I - A) - q^T B \leq 0, \quad (p^T(I - A) - q^T B)x = 0, \quad \text{and} \qquad q^T(\omega - Bx) = 0.$$

Conversely, if for some feasible x such prices exist and $p > 0$, then x is efficient. Note that the first condition says that no production activity is strictly profitable, while the second condition implies that $p_j = \sum_i p_i a_{ij} + \sum_k q_k b_{kj}$ if $x_j > 0$. That is, any industry that produces a strictly positive amount of output, makes 0 profits. The last condition implies that $q_k = 0$ when $\sum_j b_{kj} x_j < \omega_k$.

Hint: You may find it easier here to use a separation argument rather than Farkas Lemma.