Math for Econ I

Problem Set 3*

Exercise 1

Let (Y, d) be a metric space.[1]

(a) True or False: If Y is bounded, $(2^Y \setminus \{\emptyset\}, d_H)$ is a semimetric space.

(b) Give an example of Y s.t. $(2^Y \setminus \{\emptyset\}, d_H)$ is not a metric space.

(c) Let $\mathbf{c}(Y)$ be the class of all nonempty compact subset of Y. Show that $(\mathbf{c}(Y), d_H)$ is a metric space.

Exercise 2

Prove that if Y is complete, then so is $(\mathbf{c}(Y), d_H)$. (Hint. Use Cantor Intersection Theorem)

Exercise 3

Let X and Y be two metric spaces. Prove that the correspondence $\Gamma : X \rightrightarrows Y$ satisfies the closed graph property if and only if $\mathrm{Gr}(\Gamma)$ is closed in the product space $X \times Y$.

Exercise 4

Let T be a compact metric space and $\mathcal{F} \subset C(T)$. Define $\Gamma : T \rightrightarrows \mathbb{R}$ by $\Gamma(t) := \cup\{f(t) : f \in \mathcal{F}\}$. Prove that if $\mathcal{F}$ is compact in $(C(T), d_\infty)$, then Γ is upper hemicontinuous and compact valued.

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*Due by **October Wed 30th, 7pm**.

[1] For all subset $A, B \subset Y$, define $w(A, B) := \sup\{d(z, B) : z \in A\}$ and the function $d_H : 2^Y \setminus \{\emptyset\} \times 2^Y \setminus \{\emptyset\} \to \bar{\mathbb{R}}$ with $d_H(A, B) := \max\{w(A, B), w(B, A)\}$.

EXERCISE 5

Prove that a subset S of a linear space X is a basis for X iff S is linearly independent and $X = \text{span}(S)$.

EXERCISE 6

Let X and Y be linear spaces. Show that $\dim(X \times Y) = \dim(X) + \dim(Y)$.[2]

EXERCISE 7

Let S be a non-empty subset of a linear space X. Show that

$$\text{aff}(S) = \bigcap \{Y \subset X |\ Y \text{ is an affine manifold of } X \text{ and } S \subseteq Y\}$$

EXERCISE 8

Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear and onto, then the image of any open set is open.

[2]Clearly, $(X \times Y, +, \cdot)$ is a linear space where the operation of sum and scalar multiplication are inherited from those in $(X, \oplus_X, \odot_X)$ and $(Y, , \oplus_Y, \odot_Y)$. E.g. $(x, y) + (x', y') := (x \oplus_X x', y \oplus_Y y')$, and $\lambda \cdot (x, y) := (\lambda \odot_X x, \lambda \odot_Y y)$.