MICROECONOMICS II.I – PS1 SOLUTIONS

New York University, a.y. 2012-2013

General Comment:

The class as a whole did a very good job. There was some confusion here and there about the uniqueness part in Kuhn's Theorem. Remember: given a behavioral strategy, if the player is active at more than one information set, you get a continuum of equivalent mixed strategies. Given a mixed strategy, if all information sets are reached with strictly positive probability, you get uniqueness of the equivalent behavioral strategy.

1. Exercise 1

Part A

Find the normal form of the extensive form game in Figure 1. Write it down as a *bimatrix*, i.e. a matrix whose entries contain payoff pairs $(u_1(s), u_2(s))$.

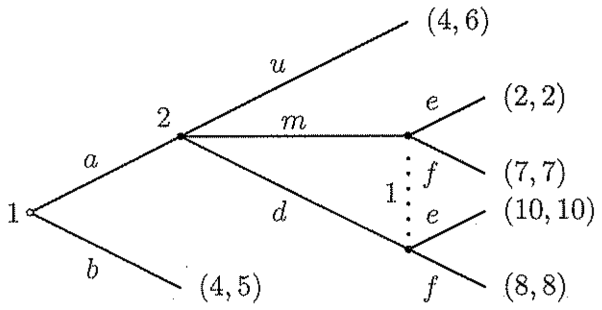

Figure 1.— An Extensive Form Game.

Solution

Recall that the normal form of an extensive form game Γ is the structure $N(\Gamma) = \langle N, (S_i, u_i)_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players, $S_i = \{s_i : \mathcal{I}_i \to A\}$ is the set of strategies and $u_i : S_i \times S_{-i} \to \mathbb{R}$ is the payoff function. Clearly, $N = \{1, 2\}$. Player 2 is active only at one history, i.e. history (a), where she has 3 available actions. Her set of strategies is then $S_2 = \{u, m, d\}$. On the other hand, Player 1 is active at the initial history and at the information set $I = \{(a, m), (a, d)\}$. In both cases, he has two actions available. Hence, he has a total of 4 different strategies, $S_1 = \{a.e, a.f, b.e, b.f\}$.[1] The normal form $N(\Gamma)$ is then the simultaneous move game reported in Figure 2.

February 20, 2013. These proposed solutions may contain minor typos. If you spot any, please e-mail me at jacopo.perego@nyu.edu.

[1] Where I use the convention $x.y$ = "Play x at $\emptyset$ and y at I".

		P2		
		u	m	d
$P1$	$a.e$	$4, 6$	$2, 2$	$10, 10$
	$a.f$	$4, 6$	$7, 7$	$8, 8$
	$b.e$	$4, 5$	$4, 5$	$4, 5$
	$b.f$	$4, 5$	$4, 5$	$4, 5$

FIGURE 2.— Normal Form of game Γ.

Part B

Find an extensive form game with **perfect information** having the normal form shown in Figure 5.

		P2
$P1$	$5, 5$	$8, 8$
	$1, 1$	$1, 1$
	$5, 5$	$6, 6$
	$1, 1$	$1, 1$
	$5, 5$	$0, 0$
	$1, 1$	$1, 1$

FIGURE 3.— Normal Form of game Γ.

Solution

First notice that $P2$ has only two strategies. This suggests she is active only at one (possibly singleton) information set, at which she has two available options, call it l and r. Similarly, Player 1 has six strategies, suggesting he is active at two (possibly singleton) information sets, in which he has 3 and 2 actions available, respectively.[2] An extensive form game that would produce the normal form in Figure 3 is the following: Player 1 moves first choosing between a or b. If b, the game ends and players receive a payoff of $(1, 1)$. If b, it's P2's turn to play. She can choose between l and r. If l, the game ends and players receive $(5, 5)$. If d, it's P1's turn again, where he can choose c, e and f. In all three cases, the game ends with payoffs $(8, 8)$, $(6, 6)$ and $(0, 0)$, respectively.

QUESTION 2

For the next items, refer to the EFG Γ_1 below.

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[2] If you are puzzled by this sentence, recall that the total number of strategies available to player i can be computed as $|S_i| = \prod_{I \in \mathcal{I}_i} |A(I)|$, where $|A(I)|$ is the number of actions available to him at the information set $I \in \mathcal{I}_i$. For example, if player i is active at 4 different information sets in which he has 2, 3, 4 and 5 available actions, respectively, he will have a total of $2 \cdot 3 \cdot 4 \cdot 5 = 120$ strategies.

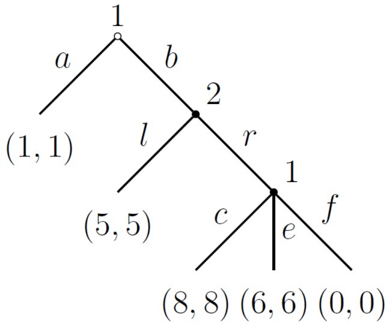

FIGURE 4.— Extended form of the game in Figure 3.

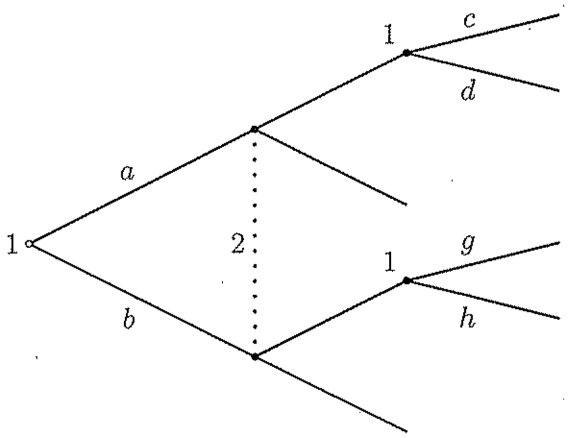

FIGURE 5.— Tree representation of Γ_1.

Part A

Find a mixed strategy for Player 1 that is equivalent in the sense of Kuhn's Theorem to the behavioral strategy that chooses a with probability 0.4, c with probability 0.6 (conditional on being reached) and g with probability 0.1 (conditional on being reached). Is it unique?

Solution

First, recall the definition of equivalent strategies. We say that the mixed strategy m_i and the behavioral strategy b_i are *equivalent* if, against **any** opponent's strategy profile s_{-i}, they induce the same probability distribution over terminal histories. Let u and u'' be the labels for P2's actions. The set of terminal histories is then

$$Z = \{(a,u,c),(a,u,d),(a,u''),(b,u,g),(b,u,h),(b,u'')\}.$$

We are given a strategy

$$b_1 = \big(b_i(a|\emptyset) = .4,\ b_i(c|a,u) = .6,\ b_i(g|b,u) = .1\big).$$

Fix any distribution $\big(b_i(u|\{a,b\})$ for player 2. Without loss of generality assume $\big(b_i(u|\{a,b\}) = 1$. This behavioral strategy profile induces a probability distribution ζ on the set of terminal histories Z. We have $\zeta(a,u,c) = .24$, $\zeta(a,u,d) = .16$, $\zeta(b,u,g) = .06$ and $\zeta(b,u,h) = .54$. We can retrieve an equivalent mixed strategy as follows:

$$\begin{aligned}
m_i(a.c.g) &= b_i(a|\emptyset) \cdot b_i(c|a,u) \cdot b_i(g|b,u) = .024\\
m_i(a.c.h) &= b_i(a|\emptyset) \cdot b_i(c|a,u) \cdot b_i(h|b,u) = .216\\
m_i(a.d.g) &= b_i(a|\emptyset) \cdot b_i(d|a,u) \cdot b_i(g|b,u) = .016\\
m_i(a.d.h) &= b_i(a|\emptyset) \cdot b_i(d|a,u) \cdot b_i(h|b,u) = .144
\end{aligned}$$

$$\begin{aligned}
m_i(b.c.g) &= b_i(b|\emptyset) \cdot b_i(c|a,u) \cdot b_i(g|b,u) = .036\\
m_i(b.c.h) &= b_i(b|\emptyset) \cdot b_i(c|a,u) \cdot b_i(h|b,u) = .324\\
m_i(b.d.g) &= b_i(b|\emptyset) \cdot b_i(d|a,u) \cdot b_i(g|b,u) = .024\\
m_i(b.d.h) &= b_i(b|\emptyset) \cdot b_i(d|a,u) \cdot b_i(h|b,u) = .216
\end{aligned}$$

Check that this mixed strategy induces the same probability ζ above. Notice that this mixed is not unique, indeed $|\mathcal{I}_1| > 1$. We have a continuum of equivalent mixed strategies. For example, $\tilde{m}_i(a.c.g) = .24$, $\tilde{m}_i(a.d.g) = .16$, $\tilde{m}_i(b.c.g) = .06$, $\tilde{m}_i(b.c.h) = .54$ and zero otherwise.

Part B

Find a behavioral strategy that is equivalent to the mixed strategy for Player 1 that assigns probability weights 0.5 on (a,c,g), 0.2 on (b,c,g), 0.2 on (b,d,g) and 0.1 on (b,d,h). Is it unique?

Solution

We have a unique m_1-equivalent behavioral strategy if, according to m_1, all non-terminal histories are reached with positive probability, for non-degenerate m_2. The mixed strategy we are given is

$$m_1 = (m_1(a.c.g) = .5,\ m_1(b.c.g) = m_1(b.d.g) = .2,\ m_1(b.d.h) = .1)$$

Notice that this strategy meets the uniqueness requirement above. Hence, the equivalent behavioral strategy we are about to find, will be unique. This given mixed strategy m_1, paired with some m_2, induces a probability measure ζ on terminal histories Z. For simplicity consider $m_2(u) = 1$. We then have, $\zeta(a,u,c) = .5$, $\zeta(a,u,d) = .0$, $\zeta(b,u,g) = .4$ and $\zeta(b,u,h) = .1$. Non-terminal histories (a,u) and (b,u) are reached with strictly positive probability. Hence we can apply Bayes' rule to get an equivalent b_1:

$$b_1(a|\emptyset) = \frac{m_1(\text{“Play } a \text{ at } \emptyset\text{”})}{m_1(\text{“Reach } \emptyset\text{”})} = \frac{.5}{1} = .5$$

$$b_1(c|(a,u)) = \frac{m_1(\text{“Play } c \text{ at } (a,u)\text{”})}{m_1(\text{“Reach } (a,u)\text{”})} = \frac{.5}{.5} = 1$$

$$b_1(g|(b,u)) = \frac{m_1(\text{“Play } g \text{ at } (b,u)\text{”})}{m_1(\text{“Reach } (b,u)\text{”})} = .8$$

Check that this strategy will indeed induce the same ζ. The above argument hold for any $m_2(u) \in [0,1]$. Of course, each different m_2 will give rise to a different distribution ζ'. But the b_1 we found will still be meeting the requirement, i.e. it will induce exactly the same ζ'.

QUESTION 3

Ten ferocious old pirates are dividing their plunder (100 gold coins) before disbanding. No single coin can be subdivided. According to pirate code, pirate number 1 $P1$ suggests a sharing rule (for instance if $P1$ suggests $(55, 0, 9, 0, 9, 0, 9, 0, 9, 9)$ he is suggesting that $P1$ gets 55 coins, $P2$ gets 0, $P3$ gets 9 and so on...). All ten pirates vote by roll call on the proposal. If a majority (even a tie is enough) accept, then the division is carried out and the game ends. If the suggestion is not accepted then the first pirate is thrown overboard (which is worse than getting no gold, because of the circling sharks) and $P2$ makes a proposal, which is subjected to majority vote, and so on. These pirates are crafty enough to perform backward induction, and so cranky that whenever a pirate is indifferent about voting for or against a proposal he votes against. Explain what happens and why.

Solution

We can solve this game by backward induction. If the game ever reaches the 10th and final stage, the only survivor P10 will retain all the 100 coins for himself. If the game ever reaches the 9th stage, P9 will offer P10 nothing, keeping for himself 100 coins. Voting on this sharing rule will lead to a tie, which is good enough for P9 to go home with the whole plunder. If the game ever reaches the 8th stage, P8 understands it's to expensive to *buy* the vote of P9. But it's cheap to buy P10's vote! P8 will offer P10 a single coin, retaining 99 for himself. In this way, his proposal will pass with two votes (P8 and P10) against one (P9). The game keeps unfolding backwardly in this fashion, as described in the table. The unique (SP) equilibrium is the one in which $P1$ offers the sharing rule $(96, 0, 1, 0, 1, 0, 1, 0, 1, 0)$. At the voting round, $P1, P3, P5, P7$ and $P9$ vote Yes while the others vote No. The proposal passes at the first stage.

$$\begin{array}{r|cccccccccc|}
 & \text{P1} & \text{P2} & \text{P3} & \text{P4} & \text{P4} & \text{P5} & \text{P6} & \text{P7} & \text{P9} & \text{P10} \\
\hline
\text{P10's offer} & & & & & & & & & & 100 \\
\text{P9's offer} & & & & & & & & & 100 & 0 \\
\text{P8's offer} & & & & & & & & 99 & 0 & 1 \\
\text{P7's offer} & & & & & & & 99 & 0 & 1 & 0 \\
\text{P6's offer} & & & & & & 98 & 0 & 1 & 0 & 1 \\
\text{P5's offer} & & & & & 98 & 0 & 1 & 0 & 1 & 0 \\
\text{P4's offer} & & & & 97 & 0 & 1 & 0 & 1 & 0 & 1 \\
\text{P3's offer} & & & 97 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\text{P2's offer} & & 96 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\text{P1's offer} & 96 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

QUESTION 4 (OPTIONAL, NOT GRADED)

Find all equilibria (in pure or mixed strategies) of the following game.

		$P2$		
		R	P	S
$P1$	R	$0, 0$	$0, 1$	$1, 0$
	P	$1, 0$	$0, 0$	$0, 1$
	S	$0, 1$	$1, 0$	$0, 0$

FIGURE 6.— Rock-paper-scissor game.

Solution

It's immediate to see there are no NE in pure strategies. Let's look for a completely mixed equilibrium, i.e. an equilibrium in which $m_i \in \text{int}\Delta(R, P, S)$ for $i \in N$. We want to find a m_2 s.t. P1 is indifferent between R, P and S. If he plays R, P1 gets an expected payoff equal to $m_2(S)$. If he plays P, he gets $m_2(R)$. Finally, if he plays S, he gets $m_2(P)$. To solve for m_2 we can set $R \sim_1 P$, $P \sim_1 S$, use transitivity and the definition of probability. We solve the linear system:

$$\begin{cases} m_2(S) = m_2(R) \\ m_2(R) = m_2(P) \\ m_2(P) = m_2(S) \\ m_2(S) + m_2(R) + m_2(P) = 1 \end{cases}$$

whose solution is $m_2(S) = m_2(R) = m_2(P) = \frac{1}{3}$. Since the game is symmetric, we can set $m_1 = m_2$ and claim that $m = (m_1, m_2)$ is a NE.

Is there any NE in mixed strategy that assign zero weight to one action? Suppose this is the case and WLOG set $m_2(S) = 0$. In this case, P1's expected payoffs of playing R, P and S are 0, $m_2(R)$ and $1 - m_2(R)$, respectively. If $m_2(R) \neq \frac{1}{2}$, then $P1$ is not indifferent between P and S. Whatever the direction, we won't have an equilibrium. Hence, $m_2(R) = \frac{1}{2}$ and $P1$ randomizes between P and S, according to some randomizing device $m_1(P)$. $P2$'s expected payoff of playing m_2 given m_1 is $\frac{1}{2}m_1(S)$. However, notice $P2$ has now a profitable deviation. Indeed, he can move probability mass from P to S and increase in this way his payoff. Hence, $m_2(S) > 0$, a contradiction.