# Math for Econ I

## Problem Set 4[*]

### Exercise 1

Let $A$ and $B$ be any two sets in $\mathbb{R}^n$. Show that:

(a) $\text{co}(A) + \text{co}(B) = \text{co}(A + B)$.

Assume now and for the rest of the exercise that $A$ and $B$ are convex.

(b) True or false: If $A \cap B \neq \emptyset$, $\text{aff}(A \cap B) = \text{aff}(A) \cap \text{aff}(B)$.

(c) If $\text{ri}(A) \cap \text{ri}(B) \neq \emptyset$, then $\text{ri}(A) \cap \text{ri}(B) = \text{ri}(A \cap B)$.

(d) If $\text{ri}(A) \cap \text{ri}(B) = \emptyset$, then $A$ and $B$ can be properly separated.

### Exercise 2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map (that is, $T(x) = A(x) + b$ for some linear function $A : \mathbb{R}^n \to \mathbb{R}^m$ and vector $b \in \mathbb{R}^m$). Show that if $C \subset \mathbb{R}^n$ is convex, then $T(C) \subset \mathbb{R}^m$ is also convex.

### Exercise 3

Let $M = T + z$ be a linear manifold in $\mathbb{R}^n$. We want to construct the orthogonal projection map $P_M : \mathbb{R}^n \to M$, that is a function such that $P_M(y) = x$ if $d(y, x) = d(y, M)$. As we showed in class, there exist $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ such that $M = \{x \in \mathbb{R}^n | Ax = b\}$.

(a) Show that $T = \text{null}(A)$ and that $T^\perp = \{A^T \lambda \mid \lambda \in \mathbb{R}^m\}$.

(b) Show that for any $y \in \mathbb{R}^n$, $x^\star = P_M(y)$ if and only if $x^\star \in M$ and $\langle y - x^\star, x - x^\star \rangle = 0$ for all $x \in M$. Conclude that $x^\star = P_M(y)$ if and only if $x^\star \in M$ and $y - x^\star \in T^\perp$.

[*]Due by **November, Wed 13th, 7pm**.

$(c)$ Assume that $m < n$ and that $A$ has rank $m$ (so $A$ is full rank). Show that there exists a matrix $B \in \mathbb{R}^{n\times n}$ and a vector $d \in \mathbb{R}^n$ such that $P_M(y) = By + d$ for all $y \in \mathbb{R}^n$. Explicitly construct $B$ and $d$.

## EXERCISE 4

**Leontief Production:** Each industry produces a single consumption good using as inputs the goods produced by other industries and raw materials. There are $n$ consumption (intermediary) goods and $m$ raw materials. The economy is endowed with $\omega_k > 0$ units of raw material $k$, $k = 1, \ldots, m$. The production of one unit of (consumption) good $j$ requires $a_{ij}$ units of good $i$ ($i = 1, \ldots, n$) and $b_{kj}$ units of raw material $k$ ($k = 1, \ldots, m$). If for each $i = 1, \ldots, n$, $x_i$ and $c_i$ denote, respectively, the total amount of good $i$ that is produced and consumed (by consumers), then

$$x = Ax + c.$$

The production schedule $x$ is feasible if $x \geq 0$, $(I - A)x \geq 0$ and $Bx \leq \omega$. The feasible schedule $x$ is efficient if there is no feasible schedule $y$ such that $(I - A)y \geq (I - A)x$ and$(I - A)y \neq (I - A)x$ . Show that if $x$ is efficient then there exist price vectors $p \in \mathbb{R}^n_+$ and $q \in \mathbb{R}^m_+$ such that $(p^T, q^T) \neq (0, 0)$,

$$p^T(I - A) - q^T B \leq 0, \quad (p^T(I - A) - q^T B)x = 0, \quad \text{and} \qquad q^T(\omega - Bx) = 0.$$

Conversely, if for some feasible $x$ such prices exist and $p > 0$, then $x$ is efficient. Note that the first condition says that no production activity is strictly profitable, while the second condition implies that $p_j = \sum_i p_i a_{ij} + \sum_k q_k b_{kj}$ if $x_j > 0$. That is, any industry that produces a strictly positive amount of output, makes $0$ profits. The last condition implies that $q_k = 0$ when $\sum_j b_{kj} x_j < \omega_k$.

**Hint**: You may find it easier here to use a separation argument rather than Farkas Lemma.