REAL ANALYSIS - PS 1 SOLUTIONS

NEW YORK UNIVERSITY, A.Y. 2013-2014

EXERCISE 1

(*i*) Prove that $\text{cl}A = \text{int}(A) \cup \text{bdry}(A)$.

(*ii*) Let (X, d) be a metric space and $(x_m), (y_m) \in X^\infty$. Show that if $x_m \to x$ and $y_m \to y$ then $d(x_m, y_m) \to d(x, y)$.

Solutions

(*i*). $x \in \text{cl}A$ iff for all $r > 0$, $B(x, r) \cap A \neq \emptyset$ iff either there exists a $r' > 0$ s.t. $B(x, r') \cap A^c = \emptyset$ or not, i.e. $B(x, r') \cap A^c \neq \emptyset$ for all $r' > 0$ iff $x \in \text{int}(A)$ or $x \in \text{bdry}(A)$. □

(*ii*) Fix $\varepsilon > 0$ arbitrarily. By definition of convergent sequence, there exists a $N \in \mathbb{N}$ s.t. for all $m \geq N$, $d(x_m, x) < \varepsilon/2$ and $d(y_m, y) < \varepsilon/2$. Using the triangle inequality twice, we can argue that $d(x_m, y_m) \leq d(x_m, x) + d(x, y) + d(y, y_m) \leq \varepsilon + d(x, y)$ and that $d(x, y) \leq d(x_m, x) + d(x_m, y_m) + d(y, y_m) \leq \varepsilon + d(x_m, y_m)$. Equivalently, $|d(x, y) - d(x_m, y_m)| \leq \varepsilon$. Since ε was arbitrary, this proves the claim. □

EXERCISE 2

Prove that if X is compact in metric space (X, d) then X is separable.

Solutions

Fix $m \in \mathbb{N}$ and consider the covering $\mathcal{U}_n := \{B(x, \frac{1}{n}) : x \in X\}$. By compactness there exists a *finite* subcollection of $\mathcal{U}_n$ covering X, that is there exists a *finite* $D_n \subset X$ s.t. $\{B(x, \frac{1}{n}) : x \in D_n\}$ covers X. Define $D := \bigcup_n D_n$ and notice that it is countable. We claim D is dense in X. Suppose not. Then, there exists a $y \in X$ and an $\varepsilon > 0$ s.t. $D \cap B(y, \varepsilon) = \emptyset$. Set m s.t. $\frac{1}{m} = \varepsilon$. Then, $\{B(x, \frac{1}{n}) : x \in D_n\}$ covers X, a contradiction. □

EXERCISE 3

Prove (l^∞, d_∞) is complete.

Solutions

Let (x^m) be Cauchy in (l^∞, d_∞). Then for arbitrary $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. for all $n, m \geq N$, $\sup_{i \in \mathbb{N}} |x_i^m - x_i^n| < \varepsilon$ or, equivalently, for all $i \in \mathbb{N}$, $|x_i^m - x_i^n| < \varepsilon$. This implies that, for all $i \in \mathbb{N}$, the real sequence (x_i^m) (with respect to the index m) is Cauchy in $\mathbb{R}$ and, by Completeness of $\mathbb{R}$, there exists a $x_i \in \mathbb{R}$. That is, for all $i \in \mathbb{N}$, the same N above is such that for all $m \geq N$, $|x_i^m - x_i| < \varepsilon$. Notice N does not depend on i, hence $\sup_{i \in \mathbb{N}} |x_i^m - x_i| < \varepsilon$. Since ε is arbitrary, $x^m \to x = (x_1, x_2, \ldots)$. Finally, we want to show $x \in l^\infty$. From above we have $\varepsilon > |x_i - x_i^m| \geq |x_i| - |x_i^m|$. Rearranging and taking the supremum, we have that $\sup_i |x_i| \leq \sup_i |x_i^m| + \varepsilon < \infty$, since $x^m \in l^\infty$. □

If you spot a typo, write me at jacopo.perego@nyu.edu.

Exercise 4

Show that (X, d) is a compact metric space if and only if for every sequence of closed subset $\{F_n\}_{n=1}^{\infty}$ of X such that $\bigcap F_n = \emptyset$ there is a finite subcollection $\{F_{n_1}, \ldots, F_{n_K}\}$ such that $\bigcap_{k=1}^{K} F_{n_k} = \emptyset$.

Solutions

(Sufficiency). Let (X, d) be compact and let $\mathcal{F}$ be a collection of closed sets s.t. $\bigcap \mathcal{F} = \emptyset$. Equivalently, $X \setminus \bigcap \mathcal{F} = X \setminus \emptyset$ or $\bigcup \mathcal{U} = X$ where each element $O \in \mathcal{U}$, being the complement of some $F \in F$, is open. By compactness, there is a finite subcollection of $\mathcal{U}$ covering X, or equivalently there is a finite subcollection of $\mathcal{F}$, say $\{F_{n_1}, \ldots, F_{n_K}\}$, with empty intersection.

(Necessity). Consider an arbitrary open cover $\mathcal{U}$ and define $\mathcal{F} := \{O^c : O \in \mathcal{U}\}$ a collection of closed sets with empty intersection. By assumption, there exists a finite subcollection $\{F_{n_1}, \ldots, F_{n_K}\} \subset \mathcal{F}$ with empty intersection. Equivalently, $\{F_{n_1}^c, \ldots, F_{n_K}^c\} \subset \mathcal{U}$ covers X. Hence, X is compact. $\square$

Exercise 5

A metric space (X, d) is complete if and only if every decreasing sequence $F_1 \supset F_2 \supset F_3 \ldots$ of nonempty closed sets with $\operatorname{diam} F_k \to 0$ is such that $\bigcap_{k \geq 1} F_k$ is a singleton.[1]

Solutions

(Necessity). Suppose not, i.e. let (X, d) not complete. By Exercise 7, we can argue (X, d) is niether compact. Notice that the assumed $\{F_k\}$ has the finite intersection property. However, not being compact, it must be the case that $\bigcap_{k \geq 1} F_k = \emptyset$, hence not a singleton, a contradiction.

(Sufficiency). The proof is similar to the one provided for Exercise 7.(ii). Read that proof before proceeding further. From a Cauchy sequence (x_m), I can define a sequence of non empty nested closed sets F_k with diameter going to zero. Each one of these sets contains the tail of the sequence (x_m). By assumption, we know that $\bigcap_{k \geq 1} F_k$ is a singleton. As in Exercise 7, show x is the limit point of (x_m). $\square$

Exercise 6

Show that (l^{∞}, d_{∞}) is not separable.

Solutions

Suppose not, that is suppose there exists a countable dense set $D \subset l^{\infty}$. Consider the set of binary sequences $\{0, 1\}^{\infty} \subset l^{\infty}$. We know already this subset has the cardinality of the continuum. Moreover, every to distinct sequences $(x_m), (y_m) \in \{0, 1\}^{\infty}$, we have $d_{\infty}((x_m), (y_m)) = 1$. Consider now the collection of open balls $\{B((x_m), \frac{1}{2}) : (x_m) \in \{0, 1\}^{\infty}\}$. By the above observation, this is an uncountable collection of *pairwise disjoint* open subsets of l^{∞}. Since, D is dense, there is one distinct element of D in each one of these balls. Hence, D must not be countable, a contradiction. $\square$

Exercise 7

Prove all the following statements:

[1] $\operatorname{diam} A = \sup_{a, b \in A} d(a, b)$

(*i*) Every sequentially compact metric space is complete.

(*ii*) Every compact metric space is complete. In showing this, do not use the known equivalence between compactness and sequential compactness.[2]

Solutions

(*i*) Let (X, d) be sequentially compact and (x_m) be Cauchy. Then, (x_m) has a subsequence (x_{m_k}) converging to $x \in X$. If a Cauchy sequence has a convergent subsequence then the sequence itself converges.[3] $\square$

(*ii*) Let (X, d) be compact and (x_m) be Cauchy. For any $k \in \mathbb{N}$, there exists a N_k such that for all $n, m \geq N_k$, $d(x_m, x_n) < \frac{1}{k}$. This means that the ball $B(x_{N_k}, \frac{1}{k})$ contains all but finitely many element of (x_m). Now let's inductively define a decreasing sequence of non empty closed nested sets $\{F_k\}$: let $F_1 := \mathrm{cl}B(x_{N_1}, 1)$ and, for all $m > 1$, $F_m := F_{m-1} \cap \mathrm{cl}B(x_{N_m}, \frac{1}{m})$. Notice that $\mathrm{diam}F_m < \frac{1}{m}$, that is $\mathrm{diam}F_m \to 0$ in m. Since X is compact and $\{F_k\}$, by construction, is a sequence of non empty closed sets with the finite intersection property, we have $\bigcap\{F_k\} \neq \emptyset$. Moreover, since $\mathrm{diam}F_m \to 0$, we have $\bigcap\{F_k\} = \{x\}$. We claim x is the limit point of (x_m). Indeed, for arbitrary $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$ such that $F_k \subset B(x, \varepsilon)$. That is, $B(x, \varepsilon)$ contains all but finitely many elements of (x_m). $\square$

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[2] You may want to follow the following hints:

a. Let (x_m) be a Cauchy sequence in a compact metric space (X, d). Argue that, for every $\varepsilon > 0$, there exists a $y \in X$ s.t. $B(y, \varepsilon)$ contains all but finitely many elements of (x_m).

b. Use the Finite Intersection Property to show that (x_m) has a limit point.

[3] *Proof*: Fix ε arbitrarily. By Cauchyness of (x_m), there exists a N such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon/2$. By the fact that (x_{m_k}) convergence we get that there exists N' such that, for all $m_k \geq N'$, $d(x, x_{m_k}) < \varepsilon/2$. Letting $M = \max\{N, N'\}$, we have that for all $m \geq M$, $d(x_m, x) \leq d(x_m, x_{m_k}) + d(x_{m_k}, x) < \varepsilon$. $\square$