MICROECONOMICS II.I – PS4 SOLUTIONS

New York University, a.y. 2012-2013

General Comment:

The average mark was 0.5 lower than in previous Problem Sets. So it's ok if your grade was lower than usual. With regard to Exercise 1, understanding the intuition behind THPE is crucial, since a direct application of its Definition was problematic in this context. In Exercise 2.b, you were asked to show that the order of voters does not affect the outcome of the election. This requires a (simple) proof. Restating the question is not a proof! Now that you have in your pocket the fact that a profile of strategies is an SPE iff it is unimprovable, you should try again to give a formal answer and appreciate the usefulness of Abreu's theorem. With regard to Pareto ranking, you gave different answers. My take is that the order you were to define is the eucledian order $\geq$, on the n-dimensional space of utility vectors $(u_i(s))_{i\in N}$. Second, the fact that $\geq$ is not complete doesn't mean we cannot Pareto rank some of the equilibria or find an efficient frontier (Think about the standard GE two-consumer example). Exercise 3 was hard and most of you crashed on it. If you are still puzzled, refer to the solution below and to the stylized example presented in Lab 6 . In Exercise 4, the majority of you didn't notice that all mixtures in $\{1, 2\}$ are rationalizable. *jp*

Exercise 1

The 100 residents of a small town are electing a mayor by secret ballot, and everyone is required to vote. There are three candidates: a Democrat (ranked first in strict reference for 60 voters), a Republican (ranked first in the other 40 voters' strict preference orderings) and a woman who has declared that she subscribes to Darwin's theory of evolution. For that reason, the town is unanimous that she is the worst candidate. The candidate with the highest number of votes wins the election. If two candidates are tied for the first place then the winner is decided by the toss of a coin. Candidates are not residents of the town and thus do not vote.[1]

(a). Consider everyone in the town voting for the evolutionist. Is this a Nash Equilibrium, and is it trembling hand perfect?

(b). Consider instead everyone in the town voting for the Republican. Is this a Nash Equilibrium, and is it trembling hand perfect?

Solution, Part (a)

There are 100 voters choose simultaneously among three candidates, $\{D, R, E\}$. 60 voters have preferences $D \succ R \succ E$. The remaining 40 have preferences $R \succ D \succ E$. We want to

March 6, 2013. These proposed solutions may contain minor typos. If you spot any, please e-mail me at jacopo.perego@nyu.edu.

[1]For (a) and (b) above your explanation should be convincing but do not need to provide the exact sequence of trembles.

know whether or not the strategy profile $(s_1 = E, \ldots, s_{100} = E)$ is a Nash equilibrium. It's easy to see the answer is affirmative. Indeed, fix player i and suppose his opponents $-i$ play the profile $s_{-i} = (E, \ldots, E)$. Irrespectively of what player i is going to do, the winner is E. Consequentially, i is going to receive the payoff associated with E. This means that player i is indifferent between $\{D, R, E\}$ and, in particular, is happy playing E, i.e. $E \in r_i(s_{-i} = E)$. Since the role of i is arbitrary, this holds for all i. Hence, $\{s_1 = E, \ldots, s_{100} = E\}$ is a Nash equilibrium.

Let's call this Nash equilibrium $s^\star$. We want to investigate whether or not it is THP. Consider **any** tremble $\{m^k\} \in (\prod_{i \in N} \text{int}\Delta(S_i))^\infty$ - recall that trembles must be completely mixed strategy profiles - and fix $i \in N$ a generic supporter of the Democrats. As usual, for all $k \in \mathbb{N}$, the profile m^k induces a distribution on terminal histories. Among all terminal histories, some are such that the decision of i is pivotal for D or R to win. Let's agree to call this subset of terminal history P. To have a THPE we need to make sure $E \in r_i(m^k_{-i})$ for all i. Notice however that by switching the choice of i from E to either D or R, player i can marginally increases is expected payoff. Indeed, for all histories in P, the one in which he is pivotal for putting E in a loosing position, he will receive a payoff which is strictly higher than if he had played E. In all the others, $Z \setminus P$, nothing is going to change. The probability of reaching those histories, no matter how small, is strictly positive since m^k is completely mixed. Hence, $E \notin r_i(m^k_{-i})$ for all k and for this reason $s^\star$ is not trembling hand perfect. Once again, since the role of i was arbitrary, this extends to all players. Hence, the NE is not a THPE.

Out of the math, the idea is very simple. Voter i is almost certain his worst preferred candidate is going to win, as he believes everyone else is going to vote for him. Still, these people can make mistakes, and for this reason there is a very small chance that someone better than E will take power "by mistake". Since player i is an VNM expected utility maximizer and since voting is costless, he prefers to vote for someone which is not E to get an infinitesimal increase in his expected utility at those histories in which, due to other voter's mistakes, his vote will result to be pivotal.

Part (b)

Now consider the second case, the one in which everyone votes Republican. Clearly, for the exact same reason as before, this is a NE, since deviating will not change in expectation the final outcome.

Discussing trembling perfection is more interesting now. In the previous case, deviating either to R or D in the worse case would have perpetuated the status quo, leaving the player indifferent. In the best case, it would have make R or D winning, making the player strictly happier. This was true for all trembles. Here, instead, everyone is voting for R. We can think of a tremble that make the histories in which player i (a democrat) is pivotal for R vs E more likely of the histories in which he is pivotal for D vs R. Deviating from R to D would make player i worse off in the former cases and better off in the latter. The net gain of the deviation is negative. That is we found a tremble for which all $R \in r_i(m^k_{-i})$, hence the outcome $\{s_1 = R, \ldots, s_{100} = R\}$ is THP.

Exercise 2: Costly Voting

The 100 residents of a small town are electing a mayor. There are two candidates: (i) a Democrat (strictly preferred by 70 voters) and (ii) a Republican (strictly preferred by 30 voters). Each person is 10 utils happier if her favored candidate is elected, but loses 1 util if she indeed votes (there is a cost to voting). Voting is NOT compulsory. The candidate with the highest number of votes wins the election. If there is a tie then the winner is decided by the toss of a coin.

(a). Voting is by secret ballot. No one can see how many others are voting. Find a Nash Equilibrium in pure strategies or prove that there isn't one.

(b). Suppose INSTEAD that voting is done at a public meeting, by roll call. Attendance at the meeting is compulsory. When your name is called, you can pass (no fee is charged) or you can declare your vote out loud (and be charged the fee). The order in which names will be called out is common knowledge before the calling starts. Using subgame perfect equilibrium as the solution concept, answer two questions:

   i. Does the order in which names are to be called out affect who is elected?

   ii. Does the order in which names are to be called out affect how many people vote? Can we Pareto rank the equilibria?

Solution, Part (a)

Let $N = [100] := 1, 2, \ldots, 100$, $A_i = \{0, 1\}$ - where we interpret $a_i = 0$ as "i abstains" and $a_i = 1$ as "i votes for his favorite candidate". Winning yields a payoff of 10 utils, whilst voting cost 1 util. The game is finite, hence utilities are trivially continuous, indeed linear in probabilities. Hence, there exists a NE in mixed strategies. The point is to see whether or not there exists a NE in pure strategies as well. Let's prove it by contradiction. Suppose there exists a NE in pure strategies and call this profile $s^\star \in S$. By definition of Nash, we must have $\forall i \in N \; s_i^\star \in r_i(s_{-i}^\star)$. Notice that best replies to pure strategy profiles are singleton sets, i.e. r_i is single-valued, hence we can impose the stricter requirement $s_i^\star = r_i(s_{-i}^\star)$. Clearly, the pure NE induces a (degenerate probability distribution on) *terminal history.* A terminal history in this static game can be of two types. Either D wins or R does. Hence,we can partition the set N in $L \subset N$, the set of agents *losing* the election, that is their most preferred candidate is not elected, and W, its complement. For notational simplicity we avoid making explicit the dependence of L and W on $s^\star$. To be a Nash equilibrium, $a^\star$ must satisfies $a_i^\star = 0$ for all $i \in L$. If this is not the case, i.e. if there is a $j \in L$ with $a_j^\star = 1$, then j has an incentive to deviate to *abstain*, getting 0 utils instead of -1, and $a^\star$ cannot be a NE. If all agent in L abstains, then NE imposes a requirement on W, too. Indeed, it must be the case that $\exists! \; i \in W$ s.t. $a_i^\star = 1$. Suppose not, i.e. let $j \in W$ be s.t. $a_j^\star = 1$. Then both i and j are not best replying to $a^\star$, hence it cannot be a NE. We found that for $a^\star$ to be a NE, all i's but one must play $a_i^\star = 0$. But then a contradiction arises. Indeed, for all $j \in L$ there is an incentive to deviate from 0 to $a_j' = 0$. If they don't, they get 0. If one deviates, he gets an expected payoff of 4. Hence, $a_i^\star \notin r_i(a_{-i}^\star)$ for all $i \in L$, a contradiction.[2]

[2]Notice that slightly changing the rule of the game would give the existence of NE in pure strategies. Indeed, it suffices to ask for 2 votes more than the opponent in order to win, and the game suddenly gets a huge amount of NE in pure strategies.

Part (b)

Now the game becomes dynamic. All voters, they are assigned a roll number, when their number is called they decide either to vote or abstain. Payoffs are as before. In what follows I am going to argue that (a) there exists an equilibrium in pure action, independently of the order, (b) the order does not affect the winner, (c) the order affects the number of voters, (d) equilibria can be Pareto ranked and (e) a comment on why using SPE is not as using NE in this game.

(a), **Existence**. Clearly there exists a Nash equilibrium in mixed strategies. More interestingly, there are also equilibria in pure strategy. This is because the game is a finite and with perfect info and we can apply BI. More importantly, payoffs are either -1, 0, 4, 5, 9 or 10 and relation btw strategies and payoffs is s.t. there are *no relevant ties.* Hence, we can pin down a **unique** SP equilibrium (clearly there are zillions of NE). To see this, let $\zeta : [100] \to N$ be a bijection assigning to each roll number a player. Then, whatever the history in which $\zeta(100)$ is playing, he has a unique best reply (see payoffs). His action ends the game. Going backwardly, at history h, player $\zeta(99)$ plays the best reply to the conjecture that $s_{\zeta(100)} = r_{\zeta(100)}(s_{\zeta(99)})$, i.e. the following player will best reply to his action and this best reply is unique. In turns, also the best reply of $\zeta(99)$ is unique. Preceding backwardly (by induction) we get that each player plays a pure action in every subgame. Of course, there can be imperfect Nash yielding the oppositive final outcome, but as long as SPE is our equilibrium concept the equilibrium is unique.

(b), **Order 1**. Does the nature of ζ affect the final winner? The answer is negative. To see this, suppose not. That is, fix an ordering ζ and suppose s is a SPE s.t. in the terminal history induced by s the Republicans win. Suppose they are winning by strictly more than one vote. Then s is not unimprovable. Indeed, the last of the Republican's voter can deviate from voting to not voting, without changing the final outcome. The only viable case is the one in which s induces a victory of the Republican by only one vote. But then you can find a history h at which a democrat is active and according to the profile s he is not voting. However, by voting for his party he can increase his payoff. That is, we have found a one-shot deviation from s. By the Proposition you saw in class and in the lab, this strategy s cannot be a SPE.

(c), **Order 2**. Does the nature of ζ affect the number of player voting? Yes, consider this example. If all Republicans are ordered first, then none of them will participate to the vote. Indeed, the final outcome is certain, i.e. Democrats will win the ballot, hence voting will lead to -1 utils instead of 0. The number of voters in equilibrium will be 1 and his name is $\zeta(100)$. On the other hand, consider a different ordering ζ' s.t. all Democrats are ordered first. Having only one Democrat voting cannot be a SPE. Actually, the unique SPE is the one in which players $\zeta'(40)$ to $\zeta'(70)$ participate to the ballot. Hence, ζ matters.

(d) **Order 3**. We can define a binary relation $\succsim_P$ on the set of SPE defined as

$$\mathscr{N} := \left\{ s \in S : \text{ is a SPE for some } \zeta \in N^{[100]} \right\}$$

The binary relation that we use when we think of a Pareto ranking is

$$a \succsim_P a' \quad \iff \quad (u_i(a))_{i \in N} \geq (u_i(a'))_{i \in N}$$

Notice that $(\mathscr{N}, \succsim_P)$ is a poset. In particular $\succsim_P$ is not complete. Whether or not you can rank outcomes according to a Pareto criterion really depends on what you mean with that. If you are looking for a complete ordering the answer is clearly no. If instead you are happy with an incomplete ordering than the answer is positive. If you consider the usual GE example of two player, there the allocations are Pareto rankable even if they live in $\mathbb{R}^2$, a set whose natural ordering is incomplete. Another legitimate way to interpret this question was whether or not there exists $\succsim_P$-maximal element in $\mathscr{N}$. We certainly have that every finite poset have a maximal element. In this case, however, it is not unique.

(e) **Comment**. This exercise is telling of the difference between NE and SPE. With the latter we are not allowing for non-credible threats off the equilibrium path. This fact squeezes the number of equilibria substantially (in fact, we get uniqueness). I noted before that for some particular ζ, $\zeta(100)$ was the unique player voting. This is true exactly because, by using SPE as a solution concept, we rule out the possibility of incredible threats. For example, player $\zeta(100)$ could threat player $\zeta(99)$ to play 0 instead of 1. An equilibrium will be $(0, \dots, 0, s_{\zeta(99)} = 1, s_{\zeta(99)} = 0)$. But this is not SPE, since is not immune to one shot deviations.

Exercise 3

Recall that in Hotelling's simultaneous location choice problem with $N = 3$ (as done in class), there is no equilibrium in pure strategies. Consider instead the sequential location problem for three players: 1 chooses a location, 2 observes this and chooses her location and 3 observes both previous choices and chooses her location. In this game assume that a player who chooses the same location as an earlier player can get the limit of nearby payoffs from the right or the left, whichever is more beneficial. Show that this game of perfect information has multiple subgame perfect equilibria with different payoffs for the third player.

Solution

We have three players, 1, 2, and 3, playing sequentially. Each player is to choose an action in $A_i = [0, 1]$. Payoffs are as in Hotelling and we allow for *limit moves,* i.e. we assume that if someone has played x before, then someone else after him can choose x^+ (the limit from right) or x^- (the limit from left). We can solve this game by backward induction.

Let's start from the last stage. The decision of Player 3 is trivial. His best reply correspondence is easy to compute as P3 knows a_1, a_2 before choosing a_3. Wlog let $a_1 \leq a_2$, then

$$r_3(a_1, a_2) = \begin{cases} a_1^- & \text{if} \quad a_1 \geq \max\{1 - a_2, \frac{a_2 - a_1}{2}\} \\ a_2^+ & \text{if} \quad 1 - a_2 \geq \max\{a_1, \frac{a_2 - a_1}{2}\} \\ \{a_1^-, a_2^+\} & \text{if} \quad 1 - a_2 = a_1 \geq \frac{a_2 - a_1}{2} \\ (a_1^+, a_2^-) & \text{if} \quad \frac{a_2 - a_1}{2} \geq \max\{a_1, 1 - a_2\} \end{cases}$$

Notice that when $1 - a_2 = a_2 \geq \frac{a_2 - a_1}{2}$, then P3 is indifferent between a continuum of action in the closed interval $[a_1^+, a_2^-]$.

Let's now move to player 2. He anticipates what Player 3 will do and he can observe what Player 1 did. Notice that focusing on best replies to $a_1 \leq \frac{1}{2}$ is without loss of generality. Indeed if Player 1 choose $a_1 > \frac{1}{2}$, we can simply flip over the beach, letting the end point be the starting point. This being said, P2's best reply function is:[3]

$$r_2(a_1) = \begin{cases} \{a_1^-, a_1 + \frac{2(1-a_1)}{3}\} & \text{if } a_1 < .25 \\ \{a_1^-, (1-a_1)\} & \text{if } a_1 = .25 \\ \{a_1^-, 1-a_1\} & \text{if } .25 < a_1 < .5 \\ \{\frac{1}{2}^-, \frac{1}{2}^+\} & \text{if } a_1 = .5 \end{cases}$$

Observe that $P2$ has always indifferences. This is because the optimal response of $P2$ not only depend on the action played by $P1$, but also on the belief of $P2$ on what $P3$ will do in case he is indifferent. We consider here the two simplest type of beliefs.

(1) *Everyone believes that, if indifferent, $P3$ will choose at the expenses of $P1$, not $P2$.* Under this common belief, let's consider Player 1 optimal strategy. Suppose he plays the action $a_1 = \frac{1}{4}$. Can this be part of a SPE? Yes it does. Our best replies predicts that $P2$ will place himself at $\frac{3}{4}$ and $P3$ will be indifferent among several options. Our common belief on $P3$ behavior tells us that $P3$ will choose a_3 to maximize his payoff at the expenses of $P1$, not $P2$. Hence $P3$ will choose in the set $\frac{1}{4}^{\pm}$, say $\frac{1}{4}^{+}$. Clearly, $P2$ and $P3$ are maximizing since their actions are consistent with their best reply correspondences we have found above. Is $P1$ best replying to his conjectures? If he plays $\frac{1}{2}$ he gets $\frac{1}{8}$. Some thinking can convince you that moving from $\frac{1}{4}$ towards $\frac{1}{2}$ monotonically decrease his payoff. Similarly, If he plays 0 he gets 0. And you can convince yourself that moving from $\frac{1}{4}$ to 0 monotonically decrease $P1$ payoff. Hence $\frac{1}{4}$ is his unique best reply given his breaking-ties belief and given r_2 and r_3. Notice, and it is important, that his belief are confirmed in equilibrium, as it is prescribed by the SPE solution concept. Summarizing, int he SPE $(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}^{+})$, payoffs are $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

(2) *Everyone believes that, if indifferent, $P3$ will choose at the expenses of $P2$, as long as $P1$ is choosing 0.* Notice, that this weird contingent belief is perfectly fine since we are working on the indifferent region of Player 3, where we can select behavior arbitrarily. Under this common belief, let's consider Player 1 optimal strategy. Suppose he plays the action $a_1 = 0$. Can this be part of a SPE? Yes it does. Our best replies predicts that $P2$ will place himself at $\frac{2}{3}$ and $P3$ will be indifferent among several options. Our common belief on $P3$ behavior tells us that $P3$ will choose a_3 to maximize his payoff at the expenses of $P2$. Hence $P3$ will choose in the set $\frac{2}{3}^{-}$. Clearly, $P2$ and $P3$ are maximizing since their actions are consistent with their best reply correspondences we have found above. Is $P1$ best replying to his conjectures? Yes, because if he deviates, he expects $P3$ to start breaking his indifferences at his expense. Hence, he prefer to stay with $a_1 = 0$. The resulting SPE is $(0, \frac{2}{3}, \frac{2}{3}^{-})$ and the payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

In the following I will try to explain in a less compact way why this second strategy profile is a SPE and what's the role of beliefs. If you have understood the solution so far, the following part is redundant and you should move to the next Question. If you are still puzzled, try to read on.

P1 needs to decides what action to take. He knows what r_2 and r_3 are, and he also knows that in some instances player 3 will be indifferent. The way $P3$ breaks his own indifferences is irrelevant for the payoff of $P3$, but it is certainly relevant for the payoff of $P1$. Hence,

[3] $a_1^{\pm} := \{a_1^{+}, a_1^{-}\}$.

$P1$ needs to have some belief on what $P3$ is going to do. Moreover, this belief must be confirmed in equilibrium, as this is implicitly required by the definition of NE and *a fortiori* in the definition of SPE. Since this belief has to be confirmed, both P1 and P2 must have the same belief about behavior of P3. To each different common belief, it will correspond a possibly different SPE. They way you choose beliefs to get to an equilibrium is arbitrary to you, as well as arbitrary it was the choice of the tie-breaking rule in the Pirate game of Problem Set 1. The belief I propose is the following. for what concern P2, P1 conjectures that $a_2 = r_2(a_1 = 0) = \frac{2}{3}$. For P3 instead, he conjectures that if indifferent he will break ties in favor of 1 rather than 2. That is to say, if P3 is indifferent between $\{a_1-, a_2+\}$, he rather chooses to locate himself at a_2+, to avoid messing around with P1. In particular, let his conjecture be ex-post confirmed, i.e. let $a_3 = r_3(a_1 = 0, r_2(a_1 = 0)) = a_2- = \frac{2}{3}-$. Given this, P2 and P3 are best replying as can be seen from the best reply function outlined above. It's important to describe belief also off the equilibrium path. In this particular case, we want P1 to believe that whenever he plays an action different than 0, then he will be "punished" by P3, whenever P3 is indifferent. Summing up, P3 minimize P2 payoff as long as P1 is at 0. As soon as he moves out from 0, P3 starts minimizing P1's payoff. Consider now the following action profile,

$$\left(0, \frac{2}{3}, \frac{2}{3}-\right)$$

Here, P1 will get a payoff of $\frac{1}{3}$, equal to the one that P2 and P3 are getting. Notice that, the fact that P3 is *targeting* P2 instead of P1, doesn't change the behavior of P2. Indeed, he cannot do anything against that. Clearly, he would be better off if P3 would *target* P1, instead: in that case, the profile of payoffs would be $(0, \frac{2}{3}, \frac{1}{3})$. P2 will be twice as happy, and P3 will be indifferent. Exactly the fact that P3 is indifferent makes *ex-post confirmed* the belief of P1 described before, and a best reply the decision to target P2. Hence, letting μ_1 is the belief of P1 on who P3 is going to target conditional on what is a_1, the structure made by

$$\langle a_1 = 0, \mu, r_2(\langle a_1 = 0, \mu), r_3(\langle a_1 = 0, \mu)\rangle$$

is a SPE, where $(0,\ \frac{2}{3},\ \frac{2}{3}-)$ is played. If P1 deviates to $\frac{1}{4}$, for example, the game would enfold as $(0, \frac{2}{3}, (\frac{1}{4}+))$, where P3 targets P1. The expected payoffs are $(\frac{1}{4}, \frac{3}{8}, \frac{1}{4})$. Clearly there is no incentive to deviate to $\frac{1}{4}$, and he can be checked that there is no incentive to deviate from 0 to anywhere else.

Exercise 4

Each of two risk-neutral firms playing a discrete Bertrand game can choose its price to be any integer between 0 and 100, inclusive. Marginal cost is 0.5 for each firm and capacities are unlimited. Consumers buy $q(p) = 100 - p$ units, where p is the lower of the two prices. If the prices are the same, demand is split evenly between the two firms. What is the set of rationalizable prices for Firm 1? Explain. Are any non degenerate mixed strategies rationalizable?

Solution

Let's characterize the set of rationalizable actions.

> **Rationality**. No player will ever play 0. All the other actions are best replies to some conjecture. For example, $p_1 = 100$ is a best reply to the conjecture $p_2 = 0$, For all $x \in 2, 100$, $p_1 = x - 1$ is the unique b.r. to the conjecture $p_2 = x$. Finally, $a_1 = 1$ is a best reply for $p_2 = 1$. Hence, $\{1, \ldots, 100\}$ is the set of justifiable actions when we assume individual Rationality.
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> **Rationality and Mutual Belief in Rationality**. Players are rational, hence they choose in $\{1, \ldots, 100\}$. Moreover the believe in the rationality of their opponents, hence they believe their opponent is choosing in $\{1, \ldots, 100\}$. Now recall that the conjecture that was "rationalizing " the choice of $p_1 = 100$ was precisely $p_2 = 0$. Since we are restricting player 1's belief to be in the set of P2's rational actions, i.e. $R_2(S)$, P1 cannot "use" conjecture $p_2 = 0$ anymore, i.e. the choice of $p_1 = 100$ cannot be possibly rationalized. More is true. Indeed, all actions $p_1 > 50$ cannot be possibly rationalized. Hence, $R^2(S) = \{1, \ldots, 50\}$
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> **Rationality and 2nd order Belief in Rationality**. Player 1 is rational, believes P2 is rational, and believes P2 believes he is rational. Hence P1 anticipates that P2 will never play $\{51, 100\}$. But then $p_1 = 50$ is never a best reply to any conjecture. In particular, it is strictly dominated by $p_1 = 49$ when the conjecture is $p_2 = 50$. Hence, 49 is never played. Same apply for P2. Hence, $R^3(S) = \{1, \ldots, 49\}$ is the set of justifiable actions when we assume individual rationality and (second order) belief in rationality.
>
> **Rationality and 3nd order Belief in Rationality**. Same argument, delete 49. $R^4(S) = \{1, \ldots, 48\}$.

Keep iterating until you get to $R^{50}(S) = R^{51}(S) = \{1, 2\}$. That is after 50 iteration the procedure stops, as 2 cannot be deleted. Indeed $(2, 2)$ is a NE.[4] As an exercise, show that $\{1, 2\}$ has the Best Reply Property.

So far, we have been considering only pure actions. Now we argue that no mixture in $\{3, \ldots, 100\}$ can be best reply of anything, hence cannot possibly be rationalizable. To see this, consider the mixture $a_\alpha = [\alpha]a + [1 - \alpha]a'$. Wlog assume $a < a'$. For simplicity, pick any deterministic conjecture $b \in \{2, \ldots, 100\}$. If $a' < b$ then $a_\alpha \prec a'$, if $b < a'$, $a_\alpha \prec a$. That is a_α is never a best reply to any conjecture.

What about mixed strategies in $\{1, 2\}$? We argue now that the whole set $\Delta(\{1, 2\}) \times \Delta(\{1, 2\})$ is rationalizable. To do that, we show that $\Delta(\{1, 2\}) \times \Delta(\{1, 2\})$ has the BRP, i.e. for each element in $\Delta(\{1, 2\})$, there exists a belief in $\Delta(\{1, 2\})$ s.t. that element is a best reply to that belief. It's easy to see that the game we are considering has reduced to a two by two game:

	$p_2 = 1$	$p_2 = 2$
$p_1 = 1$	x, x	$m, 0$
$p_1 = 2$	$0, m$	y, y

where $x = 24.75$, $m = 49.5$ and $y = 73.5$. Notice that, since NE are a subset of strategy profiles which are rationalizable, we can argue that this Bertrand game has only 2 two NE in

[4] Recall, all equilibrium concept we have seen in class are rationalizable.

pure strategies. More importantly, it has a unique completely mixed equilibrium m. Hence, given the belief m_2, player 1 is indifferent between $p_1 = 1$ and $p_1 = 2$ (by the definition of Nash equilibrium in mixed strategies itself). Hence, all mixtures in $\Delta(\{1,2\})$ are best replies to some conjecture, in particular m_2. Hence, $\Delta(\{1,2\}) \times \Delta(\{1,2\})$ has the BRP and a fortiori is the set of rationalizable strategies.