MICROECONOMICS II.I – PS5 SOLUTIONS

New York University, a.y. 2012-2013

Exercise 1

Two quantity-setting firms with identical marginal cost $c > 0$ face an inverse demand curve $p = a - q_1 - q_2$. Firm 1 chooses q_1 knowing that Firm 2 will observe q_1 before choosing q_2.

(*a*). What is the unique subgame perfect equilibrium of this game? Either prove there are no other Nash equilibria of the game, or give an example of an NE of this game that is not an SPE.

(*b*). Suppose instead that it is commonly known that after Firm 1 chooses q_1, Firm 2 doesn't observe q_1 directly, but gets a signal x distributed as follows: with probability p the signal is correct, that is $x = q_1$; with probability $1-p$ the signal is uninformative, that is x is uniformly distributed on $(0, a-c)$. Find a perfect Bayesian equilibrium **in pure strategies** for this model. How does it depend on p?

Solution, Complete Information

This is a standard Stackelberg problem. We solve it by backward induction. Player 2, plays after 1 and observe q_1. He faces the program:

$$r_2(q_1) = \arg\max_{q_2} \Big(a - c - q_1 - q_2\Big) q_2$$

which yields to $r_2(q_1) = \frac{a-c-q_1}{2}$.

Player 1, who believes in the rationality of 2, solves:

$$r_1(\mu_1 = r_2(q_1)) = \arg\max_{q_1} \left(a - c - q_1 - \frac{a-c-q_1}{2}\right) q_1$$

which yields to $r_1(\mu_1 = r_2(q_1)) = \frac{a-c}{2}$.

Hence, the equilibrium is the solution of the system induced by $r_1(r_2(q_1)) = q_1$ and $r_2(r_1(r_2(q_1))) = q_2$ that is

$$\left(q_1 = \frac{a-c}{2}, q_2 = \frac{a-c}{4}\right)$$

with equilibrium price equal to $p^\star = \frac{a-c}{4}$ and equilibrium payoffs equal to

$$\left(u_1(q) = \frac{(a-c)^2}{8}, u_2(q) = \frac{(a-c)^2}{16}\right)$$

March 13, 2013. These proposed solutions may contain minor typos. If you spot any, please e-mail me at jacopo.perego@nyu.edu.

Solution, Uniqueness

Proving uniqueness here amounts to prove that Backwards Induction, when no relevant ties are in order, gives the unique SPE strategy profile of the game. Recall that a strategy profile is a SPE iff it is unimprobable, or in other words, it is *immune* to one shot deviations. In this game there are only two subgames: the one starting at the empty history, $\emptyset$, and the one starting at $(\emptyset, q_1)$. We showed above that r_2 is a function, meaning that given all possible histories there exists a unique best reply. This best reply is trivially immune to one shot deviation. Since $r_2(q_1)$ is a function, we can define a continuation value *function* that maps $q_1 \mapsto u_1(q_1, r_2(q_1))$. As we found above, the optimal choice of q_1 is unique that is r_1 is a function. Hence, the equilibrium is unique.

A formal proof: Suppose not, that is let q' be a SPE s.t. $q' \neq q$. Suppose $q'_2 \neq q_2$. Since r_2 is a strictly decreasing function, this requires $q'_1 \neq q_1$. If $u_1(q_1, r_2(q_1)) > u_1(q'_1, r_2(q'_1))$ we have a contradiction, because q' is not immune to one shot deviations (i.e. is not unimprovable). If instead $u_1(q_1, r_2(q_1)) = u_1(q'_1, r_2(q'_1))$ we have another contradiction since $q_1 \mapsto u_1(q_1, r_2(q_1))$ is strictly quasi-concave, hence $q' = q$.

Example of Subgame Imperfect Nash. Consider the incredible threat for which Player 2, plays the following:

$$r_2(q_1) = \begin{cases} \frac{a-c}{2} & \text{if } q_1 = 0 \\ a-c & \text{if } q_1 > 0 \end{cases}$$

and P1 best reply with $\frac{a-c}{4}$.

Solution, Incomplete Information

As in a previous problem set let's normalize $a - c = 1$. Player 1 plays q_1. Player 2 receives a signal x which is equal to q_1 with probability p and uniformly distributed on $(0, 1)$ with probability $(1 - p)$. They key point is that Player 2 has no way to discern between a good signal and a bad signal, i.e. she doesn't know whether the signal she sees is actually coming from the realization of the probability p event of observing a truthful signal or if it is the realization of random uniform draw. This being said, the simplest epistemic assumption we could do is to think that Player 2 always holds a determinist belief s.t. $\mu_2(q_1) = x$ is true with probability one. That is she always interpret the signal x as truthful. If this is the case, the best reply will be always equal to

$$r_2(x) = \frac{a-c-x}{2}$$

However, the basic assumption that we are implicitly carrying on is that the structure of the game is common knowledge and that there is common belief in rationality. Since this is the case, let's consider the effect of *belief in rationality.*

Notice that Rationality of P2 and her belief in the rationality of Player 1 implies that $P1$ will never play $q_1 > \frac{1}{2}$ and that all signals higher than $\frac{1}{2}$ will not be trusted. Similarly, Player 1 understands that Player 2 will at least best reply to her worst case conjecture, i.e. the one in which $q_1 = \frac{1}{2}$ with the stackelberg quantity $q_2 = \frac{1}{4}$, that is $q_2 \geq \frac{1}{4}$. In turns, the best reply of P1 to $q_2 = \frac{1}{4}$ is $q_1 = \frac{3}{8}$. Hence, P2 will never believe signals that are outside $[\frac{3}{8}, \frac{1}{2}]$. The procedure goes on like in the standard application of Rationalizability in the standard

Cournot oligopoly. Not suprisingly, we get the same result, the Cournot equilibrium. More precisely the equilibrium we found is

$$(q_1 = \frac{1}{3}, q_2 = \frac{1}{3}, \mu_2 = \delta_{13})$$

where δ_{13} is the Dirac mesure. Notice that, in order to get to this particular equilibrium in pure strategies, we have made no intial assumption on the beliefs of P2. We have just used rationality and common belief in rationality. We this technique we have found a unique belief structure, i.e. δ_{13}, that is consistent with it. This suggests that the solution we have found is indeed unique among the pure strategy subset of equilibria.

To argue a bit more on that suppose we are trying to find a different equilibrium, and to do so we posit that the beliefs of $P2$ are such that she never trust the signal. Still, she will give zero probability to $x \in (1/2, 1]$. Whenever $x \in R_1(Q_1)$, instead, she will hold a uniform belief on $[0, \frac{1}{2}]$. Being an expected utility maximizer, and being the expectation functional a linear operator he will best reply with $q_2 = 1/4$. At that point, $P1$, who believes that $P2$ is not trusting any signal, will best reply exactly to $q_2 = 1/4$, hence he will play the Stackelberg solution. However, in this way, not trusting the signal will give $P2$ his Stackelberg payoff, which is lower than the Cournot one. Hence, in equilibrium, $P2$ prefers to trust the signal, I prefer $P1$ to believe that he is trusting the signal and so on so forth.

Comment: The analysis of pure strategy equilibria lead us to the following conclusion. p is irrelevant for the equilibrium. That is, no matter how infinitesimally small $1 - p$ is, no matter the signal being almost surely correct, the equilibrium will never be close to the the Stackelberg one. That is there is a large discontinuity at $p = 1$ in the equilibrium behavior.

Exercise 2

Evaluating an infinite sequence of payoffs by its present discounted value captures the impatience of the decision maker. Koopmans (1960) showed axiomatically that there is no satisfactory way to model "perfect patience". A crude way to do this is by the limit of means criterion, which says a player cares only about the long-run average of the payoffs she receives. Suppose that each period $t = 1, 2, \ldots$ she gets some payoff x_t from the same bounded subset of $\mathbb{R}$. Define the associated sequence of averages by

$$a_t = \frac{1}{t}\sum_{j=1}^{t} x_j$$

(a). Give an example of a sequence that does not converge, although its sequence of averages does. Now give an example of a bounded sequence whose sequence of averages doesn't converge.

(b). To get around the problem that the "long-run average" may not be well-defined, the limit of means of the sequence (x_t) is defined to be the limit inferior

$$\lim_{T\to\infty} \inf_{t\geq T} x_t$$

Explain why this is always well-defined (assuming as above that the payoffs are bounded).

(*c*). Recall the definition of unimprovable strategy: A strategy σ_i for a player i is called unimprovable if there is no history of play after which player i can do strictly better by deviating from σ_i in the current period only (and conforming to σ_i thereafter) than by conforming with σ_i in the current period and thereafter. Now consider the following problem.

Samantha loves cookies. She gets 0 utils from eating no cookies at lunch, 1 from eating 1, and 2 from eating 2. Her friend Enabla is willing to give her up to 2 cookies each day to eat at lunch. Cookies cannot be stored for future consumption. Samantha's mother approves pleasure, but only in moderation. She gives Samantha the following instructions: "On any day t, eat one cookie at lunch, unless you ate two cookies the day before, in which case you eat no cookie on day t". View this as a strategy for Samantha in a one-person infinitely repeated game. Suppose that Samantha evaluates her payoff stream using the limit of means criterion. Show that although the strategy is not optimal, it is unimprovable.

Solution, Part (a.)

An example of a non converging sequence with converging mean is the following

$$x_t = \begin{cases} 1 & \text{if } t \text{ odd} \\ -1 & \text{if } t \text{ even} \end{cases}$$

The sequence doesn't converge, but its sequence of mean does converge to 0.

To see an example of the second kind consider the infinite sequence in $\{0, 1\}$ made as follows. First element is one, then *two* zeros, then *four* ones, then *eight* zeros, then *sixteen* ones and so on. It's easy to see that such a sequence does not have converging sequence of means.

Solution, Part (b.)

The $\liminf$ always exists because $\mathbb{R}$ is *conditionally complete*, i.e. is a field where the completeness axiom hold (every bounded set has an $\inf$ and a $\sup$). Now, the sequence $y_T = \inf_{t \geq T} a_t$ is an increasing sequence in a bounded space. But once we have observed this, it is immediate to show that any monotone bounded sequence converges (this is a result you should know from Math I, I am not proving it).

Notice that in my previous example, the $\lim$ of the mean didn't exists but the $\liminf$ did, it was $\frac{1}{3}$.

Solution, Part (c.)

Player's action set is $A = \{0, 1, 2\}$. Payoffs in the *stage game* is $u(a) = a$ for all $a \in A$.

This example is meant to show that the equivalence between SPE and improvability **is no longer true** when we compute the value of some strategy s using the limit of the means criterion. From now on, suppose we abide to such a criterion. Now consider the strategy proposed: $s_t = 1$ for if $s_{t-1} < 2$ for all t. If $s_{t-1} = 2$ then $s_t = 0$. Notice that I used time subscript just out of simplicity, but we agree on the fact that the notation s_t is meaningless, since for each time period t there is a multiplicity of different histories, and a strategy must give an instruction to all of them. [1]

In this game there is an obvious unique SPE $s^\star$ which is the one that plays 2 at all $h \in H$. The value of such a SPE is

$$V(s^\star) = \lim_t \frac{1}{t} \sum_1^t s_t^\star = 2$$

Now, take an arbitrary $h \in H$, and consider the following strategy s'. Suppose that for all $h' \prec h$, we had $s'(h') = 1$. That is there was no deviation wrt to s until h. Now suppose that the player wants to deviate to 2. In that case, $s'(h) = 2$. Since we want to consider only one shot deviation, we want the strategy s' to be as close as possible to s, that is, to play at $(h, 2)$ what s would have played. That is we want $s'(h, a) = 0$. From $(h, 2)$ onward the player sticks to s, i.e. $s'(h'') = s(h'') = 1$ for all $(h, 2) \prec h''$.

This being said, what is the value of s and s'. Clearly, the value of s is 1 and the one of s' is 1 as well. That is deviating leaves the player indifferent, i.e. s is unimprovable at every history. However, it is not a SPE.

As a further exercise, notice that if the criterion with which we compute the value of a strategy is the usual Discounting we are back to the standard framework where the equivalence between SPE and unimprovability holds. To check that this is really the case let s be the strategy suggested by Samantha's mom and consider the same deviation we considered beforel. We would have

$$\sum_t^\infty \delta^i u(s_i) < u(s'_t) + \delta(s'_{t+1}) + \sum_{t+2}^\infty \delta^i u(s'_i)$$

where on the LHS there is the PDV of sticking to s and on the RHS the PDV of deviating at t only.

$$\sum_t^\infty \delta^i 1 < 2 + 0 + \sum_{t+2}^\infty \delta^i 1 \qquad \Rightarrow \delta^t + \delta^{t+2} < 2$$

That is, we have found one profitable deviation, hence s is not immune to one shot deviations.

[1] A proper, but heavier notation, I think would be the following: an history h is a sequence of actions taken by the player. The set of histories is H which in this simple case is $H := A^\infty$. Then our strategy is described as follows: for any given history $h \in H$, $s(h) = 1$ if $s(h') \neq 2$, where h' is the immediate predecessor of h, that is the longest $h' \in H$ s.t. $h' \prec h$.