Games with Information Constraints: Seeds and Spillovers

Simone Galperti
UC San Diego

Jacopo Perego
Columbia University

July 19, 2023

Abstract

We study equilibrium behavior in incomplete-information games under two information constraints: seeds and spillovers. The former restricts which agents can initially receive information. The latter specifies how this information spills over to other agents. Our main result characterizes the equilibrium outcomes under these constraints, without making additional assumptions about the agents' initial information. This involves deriving a "revelation-principle" result for settings in which a mediator cannot communicate directly or privately with the agents. Our model identifies which spillovers are more restrictive and which seeds are more influential. Moreover, it generates predictions that hold robustly under a general class of spillover processes, which includes strategic communication. We apply our results to a problem of optimal organization design.

JEL Codes: C72, D82, D83, D85, M3

E-mails: sgalperti@ucsd.edu, jacopo.perego@columbia.edu. We thank Charles Angelucci, Odilon Câmara, Yeon-Koo Che, Laura Doval, Ben Golub, Emir Kamenica, Elliot Lipnowski, Stephen Morris, Xiaosheng Mu, Andrea Prat, Joel Sobel, Alex Wolitzky, and seminar participants at Pittsburgh, Yale, USC, Pompeu Fabra, UCSD, MIT, Harvard, Chicago, Stanford, UCLA, Queen Mary, Duke, Brown, PSE, Bonn, Georgetown, Maryland, Caltech, Carnegie Mellon, CIDE, ITAM, CETC 2019, SAET 2019, ESSET 2019, and SWET 2019, for comments and suggestions. We are grateful to the Cowles Foundation for its hospitality. Tianhao Liu and Aleksandr Levkun provided excellent research assistance. Earlier versions of this paper circulated under the title "Belief Meddling in Social Networks" and "Information Systems."

1 Introduction

An analyst wants to predict the behavior of a group of agents who interact in a game of incomplete information. In general, this behavior will hinge on details of the agents' information that the analyst can hardly observe. A cautious analyst may then be reluctant to make assumptions about this information, which comes at the cost of being left with coarse predictions. In many practical settings, however, the analyst is not entirely in the dark. The social context where agents interact can provide insights into what they may know about each other's information. For instance, the analyst may infer that an agent is always more informed than another, like a supervisor relative to her subordinate, or that some agents consistently share information with each other, like co-workers in an office. Although the analyst remains agnostic about what exactly this information is, she can leverage such observables to improve her predictions. This paper studies how to do so.

To illustrate, consider an organization that develops software products for its clients. The software is divided into modules that are designed by different teams. These teams need to coordinate the modules to ensure their compatibility, while also tailoring them to the needs of each client, which are ex-ante unknown. An analyst—for example, the organization's manager—wants to predict the probability of a coordination failure among the teams, which may occur as a result of the information they will receive from each client. Realistically, the manager lacks detailed knowledge of this information. Yet, she knows the organization well. First, she knows which teams are client-facing and thus can obtain some information about the clients' needs. These are the only teams who can seed information into the organization. Second, she knows which teams must report to others and which do not and thus how the seeded information spills over to the other teams. How does this knowledge about the organization's structure help the manager make better predictions?

To address these questions, we model a group of agents who interact in a game of incomplete information. Before the game begins, some agents—referred to as *seeds*—receive a signal about a payoff-relevant state. These signals then spill over from the seeds to other agents, following the links of a *spillover network*. Specifically, if a path connects one agent to another in this network, the latter will observe the signal of the former. Finally, the agents participate in the game using all the information they have thus obtained. In our model, the seeds and the spillover network act as an exogenous constraint on what the agents know about each other's information. They restrict the set of information structures the analyst should consider when predicting the outcome of the game. Our main goal is to characterize the outcomes that can arise in a Bayes-Nash equilibrium given a set of seeds and a spillover network.

The typical approach to achieve such a goal is to imagine a mediator who can flexibly provide information to the agents about the state (Bergemann and Morris, 2016). This approach is powerful when the mediator is unconstrained—i.e., when every agent can be seeded and there are no spillovers. Rather than working with information structures, one can conveniently focus on obedient recommendation mechanisms, in which each agent is recommended an action by the mediator and is willing to follow it. However, when the mediator faces seeding or spillover constraints, this approach is no longer valid. We show that, fixing those constraints, there can be outcomes that the mediator can induce using information structures but not using obedient recommendation mechanisms. To see why, note that in our setting the mediator can only directly communicate with the seeds and must rely on spillovers to indirectly communicate with the other agents. Thus, she cannot communicate directly and privately with all agents. In these cases, the richness of information structures grants the mediator more flexibility compared to the narrower class of recommendation mechanisms. As a consequence, seeding and spillover constraints significantly complicate the analyst's problem.

To overcome these challenges, we show how to recast the problem in a way that enables the characterization of all feasible outcomes in terms of obedient-recommendation mechanisms, despite the presence of constraints. Although more general, our approach retains much of the analytical convenience of the unconstrained characterization of Bergemann and Morris (2016), including a representation in terms of linear inequalities. We achieve this in two steps. First, we fully relax the seeding constraint by allowing the mediator to directly convey information to *all* agents. At the same time, we tighten the spillover constraint by expanding the original spillover network with new links. In Theorem 1, we show how to expand the original network so as to exactly offset the removal of the seeding constraint, in the sense that the set of feasible outcomes is unchanged. Second, under these modified constraints, we show in Theorem 2 that we can characterize all these outcomes by focusing on a mediator who directly recommends a (possibly mixed) action to each agent. The recommendations must be obedient in a robust sense: Each agent follows the recommended action conditional on knowing not only her recommendation but also the recommendations of those agents from whom there is a path to her in the expanded network.

These two results enable us to study how equilibrium outcomes change when we modify the seeds or the spillover network. For instance, in the context of our leading example, we may ask how the probability of a coordination failure changes if an additional team is required to report to another—i.e., if we add a new link to the network. Adding links can have ambiguous effects on the set of feasible outcomes. On the one hand, it makes obedience more demanding, as some agents know more about others; on the other hand, it can relax obedience by changing the expanded network, as the mediator can reach the same agent through more channels and hence with a higher degree of privacy. To characterize this trade-off, we introduce an order on seeds and networks that explicitly builds on the concept of network expansions, which underpins our notion of obedience. We then show in Lemma 1 that when the seeds and the network become "more connected," according to this order, the set of feasible outcomes shrinks for all games; the converse holds as well. Thus, our order exactly characterizes when seeding and spillover constraints become tighter. As an application of this order, we identify conditions under which one group of agents is more influential than another—in the sense that, if the former group is seeded, it induces a larger set of outcomes than that induced by the latter, for all games.

To illustrate our results, we study a problem of organization design. We consider an effort-provision game among various teams in an organization. Its manager can choose once and for all which teams are tasked with sourcing information from the outside (the seeds) and which teams have to report their information to which other teams (the spillovers). The manager's goal is to design an organization that performs well across all possible outcomes of the teams' interactions as driven by the information they obtain on a daily basis. Our exercise offers insights into when it is optimal to mandate full transparency between teams, or to institute a "firewall" that prevents them from sharing information, or a to impose a hierarchy in which lower teams must report their information to higher teams.[1]

In the last section of the paper, we revisit our main assumption that information spillovers are deterministic. What if instead information spills over randomly along the links of the network, or if the agents communicate strategically with their neighbors? Returning to our leading example, the manager may wonder what happens if, despite the firewall instituted between two teams, information leaks with some small probability. Under these more complex spillovers, fully characterizing the set of feasible outcomes becomes challenging. Nevertheless, Propositions 2 and 3 show that our results are still useful to characterize certain properties of these feasible outcomes. For example, one can use them to establish bounds for the probability a specific action profile is played, or for the payoff an agent can obtain.

Related Literature. Our work introduces information constraints—specifically, seeding and spillovers—in a setting that builds on Bergemann and Morris (2016). Conveniently, our characterization of feasible outcomes reduces to theirs when these constraints are absent. An alternative interpretation of our contribution is to provide tools to study "constrained" information-design problems (see Bergemann and Morris (2019) and Kamenica (2019) for surveys). From this perspective, our results can be useful to assess the robustness of unconstrained solutions to the possibility that some agents share their information. Mathevet and Taneva (2022) also characterize feasible outcomes under information constraints but focus on constraints—single-meeting schemes and delegated hierarchies—that differ from ours. They consider strategic incentives to share information and identify a class of games where constrained and uncon-

[1] In practice, organizations manage information flows in starkly different ways. Examples include Nintendo, which imposed an information firewall between the marketing and the game-development divisions (Brandenburger et al., 1995), and Capital One, which imposed full transparency between the marketing and the risk-analysis divisions (Lattin and Rierson, 2007).

strained solutions coincide. Candogan (2020) and Babichenko et al. (2022) study optimal information structures under information spillovers. They characterize when spillovers make finding the optimal solution computationally hard. They introduce algorithms that efficiently find solutions for certain spillover networks. Finally, our seeding and spillover constraints induce a particular "information hierarchy" by which each agent is more informed than all of her sources; Brooks et al. (2022) provide a general characterization of these hierarchies.

In the literature on secure information transmission, Renault et al. (2014) study the problem of a sender who wishes to communicate a secret to a receiver through a network of adversaries, while preventing the latter from learning and tampering with the secret. They identify necessary and sufficient conditions on the network for a secure communication protocol to exist.[2] Closer to our work, Renou and Tomala (2012) and Rivera (2018) study mechanism design problems in which the communication between the mediator and the agents occurs on a network that may be incomplete. As in our setting, they observe that the revelation principle can fail. They find conditions on the network that ensure secure communications and thus guarantee the applicability of the revelation principle. From this perspective, our goal is opposite to theirs: We seek to characterize feasible outcomes for networks where the revelation principle fails.

Our work also relates to the literature on strategic communication in networks, e.g., Hagenbach and Koessler (2010), Galeotti et al. (2013), and Calvó-Armengol et al. (2015). These papers study agents who receive exogenous signals and then strategically communicate with each other before participating in a final incomplete-information game. They seek to characterize the equilibria of this game. In general, this is a challenging problem, which they tackle by focusing on a specific game and initial information structure. Our approach differs in that we consider nonstrategic communication but allow for arbitrary games and information structures.

Finally, our main application is inspired by a literature that studies information flows within organizations (e.g., see seminal contributions of Radner (1992), Radner (1993), and Bolton and Dewatripont (1994)). In this tradition, Dessein and Santos (2006), Dessein et al. (2016), and Matouschek et al. (2023) study a team-theoretic model and assume the manager can dictate the information flows. In these papers, the optimal organization design is nontrivial due to external constraints, such as the costs of establishing a link. In our case, it is nontrivial due to incentive conflicts between teams. Additionally, these papers study organization design under a specific initial information structure, which is given exogenously. By contrast, we take a robust approach. Our manager does not know which distribution describes the teams' information and aims to design an organization that performs adequately across all of them.

[2] Earlier work in this literature includes Dolev et al. (1993), Franklin and Wright (2000), and Desmedt and Wang (2002).

2 Model

We are interested in studying the behavior of a group of players who interact in a game of incomplete information. Before the game begins, some players (the seeds) privately receive a signal about the state of the world. These signals then spill over to other players following the links of a given network. Finally, using all information obtained, either from the initial signal or from others, players interact in the game.

Let I be a finite set of players and Ω be a finite set of states of the world. Players have a common, full-support, prior belief about the state, denoted by $\mu \in \Delta(\Omega)$. An information structure is a pair (T, π) consisting of a finite signal space $T = \times_{i\in I} T_i$ and a function $\pi : \Omega \to \Delta(T)$. For convenience, we assume that each T_i is a subset of an infinite set $\bar{T}$ and denote the set of all information structures by $\mathcal{P}$.[3]

The information that players have before playing the game is constrained in two ways. First, only the players in the set $S \subseteq I$—called *seeds*—can receive initial information. We model this by requiring that the initial information structure (T, π) satisfy $|T_i| = 1$ for all $i \notin S$. We denote the set of such information structures by $\mathcal{P}_S \subseteq \mathcal{P}$. Second, a *spillover network* N determines how the initial signal realizations spill over to other players. We assume that player i learns j's signal t_j if there is a path from j to i in the network N.[4] In this case, we call j a *source* of i. We denote by $N_i \subseteq I$ the set that contains i and all of i's sources. Therefore, given any signal profile t from (T, π), player i learns $t_{N_i} := (t_j)_{j\in N_i}$.

Hereafter, we refer to the pair (N, S) as a *network–seed system*. Throughout, we maintain the assumption that such a system is *connected*, in the sense that every player has at least one seeded source, i.e., $N_i \cap S \neq \varnothing$ for all i. This implies that every player can receive some information, either directly or indirectly.[5]

A network–seed system (N, S) transforms every initial information structure $(T, \pi) \in \mathcal{P}_S$ into a final one $(T', \pi') \in \mathcal{P}$, which is defined by $T'_i = \times_{j\in N_i} T_j$ for all i and, for all ω, $\pi'(t'|\omega) = \pi(t|\omega)$ when $t'_i = t_{N_i}$ for all i. Denote by $\mathcal{P}_{(N,S)} \subseteq \mathcal{P}$ the set of final information structures that can arise under the system (N, S).

Given a final information structure (T', π'), the players then interact in a game. Let A_i be a finite set of actions for player i and let her utility function be $u_i : A \times \Omega \to \mathbb{R}$, where $A = \times_{i\in I} A_i$. Let $G = (\Omega, \mu, (A_i, u_i)_{i\in I})$ denote the *base game*. The base game G and a final information structure $(T', \pi') \in \mathcal{P}_{(N,S)}$ define a Bayesian game $(G, (T', \pi'))$. Given such a game, we denote

[3] The restriction $T_i \subset \bar{T}$ is expositional. It guarantees that the set $\mathcal{P}$ is well-defined, thus avoiding set-theoretic issues related to self-referential sets (i.e., Russell's paradox).

[4] A path from j to i in the network N is a sequence of players $(\iota_1, \ldots, \iota_m)$ such that $\iota_1 = j$, $\iota_m = i$, and $(\iota_k, \iota_{k+1}) \in N$ for all $k = 1, \ldots, m-1$.

[5] We discuss how to relax this assumption in Online Appendix D.1.

a strategy of player i by $\sigma_i : T'_i \to \Delta(A_i)$ and its Bayes–Nash equilibria by $\mathrm{BNE}(G, (T', \pi'))$.

The main goal of this paper is to characterize the set of all possible Bayes–Nash equilibria of a base game G given the restrictions imposed by a network–seed system (N, S) on what information the players have. In other words, we characterize the equilibria that can arise from any information structure $(T', \pi') \in \mathcal{P}_{(N,S)}$.

Discussion. Before proceeding, it is instructive to consider two extreme cases of network–seed systems. First, suppose there are no information spillovers and every player is a seed; that is, $N = \varnothing$ and $S = I$. Clearly, this system does not constrain the final information structure in any way. In particular, it allows for arbitrary correlation between the players' signals. In this case, the set of possible Bayes-Nash equilibrium outcomes is equal to the set of Bayes correlated equilibria (BCE) defined in Bergemann and Morris (2016). Conversely, suppose that the network is complete and at least one player is seeded; that is, $N = I^2$ and $S \neq \varnothing$. This system only allows for final information structures that are "public," in the sense that the players' signals are perfectly correlated. In this case, the only possible Bayes–Nash equilibrium outcomes are those corresponding to public information. This paper considers any network–seed system defined by combinations of N and S between these extreme cases, which is a flexible way of restricting how players' information is correlated. While arbitrary restrictions may render the equilibrium analysis intractable, we show in the next section that the restrictions imposed by network–seed systems preserve the tractability of the problem.

3 Constrained Feasible Outcomes

This section characterizes the equilibrium behavior in a base game G that is feasible given a network–seed system (N, S).

We begin by defining the notion of a feasible outcome. Fix a final information structure $(T', \pi') \in \mathcal{P}_{(N,S)}$, an equilibrium of the ensuing Bayesian game, and a state ω. Note that each realization $t' \in T'$ induces an equilibrium profile of possibly mixed actions of the players. Let $\mathcal{A} := \times_{i \in I} \Delta(A_i)$ be the set of mixed-action profiles and α a generic element. Since t' is random and T' is finite, the information structure induces a finite-support distribution over these mixed-action profiles. We call the mapping from states to such distributions an "outcome."

Definition 1 (Outcome)**.** An *outcome* for G is a mapping $x : \Omega \to \Delta(\mathcal{A})$, where $x(\cdot|\omega)$ has finite support for every $\omega \in \Omega$.

An outcome is feasible if it can arise as equilibrium play given some initial information structure and how it is transformed by the system (N, S).

Definition 2 (Feasible Outcome)**.** An outcome x is *feasible* for a base game G and a network–seed system (N, S) if there is an information structure $(T', \pi') \in \mathcal{P}_{(N,S)}$ and an equilibrium $\sigma \in \text{BNE}(G, (T', \pi'))$ such that

$$x(\alpha|\omega) = \sum_{t' \in T'} \pi'(t'|\omega) \prod_{i \in I} \mathbb{I}\{\sigma_i(t'_{N_i}) = \alpha_i\}, \qquad \forall\, \omega \in \Omega, \alpha \in \mathcal{A}, \tag{1}$$

where $\mathbb{I}\{\cdot\}$ is the indicator function. Let $X(G, N, S)$ denote the set of feasible outcomes for G and (N, S).

The goal of this section is to characterize $X(G, N, S)$. As observed earlier, when $N = \varnothing$ and $S = I$, the network–seed system imposes no constraints on the final information structure. In this case, $X(G, N, S)$ is the set of BCE outcomes, which can be conveniently characterized via incentive-compatible pure-action recommendations that a mediator sends directly to the players (Bergemann and Morris (2016)).

In general, however, network–seed systems impose constraints that make the characterization of $X(G, N, S)$ challenging. The standard approach based on recommendation mechanisms is not directly applicable because the mediator may be unable to communicate privately (if $N \neq \varnothing$) or directly (if $S \subsetneq I$) with some players. Section 3.1 illustrates these challenges. To address them, we develop an alternative approach, which has two parts. First, we show that any network–seed system is equivalent to an auxiliary one that relaxes the seeding constraint but features additional information spillovers (Section 3.2). Second, we characterize all feasible outcomes for this auxiliary system in terms of mixed-action recommendation mechanisms that are robust to information spillovers (Section 3.3).

3.1 Challenges with the Standard Approach

The constraints imposed by a network–seed system create two challenges when trying to characterize feasible outcomes in terms of incentive-compatible, pure-action recommendations. The first is due to information spillovers: When $N \neq \varnothing$, restricting the mediator to recommending only pure actions is with loss of generality. The following example illustrates this in intentionally simple terms.

Example 1. Consider a two-player, two-action, "matching pennies" game with complete information (i.e., $|\Omega| = 1$). Suppose $S = I$. In the unique feasible outcome, each player mixes uniformly and independently between the two actions. With no information spillovers ($N = \varnothing$), the mediator can replicate this outcome by flipping a coin on behalf of each player and recommending to her a different pure action for each side of her coin. Such recommendations are obedient. By contrast, if the spillover network is $N = \{(1, 2), (2, 1)\}$, no incentive-compatible, pure-action recommendation mechanism can replicate the unique feasible outcome of the game.

Given N, any recommended action would become common knowledge, causing at least one of the players to disobey the mediator's recommendation. $\triangle$

Without information spillovers, it is well-known that the mediator can always randomize on behalf of the players, so focusing on pure-action recommendations is without loss. This is no longer true with information spillovers because recommendations may not be *private*. To address this challenge, we allow the mediator to recommend mixed actions, thus delegating some of the randomizations to the players. Therefore, in our setting, a recommendation mechanism maps states into distributions over mixed-action profiles. Conveniently, such mechanisms coincide with how we defined outcomes (Definition 1).

The second challenge when trying to characterize feasible outcomes using recommendation mechanisms is more substantive. Due to the seeding constraint, the mediator cannot *directly* recommend actions to non-seed players but has to *indirectly* deliver such recommendations through the seeds (who then observe them). As a consequence, there can be outcomes the mediator can induce with information structures but not with the narrower class of obedient recommendation mechanisms.

Example 2. As in Figure 1(a), let $I = \{1, 2, 3\}$, $S = \{1, 2\}$, and $N = \{(1, 3), (2, 3)\}$. Let the state $\omega = (\omega_1, \omega_2) \in \Omega = \{0, 1\}^2$ be the outcome of two independent tosses of a fair coin. Each player wants her action to match the state: $A = \Omega$ and $u_i(a_i, a_{-i}; \omega) = -\sum_k (\omega_k - a_{ik})^2$. Consider an initial information structure $(T, \pi) \in \mathcal{P}_S$ such that, for all ω, player 1 learns ω_1 and player 2 learns ω_2. Given N, player 3 always learns ω. The following is then a feasible outcome: $a_1 = (\omega_1, 0)$, $a_2 = (0, \omega_2)$, and $a_3 = (\omega_1, \omega_2)$ for all ω. Note that keeping players 1 and 2 uncertain about a_3 is necessary to sustain this outcome. Therefore, an incentive-compatible recommendation mechanism that is constrained by (N, S) cannot achieve this outcome. The mediator would need to deliver player 3's recommendation via, say, player 1. This would reveal information to player 1, who then would want to deviate from the recommended $a_1 = (\omega_1, 0)$ when $\omega_2 = 1$. $\triangle$

Example 2 illustrates how a non-seed player j can receive from each of her sources only part of the information determining her behavior. By delivering j's recommendation through her sources, the mediator may reveal too much information to them, thereby changing their behavior. In other words, the language of recommendations is not rich enough to replicate all feasible outcomes.

3.2 Network Expansion and Outcome Equivalence

This section explains how we address the challenge illustrated by Example 2. We show that, for any network–seed system (N, S), there is an auxiliary system (N', I) that allows all players to

Figure 1: Three network–seed systems. Seed players are depicted in gray.

be seeded and yet, it induces the same outcomes as (N, S). To guarantee that (N, S) and (N', I) are outcome-equivalent, we need to appropriately tighten the spillover constraint by adding links to N, i.e., $N' \supseteq N$. We will show that, instead of focusing on (N, S), we can equivalently study (N', I) and let the mediator directly recommend actions to each player, without the need for intermediaries. We introduce the logic of this construction in two illustrative steps and then present the general treatment.

The first step illustrates that, for some (N, S), expanding S to I does not change the feasible outcomes (i.e., $X(G, N, S) = X(G, N, I)$). For example, consider again the system (N, S) of Figure 1(a). The mediator can deliver player 3's recommendation via players 1 and 2 in such a way that neither 1 nor 2 learns that recommendation. In other words, it is as if the mediator could communicate directly with player 3, even if she is not a seed. This can be done via the standard technique of "secret sharing" (Shamir, 1979), an instance of which appears in the next example.[6]

Example 3. Let (N, S) be as in Figure 1(a). Let $\Omega = \{0, 1\}$. We construct an initial information structure $(T, \pi) \in \mathcal{P}_S$ such that both t_1 and t_2 are uninformative, while the pair (t_1, t_2) fully reveals ω. Let $T_1 = T_2 = \{0, 1\}$ and π be such that t_1 is distributed uniformly and independently of ω and t_2, whereas $t_2 = 1$ if and only if $\omega + t_1 = 1$. Clearly, t_1 is uninformative. Signal t_2 is uninformative because, for every ω, t_2 has an equal chance of being 0 or 1. Therefore, players 1 and 2 learn nothing about ω from their signals. Instead, player 3 observes (t_1, t_2) and learns the state: $\omega = 1$ if and only if $t_1 \neq t_2$. In words, t_2 is an encrypted version of ω, and t_1 is the key to deciphering it. △

Unfortunately, it is not true that for all (N, S) we can expand S without changing the set of feasible outcomes. The next example illustrates this problem but also indicates a solution.

[6]These techniques are commonly used in computer science and economics, see, e.g., Dolev et al. (1993), Franklin and Wright (2000), and Desmedt and Wang (2002), Renou and Tomala (2012), Renault et al. (2014), and Rivera (2018).

Example 4. Let $I = \{1, 2, 3\}$, $S = \{1, 2\}$, and $N = \{(1, 3), (2, 3), (2, 1)\}$ as in Figure 1(b). The information structure we constructed in the previous example no longer allows player 3 to learn something player 1 does not. Therefore, for some G, adding player 3 to the seed set would allow the mediator to achieve outcomes that are infeasible under (N, S), i.e., $X(G, N, S) \subsetneq X(G, N, I)$. For example, under (N, I) the mediator can reveal the state to 3 while leaving 1 in the dark, which is unfeasible under (N, S). However, imagine that we not only add player 3 to S, but also add link $(3, 1)$ to N (see Figure 1(c)). Let us denote this expanded network by N^S. Under (N^S, I), players 1 and 3 are both seeded; yet, since $(3, 1) \in N^S$, they always share the same final information, as they did in the original (N, S). One may then expect that (N, S) and (N^S, I) induce the same outcomes for all G, as we will show shortly. △

We now formalize and generalize these ideas.

Definition 3 (S-expansion)**.** The S-expansion of N is the network N^S that contains N and is obtained as follows: If $i \notin N_j$, we add link (i, j) to N if and only if $N_i \cap S \subseteq N_j$.

The logic is that if all seeded sources of i (i.e., $N_i \cap S$) are also sources of j, then j must infer all the information i could ever get. Adding a link from i to j should not affect j's behavior, and thus the feasible outcome (i.e., $X(G, N, S) = X(G, N^S, S)$).

Our first main result follows. It shows that, for all base games G, we can fully relax the seeding constraint provided that we appropriately tighten the spillover constraint. This equivalence is crucial for the rest of our analysis.

Theorem 1 (Equivalence)**.** *Fix a base game G and a network–seed system (N, S). The set of feasible outcomes for G under (N, S) is equal to the set of feasible outcomes for G under the auxiliary system (N^S, I)—i.e., $X(G, N, S) = X(G, N^S, I)$.*

The proof builds on the insights discussed above. We first show that the S-expansion does not change the seeded sources of any player and, hence, the information on which she can ultimately act. Therefore, $X(G, N, S) = X(G, N^S, S)$. We then show by induction that under N^S we can expand S to I (i.e., $X(G, N^S, S) = X(G, N^S, I)$). This step uses a generalized version of the secret sharing technique illustrated in Example 3.

3.3 Spillover-Robust Obedience

We can now describe the second part of our approach. Using Theorem 1, we can restrict attention to network–seed systems where all players are seeded. We show that it is possible to characterize the feasible outcomes for such systems via obedient recommendations that are robust to information spillovers.

To do so, we first need to introduce some notation. Define the mixed-action extension of the utility function as $u_i(\alpha_i, \alpha_{-i}; \omega) = \sum_{a \in A} u_i(a; \omega) \prod_{j \in I} \alpha_j(a_j)$, for all $\alpha \in \mathcal{A}$. Given an outcome $x : \Omega \to \Delta(\mathcal{A})$, define its supported mixed-action profiles as $\operatorname{supp} x = \{\alpha : \exists\, \omega \in \Omega \text{ s.t. } x(\alpha|\omega) > 0\} \subseteq \mathcal{A}$. Let $\mathcal{A}_{N_i} = \times_{j \in N_i} \Delta(A_j)$ and define the projection of $\operatorname{supp} x$ on $\mathcal{A}_{N_i}$ as $\operatorname{supp}_{N_i} x = \{\alpha_{N_i} \in \mathcal{A}_{N_i} : \exists\, \alpha_{-N_i} \in \mathcal{A}_{-N_i} \text{ s.t. } (\alpha_{N_i}, \alpha_{-N_i}) \in \operatorname{supp} x\}$.

Definition 4 (Spillover-Robust Obedience)**.** An outcome x is *spillover-robust obedient* for a base game G given a spillover network N if, for all i and $\alpha_{N_i} \in \operatorname{supp}_{N_i} x$,

$$\sum_{\substack{\omega \in \Omega \\ \alpha_{-N_i} \in \operatorname{supp}_{-N_i} x}} \Big(u_i(\alpha_i, \alpha_{-i}; \omega) - u_i(a_i, \alpha_{-i}; \omega)\Big) x(\alpha_i, \alpha_{-i}|\omega)\mu(\omega) \geq 0, \quad \forall\, a_i \in A_i. \tag{2}$$

To interpret condition (2), imagine dividing both sides by the total probability that α_{N_i} arises under x and μ. The resulting condition requires that after observing α_{N_i}—namely, the recommendations for herself and her sources—player i be willing to play α_i rather than deviating to action a_i.

This leads to our second main result.

Theorem 2 (Feasibility)**.** *An outcome x is feasible for a base game G and a network–seed system (N, I)—i.e., $x \in X(G, N, I)$—if and only if x is spillover-robust obedient for G given N.*

Robust obedience captures the basic economic trade-off caused by information spillovers. The signal for each player not only directly influences her beliefs—like in Bergemann and Morris (2016)—but can also influence the beliefs of her followers in the network. This curbs the scope for keeping them uncertain about that player's behavior. Thus, spillovers render it harder—in the sense of incentive compatibility captured by (2)—to implement joint behaviors that require some dependence on ω and mutual uncertainty among players.

The intuition for Theorem 2 is as follows. Suppose x is feasible. Note that by learning her sources' signals through N, player i also learns the signals of her sources' sources and so on. Since in equilibrium i knows σ, she can predict the mixed action of all her sources. In equilibrium, she must best respond to this behavior as well as to her belief about all other players' behavior and the state. But this property is robust obedience. Conversely, suppose x is robust obedient. We can view the outcome itself as an information structure, where $T = \operatorname{supp} x$ and $\pi = x : \Omega \to \Delta(\mathcal{A})$. It is then a BNE for each player to follow her recommendation, given what she learns through the spillovers and given that the others follow their recommendations.

The combination of Theorem 1 and 2 provides a "revelation-principle" characterization of the feasible outcomes of game G under any system (N, S).

Corollary 1. *$x \in X(G, N, S)$ if and only if x is spillover-robust obedient for G given N^S.*

This result allows us to study feasible outcomes as if the mediator could directly recommend to each player how to play in G subjects to appropriately defined obedience constraints. Importantly, the linearity of these constraints opens the door to using powerful linear-programming methods to characterize feasible outcomes, as illustrated in the next section.[7]

4 The Effects of Network–Seed Systems

In this section, we put our theorems to work and analyze several applications. Section 4.1 studies how feasible outcomes change as we modify the seeding and spillover constraints. Building on this, Section 4.2 proposes a notion of "influence" of a group of players and offers insights into optimal seeding. Finally, Section 4.3 applies our results to study a problem of organization design.

4.1 More-Connected Systems

What happens to the set of feasible outcomes when the network–seed system changes, e.g., when new players join the seed set S or new links form in the network N? These changes can give rise to nontrivial trade-offs. For instance, suppose $S \subsetneq I$. Richer spillovers can curb the mediator's ability to influence players' behavior, shrinking the set of feasible outcomes; but they can also open new channels for the mediator to reach the players, expanding the set of feasible outcomes. To organize these trade-offs, we introduce the notion of "more-connected" network–seed systems, building on our previous notion of network expansion.

Definition 5. (N, S) is *more connected* than $(\hat{N}, \hat{S})$—denoted by $(N, S) \trianglerighteq (\hat{N}, \hat{S})$—if i's sources in $\hat{N}^{\hat{S}}$ are also i's sources in N^S for all $i \in I$; that is, if $\hat{N}_i^{\hat{S}} \subseteq N_i^S$ for all $i \in I$.

This order accounts for the constraints imposed by both the information spillovers and the limited seeds. To build intuition, suppose first that $S = \hat{S} = I$. In this case, the expansions of N and $\hat{N}$ are equal to the original networks: $N^S = N$ and $\hat{N}^{\hat{S}} = \hat{N}$. Therefore, (N, I) is more connected than $(\hat{N}, I)$ if $\hat{N}_i \subseteq N_i$ for all i. For any base game G, fewer outcomes should then be feasible under (N, I) than $(\hat{N}, I)$, as in the former each player observes the recommendations sent to a larger set of other players, thus making obedience more demanding. Consider now the case of $S \subsetneq I$. To learn about the set of feasible outcomes, it is no longer sufficient to check whether $\hat{N}_i \subseteq N_i$ for all i. For example, the systems in Figure 2 satisfy $\hat{N}_i \subseteq N_i$ for all i, yet

[7] We leave to Appendix A.2 the discussion of two additional and more technical properties of $X(G, N, S)$. First, it is without loss of generality to focus on outcomes x whose support has cardinality no larger than a finite exogenous bound, which depends only on A. Second, information-design problems over $X(G, N, S)$ always have a solution.

Figure 2: Network–seed systems that are not ranked

Example 3 indicates there could be outcomes under (N, S) that are not feasible under $(\hat{N}, S)$. This suggests that, in addition to the spillovers N, we need to consider the informational role played by the seeds S. Definition 5 does so by using the notion of network expansion. As it turns out, this order exactly characterizes when changes in the network–seed system shrink the set of feasible outcomes.

Lemma 1. *$X(G, N, S) \subseteq X(G, \hat{N}, \hat{S})$ for all G if and only if $(N, S) \trianglerighteq (\hat{N}, \hat{S})$.*

When a network–seed system becomes more connected in the sense of Definition 5, "local" information received by the seeds can more easily spread "globally." This shrinks the set of equilibria that can be achieved, irrespective of the game being played.[8]

Finally, we clarify how Definition 5 relates to the primitives of our model.

Proposition 1. *$(N, S) \trianglerighteq (\hat{N}, \hat{S})$ if and only if $\hat{N}_i \cap \hat{S} \subseteq \hat{N}_j$ implies $N_i \cap S \subseteq N_j$ for all $i, j \in I$.*

Intuitively, if in $(\hat{N}, \hat{S})$ all seeded sources of player i are also sources of player j, then j knows i's information. This should also be true in a more connected system (N, S). Therefore, in (N, S) either i is already a source of j, or all seeded sources of i must again be sources of j.

4.2 Seeds' Influence

When is a group of players more influential than another, in the sense of inducing a larger set of feasible outcomes for a given game? We can study this question through the lens of our model and develop a notion of group influence. More precisely, let us fix a spillover network N. We would like to know when a set of seeds is more influential than another set in the following sense:

Definition 6 (Influence)**.** Fix N. S is *more influential* than S' if $X(G, N, S) \supseteq X(G, N, S')$ for all G.

[8] Related to this, it can be shown that $(N, S) \trianglerighteq (\hat{N}, \hat{S})$ if and only if (N, S) "better aggregates" the information received by the players. We formalize this point in Appendix D.2.

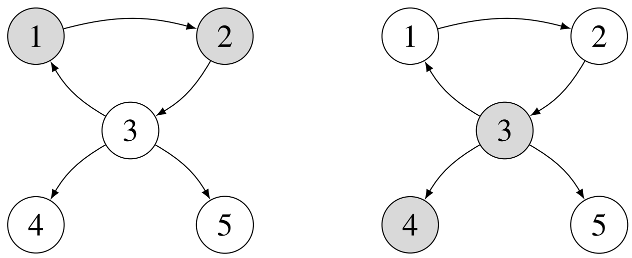

Figure 3: $S = \{3, 4\}$ is more influential than $S' = \{1, 2\}$.

In other words, S is more influential than S' if, for any base game, nothing that could occur under (N, S') is precluded under (N, S). This notion of influence is absolute: It cannot depend on the details of the strategic interactions among players, but can only depend on the information constraints induced by the systems (N, S) and (N, S'). It is easy to see that if $S \supseteq S'$, then S is more influential than S'. However, S and S' can be ranked even if S' is not included in S. Lemma 10 allows us to provide a tight characterization of these cases.

Corollary 2. *Fix N. S is more influential than S' if and only if $(N, S') \trianglerighteq (N, S)$.*

The question of which seed set is more influential boils down to which leads to a less connected system. For example, consider Figure 3. On the left panel, $S' = \{1, 2\}$ and $N^{S'} = I^2$; on the right panel, $S = \{3, 4\}$ and $N^S \subsetneq I^2$. Therefore, $(N, S') \trianglerighteq (N, S)$ and players 3 and 4 are more influential than players 1 and 2.

This result has implications for the design of optimal network–seed systems. For example, a manager may need to choose which divisions in an organization should be assigned the task of obtaining outside information, e.g., by interacting with the client. This choice can depend on complex aspects of the organization (such as the incentives of its members—i.e., G). However, if the manager can establish that divisions in S are more influential than divisions in S', then her choice is greatly simplified. We will analyze a related problem of optimal design in the next subsection.

This perspective is also related to the "seeding problem," which has received considerable attention in the network literature—in economics and beyond.[9] Similarly to this paper, this literature has introduced several notions of nodes' influence that depend only on properties of the network N (e.g., centrality). However, unlike this paper—which focuses on all feasible outcomes for all games—this literature has typically focused on specific games or specific objectives (e.g., maximizing diffusion). Unsurprisingly, our notion of influence can disagree with notions based on network centrality. As a simple example, in Figure 4, player 2 is strictly more central than player 1 according to Bonacich centrality, but players 1 and 2 are equally

[9] See, for example, Morris (2000), Ballester et al. (2006), Banerjee et al. (2013), Akbarpour et al. (2018), Galeotti et al. (2020), and Sadler (2020). Valente (2012) provides a review of the literature outside economics.

influential according to our notion.

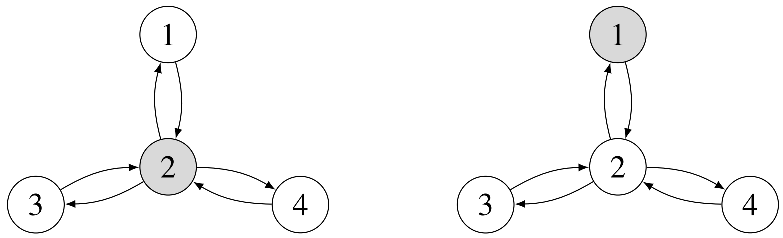

Figure 4: Seeds' influence and Bonacich centrality

4.3 An Application to Organization Design

The last two results offered absolute comparisons of network–seed systems that hold independently of the specifics of the base game. In some cases, however, an analyst may be interested in a particular base game and have a specific objective in mind to further refine the ranking of network–seed systems. For instance, she may want to identify which system guarantees the highest probability that an action profile is played. We can use our results to study such problems.

To illustrate, fix a base game G and let $v : \Omega \times A \to \mathbb{R}_+$ be the objective function the analyst uses to evaluate outcomes. Given a system (N, S), let the expected value of v given an outcome $x \in X(G, N, S)$ be

$$\mathbb{E}_x(v) = \sum_{\substack{\alpha \in \operatorname{supp} x \\ \omega \in \Omega, a \in A}} v(a, \omega)\alpha(a)x(\alpha|\omega)\mu(\omega).$$

The analyst may be interested in computing the highest (or lowest) value of $\mathbb{E}_x(v)$ across all feasible outcomes. Conveniently, this problem has a linear-programming formulation due to the structure of $X(G, N, S)$ as characterized by Theorem 2.[10] Moreover, this problem offers another criterion for ranking network–seed systems. This is useful in applications, as illustrated by the next example.

Consider the manager (the analyst) of an organization with two divisions (the players), denoted by $I = \{1, 2\}$. On a daily basis, each division has to perform a distinct task. Division i chooses whether to exert observable effort in its task, denoted by $a_i \in \{y, n\}$. The cost of effort, denoted by $c_i(\omega)$, depends on both the task and the state of the environment where the organization operates: $\omega \in \{B, H\}$, where B stands for "benign" and H for "hostile." Suppose that if

[10]Information design is an example of this problem. In this paradigm, it is as if an information designer could choose to implement the outcome that maximizes $\mathbb{E}_x(v)$. Relative to the information design literature, the novelty of our setting is that the designer cannot freely choose any information structure in $\mathcal{P}$. Instead, she is constrained by the network–seed system to choose in $\mathcal{P}_{(N,S)}$.

	$a_2 = y$	$a_2 = n$
$a_1 = y$	$1-\varepsilon-\underline{c}_1\ ,\ 1-\varepsilon-\underline{c}_2$	$1-\underline{c}_1, 0$
$a_1 = n$	$0\ ,\ 1-\underline{c}_2$	$0\ ,\ 0$

ω = Benign

	$a_2 = y$	$a_2 = n$
$a_1 = y$	$-1-\varepsilon\ ,\ -1-\varepsilon$	$-1\ ,\ 0$
$a_1 = n$	$0\ ,\ -1$	$0\ ,\ 0$

ω = Hostile

Table 1: Effort-Game Payoffs

$\omega = H$, both tasks are equally hard to perform and $c_i(H) = 2$ for both i. Instead, if $\omega = B$, the task of division 1 is easier to perform: $c_i(B) = \underline{c}_i$, where $0 < \underline{c}_1 < \underline{c}_2 < 1$. Exerting no effort costs zero. Division i gets a bonus of 1 if and only if it exerts effort ($a_i = y$). In addition, suppose each division likes to stand out in the eyes of the manager so that, if both exert effort, then each suffers a small disutility ε.[11] Table 1 summarizes these payoffs. Before choosing whether to exert effort, the divisions obtain information about ω; in the language of our model, they are both seeds, $S = I$.

The manager would like each division to exert effort, which we can capture with the function $v(a_1, a_2, \omega) = \sum_i \mathbb{I}\{a_i = y\}$ for all ω. The manager can design her organization to incentivize effort provision. In particular, she can specify whether one division has to share its information with the other or not (the spillover network). If $N = \varnothing$, she institutes a "firewall": No information can leak between divisions. If $N = \{(1,2),(2,1)\}$, she mandates full transparency: All information must be shared between divisions. If $N = \{(i, -i)\}$, she places division i under the oversight of division $-i$ so that the latter automatically observes the former's information. Which organizational design best serves the manager's goals?[12]

To answer this question, we use Theorem 2 to derive the set of feasible outcomes $X(G, N, I)$ for each N. We then project it on the two dimensions that matter for the manager, namely the probability that each division exerts effort: $\Pr(a_i = y) = \sum_{\omega, a_{-i}} x(a_i = y, a_{-i}|\omega)\mu(\omega)$ for $i \in I$. Figure 5 shows the resulting sets for different prior probabilities that the environment is benign: $\mu(B) = 1/2$ (left panel) and $\mu(B) = 4/5$ (right panel). Appendix C contains the formal

[11] Specifically, $0 < \varepsilon < \min\{1 - \underline{c}_2, \frac{\underline{c}_2 - \underline{c}_1}{\underline{c}_2}\}$. Since $\varepsilon > 0$, effort provision is a strategic substitute. The case of strategic complements, $\varepsilon < 0$, can be analyzed following similar steps.

[12] This question stems from a literature in organizational economics that studies how a manager should design the information flows among the divisions of an organization (see, e.g., Dessein and Santos (2006), Dessein et al. (2016), and Matouschek et al. (2023)). We depart from this literature in two ways: we consider incentive conflicts between the divisions and take a robustness approach by making minimal assumptions about what initial information divisions have.

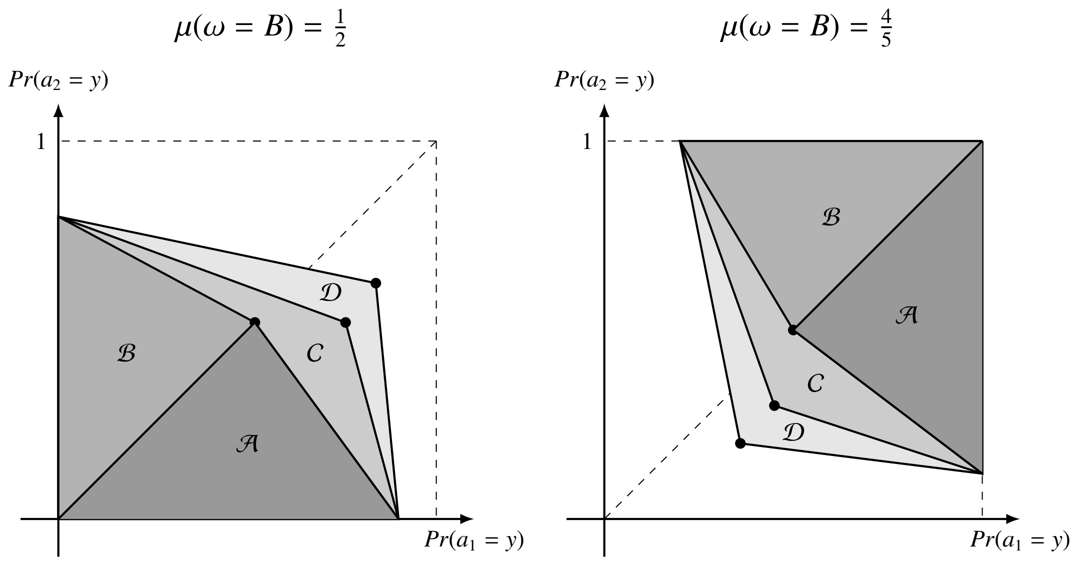

Figure 5: Feasible effort probabilities under different priors and networks.
Left panel: $N = \emptyset$: $\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}$; $N = \{(1,2)\}$: $\mathcal{A}+\mathcal{B}+\mathcal{C}$; $N = \{(2,1)\}$: $\mathcal{A}+\mathcal{B}$; $N = I^2$: $\mathcal{A}$.
Right panel: $N = \emptyset$: $\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}$; $N = \{(1,2)\}$: $\mathcal{A}+\mathcal{B}$; $N = \{(2,1)\}$: $\mathcal{A}+\mathcal{B}+\mathcal{C}$; $N = I^2$: $\mathcal{A}$.

derivations.

To gain intuition, let us discuss a few properties of these sets. First, the sets corresponding to the full and empty networks are nested as a consequence of Lemma 1. This result, however, does not rank the other two networks, $\{(1,2)\}$ and $\{(2,1)\}$. Second, the sets in the two panels are "flipped" because, when the divisions get no information, in the unique equilibrium of the ensuing game they exert effort if and only if the prior $\mu(B)$ is high. Third, there is a trade-off between increasing $\Pr(a_1 = y)$ and $\Pr(a_2 = y)$ along the boundaries of these sets due to strategic substitutability ($\varepsilon > 0$). Lastly, when the prior is low (resp. high) network $N = \{(2,1)\}$ shrinks the feasible set more (resp. less) than $N = \{(1,2)\}$ does. Since $\underline{c}_2 > \underline{c}_1$, inducing $a_2 = y$ (resp. $a_1 = n$) requires a more informative signal, which when leaked constrains behavior more.

Figure 5 shows the range of outcomes that can occur under each organization design. If the manager is reluctant to make assumptions about the information the divisions will obtain, she will choose N in a way that is robust to the ambiguity implied by all these feasible outcomes. One way to do so is to use the following max-min criterion (see, e.g., Ghirardato et al. (2004)):

$$\gamma \max_{x \in X(G,N,S)} \mathbb{E}_x(v) + (1-\gamma) \min_{x \in X(G,N,S)} \mathbb{E}_x(v), \tag{3}$$

where $0 < \gamma < 1$. When $\mu(B)$ is small (Figure 5, left panel), the manager then prefers $N = \varnothing$ to $N = \{(1,2)\}$ and the latter to either $N = \{(2,1)\}$ or $N = \{(1,2),(2,1)\}$. When $\mu(B)$ is large (Figure 5, right panel), she instead prefers $N = \{(1,2),(2,1)\}$ or $N = \{(1,2)\}$ to $N = \{(2,1)\}$ and the latter to $N = \varnothing$. The reason is twofold. First, due to strategic substitutability, knowing that division $-i$ will (not) exert effort weakens i's incentives to (not) exert effort. Second, since the manager cares only about effort (and not the state), the best scenario is when each division gets just enough positive news to weakly prefer exerting effort. However, this threshold is higher for division 2, which renders spillovers of good news from 2 to 1 worse than those in the opposite direction. When $\mu(B)$ is low, the divisions are ex-ante pessimistic and they would not exert effort without information. In this case, the manager is concerned about their becoming too optimistic, which explains the first ranking. By contrast, when $\mu(B)$ is high, the divisions are ex-ante optimistic and they would exert effort without information. In this case, the manager is concerned about their becoming too pessimistic, which explains the second ranking.

These results imply that, in both cases, the manager is better off when division 1 reports to division 2, rather than the opposite. An insight from this discussion is that, when effort is a strategic substitute, divisions handling harder tasks (higher cost) should oversee divisions handling easier ones (lower cost), to better leverage the resulting information flows and robustly induce effort.

Before concluding, we note that these rankings of networks can be useful also in settings in which the manager has limited control over her organization's hierarchy, but she can choose the task allocation. For instance, suppose that in the organization division 1 oversees division 2 (i.e., $N = \{(2, 1)\}$) and this cannot be changed. She can, however, assign tasks to the divisions. In particular, it is optimal to allocate the harder task to division 1 and the easier one to division 2. This is isomorphic to changing the spillover network, as considered above.

5 More General Spillover Processes

In this section, we discuss how our results can help analyze the effects of spillovers that are more complex than those we considered so far. Throughout, we focus on the case where $S = I$, but study general information spillovers that include strategic communication and random spillovers. For such spillovers, computing the entire set of feasible outcomes can be challenging. Yet, our results can be used to provide a partial characterization of these outcomes. For example, consider an analyst interested in the highest probability that an action profile can be played in equilibrium for some game. This section's main result shows that our baseline model can be used to bound this probability.

We consider the following model of communication on a network N, which subsumes the mechanical spillovers introduced in Section 2. For each player i, let the set of i's direct sources be $\bar{N}_i = \{j \in I : (j, i) \in N\}$ and the set of i's direct followers be ${}_i\bar{N} = \{j \in I : (i, j) \in N\}$. Let K be a finite number of communication rounds. In every round, player i sends a message to each of her direct followers. For every initial information structure $(T, \pi) \in \mathcal{P}_S$, let M be the set of messages i can send to j at every round. We assume that M is large enough so that each player can convey all her initial information if she wants to. One way to ensure this is to assume that $M = \bar{T} \supsetneq T$ (recall footnote 3). Let i's initial histories be of the form $h_i^0 = ((T,\pi), t_i)$, where t_i is privately observed by i.[13] Thus, i's set of initial histories is $H_i^0 = \{((T,\pi), t_i) : t_i \text{ s.t. } \pi(t|\omega) > 0 \text{ for some } \omega\}$. For every round $k \leq K$, define i's histories at round k recursively by $H_i^k = \{(h_i^{k-1}, ({}_i m, m_i)) : h_i^{k-1} \in H_i^{k-1}, {}_i m \in {}_i M, m_i \in M_i\}$, where ${}_i M = \times_{j \in {}_i \bar{N}} M$ are the messages i can send to her direct followers and $M_i = \times_{j \in \bar{N}_i} M$ are those she can receive from her direct sources. Let the set of all i's histories be $\mathcal{H}_i = \cup_{k=0}^K H_i^k$. Player i's communication is described by a function $\xi_i : \mathcal{H}_i \to \Delta({}_i M)$, where $\xi_i(h_i^k)$ has finite support for all i and $h_i^k \in \mathcal{H}_i$. The profile $\xi = (\xi_i)_{i=1}^I$ defines a spillover process. We assume that ξ is common knowledge among the players.

This model of spillover processes accommodates two important cases. At one extreme, when every player truthfully reports her history to her direct followers and K is sufficiently large, ξ coincides with our baseline assumption of mechanical spillovers. At the other extreme, when players choose what message to send to maximize their utilities in the base game, which is played after the K communication rounds, ξ captures strategic cheap-talk communication.

Every process ξ transforms the initial information structure (T,π) into a final information structure (T',π'). Indeed, for every ω, (T,π) induces a distribution over finitely many profiles of initial histories $h^0 = (h_i^0)_{i=1}^I$. For every h^0, ξ induces a distribution over finitely many profiles of histories $h^K = (h_i^K)_{i=1}^I$. Interpreting h_i^K as i's final signal realization from these compounded distributions, we get that every ω induces a distribution over such signal profiles. Therefore, ξ maps every initial (T,π) into a final (T',π'). Let the set of final information stuctures induced by ξ be $\mathcal{P}_{(N,S)}^{\xi} \subseteq \mathcal{P}$. Every $(T',\pi') \in \mathcal{P}_{(N,S)}^{\xi}$ and base game G define a Bayesian game, for which we continue to assume that the players will play a BNE.

In the spirit of Section 4.3, consider an analyst who evaluates outcomes of a base game G according to the function $v : \Omega \times A \to \mathbb{R}$. As a result, she is not directly interested in all feasible outcomes but in those for which v takes extreme values. That is, for any final $(T',\pi') \in \mathcal{P}_{(N,S)}^{\xi}$, denote the highest expected value that v takes in a BNE of the resulting Bayesian game by $V(G,(T',\pi'))$, and let

$$V_{\xi}^*(G,N,S) = \sup_{(T',\pi') \in \mathcal{P}_{(N,S)}^{\xi}} V(G,(T',\pi')). \tag{4}$$

For example, if $v(\omega,a) = \mathbb{I}\{a = \bar{a}\}$ for some action profile $\bar{a}$, then $V_{\xi}^*(G,N,S)$ is the highest probability that $\bar{a}$ is played in some equilibrium of G when initial information spills over network N according to ξ.

In general, computing $V_{\xi}^*(G,N,S)$ can be challenging. The next result, however, shows that we can bound $V_{\xi}^*(G,N,S)$ using our baseline model of mechanical spillovers. These bounds

[13] We include (T,π) in the initial history so that the spillover process can depend on the initial information structure and to easily keep track of this dependence.

are the extreme values that v takes over the set of feasible outcomes in Section 3, which can be computed using linear-programming techniques.

Proposition 2 (Payoff Bounds for General Spillovers)**.** *Fix a base game G and a system (N, S) with $S = I$. For every spillover process ξ,*

$$V^*(G, N, S) \leq V^*_\xi(G, N, S) \leq V^*(G, \emptyset, S).$$

Intuitively, the first inequality holds because there are (weakly) fewer spillovers under ξ than in the baseline model: Under ξ, information may not leak between two connected players, while it will leak in our baseline model. When $S = I$, this can only be detrimental for the analyst. Through the lens of this result, deterministic spillovers represent a worst-case scenario from the analyst's point of view. The second inequality arises because, in a network–seed system with no spillovers ($N = \emptyset$), a mediator could always replicate the spillovers induced by ξ by choosing the initial information accordingly.

It is worth noting that Proposition 2 does not hold when $S \subsetneq I$. For example, consider a situation where $I = \{1, 2\}$, $N = \{(1, 2)\}$, $S = \{1\}$, player 2 wants to choose an action that matches the state, player 1 chooses no action, and $u_1 = -u_2$. Then, player 1 will never share any signal with player 2, and this can be captured by an appropriately defined ξ. Thus, if $v = u_2$, we have $V^*_\xi(G, N, S) < V^*(G, N, S)$.

The restriction $S = I$ can be relaxed when ξ captures the special case of random spillovers. In this case, if $(i, j) \in N$, information flows from i to j with some exogenous probability. This is akin to a model where the spillover network is random.[14] Proposition 3 in Appendix A.1 shows that Theorems 1 and 2 can be used to bound $V^*_\xi(G, N, S)$ even when $S \subsetneq I$.

Beyond the computational challenges of calculating $V^*_\xi(G, N, I)$, strategic spillovers also raise some conceptual issues related to equilibrium selection. To see this, consider the case of cheap talk, which can lead to multiple equilibria in the communication phase of the overall game. By selecting the analyst-preferred equilibrium, as it is common in the information-design literature, the optimal initial information structure would be followed by a babbling equilibrium. This would thus eliminate all spillovers among players, defeating the purpose of studying their effects in the first place. To avoid this, one could select the worst equilibrium in the communication phase and the ensuing final game. However, this max-min approach may raise other challenges by preventing a linear-programming formulation of the problem (Mathevet et al. (2020)). For these reasons, focusing on robust payoff bounds, as we have done in Proposition 2, may be an effective approach to studying the effects of strategic spillovers.

[14]Random networks are commonly studied in the social learning literature. See, for example, Acemoglu et al. (2011), Lobel and Sadler (2015), Sadler (2020), and Board and Meyer-ter-Vehn (2021). Closer to our work, random information spillovers have been considered in the organizational economics literature (e.g., Dessein and Santos (2006) and Dessein et al. (2016)).

6 Concluding Remarks

This paper studies equilibrium behavior in incomplete-information games under two information constraints: seeding and spillovers. These constraints offer a flexible, yet tractable, way to encode restrictions on what agents know about each other's information. Our framework is especially suited to applications in which the analyst can observe the bare bones of the informational environment in which agents interact: Who can get information and how it spills over to others. In particular, it can be used to revisit the classic question of how information flows affect the performance of an organization and their optimal design.

Our main results provide a characterization of all feasible equilibrium outcomes. We achieve this in two steps. First, we show how to replace the seeding constraint with richer information spillovers while leaving the set of feasible outcomes unchanged. This result—which relies on our notion of network expansion—enables the use of direct recommendation mechanisms. Second, we introduce a notion of spillover-robust obedience, which allows us to conveniently characterize feasible outcomes in terms of linear inequalities. Building on these results, we study how changing the seeds and the spillovers affect equilibrium outcomes. We introduce an order on network–seed systems that identifies when spillovers are more constraining and seeds are more influential. Finally, although our methodology has been developed for the case of deterministic spillovers, we show it also proves useful to analyzing more general spillover processes.

We hope our approach can be useful in empirical work by helping the econometrician better exploit observables in the data, while still making minimal assumptions about the information that agents might have. For example, in an application like that of Magnolfi and Roncoroni (2023)—who model competition in the supermarket industry—it may be possible to assume that stores belonging to the same chain share the same information, whereas stores belonging to different chains do not. Our results could then be used to sharpen predictions and, thus, facilitate identification.

References

Acemoglu, D., M. A. Dahleh, I. Lobel, and A. Ozdaglar (2011): "Bayesian learning in social networks," *Review of Economic Studies*, 78, 1201–1236.

Akbarpour, M., S. Malladi, and A. Saberi (2018): "Just a Few Seeds More: Value of Network Information for Diffusion," *Working Paper*.

Babichenko, Y., I. Talgam-Cohen, H. Xu, and K. Zabarnyi (2022): "Multi-Channel Bayesian Persuasion," *Working Paper*.

BALLESTER, C., A. CALVÓ-ARMENGOL, AND Y. ZENOU (2006): "Who's Who in Networks. Wanted: The Key Player," *Econometrica*, 74, 1403–1417.

BANERJEE, A., A. G. CHANDRASEKHAR, E. DUFLO, AND M. O. JACKSON (2013): "The diffusion of microfinance," *Science*, 341, 1236498.

BERGEMANN, D. AND S. MORRIS (2016): "Bayes Correlated Equilibrium and the Comparison of Information Structures in Games," *Theoretical Economics*, 11, 487–522.

——— (2019): "Information Design: A Unified Perspective," *Journal of Economic Literature*, 57(1), pp. 44-95).

BLACKWELL, D. (1951): "Comparison of experiments," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*.

BOARD, S. AND M. MEYER-TER-VEHN (2021): "Learning Dynamics in Social Networks," *Econometrica*, 89, 2601–2635.

BOLTON, P. AND M. DEWATRIPONT (1994): "The Firm as a Communication Network," *The Quarterly Journal of Economics*, Volume 109, Issue 4, 809–839.

BRANDENBURGER, A., J. KOU, AND M. BURNETT (1995): "Power Play: Nintendo in 8-bit Video Games," *Case Study*, Harvard Business School Publishing, 795102.

BROOKS, B., A. P. FRANKEL, AND E. KAMENICA (2022): "Information Hierarchies," *Econometrica*, 90, 2187–2214.

CALVÓ-ARMENGOL, A., J. DE MARTI BELTRAN, AND A. PRAT (2015): "Communication and influence," *Theoretical Economics*, 10, 649–690.

CANDOGAN, O. (2020): "On Information Design with Spillovers," *Working Paper*.

DESMEDT, Y. AND Y. WANG (2002): "Perfectly secure message transmission revisited," in *Advances in Cryptology*, Berlin: Springer, 502–517.

DESSEIN, W., A. GALEOTTI, AND T. SANTOS (2016): "Rational Inattention and Organizational Focus," *American Economic Review*, 106, 1522–36.

DESSEIN, W. AND T. SANTOS (2006): "Adaptive Organizations," *Journal of Political Economy*, 114, 956–995.

DOLEV, D., C. DWORK, O. WAARTS, AND M. YUNG (1993): "Perfectly secure message transmission," *Journal of the ACM (JACM)*, 40, 17–47.

FRANKLIN, M. AND R. N. WRIGHT (2000): "Secure communication in minimal connectivity models," *Journal of Cryptology*, 13, 9–30.

GALEOTTI, A., C. GHIGLINO, AND F. SQUINTANI (2013): "Strategic information transmission networks," *Journal of Economic Theory*, 148, 1751–1769.

GALEOTTI, A., B. GOLUB, AND S. GOYAL (2020): "Targeting Interventions in Networks," *Econometrica*, 88, 2445–2471.

GALPERTI, S. AND J. PEREGO (2018): "A Dual Perspective on Information Design," *Working*

Paper.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004): "Differentiating ambiguity and ambiguity attitude," *Journal of Economic Theory*, 118, 133–173.

Hagenbach, J. and F. Koessler (2010): "Strategic Communication Networks," *The Review of Economic Studies*, 77, 1072–1099.

Kamenica, E. (2019): "Bayesian persuasion and information design," *Annual Review of Economics*, 11, 249–272.

Lattin, J. and M. Rierson (2007): "Capital One: Leveraging Information-Based Marketing," *Case Study*, Harvard Business School Publishing, M316.

Lobel, I. and E. Sadler (2015): "Information diffusion in networks through social learning," *Theoretical Economics*, 10, 807–851.

Magnolfi, L. and C. Roncoroni (2023): "Estimation of discrete games with weak assumptions on information," *Review of Economic Studies*, 90(4), 2006–2041.

Mathevet, L., J. Perego, and I. Taneva (2020): "On Information Design in Games," *Journal of Political Economy*, 128(4), 1370–1404.

Mathevet, L. and I. Taneva (2022): "Organized Information Transmission," *Working Paper*.

Matouschek, N., M. Powell, and B. Reich (2023): "Organizing Modular Production," *Working Paper*.

Morris, S. (2000): "Contagion," *Review of Economic Studies*, 67, 57–78.

Radner, R. (1992): "Hierarchy: The Economics of Management," *Journal of Economic Literature*, 30, 1382–415.

——— (1993): "The Organization of Decentralized Information Processing," *Econometrica*, 61, 1109–1146.

Renault, J., L. Renou, and T. Tomala (2014): "Secure message transmission on directed networks," *Games and Economic Behavior*, 85, 1–18.

Renou, L. and T. Tomala (2012): "Mechanism design and communication networks," *Theoretical Economics*, 7, 489–533.

Rivera, T. J. (2018): "Incentives and the Structure of Communication," *Journal of Economic Theory*, 175, 201–247.

Sadler, E. (2020): "Diffusion games," *American Economic Review*, 110, 225–270.

Shamir, A. (1979): "How to share a secret," *Communications of the ACM*, 22(11), 612–613.

Valente, T. (2012): "Network Interventions," *Science*, 337, 49–52.

Appendix

A Additional Material

A.1 Random Spillovers

Building on our work from Section 5, we now focus attention on a particular spillover process ξ. We assume that spillovers are mechanical, as in our baseline model, but occur on a random network. The analyst may not know the network and each player may only know the part that directly involves him. To model this, fix players I and seeds S. Unlike in Section 5, we allow $S \subsetneq I$. Let φ be a commonly known probability distribution over networks with nodes I, whose support is $\Phi = \operatorname{supp} \varphi$. We assume that every player i learns the realization of her information sources N_i and direct followers ${}_i\bar{N} = \{j \in I : (i, j) \in N\}$. For convenience, we write $\nu_i = (N_i, {}_i\bar{N})$ and interpret ν_i as a signal that player i receives about the realized N, whose distribution follows from φ. We assume that φ is independent of ω to avoid that ν_i conveys exogenous information about ω, which—although possible—only distracts from the focus of the paper.

As in our baseline model, spillovers occur mechanically: If the realized network N has a path from player i to player j, then j learns all the information i initially gets, namely, (t_i, ν_i).[15] Since spillovers are mechanical, albeit random, an equivalent interpretation for this model of network–seed systems with random networks is that player i always attempts to share information with every other player j, but succeeds only if link (i, j) exists in the realized N.

Before proceeding, we need to extend our maintained assumption that (N, S) is connected to the case of random networks. Recall from Section 2 that (N, S) is connected if every player has at least one seeded source: $N_i \cap S \neq \emptyset$ for all i. In this section, we maintain the following assumption on the support of φ:

Assumption 1. *For all $N \in \Phi$, (N, S) is connected.*

This restricts the uncertainty about N. With probability one, every player has a seeded source and, therefore, can receive nontrivial information. However, the identity of this source is ex-ante uncertain.

As in Proposition 2, we want to bound the value that an objective function $v : \Omega \times A \to \mathbb{R}$ can take in an equilibrium. Since the initial information that players obtain is independent of the realized network N, we can view φ and the mechanical spillovers as inducing a specific spillover process ξ^φ, as defined in Section 5, over the deterministic network $N^\varphi = \cup_{N \in \Phi} N$

[15] Note that the more each player knows about the network, the better he can predict other players' behavior, which tightens incentive constraints. Thus, letting the spillovers include the information about N provides a conservative analysis.

(Appendix D.4 explains the construction of ξ^φ in detail). Given ξ^φ, the value of the objective function is $V^*_{\xi^\varphi}(G, N^\varphi, S)$ (recall the definition in Equation 4).

To construct our bounds, we need to define two sets of outcomes. An outcome is now described by $x^\varphi : \Omega \times \Phi \to \Delta(\mathcal{A})$, where the probability assigned to α can depend on N. Let $\bar{X}(G, \varphi, I)$ be the set of outcomes that satisfy the following property: For every $N \in \Phi$, $i \in I$, $\alpha_{N_i} \in \mathrm{supp}_{N_i} x^\varphi$, and $a_i \in A_i$

$$\sum_{\substack{\omega \in \Omega, N \in \Phi \\ \alpha_{-N_i} \in \mathrm{supp}_{-N_i} x^\varphi}} \Big(u_i(\alpha_i, \alpha_{-i}, \omega) - u_i(a_i, \alpha_{-i}, \omega) \Big) x^\varphi(\alpha_i, \alpha_{-i} | N, \omega) \mu(\omega) \hat{\varphi}(N | \nu_{N_i}) \geq 0, \tag{A.1}$$

where $\hat{\varphi}(\cdot | \nu_{N_i})$ is i's belief about the realized N conditional on $\nu_{N_i} = (\nu_j)_{j \in N_i}$. Therefore, the set $\bar{X}(G, \varphi, I)$ contains outcomes for which it is optimal for every player to follow her recommendation, knowing the recommendations for her sources in the realized N and her exogenous information about N. Also, let

$$\underline{X}(G, \varphi, S) = \cap_{N \in \Phi} X(G, N, S),$$

where $X(G, N, S)$ is the set of feasible outcomes of the baseline problem (G, N, S). Since each $X(G, N, S)$ contains BNE outcomes where players get no information beyond the prior and since Φ is finite, $\underline{X}(G, \varphi, S)$ is nonempty. The set $\underline{X}(G, \varphi, S)$ contains outcomes that are feasible for all network–seed systems (N, S) where network $N \in \Phi$ is deterministic, as in the baseline model.

Proposition 3. *Fix G and network distribution φ. We have $\underline{V}(G, \varphi, S) \leq V^*_{\xi^\varphi}(G, N^\varphi, S) \leq \bar{V}(G, \varphi, I)$, where*

$$\underline{V}(G, \varphi, S) = \begin{cases} V^*(G, N^\varphi, I) & \text{if } S = I, \\ \sup\limits_{x^\varphi \in \underline{X}(G, \varphi, S)} \sum\limits_{\substack{\omega \in \Omega, N \in \Phi \\ \alpha \in \mathrm{supp}\, x^\varphi}} v(\alpha, \omega) x^\varphi(\alpha | N, \omega) \mu(\omega) \varphi(N) & \text{if } S \subsetneq I, \end{cases}$$

and

$$\bar{V}(G, \varphi, I) = \sup_{x^\varphi \in \bar{X}(G, \varphi, I)} \sum_{\substack{\omega \in \Omega, N \in \Phi \\ \alpha \in \mathrm{supp}\, x^\varphi}} v(\alpha, \omega) x^\varphi(\alpha | N, \omega) \mu(\omega) \varphi(N).$$

To see the intuition, consider first $S = I$. The lower bound follows immediately from Proposition 2. The upper bound follows because condition (A.1) corresponds to robust obedience in the fictitious setting where the mediator issues her recommendations after learning N. By contrast, in our setting, the players' initial information cannot depend on N. Thus, the mediator's ability to correlate the players' behavior with N is limited to how the network realizations *exogenously* shape the spillovers and consequently the players' final information. Now consider $S \subsetneq I$. Since seeding constrains the initial information structures, $\bar{V}(G, \varphi, I)$ remains a valid upper bound. Regarding the lower bound, outcomes that are feasible for each realized N must be implemented by some "common" initial information that cannot rely on differences between the possible realizations of N. Such initial information then also works when N is uncertain.

To conclude, we emphasize that Proposition 3 exploits the method and insights of our baseline analysis to provide useful bounds for an otherwise challenging problem. All bounds are again expressed in terms of recommendation mechanisms (as opposed to information structures) and involve solving a linear program.

A.2 Technical Remarks

A.2.1 Finite-Support Outcomes x

In this paper, we focused attention on information structures (T,π) where T is finite. Because of this, the outcome they induce must have finite support (Definition 1). How large does this support need to be? The answer to this question is important for computing the set of feasible outcomes. The answer is simple in the standard case studied by the literature, namely when $S = I$ and $N = \varnothing$. In this case, we know it is without loss of generality to focus on outcomes x that involve only pure-action recommendations. Therefore, the support of an outcome has at most cardinality $|A|$. By contrast, when $N \neq \varnothing$, we argue that recommendations need to belong to the set $\mathcal{A} = \times_{i\in I}\Delta(A_i)$, which is not finite.

In the following, we use Theorem 2 to show that we can identify a finite, exogenous, upper bound on the outcomes' support, which only depends on G. Fix (G,N,S). By Theorem 1, it is without loss of generality to focus attention on the case $S = I$. It is convenient to rewrite the robust-obedience condition (2) as follows: For every $i \in I$, $\alpha_{N_i} \in \operatorname{supp}_{N_i} x$, and $a_i, a_i' \in A_i$,

$$\sum_{\substack{\omega\in\Omega \\ \alpha_{-N_i}\in \operatorname{supp}_{-N_i} x}} \Big(u_i(a_i,\alpha_{-i},\omega) - u_i(a_i',\alpha_{-i},\omega)\Big)\alpha_i(a_i)x(\alpha_i,\alpha_{-i}|\omega)\mu(\omega) \geq 0. \tag{A.2}$$

This highlights that for each player i robust obedience ultimately involves her primitive pure actions. To willingly implement any mixed action, i must deem all pure actions in its support optimal given her information. This information is provided by the realization of α_{N_i}. Each α_{N_i} then pins down a subset of optimal actions for i. Since A_i is finite, there can only be finitely many such subsets.

Lemma 2. *Suppose $x \in X(G,N,I)$. There exists $x' \in X(G,N,I)$ such that $|\operatorname{supp} x_i'| \leq |2^{A_i}|$ for every $i \in I$ and x and x' induce the same joint distribution over A for every $\omega \in \Omega$:*

$$\sum_{\alpha'\in\operatorname{supp} x'} \alpha'(a)x'(\alpha'|\omega) = \sum_{\alpha\in\operatorname{supp} x} \alpha(a)x(\alpha|\omega), \quad a \in A.$$

Proof. See Online Appendix D.3. ◻

This also implies that it is without loss of generality to focus on information structures such that $|T_i| \leq |2^{A_i}|$ for all $i \in I$.

A.2.2 Existence of an optimal information structure

We now turn to the existence of outcomes that maximize a given objective $v : \Omega \times A \to \mathbb{R}$. Consider the information-design problem

$$V^*(G, N, S) = \sup_{x \in X(G,N,S)} \sum_{\substack{\omega \in \Omega, a \in A \\ \alpha \in \operatorname{supp} x}} v(\omega, a)\alpha(a)x(\alpha|\omega)\mu(\omega). \tag{A.3}$$

When the network–seed system does not impose any meaningful constraint (i.e. when $N = \varnothing$ and $S = I$), $X(G, N, S)$ can be written as a compact subset of $\mathbb{R}^{\Omega \times A}$. Therefore, the existence of a solution to (A.3) follows from standard arguments. In general, however, the network–seed system is constraining and, therefore, $X(G, N, S)$ has a more complex structure. Despite the exogenous bound on the outcomes' support, its elements are profiles of mixed actions, which form an uncountable space. Nonetheless, the next result establishes the existence of a maximizer.

Lemma 3 (Existence)**.** *For every base game G, objective v, and network–seed system (N, S), there exists a feasible outcome $x^* \in X(G, N, S)$ such that*

$$\sum_{\substack{\omega \in \Omega, a \in A \\ \alpha \in \operatorname{supp} x^*}} v(\omega, a)\alpha(a)x^*(\alpha|\omega)\mu(\omega) = V^*(G, N, S)$$

Proof. See Online Appendix D.3. □

In some settings with information spillovers (i.e., $N \neq \varnothing$), the solution to the designer's problem may involve only pure-action recommendations. One simple example of this is when the base game is actually a collection of single-agent decision problems: $u_i(a_i, a_{-i}, \omega)$ does not depend on a_{-i} for all $i \in I$. Intuitively, in this case we cannot relax condition (A.2) by keeping player i uncertain about other players' behavior whose randomness is independent of the state. Thus, mixed-action recommendations are useless. More generally, we can always search for a candidate solution within the space of outcomes that only recommend pure actions. Galperti and Perego (2018) show how to verify that this candidate solves the overall problem using linear-programming duality.

B Main Proofs

To prove Theorem 1, we first introduce and prove Lemmas 4, 6, and 7, and the intermediate equivalence result of Lemma 5. Lemma 4 characterizes N^S. It shows that in N^S, while each player may have new sources relative to N (formally, $N_i \subseteq N_i^S$), none of these new sources is a seed (i.e., $N_i^S \cap S = N_i \cap S$).

Lemma 4. *Fix (N, S). For all i, $N_i^S \cap S = N_i \cap S$.*

Proof of Lemma 4. Fix N and i. First, we show that $N_i \cap S \subseteq N_i^S \cap S$. To see this, note that $N \subseteq N^S$, by definition of S-expansion. This implies that $N_i \subseteq N_i^S$. Hence, $N_i \cap S \subseteq N_i^S \cap S$. Second, we show that $N_i^S \cap S \subseteq N_i \cap S$. Note that it is enough to show that $N_i^S \cap S \subseteq N_i$. Suppose not, $N_i^S \cap S \nsubseteq N_i$, there is $j \in N_i^S \cap S$ such that $j \notin N_i$. Since $j \in N_i^S$, there exists a path in N^S from j to i. That is, a sequence $P = (k_1, \dots k_m)$ of distinct k_l for $1 \leq l \leq m$, such that $k_1 = j$, $k_m = i$, and $(k_l, k_{l+1}) \in N^S$, for all $l \leq m-1$. Since $j \notin N_i$, it must be that $(k_l, k_{l+1}) \notin N$, for at least one $l \leq m-1$. We refer to these l's as the *gaps* of P. Let $\underline{P} = (\underline{k}_1, \dots \underline{k}_m)$ be a path from j to i in N^S with the property that its number of gaps is smaller or equal than the number of gaps in any other path P from j to i in N^S. Note that $\underline{P}$ is well-defined since I is finite. Denote $\underline{l}$ the gap in $\underline{P}$ with the smallest index. By construction, we have that (1) $j \in S$, (2) $j \in N_{\underline{k}_{\underline{l}}}$, (3) $(\underline{k}_{\underline{l}}, \underline{k}_{\underline{l}+1}) \in N^S$, (4) $j \notin N_{\underline{k}_{\underline{l}+1}}$, and (5) $(\underline{k}_{\underline{l}}, \underline{k}_{\underline{l}+1}) \notin N$. Points (1) and (2) imply that $j \in N_{\underline{k}_{\underline{l}}} \cap S$. By Definition 3, Point (3) implies that $N_{\underline{k}_{\underline{l}}} \cap S \subseteq N_{\underline{k}_{\underline{l}+1}}$. However, $j \notin N_{\underline{k}_{\underline{l}+1}}$, by point (4). Thus, $N_{\underline{k}_{\underline{l}}} \cap S \nsubseteq N_{\underline{k}_{\underline{l}+1}}$. Finally, by (5), $(\underline{k}_{\underline{l}}, \underline{k}_{\underline{l}+1}) \notin N$. We conclude that N^S is not the expansion of N, a contradiction. □

Lemma 4 shows that the set of i's seeded sources is the same in N and N^S. Therefore, any initial information structure will lead to the same final information structure in these two cases. Thus, they should induce the same outcome, as the next result shows.

Lemma 5. *Fix (N, S). Then for all G, $X(G, N, S) = X(G, N^S, S)$.*

Proof of Lemma 5. Fix G, i and the information structure $(T, \pi) \in \mathcal{P}_S$. Note that $(N_i^S \setminus N_i) \cap S = (N_i^S \cap S) \setminus (N_i \cap S) = \emptyset$. The first equality derives from the distributive property of set intersection over set difference. The second equality derives from Lemma 4. This implies that $T_{N_i^S \setminus N_i}$ is a singleton. Fix $t := (t_1, \dots, t_I)$, such that $\sum_\omega \mu(\omega)\pi(t|\omega) > 0$. We want to show that $\Pr_\pi(t|t_{N_i}) = \Pr_\pi(t|t_{N_i^S})$. Namely, conditioning on $t_{N_i^S}$ rather than t_{N_i} does not change the probability assessment over t. Thus, vectors t_{N_i} and $t_{N_i^S}$ are identical up to $t_{N_i^S \setminus N_i}$, which realizes with probability 1 under π, since $T_{N_i^S \setminus N_i}$ is a singleton. Hence $\Pr_\pi(t|t_{N_i}) = \Pr_\pi(t|t_{N_i^S})$. Since i, t, and π were arbitrary, we have that $\mathcal{P}_{(N,S)} = \mathcal{P}_{(N^S,S)}$ and, thus $X(G, N, S) = X(G, N^S, S)$. □

The next result shows that N^S is the S-expansion of itself, thus proving the uniqueness of the expansion of a network.

Lemma 6. *$(i, j) \in N^S$ if and only if $N_i^S \cap S \subseteq N_j^S$.*

Proof of Lemma 6 *Only if.* Let $(i, j) \in N^S$. Then, $N_i^S \subseteq N_j^S$, hence $N_i^S \cap S \subseteq N_j^S$. *If.* Suppose $N_i^S \cap S \subseteq N_j^S$. Then, $N_i^S \cap S \subseteq N_j^S \cap S$. By Lemma 4, $N_i \cap S \subseteq N_j \cap S$. Thus, $N_i \cap S \subseteq N_j$. By Definition 3, this implies $(i, j) \in N^S$. □

The next result will play an important role in the proof of Theorem 1.

Lemma 7. *Fix (G, N, S) and $S \subseteq S' \subseteq I$. Let $i \in S'$ and $(i, j) \in N^S$. Then $X(G, N^S, S') = X(G, N^S, S' \cup \{j\})$.*

Proof of Lemma 7.
($\subseteq$). This direction is immediate since, by the definition of $\mathcal{P}_S$, it follows that $\mathcal{P}_{S'} \subseteq \mathcal{P}_{S' \cup \{j\}}$. Therefore, $\mathcal{P}_{(N^S, S')} \subseteq \mathcal{P}_{(N^S, S' \cup \{j\})}$.
($\supseteq$). If $j \in S'$ there is nothing to prove since, in such case, $S' \cup \{j\} = S'$. Therefore, let $j \notin S'$. Fix any $(T, \pi) \in \mathcal{P}_{S' \cup \{j\}}$. Using a secret-sharing technique (Shamir, 1979), we will construct a $(\hat{T}, \hat{\pi}) \in \mathcal{P}_{S'}$ such that (T, π) and $(\hat{T}, \hat{\pi})$ induce the same set of equilibria. Let $B(\kappa) := \{0, 1\}^{\kappa}$ and $\underline{\kappa} := \min\{\kappa \in \mathbb{N} : |T_j| \leq |B(\kappa)|\}$. For notational convenience, denote $B := B(\underline{\kappa})$. Let $\mathcal{Z} : T_j \to B$ be an arbitrary injective function. It represents a "public key," that univocally transforms j's signals into binary numbers. Denote $\oplus$ the following logical operator (exclusive or). Bits are added according to the following rule: $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$. For all $b, b' \in B$, $b \oplus b' = (b_1 \oplus b'_1, \ldots, b_{\underline{\kappa}} \oplus b'_{\underline{\kappa}}) \in B$. For notational convenience, denote $Q := N_j^S \cap S'$, the seeded sources of player j. Recall that, by assumption, $j \notin Q$. We now construct the type space of $(\hat{T}, \hat{\pi})$. Let $\hat{T}_i := T_i$ for all $i \notin Q$ and $\hat{T}_i := T_i \times B \times \{\mathcal{Z}\}$ for all $i \in Q$. Note that, by construction, $\hat{T}$ is such that $\hat{T}_i = \{\hat{t}_i\}$ for all $i \notin S'$. That is $(\hat{T}, \hat{\pi})$ seeds only players in S'. Next, we construct $\hat{\pi}$ from π. Let $q := \max\{i : i \in Q\}$. For each realization $t \in T$ under π, the realization $\hat{t} \in \hat{T}$ under $\hat{\pi}$ is such that $\hat{t}_i = t_i$ for $i \notin Q$, with the same conditional distribution of π. Instead, if $i \in Q$, $\hat{t}_i = (t_i, b_i, \mathcal{Z})$. More specifically, if $i \in Q \setminus \{q\}$, $b_i \in B$ is drawn at uniform random from B, independently of (ω, t); instead, if $i = q$, $b_q := \mathcal{Z}(t_j) \oplus (\oplus_{i \in Q \setminus \{q\}} b_i)$. There are two cases to consider, $|Q| = 1$ and $|Q| > 1$.

- If $|Q| = 1$, $b_q = \mathcal{Z}(t_j)$ and observing $\hat{t}_q$ reveals t_j. Thus, player i learns t_j if and only if $\{q\} = Q \subseteq N_i^S$.

- If $|Q| > 1$, instead, observing all but one element in $(b_i)_{i \in Q}$ carries no information about t_j. Instead, observing the whole sequence $(b_i)_{i \in Q}$ fully reveals t_j. This is because:

$$\begin{aligned}
\mathcal{Z}^{-1}\Big(\oplus_{i \in Q} b_i \Big) &= \mathcal{Z}^{-1}\Big((\oplus_{i \in Q \setminus \{q\}} b_i) \oplus b_q \Big) \\
&= \mathcal{Z}^{-1}\Big((\oplus_{i \in Q \setminus \{q\}} b_i) \oplus (\mathcal{Z}(t_j) \oplus (\oplus_{i \in Q \setminus \{q\}} b_i)) \Big) \\
&= \mathcal{Z}^{-1}\Big(\mathcal{Z}(t_j) \Big) \\
&= t_j.
\end{aligned}$$

  The third equality comes from the fact that $b_i \oplus b_i = \mathbf{0}$ and $\mathcal{Z}(t_j) \oplus \mathbf{0} = \mathcal{Z}(t_j)$. Thus, player i learns t_j if and only if $Q \subseteq N_i^S$.

Therefore, irrespective of whether or not Q is a singleton, player i learns t_j if and only if $Q \subseteq N_i^S$. However, note that $Q \subseteq N_i^S$ if and only if $j \in N_i^S$. In fact, if $Q \subseteq N_i^S$, $N_j^S \cap S \subseteq$

$N_j^S \cap S' = Q \subseteq N_i^S$ and, by Lemma 6, $(j,i) \in N^S$ and $j \in N_i^S$. Conversely, if $j \in N_i^S$, then $N_j^S \subseteq N_i^S$, and therefore $Q \subseteq N_i^S$. We conclude that under the constructed $(\hat{T}, \hat{\pi})$ player i learns t_j if and only if $j \in N_i^S$, just like under the original (T, π). Therefore, any outcome x induced by (T, π) can be also induced by $(\hat{T}, \hat{\pi})$. Since (T, π) was arbitrary, this shows that $X(G, N^S, S' \cup \{j\}) \subseteq X(G, N^S, S')$. □

Proof of Theorem 1. Fix (G, N, S). By Lemma 5, $X(G, N, S) = X(G, N^S, S)$. We are left to show that $X(G, N^S, S) = X(G, N^S, I)$. If $S = I$ there is nothing to prove, so let $S \subsetneq I$. The following induction argument proves the claim.

Basis Step. Let $S_1 = S$. By assumption, (N, S) is connected. Therefore, there exist $i, j \in I$ such that $i \in S_1$, $j \notin S_1$, and $(i, j) \in N$. Since $N \subseteq N^S$, $(i, j) \in N^S$. Let $S_2 := S_1 \cup \{j\}$. Since $S_1 \subseteq S_2 \subseteq I$, $i \in S_2$ and $(i, j) \in N^S$, we can invoke Lemma 7 to show that $X(G, N^S, S_1) = X(G, N^S, S_2)$. Finally, it is straightforward to see that (N, S_2) is connected.

Inductive Step. Suppose that $X(G, N^S, S_1) = X(G, N^S, S_k)$ for $S_k := S_1 \cup \{j_1, \dots, j_k\}$. If $S_k = I$ there is nothing to prove. Hence, let $S_k \subsetneq I$. (N, S_k) is connected. Hence, there are $i, j \in I$ such that $i \in S_k$, $j \notin S_k$, and $(i, j) \in N$. Since $N \subseteq N^S$, $(i, j) \in N^S$. Denote $S_{k+1} := S_k \cup \{j\}$. Since $S_{k+1} \supseteq S_k$, $i \in S_{k+1}$, and $(i, j) \in N^S$, we can invoke Lemma 7 to show that $X(G, N^S, S_{k+1}) = X(G, N^S, S_k) = X(G, N^S, S_1)$.

Since I is finite, this procedure stops after $\bar{k} = |I \setminus S|$ steps. We conclude that $X(G, N^S, S) = X(G, N^S, I)$. □

In what follows, a (behavioral) strategy of player i in $(G, (T, \pi))$ is $\sigma_i : T_i \to \Delta(A_i)$. We write $\sigma_i(a_i|t_i)$ instead of $\sigma(t_i)[a_i]$. A profile $\sigma = (\sigma_i)_{i \in I}$ belongs to $BNE(G, (T, \pi))$ if for each i, $t_i \in T_i$, and $a_i \in A_i$ with $\sigma_i(a_i|t_i) > 0$,

$$\sum_{a_{-i}, t_{-i}, \omega} \Big(u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)\Big)\sigma(a_i, a_{-i}|t_i, t_{-i})\pi(t_i, t_{-i}|\omega)\mu(\omega) \geq 0$$

for all $a_i' \in A_i$, where $\sigma(a_i, a_{-i}|t_i, t_{-i}) := \prod_{j=1}^{I} \sigma_j(a_j|t_j)$.

Proof of Theorem 2. **Part 1** **(⇒)**: Suppose $(T, \pi) \in \mathcal{P}$ and $\sigma \in \text{BNE}(G, (T, \pi))$ induce x. Then, for every i and $t_{N_i} \in T_{N_i}$,

$$\sum_{\omega, t'} \Big(u_i(\sigma_i(t_{N_i}), \sigma_{-i}(t'_{N_{-i}}), \omega) - u_i(a_i, \sigma_{-i}(t'_{N_{-i}}), \omega)\Big)\Pr_\pi(\omega, t'|t_{N_i}) \geq 0, \quad a_i \in A_i.$$

where $\sigma_{-i}(t'_{N_{-i}}) = (\sigma_j(t'_{N_j}))_{j \neq i}$. Using π, we can write this condition as, for every i and t_{N_i},

$$\sum_{\omega, (t'_{N_j})_{j \neq i}} \Big(u_i(\sigma_i(t_{N_i}), \sigma_{-i}(t'_{N_{-i}}), \omega) - u_i(a_i, \sigma_{-i}(t'_{N_{-i}}), \omega)\Big) \frac{\pi(t_{N_i}, (t'_{N_j})_{j \neq i}|\omega)\mu(\omega)}{\sum_{\omega', (t''_{N_j})_{j \neq i}} \pi(t_{N_i}, (t''_{N_j})_{j \neq i}|\omega')\mu(\omega')} \geq 0,$$

for all $a_i \in A_i$, or equivalently,

$$\sum_{\omega, (t'_{N_j})_{j \neq i}} \Big(u_i(\sigma_i(t_{N_i}), \sigma_{-i}(t'_{N_{-i}}), \omega) - u_i(a_i, \sigma_{-i}(t'_{N_{-i}}), \omega)\Big)\pi(t_{N_i}, (t'_{N_j})_{j \neq i}|\omega)\mu(\omega) \geq 0,$$

for all $a_i \in A_i$. Note that, for every i and t, by knowing t_{N_i} player i knows the mixed action $\sigma_j(t_{N_j})$ for all $j \in N_i$.

Given this and using the definition of x in (1), the last family of inequalities can be written as follows: For all i and α_{N_i},

$$\sum_{\omega,\alpha_{-N_i}} \Big(u_i(\alpha_i,\alpha_{-i},\omega) - u_i(a_i,\alpha_{-i},\omega)\Big)x(\alpha_i,\alpha_{-i}|\omega)\mu(\omega) \geq 0, \quad a_i \in A_i.$$

Thus, we conclude that if x is feasible, then it is robustly obedient.

Part 2 ($\Leftarrow$): Suppose x is robustly obedient. Recall that $\operatorname{supp} x = \{\alpha : \exists\, \omega \in \Omega \text{ s.t. } x(\alpha|\omega) > 0\} \subseteq \mathcal{A}$ is finite. Note that $(\operatorname{supp} x, x) \in \mathcal{P}$. Given this, for every i, consider the strategy $\sigma_i : \operatorname{supp}_{N_i} \to \Delta(A_i)$ defined as $\sigma_i(\alpha_{N_i}) = \alpha_i$, for all $\alpha_{N_i} \in \operatorname{supp}_{N_i}$. Optimality for each i requires that, for every α_{N_i},

$$\sum_{\omega,\alpha'_{-N_i}} \Big(u_i(\sigma_i(\alpha_{N_i}),\sigma_{-i}(\alpha'_{N_{-i}}),\omega) - u_i(a_i,\sigma_{-i}(\alpha'_{N_{-i}})),\omega)\Big)\mathrm{Pr}_x(\omega,\alpha'|\alpha_{N_i}) \geq 0, \quad a_i \in A_i,$$

where $\sigma_{-i}(\alpha'_{N_{-i}}) = (\sigma_j(\alpha'_{N_j}))_{j\neq i}$. Given our construction of σ, this is equivalent to, for every α_{N_i} and $a_i \in A_i$,

$$\sum_{\omega,\alpha'_{-N_i}} \Big(u_i(\alpha_{N_i},\alpha'_{-N_i},\omega) - u_i(a_i,\alpha_{N_i\setminus i},\alpha'_{-N_i},\omega)\Big)\frac{x(\alpha_{N_i},\alpha'_{-N_i}|\omega)\mu(\omega)}{\sum_{\omega',\alpha''_{-N_i}} x(\alpha_{N_i},\alpha''_{-N_i}|\omega')\mu(\omega')} \geq 0,$$

which holds because x is robustly obedient. $\square$

Proof of Lemma 1. We begin with two preliminary observations. First, note that by Theorem 1 we only need to show that $X(G,N,I) \subseteq X(G,N',I)$ for all G if and only if $(N,I) \trianglerighteq (N',I)$. Second, spillover-robust obedience is equivalent to requiring that, for every i and $\delta_i : \mathcal{A}_{N_i} \to A_i$,

$$\sum_{\omega\in\Omega,\alpha\in\operatorname{supp} x} \Big(u_i(\alpha_i,\alpha_{-i};\omega) - u_i(\delta_i(\alpha_{N_i}),\alpha_{-i};\omega)\Big)x(\alpha_i,\alpha_{-i}|\omega)\mu(\omega) \geq 0. \tag{B.1}$$

Part 1 ($\Leftarrow$): Suppose $(N,I) \trianglerighteq (N',I)$ and $x \in X(G,N,I)$ for some G. Then, by Theorem 2, x satisfies (B.1) for all i and $\delta_i \in D_i = \{\hat{\delta}_i : \mathcal{A}_{N_i} \to A_i\}$. Let $D'_i = \{\delta_i : \mathcal{A}_{N'_i} \to A_i\}$. To prove that $x \in X(G,N',I)$, it suffices to show that the set of available deviations D'_i is smaller than D_i, for all $i \in N$. To show this, consider any $\delta_i \in D'_i$ and define $\hat{\delta}_i : \mathcal{A}_{N_i} \to A_i$ as $\hat{\delta}_i(\alpha_{N'_i},\alpha_{N_i\setminus N'_i}) = \delta_i(\alpha_{N'_i})$, for all $\alpha_{N_i} \in \mathcal{A}_{N_i}$. Since $N_i \supseteq N'_i$ for all i, $\hat{\delta}_i$ is a well-defined function and $\hat{\delta}_i \in D_i$.

Part 2 ($\Rightarrow$): We prove this with a contrapositive argument. The only relevant case to consider is that $(N,I) \ntrianglerighteq (N',I)$ and $(N,I) \ntrianglelefteq (N',I)$. This implies that for some i, there exists a k such that $k \in N'_i$ and $k \notin N_i$, and for some j (possibly $i = j$), there exists m such that $m \in N_j$ and $m \notin N'_j$. It follows that there exists a player i_k such that $i_k \neq k$ and there is a direct link from k to i_k in N' but not in N, and there exists a player i_m such that $i_m \neq m$ there is a direct link from m to i_m in N but not in N'. Now consider the following game G. Let $\Omega = \{0,1\}$ and $\mu(0) = \mu(1) = \frac{1}{2}$. Let $A_i = \{0,\frac{1}{2},1\}$ for all $i \in N$. For all $j \notin \{k,m,i_k,i_m\}$, let the payoff function u_j be such that action

$a_j = \frac{1}{2}$ is strictly dominant. For $j \in \{k, m, i_k, i_m\}$, the payoff function is $u_j(a, \omega) = -(a_j - \omega)^2$. Consider the following two cases.

Case 1: Suppose that all players in $\{k, m, i_k, i_m\}$ are distinct. Consider x such that player k always matches the state, while all other players choose $a = \frac{1}{2}$. Thus, $x \in X(G, N, I)$, but clearly does not belong to $X(G, N', I)$. This is because in N' player i_k has to choose $a_{i_k} = \frac{1}{2}$ after learning $a_k = \omega$, which renders $a_{i_k} = \frac{1}{2}$ strictly suboptimal. Thus, x violates robust obedience for (G, N', I). Now consider x' such that player m always matches the state, while all the other players choose $a = \frac{1}{2}$. This x' belongs to $X(G, N', I)$, but clearly does not belong to $X(G, N, I)$. This is because in N player i_m has to choose $a = \frac{1}{2}$ after learning $a_m = \omega$, which renders $a = \frac{1}{2}$ strictly suboptimal and so x' violates obedience. The same arguments work for the following four alternative configurations of the network that satisfy the aforementioned properties: (1) $m = i_k$ and $k \neq i_m$; (2) $m \neq k$ and $i_k = i_m$; (3) $k = i_m$ and $m = i_k$; (4) $i_m = k$ and $m \neq i_k$.

Case 2: Suppose that $m = k$ and $i_k \neq i_m$. Consider x such that m and i_m always match the state, while all other players choose $a = \frac{1}{2}$. This x belongs to $X(G, N, I)$, but clearly does not belong to $X(G, N', I)$. This is because in N' player i_k has to choose $a_{i_k} = \frac{1}{2}$ after learning $a_k = \omega$, which renders $a_{i_k} = \frac{1}{2}$ strictly suboptimal. Thus, x violates obedience for (G, N', I). Alternatively, consider x' such that player m and i_k always match the state, while all the other players choose $a = \frac{1}{2}$. This x' belongs to $X(G, N', I)$, but clearly does not belong to $X(G, N, I)$. This is because in N player i_m has to choose $a_{i_m} = \frac{1}{2}$ after learning that $a_m = \omega$, which renders $a_{i_m} = \frac{1}{2}$ strictly suboptimal. Thus, x' violates obedience for (G, N, I). $\square$

Proof of Proposition 1. **Part 1 (⇒)**: Suppose $\hat{N}_i^{\hat{S}} \subseteq N_i^S$ for all i. Consider $i, j \in I$ that satisfy $\hat{N}_i \cap \hat{S} \subseteq \hat{N}_j$ and so $i \in \hat{N}_j^{\hat{S}}$ by Definition 3. It follows that $i \in N_j^S$. If $i \in N_j$, then $N_i \cap S \subseteq N_j$ holds automatically. If $i \notin N_j$, we must have added links to N according to Definition 3 that result in $i \in N_j^S$. For this to be the case, there must exist some sequence $\{j_k\}_{k=0}^m$ which satisfies $j_0 = i$, $j_m = j$, and $N_{j_k} \cap S \subseteq N_{j_{k+1}}$. This implies $N_i \cap S \subseteq N_j$.

Part 2 (⇐): Now suppose that $\hat{N}_i \cap \hat{S} \subseteq \hat{N}_j$ implies $N_i \cap S \subseteq N_j$ for all $i, j \in I$. We need to show that $i \in \hat{N}_j^{\hat{S}}$ implies $i \in N_j^S$. Fix any i and j that satisfy $i \in \hat{N}_j^{\hat{S}}$. If $i \in \hat{N}_j$, then we automatically have $\hat{N}_i \cap \hat{S} \subseteq \hat{N}_j$ and so $N_i \cap S \subseteq N_j$ by assumption. Thus, if $i \notin N_j$ (which is the only relevant case), we must add (i, j) to N according to Definition 3, implying $i \in N_j^S$. Next, suppose that $i \in \hat{N}_j^{\hat{S}} \setminus \hat{N}_j$. By Definition 3, there must exist a sequence $\{j_k\}_{k=0}^m$ which satisfies $j_0 = i$, $j_m = j$, and $\hat{N}_{j_k} \cap \hat{S} \subseteq \hat{N}_{j_{k+1}}$. Therefore, it must be that $\hat{N}_i \cap \hat{S} \subseteq \hat{N}_j$ and, by assumption, $N_i \cap S \subseteq N_j$. If again $i \notin N_j$, it follows that $(i, j) \in N^S$, which implies $i \in N_j^S$. $\square$

Proof of Proposition 2. Let us simplify notation and write π in place of $(T, \pi) \in \mathcal{P}$. Moreover, let $f_N(\pi) \in \mathcal{P}_{(N,I)}$ be the information structure induced by π under N and deterministic spillovers. Finally, let $f_{\xi,N}(\pi) \in \mathcal{P}_{(N,I)}^{\xi}$ be the information structure induced by π under N and spillover process ξ. We begin by proving the upper bound, i.e., $V_{\xi}^*(G, N, I) \leq V^*(G, \varnothing, I)$. Fix

ξ. For every π, $f_{\xi,N}(\pi)$ is itself an information structure, possibly seeding all players in I. It follows that,

$$V^*_\xi(G,N,I) = \sup_\pi V(G, f_{\xi,N}(\pi)) \le \sup_\pi V(G,\pi) = V^*(G,\varnothing,I),$$

where the inequality follows from the fact that $f_{\xi,N}(\pi) \in \mathcal{P}$.

Next, we prove the lower bound, i.e., $V^*(G,N,I) \le V^*_\xi(G,N,I)$. Fix π and consider the information structure $(T, f_N(\pi))$. It has the property that, for every player i and signal $t \in T$, Bayesian posteriors satisfy,

$$\Pr_{f_N(\pi)}(\omega, t|t_{N_i}) = \Pr_{f_N(\pi)}(\omega, t|t_i);$$

that is, there is nothing that player i can learn from her sources that is not already contained in her private signal t_i. Therefore, for every spillover process ξ, we have

$$\Pr_{f_{\xi,N}(f_N(\pi))}(\omega, t|h_i) = \Pr_{f_{\xi,N}(f_N(\pi))}(\omega, t|h^0_i) = \Pr_{f_N(\pi)}(\omega, t|t_i).$$

for all players i, histories h_i and h^0_i. That is, additional communication according to ξ adds nothing to what players already know. Therefore, for all π,

$$\sigma \in \mathrm{BNE}(G, f_N(\pi)) \qquad \Longrightarrow \qquad \sigma \in \mathrm{BNE}(G, f_{\xi,N}(f_N(\pi))).$$

Therefore,

$$\bigcup_\pi \mathrm{BNE}(G, f_N(\pi)) \subseteq \bigcup_\pi \mathrm{BNE}(G, f_{\xi,N}(f_N(\pi)) \subseteq \bigcup_\pi \mathrm{BNE}(G, f_{\xi,N}(\pi))$$

from which it follows that $V^*(G,N,I) \le V^*_\xi(G,N,I)$. □

Proof of Proposition 3. As in the proof of Proposition 2, let us simplify notation and write π_I in place of $(T,\pi) \in \mathcal{P}$ and π_S in place of $(T,\pi) \in \mathcal{P}_S$. Moreover, let $f_{\xi^\varphi,N^\varphi}(\pi)$ be the information structure induced π under spillover process ξ^φ and network N^φ.

Case 1: Direct Provision ($S = I$). Theorem 2 immediately implies the inequality $V^*(G,N^\varphi,I) \le V^*(G,\varphi,I)$. The other inequality $V^*(G,N^\varphi,I) \le \bar{V}(G,\varphi,I)$ follows from showing that every π_I and $\sigma \in \mathrm{BNE}(G, f_{\xi^\varphi,N^\varphi}(\pi_I))$ leads to an outcome x^φ that must satisfy condition (A.1) and therefore belongs to $\bar{X}(G,\varphi,I)$. First, given such π_I and σ, let

$$x^\varphi(\alpha_1,\ldots,\alpha_I|\omega,N) = \sum_{t\in T}\pi_I(t|\omega)\prod_{i\in I}\mathbb{I}\{\sigma_i(\nu_i,\hat{t}_{N_i}) = \alpha_i\}, \quad (\alpha,N,\omega)\in\mathcal{A}\times\Phi\times\Omega,$$

where $\hat{t}_{N_i} = (t_j,\nu_j)_{j\in N_i}$. Let $\pi^\varphi = f_{\xi^\varphi,N^\varphi}(\pi_I)$. Since σ is a BNE, for all $N \in \Phi$, $i \in I$, $\hat{t}_{N_i}$, and $a_i \in A_i$,

$$\begin{aligned}&\sum_{\omega\in\Omega,N\in\Phi,t\in T} u_i(\sigma_i(\nu_i,\hat{t}_{N_i}),(\sigma_j(\nu_j,\hat{t}_{N_j}))_{j\neq i},\omega)\Pr_{\pi^\varphi,\mu}(\omega,t|t_{N_i})\hat{\varphi}(N|\nu_{N_i})\\ \ge &\sum_{\omega\in\Omega,N\in\Phi,t\in T} u_i(a_i(\sigma_j(\nu_j,\hat{t}_{N_j}))_{j\neq i},\omega)\Pr_{\pi^\varphi,\mu}(\omega,t|t_{N_i})\hat{\varphi}(N|\nu_{N_i}),\end{aligned}$$

which is equivalent to

$$\sum_{\omega\in\Omega,N\in\Phi,t\in T,\alpha_{-i}} u_i(\alpha,\omega)\prod_{j\in I}\mathbb{I}\{\sigma_j(\nu_j,\hat{t}_{N_j})=\alpha_j\}\Pr_{\pi^\varphi,\mu}(\omega,t|t_{N_i})\hat{\varphi}(N|\nu_{N_i})$$
$$\geq \sum_{\omega\in\Omega,N\in\Phi,t\in T,\alpha_{-i}} u_i(a_i,\alpha_{-i},\omega)\prod_{j\in I}\mathbb{I}\{\sigma_j(\nu_j,\hat{t}_{N_j})=\alpha_j\}\Pr_{\pi^\varphi,\mu}(\omega,t|t_{N_i})\hat{\varphi}(N|\nu_{N_i}),$$

which is equivalent (once we multiply both sides by the total probability of t_{N_i}) to

$$\sum_{\omega\in\Omega,N\in\Phi,t\in T,\alpha_{-i}} u_i(\alpha,\omega)\prod_{j\in I}\mathbb{I}\{\sigma_j(\nu_j,\hat{t}_{N_j})=\alpha_j\}\Pr_{\pi^\varphi,\mu}(\omega,t_{N_i}(t_{N_j})_{j\neq i})\hat{\varphi}(N|\nu_{N_i})$$
$$\geq \sum_{\omega\in\Omega,N\in\Phi,t\in T,\alpha_{-i}} u_i(a_i,\alpha_{-i},\omega)\prod_{j\in I}\mathbb{I}\{\sigma_j(\nu_j,\hat{t}_{N_j})=\alpha_j\}\Pr_{\pi^\varphi,\mu}(\omega,t_{N_i}(t_{N_j})_{j\neq i})\hat{\varphi}(N|\nu_{N_i}),$$

which is equivalent (once we explicitly write $\Pr_{\pi^\varphi,\mu}(\cdot)$) to

$$\sum_{\omega\in\Omega,N\in\Phi,\alpha_{-i}} u_i(\alpha,\omega)\left\{\sum_{t\in T}\prod_{j\in I}\mathbb{I}\{\sigma_j(\nu_j,\hat{t}_{N_j})=\alpha_j\}\pi_I(t_{N_i},t_{-N_i}|\omega)\mu(\omega)\right\}\hat{\varphi}(N|\nu_{N_i})$$
$$\geq \sum_{\omega\in\Omega,N\in\Phi,\alpha_{-i}} u_i(a_i,\alpha_{-i},\omega)\left\{\sum_{t\in T}\prod_{j\in I}\mathbb{I}\{\sigma_j(\nu_j,\hat{t}_{N_j})=\alpha_j\}\pi_I(t_{N_i},t_{-N_i}|\omega)\mu(\omega)\right\}\hat{\varphi}(N|\nu_{N_i}),$$

By knowing N_i and $\hat{t}_{N_i}$, player i also knows ν_j and $\hat{t}_{N_j}$ for all $j\in N_i$ and so also $\alpha_j=\sigma_j(\nu_j,\hat{t}_{N_j})$. That is, conditioning on $\hat{t}_{N_i}$ implies that α_{N_i} is fixed. Using the definition of x^φ, the last inequality requires that for all $N\in\Phi$, $i\in I$, $\alpha_{N_i}\in\mathbf{x}^\varphi_{N_i}$, and $a_i\in A_i$,

$$\sum_{\omega\in\Omega,\alpha_{-N_i}\in\mathbf{x}^\varphi_{-N_i},N\in\Phi}\Big(u_i(\alpha_i,\alpha_{-i},\omega)-u_i(a_i,\alpha_{-i},\omega)\Big)x^\varphi(\alpha_i,\alpha_{-i}|N,\omega)\mu(\omega)\hat{\varphi}(N|\nu_{N_i})\geq 0.$$

Case 2: Indirect Provision ($S\subsetneq I$). The inequality $V^*(G,\varphi,S)\leq\bar{V}(G,\varphi,I)$ follows from observing that every outcome x^φ the designer can induce with some π_S followed by $\sigma\in\mathrm{BNE}(G,f_{\xi^\varphi,N^\varphi}(\pi_S))$ is also feasible when she can choose a π_I, since $S\subsetneq I$. Therefore, $x^\varphi\in\bar{X}(G,\varphi,I)$ and, hence, the designer does weakly better when maximizing over $\bar{X}(G,\varphi,I)$ relative to the original problem.

Now consider the inequality $\underline{V}(G,\varphi,I)\leq V^*(G,\varphi,S)$. Let $\underline{x}^\varphi\in\underline{X}(G,\varphi,S)$ achieve $\underline{V}(G,\varphi,I)$.[16] It suffices to argue that there exists π_S and $\sigma\in\mathrm{BNE}(G,f_{\xi^\varphi,N^\varphi}(\pi_S))$ such that (π_S,σ) induces $\underline{x}^\varphi$, that is,

$$\underline{x}^\varphi(\alpha_1,\ldots,\alpha_I|\omega,N)=\sum_{t\in T}\pi_S(t,\omega)\prod_{i\in I}\mathbb{I}\{\sigma_i(\nu_i,\hat{t}_{N_i})=\alpha_i\},\quad (\alpha,N,\omega)\in R\times\Phi\times\Omega,$$

where $\hat{t}_{N_i}=(t_j,\nu_j)_{j\in N_i}$. Since $\underline{x}^\varphi\in X(G,N,S)$ for all $N\in\Phi$, $\underline{x}^\varphi$ can depend at most on the network aspects that are common to all $N\in\Phi$, namely, the set of connections in $\cap_{N\in\Phi}N$. Since $\cap_{N\in\Phi}N$ is non-random, $\underline{x}^\varphi$ statistically depends only on ω, i.e., $\underline{x}^\varphi:\Omega\to\Delta(R)$.

By Theorem 1 and Assumption 1, this implies that $\underline{x}^\varphi\in X(G,N^S,I)$ for all $N\in\Phi$.

[16] One can show the existence of a solution $\underline{x}^\varphi$ using an argument similar to the proof of Lemma 3.

By Theorem 2, there exists a pair $(\underline{\pi}_I, \sigma^{N^S})$ for every $N \in \Phi$ such that $\sigma^{N^S} \in \text{BNE}(G, f_{N^S}(\underline{\pi}_I))$ and the pair induces $\underline{x}^{\varphi}$. Namely, $\underline{\pi}_I = \underline{x}^{\varphi}$ and σ^{N^S} is obedient given $(G, f_{N^S}(\underline{\pi}_I))$ for all $N \in \Phi$. But then there also exists a common $\pi' \in \Pi_S$ that conveys as much information to every player as does $\underline{\pi}_I$ for every $N \in \Phi$ (by Theorem 1 and Assumption 1) and a profile $\sigma^N \in \text{BNE}(G, f_N(\pi'))$ such that, for every $N \in \Phi$,

$$\underline{x}^{\varphi}(\alpha_1, \ldots, \alpha_I|\omega) = \sum_{t \in T} \pi'(t|\omega) \prod_{i \in I} \mathbb{I}\{\sigma_i^N(t_{N_i}) = \alpha_i\}, \quad (\alpha, \omega) \in R \times \Omega.$$

Given this, we can let σ' be defined, for every $i \in I$ and $N \in \Phi$, by

$$\sigma_i'(\nu_i, \hat{t}_{N_i}) = \sigma_i^N(t_{N_i}), \quad t \in T.$$

By construction, σ' is measurable with respect to the information each player i can receive from π' for each realization of N. Also, for each N and t, the realization of (ν_i, t_{N_i}) provides player i weakly less information than (N, t_{N_i}). Therefore, if σ_i^N is a best reply for every N in $(G, f_N(\pi'))$, so is σ' in $(G, f_{\xi^{\varphi}, N^{\varphi}}(\pi'))$. That is, $\sigma \in \text{BNE}(G, f_{\xi^{\varphi}, N^{\varphi}}(\pi'))$.

□

C Analysis for the Application of Section 4.3

In this Appendix, we characterize the feasible outcomes depicted in Figure 5. For notational convenience, let $\gamma_i := 1 - \underline{c}_i$ and $\chi_i := \Pr(a_i = y)$. Given this, note that $0 < \gamma_2 < \gamma_1 < 1$ and $\varepsilon < \min\{\gamma_2, \frac{\gamma_1 - \gamma_2}{1 + \gamma_2}\}$. Moreover, we abuse notation and write y_i and n_i instead of $a_i = y$ and $a_i = n$, respectively. We begin by focusing on recommendation mechanisms that only recommend pure actions. In Section C.5, we show this is without loss of generality.

C.1 Empty Network: $N = \varnothing$

In this case, players observe only their own recommendations. Their obedience constraints given y_i and n_i are, respectively:

$$\begin{aligned}
\Big((\gamma_i - \varepsilon)x(y_i, y_{-i}|B) + \gamma_i x(y_i, n_{-i}|B)\Big)\mu(B) - \Big((1 + \varepsilon)x(y_i, y_{-i}|H) + x(y_i, n_{-i}|H)\Big)\mu(H) &\geq 0, \\
-\Big((\gamma_i - \varepsilon)x(n_i, y_{-i}|B) + \gamma_i x(n_i, n_{-i}|B)\Big)\mu(B) + \Big((1 + \varepsilon)x(n_i, y_{-i}|H) + x(n_i, n_{-i}|H)\Big)\mu(H) &\geq 0.
\end{aligned}$$

Letting $x(a_i|\omega) := x(a_i, y_{-i}|\omega) + x(a_i, n_{-i}|\omega)$ and using the fact that $x(\cdot|\omega)$ is a probability distribution, we can rewrite these two obedience constraints as:

$$\chi_i \leq (\gamma_i + 1)x(y_i|B)\mu(B) - \varepsilon\Big(x(y_1, y_2|B)\mu(B) + x(y_1, y_2|H)\mu(H)\Big) \tag{C.1}$$

and

$$\begin{aligned}
&\chi_i - \varepsilon\Big(x(y_{-i}|B)\mu(B) + x(y_{-i}|H)\mu(H)\Big) \\
\leq \quad &(\gamma_i + 1)x(y_i|B)\mu(B) - \varepsilon\Big(x(y_1, y_2|B)\mu(B) + x(y_1, y_2|H)\mu(H)\Big) + \mu(H) - \gamma_i\mu(B)
\end{aligned} \tag{C.2}$$

When $\mu(B) = \mu(H) = 1/2$, which we refer to as the low-prior case, (C.2) is slack because $\varepsilon > 0$; When $\gamma_2\mu(B) \geq \mu(H) + \varepsilon$, which we refer to as the high-prior case, (C.1) is slack. We will discuss the two cases separately below.

C.1.1 Low-Prior Case

We will prove that the four extreme points in Figure 5 (Left Panel) that refer to $N = \emptyset$ are $P_1 = (0, 0)$, $P_2 = (\frac{1+\gamma_1}{2}, 0)$, $P_3 = (0, \frac{1+\gamma_2}{2})$, and $P_4 = (\frac{1+\gamma_1-\varepsilon}{2}, \frac{1+\gamma_2-\varepsilon}{2})$. We first show that these points are feasible. To do so, note that the following recommendation mechanisms induce these probabilities and satisfy (C.1) hold.

- P_1: $x(n_1, n_2|B) = x(n_1, n_2|H) = 1$;
- P_2: $x(y_1, n_2|B) = 1, x(y_1, n_2|H) = \gamma_1, x(n_2|H) = 1$;
- P_3: $x(n_1, y_2|B) = 1, x(n_1, y_2|H) = \gamma_2, x(n_1|H) = 1$;
- P_4: $x(y_1, y_2|B) = 1$, $x(y_1, y_2|H) = 0$, $x(n_1, y_2|H) = \gamma_2 - \varepsilon$, $x(y_1, n_2|H) = \gamma_1 - \varepsilon$.

Next, we argue that the feasible set cannot be larger than the convex hull of these points. From (C.1), we have that:

$$\chi_1 \leq \frac{1 + \gamma_1 - \varepsilon}{2} x(y_1, y_2|B) + \frac{1 + \gamma_1}{2} x(y_1, n_2|B), \tag{C.3}$$

$$\begin{aligned} \chi_2 \leq& \frac{1 + \gamma_2 - \varepsilon}{2} x(y_1, y_2|B) + \frac{1 + \gamma_2}{2} x(n_1, y_2|B) \\ \leq& \frac{1 + \gamma_2}{2} - \frac{\varepsilon}{2} x(y_1, y_2|B) - \frac{1 + \gamma_2}{2} x(y_1, n_2|B). \end{aligned} \tag{C.4}$$

The last inequality uses the fact that $x(n_1, y_2|B) \leq 1 - x(y_1, y_2|B) - x(y_1, n_2|B)$.

Fixing any $0 \leq \chi_1 \leq \frac{1+\gamma_1}{2}$, we want to choose x to maximize the RHS of (C.4) while satisfying (C.3). Since ε is small (satisfying our maintained assumptions), to maximize the RHS of (C.4) while satisfying (C.3), one should first increase $x(y_1, y_2|B)$ until either (C.3) is satisfied or $x(y_1, y_2|B) = 1$. In the former case, the RHS of (C.4) is one line segment connecting P_3 and P_4; In the latter case, $x(y_1, n_2|B)$ should be increased until (C.3) is satisfied. In this case, the RHS of (C.4) is one line segment connecting P_2 and P_4.

C.1.2 High-Prior Case

We will prove that the four extreme points in Figure 5 (Right Panel) that refer to $N = \emptyset$ are $Q_1 = (1, 1)$, $Q_2 = (1, \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{\gamma_2} - \varepsilon)$, $Q_3 = (\frac{\gamma_1\mu(B)-\mu(H)-\varepsilon}{\gamma_1-\varepsilon}, 1)$, $Q_4 = (\frac{\mu(B)(\gamma_1-\varepsilon)-\mu(H)}{\gamma_1-\varepsilon}, \frac{\mu(B)(\gamma_2-\varepsilon)-\mu(H)}{\gamma_2-\varepsilon})$. The following recommendation mechanisms induce these probabilities and satisfy (C.2).

- Q_1: $x(y_1, y_2|B) = x(y_1, y_2|H) = 1$;

– Q_2: $x(y_1, n_2|H) = 1$, $x(y_1, y_2|B) = \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{(\gamma_2-\varepsilon)\mu(B)}$, $x(y_1|B) = 1$;[17]

– Q_3: $x(n_1, y_2|H) = 1$, $x(y_1, y_2|B) = \frac{\gamma_1\mu(B)-\mu(H)-\varepsilon}{(\gamma_1-\varepsilon)\mu(B)}$, $x(y_2|B) = 1$;

– Q_4: $x(n_1, n_2|H) = 1$, $x(n_1, n_2|B) = 0$, $x(n_i|B) = \frac{\mu(H)}{\mu(B)(\gamma_i-\varepsilon)}$.

To show these are the extreme points, rewrite (C.2) as follows:

$$\begin{aligned}(\gamma_i - \varepsilon)(1 - \chi_i) &\leq (1 + \gamma_i)x(n_i|H)\mu(H) - \varepsilon\Big(x(n_1, n_2|B)\mu(B) + x(n_1, n_2|H)\mu(H)\Big)\\ &\leq (1 + \gamma_i)x(n_i|H)\mu(H) - \varepsilon x(n_1, n_2|H)\mu(H),\end{aligned}$$

which implies:

$$1 - \chi_1 \leq \frac{(1 + \gamma_1)\mu(H)}{\gamma_1 - \varepsilon}x(n_1, y_2|H) + \frac{(1 + \gamma_1 - \varepsilon)\mu(H)}{\gamma_1 - \varepsilon}x(n_1, n_2|H), \tag{C.5}$$

$$\begin{aligned}1 - \chi_2 \leq &\frac{(1 + \gamma_2)\mu(H)}{\gamma_2 - \varepsilon}x(y_1, n_2|H) + \frac{(1 + \gamma_2 - \varepsilon)\mu(H)}{\gamma_2 - \varepsilon}x(n_1, n_2|H)\\ \leq &\frac{(1 + \gamma_2)\mu(H)}{\gamma_2 - \varepsilon} - \frac{(1 + \gamma_2)\mu(H)}{\gamma_2 - \varepsilon}x(n_1, y_2|H) - \frac{\varepsilon\mu(H)}{\gamma_2 - \varepsilon}x(n_1, n_2|H).\end{aligned} \tag{C.6}$$

Fixing any $1 \geq \chi_1 \geq \frac{\gamma_1\mu(B)-\mu(H)-\varepsilon}{\gamma_1-\varepsilon}$, we want to choose x to maximize the RHS of (C.6) while satisfying (C.5). Since ε is small (satisfying our maintained assumptions), to maximize the RHS of (C.6) while satisfying (C.5), one should first increase $x(n_1, n_2|H)$ until either (C.5) is satisfied or $x(y_1, y_2|B) = 1$. In the former case, the RHS of (C.6) is one line segment connecting Q_3 and Q_4; In the latter case, $x(n_1, y_2|B)$ should be increased until (C.5) is satisfied. In this case, the RHS of (C.6) is one line segment connecting Q_2 and Q_4.

C.2 Partial Network: $N = \{(1, 2)\}$

In this case, the obedience constraints for player 1 are still (C.1) and (C.2). Player 2's obedient constraints, instead, are:

$$\begin{aligned}(\gamma_2 - \varepsilon)x(y_1, y_2|B)\mu(B) - (1 + \varepsilon)x(y_1, y_2|H)\mu(H) &\geq 0\\ \gamma_2 x(n_1, y_2|B)\mu(B) - x(n_1, y_2|H)\mu(H) &\geq 0\\ -(\gamma_2 - \varepsilon)x(y_1, n_2|B)\mu(B) + (1 + \varepsilon)x(y_1, n_2|H)\mu(H) &\geq 0\\ -\gamma_2 x(n_1, n_2|B)\mu(B) + x(n_1, n_2|H)\mu(H) &\geq 0\end{aligned} \tag{C.7}$$

[17]We note that $0 \leq \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{(\gamma_2-\varepsilon)\mu(B)} \leq 1$ because of our restriction $\varepsilon \leq \gamma_i\mu(B) - \mu(H)$ and $\varepsilon > 0$.

C.2.1 Low-Prior Case

We will prove that the four extreme points in Figure 5 (Left Panel) when $N = \{(1,2)\}$ are $P_1 = (0,0)$, $P_2 = (\frac{1+\gamma_1}{2}, 0)$, $P_3 = (0, \frac{1+\gamma_2}{2})$, and $P_5 = (\frac{1+\gamma_1-\varepsilon}{2} - \frac{\varepsilon(\gamma_2-\varepsilon)}{2(1+\varepsilon)}, \frac{1+\gamma_2}{2(1+\varepsilon)})$. The constructions for P_1, P_2, and P_3 are the same as in Section C.1.1. To induce P_5, instead, let $x(y_1, y_2|B) = 1$, $x(y_1, y_2|H) = \frac{\gamma_2-\varepsilon}{1+\varepsilon}$, $x(n_1, y_2|H) = 0$, and $x(y_1, n_2|H) = \gamma_1 - \gamma_2$.

Next, we show that the feasible set is no larger than the convex hull of these points. For player 1, (C.1) implies

$$\chi_1 \leq \frac{1+\gamma_1}{2} x(y_1, n_2|B) + \frac{1+\gamma_1-\varepsilon}{2} x(y_1, y_2|B) - \frac{\varepsilon}{2} x(y_1, y_2|H). \tag{C.8}$$

For player 2, the first two inequalities of (C.7) imply

$$x(y_1, y_2|H) \leq \frac{\gamma_2 - \varepsilon}{1+\varepsilon} x(y_1, y_2|B), \tag{C.9}$$

$$\begin{aligned} \chi_2 \leq& \frac{1+\gamma_2}{2} x(n_1, y_2|B) + \frac{1}{2} x(y_1, y_2|B) + \frac{1}{2} x(y_1, y_2|H) \\ \leq& \frac{1+\gamma_2}{2} - \frac{1+\gamma_2}{2} x(y_1, n_2|B) - \frac{\gamma_2}{2} x(y_1, y_2|B) + \frac{1}{2} x(y_1, y_2|H). \end{aligned} \tag{C.10}$$

Fixing any $0 \leq \chi_1 \leq \frac{1+\gamma_1}{2}$, we want to choose x to maximize the RHS of (C.10) while satisfying (C.8). First, we observe that it is optimal to choose x such that (C.9) binds. This is because if (C.9) does not bind, we can increase $x(y_1, y_2|H)$ by a small amount, and then decrease $x(y_1, y_2|B)$ and increase $x(y_1, n_2|B)$ by the same amount. In this way, the RHS of both (C.8) and (C.10) will remain unchanged. We can plug the binding constraint (C.9) in (C.8) and (C.10) to obtain:

$$\chi_1 \leq \frac{1+\gamma_1}{2} x(y_1, n_2|B) + (\frac{1+\gamma_1-\varepsilon}{2} - \frac{\varepsilon(\gamma_2-\varepsilon)}{2(1+\varepsilon)}) x(y_1, y_2|B), \tag{C.11}$$

$$\chi_2 \leq \frac{1+\gamma_2}{2} - \frac{1+\gamma_2}{2} x(y_1, n_2|B) - \varepsilon \frac{1+\gamma_2}{2(1+\varepsilon)} x(y_1, y_2|B). \tag{C.12}$$

Since ε is small (satisfying our maintained assumptions), to maximize the RHS of (C.12) while satisfying (C.11), one should first increase $x(y_1, y_2|B)$ until either (C.11) is satisfied or $x(y_1, y_2|B) = 1$. In the former case, the RHS of (C.12) is one line segment connecting P_3 and P_5; In the latter case, $x(y_1, n_2|B)$ should be increased until (C.11) is satisfied. In this case, the RHS of (C.12) is one line segment connecting P_2 and P_5.

C.2.2 High-Prior Case

We will prove that the four extreme points in Figure 5 (Right Panel) when $N = \{(1,2)\}$ are $Q_1 = (1,1)$, $Q_2 = (1, \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{\gamma_2} - \varepsilon)$, $Q_3 = (\frac{\gamma_1\mu(B)-\mu(H)-\varepsilon}{\gamma_1-\varepsilon}, 1)$, and $Q_5 = (1-(1+\frac{1}{\gamma_1})\mu(H), 1-(1+\frac{1}{\gamma_1})\mu(H))$. The constructions for points Q_1, Q_2, and Q_3 are the same as in Section C.1.2. To induce Q_5, instead, let $x(n_1, n_2|H) = 1$, $x(n_1, n_2|B) = \frac{\mu(H)}{\gamma_1\mu(B)}$, $x(y_1, n_2|B) = x(n_1, y_2|B) = 0$.

Next, we show that the feasible set is no larger than the convex hull of these points. From (C.2), we obtain,

$$\chi_1 \geq 1 - \frac{1+\gamma_1}{\gamma_1-\varepsilon} x(n_1, y_2|H)\mu(H) - \frac{1+\gamma_1-\varepsilon}{\gamma_1-\varepsilon} x(n_1, n_2|H)\mu(H) + \frac{\varepsilon}{\gamma_1-\varepsilon} x(n_1, n_2|B)\mu(B). \quad \text{(C.13)}$$

Moreover, notice that $\chi_1 \leq 1 - x(n_1, n_2|B)\mu(B) - x(n_1, n_2|H)\mu(H)$. Putting these two constraints together, we get:

$$x(n_1, n_2|B)\mu(B) \leq \frac{1}{\gamma_1} x(n_1, n_2|H)\mu(H). \quad \text{(C.14)}$$

From the third inequality of (C.7) we get:

$$\begin{aligned}\chi_2 \geq & 1 - \frac{1+\gamma_2}{\gamma_2-\varepsilon} x(y_1, n_2|H)\mu(H) - x(n_1, n_2|B)\mu(B) - x(n_1, n_2|H)\mu(H) \\ \geq & 1 - \frac{1+\gamma_2}{\gamma_2-\varepsilon}\mu(H) + \frac{1+\gamma_2}{\gamma_2-\varepsilon} x(n_1, y_2|H)\mu(H) + \frac{1+\varepsilon}{\gamma_2-\varepsilon} x(n_1, n_2|H)\mu(H) - x(n_1, n_2|B)\mu(B),\end{aligned} \quad \text{(C.15)}$$

Fixing any $1 \geq \chi_1 \geq \frac{\gamma_1\mu(B)-\mu(H)-\varepsilon}{\gamma_1-\varepsilon}$, we want to choose x to minimize the RHS of (C.15) while satisfying (C.13). First, we observe that it is optimal to choose x such that (C.14) binds. This is because if (C.14) does not bind, we can increase $x(n_1, n_2|B)\mu(B)$ by a small amount, and then decrease $x(n_1, n_2|H)\mu(H)$ and increase $x(n_1, y_2|H)\mu(H)$ by the same amount. In this way, the RHS of both (C.13) and (C.15) will remain unchanged. We can plug the binding constraint (C.14) in (C.13) and (C.15) to obtain:

$$\chi_1 \geq 1 - \frac{1+\gamma_1}{\gamma_1-\varepsilon} x(n_1, y_2|H)\mu(H) - \left(\frac{1+\gamma_1-\varepsilon}{\gamma_1-\varepsilon} - \frac{\varepsilon}{\gamma_1(\gamma_1-\varepsilon)}\right) x(n_1, n_2|H)\mu(H), \quad \text{(C.16)}$$

$$\chi_2 \geq 1 - \frac{1+\gamma_2}{\gamma_2-\varepsilon}\mu(H) + \frac{1+\gamma_2}{\gamma_2-\varepsilon} x(n_1, y_2|H)\mu(H) + \left(\frac{1+\varepsilon}{\gamma_2-\varepsilon} - \frac{1}{\gamma_1}\right) x(n_1, n_2|H)\mu(H), \quad \text{(C.17)}$$

Since ε is small (satisfying our maintained assumptions), to minimize the RHS of (C.17) while satisfying (C.16), one should first increase $x(n_1, n_2|H)$ until either (C.16) is satisfied or $x(n_1, n_2|H) = 1$. In the former case, the RHS of (C.17) is one line segment connecting Q_2 and Q_5; In the latter case, $x(n_1, y_2|H)$ should be increased until (C.16) is satisfied. In this case, the RHS of (C.12) is one line segment connecting Q_3 and Q_5.

C.3 Partial Network: $N = \{(2, 1)\}$

In this case, the obedience constraints for player 2 are still (C.1) and (C.2). Player 1's obedient constraints, instead, are:

$$\begin{aligned}(\gamma_1-\varepsilon)x(y_1, y_2|B)\mu(B) - (1+\varepsilon)x(y_1, y_2|H)\mu(H) &\geq 0 \\ \gamma_1 x(y_1, n_2|B)\mu(B) - x(y_1, n_2|H)\mu(H) &\geq 0 \\ -(\gamma_1-\varepsilon)x(n_1, y_2|B)\mu(B) + (1+\varepsilon)x(n_1, y_2|H)\mu(H) &\geq 0 \\ -\gamma_1 x(n_1, n_2|B)\mu(B) + x(n_1, n_2|H)\mu(H) &\geq 0\end{aligned} \quad \text{(C.18)}$$

C.3.1 Low-Prior Case

We will prove that the four extreme points in Figure 5 (Left Panel) when $N = \{(2,1)\}$ are $P_1 = (0,0)$, $P_2 = (\frac{1+\gamma_1}{2}, 0)$, $P_3 = (0, \frac{1+\gamma_2}{2})$, and $P_6 = (\frac{1+\gamma_2}{2(1+\varepsilon)}, \frac{1+\gamma_2}{2(1+\varepsilon)})$. The constructions for points P_1, P_2, and P_3 are the same as in Section C.1.1. To induce P_6, instead, let $x(y_1, y_2|B) = 1$, $x(y_1, y_2|H) = \frac{\gamma_2-\varepsilon}{1+\varepsilon}$, and $x(y_1, n_2|H) = x(n_1, y_2|H) = 0$.

Next, we show that the feasible set is no larger than the convex hull of these points. For player 2, (C.1) implies

$$\chi_2 \leq \frac{1+\gamma_2}{2} x(n_1, y_2|B) + \frac{1+\gamma_2-\varepsilon}{2} x(y_1, y_2|B) - \frac{\varepsilon}{2} x(y_1, y_2|H). \tag{C.19}$$

Moreover, non-negativity of $x(n_1, y_2|\omega)$ implies that $\frac{1}{2}(x(y_1, y_2|B) + x(y_1, y_2|H)) \leq \chi_2$. Putting these two constraints together, we obtain:

$$x(y_1, y_2|H) \leq \frac{\gamma_2-\varepsilon}{1+\varepsilon} x(y_1, y_2|B). \tag{C.20}$$

The second inequality of (C.18) implies:

$$\begin{aligned} \chi_1 \leq & \frac{1+\gamma_1}{2} x(y_1, n_2|B) + \frac{1}{2} x(y_1, y_2|B) + \frac{1}{2} x(y_1, y_2|H) \\ \leq & \frac{1+\gamma_1}{2} - \frac{1+\gamma_1}{2} x(n_1, y_2|B) - \frac{\gamma_1}{2} x(y_1, y_2|B) + \frac{1}{2} x(y_1, y_2|H). \end{aligned} \tag{C.21}$$

Fixing any $0 \leq \chi_2 \leq \frac{1+\gamma_2}{2}$, we want to choose x to maximize the RHS of (C.21) while satisfying (C.19). First, we observe that it is optimal to choose x such that (C.20) binds. This is because if (C.20) does not bind, we can increase $x(y_1, y_2|H)$ by a small amount, and then decrease $x(y_1, y_2|B)$ and increase $x(n_1, y_2|B)$ by the same amount. In this way, the RHS of both (C.19) and (C.21) will remain unchanged. We can plug the binding constraint (C.20) in (C.19) and (C.21) to obtain:

$$\chi_2 \leq \frac{1+\gamma_2}{2} x(n_1, y_2|B) + \frac{1+\gamma_2}{2(1+\varepsilon)} x(y_1, y_2|B), \tag{C.22}$$

$$\chi_1 \leq \frac{1+\gamma_1}{2} - \frac{1+\gamma_1}{2} x(n_1, y_2|B) - (\frac{\gamma_1}{2} - \frac{\gamma_2-\varepsilon}{2(1+\varepsilon)}) x(y_1, y_2|B). \tag{C.23}$$

Since ε is small (satisfying our maintained assumptions), to maximize the RHS of (C.23) while satisfying (C.22), one should first increase $x(y_1, y_2|B)$ until either (C.22) is satisfied or $x(y_1, y_2|B) = 1$. In the former case, the RHS of (C.23) is one line segment connecting P_2 and P_6; In the latter case, $x(n_1, y_2|B)$ should be increased until (C.22) is satisfied. In this case, the RHS of (C.23) is one line segment connecting P_3 and P_6.

C.3.2 High-Prior Case

We will prove that the four extreme points in Figure 5 (Right Panel) when $N = \{(2,1)\}$ are $Q_1 = (1,1)$, $Q_2 = (1, \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{\gamma_2} - \varepsilon)$, $Q_3 = (\frac{\gamma_1\mu(B)-\mu(H)-\varepsilon}{\gamma_1-\varepsilon}, 1)$, and $Q_6 = (1-(1+\frac{1}{\gamma_1})\mu(H), 1-(\frac{1+\gamma_2-\varepsilon}{\gamma_2-\varepsilon} -$

$\frac{\varepsilon}{\gamma_1(\gamma_2-\varepsilon)})\mu(H))$. The constructions for points Q_1, Q_2, and Q_3 are the same as in Section C.1.2. To induce Q_6, instead, let x be such that $x(n_1, n_2|H) = 1$, $x(n_1, n_2|B) = \frac{\mu(H)}{\gamma_1\mu(B)}$, $x(n_1, y_2|B) = 0$, and $x(y_1, n_2|B) = \frac{(\gamma_1-\gamma_2)\mu(H)}{\gamma_1(\gamma_2-\varepsilon)\mu(B)}$.

Next, we show that the feasible set is no larger than the convex hull of these points. From (C.2), we get:

$$\chi_2 \geq 1 - \frac{1+\gamma_2}{\gamma_2-\varepsilon}x(y_1, n_2|H)\mu(H) - \frac{1+\gamma_2-\varepsilon}{\gamma_2-\varepsilon}x(n_1, n_2|H)\mu(H) + \frac{\varepsilon}{\gamma_2-\varepsilon}x(n_1, n_2|B)\mu(B). \tag{C.24}$$

From the last two inequalities of (C.18) we get:

$$\begin{aligned}\chi_1 \geq& 1 - \frac{1+\gamma_1}{\gamma_1-\varepsilon}x(n_1, y_2|H)\mu(H) - x(n_1, n_2|B)\mu(B) - x(n_1, n_2|H)\mu(H)\\ \geq& 1 - \frac{1+\gamma_1}{\gamma_1-\varepsilon}\mu(H) + \frac{1+\gamma_1}{\gamma_1-\varepsilon}x(y_1, n_2|H)\mu(H) + \frac{1+\varepsilon}{\gamma_1-\varepsilon}x(n_1, n_2|H)\mu(H) - x(n_1, n_2|B)\mu(B),\end{aligned} \tag{C.25}$$

$$x(n_1, n_2|B)\mu(B) \leq \frac{1}{\gamma_1}x(n_1, n_2|H)\mu(H). \tag{C.26}$$

Fixing any $1 \geq \chi_2 \geq \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{\gamma_2-\varepsilon}$, we want to choose x to minimize the RHS of (C.25) while satisfying (C.24). First, we observe that it is optimal to choose x such that (C.26) binds. This is because if (C.26) does not bind, we can increase $x(n_1, n_2|B)\mu(B)$ by a small amount, and then decrease $x(n_1, n_2|H)\mu(H)$ and increase $x(y_1, n_2|H)\mu(H)$ by the same amount. In this way, the RHS of both (C.24) and (C.25) will remain unchanged. We can plug the binding constraint (C.26) in (C.24) and (C.25) to obtain:

$$\chi_2 \geq 1 - \frac{1+\gamma_2}{\gamma_2-\varepsilon}x(y_1, n_2|H)\mu(H) - (\frac{1+\gamma_2-\varepsilon}{\gamma_2-\varepsilon} - \frac{\varepsilon}{\gamma_1(\gamma_2-\varepsilon)})x(n_1, n_2|H)\mu(H), \tag{C.27}$$

$$\chi_1 \geq 1 - \frac{1+\gamma_1}{\gamma_1-\varepsilon}\mu(H) + \frac{1+\gamma_1}{\gamma_1-\varepsilon}x(y_1, n_2|H)\mu(H) + \frac{\varepsilon(1+\gamma_1)}{\gamma_1(\gamma_1-\varepsilon)}x(n_1, n_2|H)\mu(H), \tag{C.28}$$

Since ε is small (satisfying our maintained assumptions), to minimize the RHS of (C.28) while satisfying (C.27), one should first increase $x(n_1, n_2|H)$ until either (C.27) is satisfied or $x(n_1, n_2|H) = 1$. In the former case, the RHS of (C.28) is one line segment connecting Q_3 and Q_6; In the latter case, $x(y_1, n_2|H)$ should be increased until (C.27) is satisfied. In this case, the RHS of (C.23) is one line segment connecting Q_2 and Q_6.

C.4 Complete Network: $N = \{(1, 2), (2, 1)\}$

In this case, the obedience constraints are (C.7) and (C.18). We make several observations to reduce the inequalities:

- The first inequality of (C.7) implies the first of (C.18);
- The last inequality of (C.18) implies the last of (C.7);

- Under our assumption on ε, the second inequality of (C.7) and the third inequality of (C.18) imply $x(n_1, y_2|B) = x(n_1, y_2|H) = 0$.

Since $x(n_1, y_2|B) = x(n_1, y_2|H) = 0$, player 1 has to take action y whenever player 2 does so. Therefore, we conclude $\chi_2 \leq \chi_1$ by definition.

C.4.1 Low-Prior Case

We will prove that the three extreme points in Figure 5 (Left Panel) when $N = I^2$ are $P_1 = (0, 0)$, $P_2 = (\frac{1+\gamma_1}{2}, 0)$, and $P_6 = (\frac{1+\gamma_2}{2(1+\varepsilon)}, \frac{1+\gamma_2}{2(1+\varepsilon)})$. The constructions are the same as Section C.3.1.

Next, we argue the feasible set is no larger than the convex hull of these points. From the first inequality of (C.7) and the second inequality of (C.18) we have:

$$\chi_2 \leq \frac{1+\gamma_2}{2(1+\varepsilon)} x(y_1, y_2|B)$$

$$\begin{aligned}\chi_1 &\leq \frac{1+\gamma_2}{2(1+\varepsilon)} x(y_1, y_2|B) + \frac{1+\gamma_1}{2} x(y_1, n_2|B) \\ &\leq \frac{1+\gamma_1}{2} - (\frac{1+\gamma_1}{2} - \frac{1+\gamma_2}{2(1+\varepsilon)}) x(y_1, y_2|B)\end{aligned}$$

Therefore, the feasible set of (χ_1, χ_2) must be a subset of the convex hull of P_1, P_2, and P_6.

C.4.2 High-Prior Case

We will prove that the three extreme points in Figure 5 (Right Panel) when $N = I^2$ are $Q_1 = (1, 1)$, $Q_2 = (1, \frac{\gamma_2\mu(B)-\mu(H)-\varepsilon}{\gamma_2} - \varepsilon)$, $Q_6 = (1 - (1 + \frac{1}{\gamma_1})\mu(H), 1 - (1 + \frac{1}{\gamma_1})\mu(H))$. The constructions are the same as Section C.2.2.

Next, we argue the feasible set is no larger than the convex hull of these points. From the last inequality of (C.18) and the third inequality of (C.7) we have:

$$1 - \chi_1 \leq (1 + \frac{1}{\gamma_1}) x(n_1, n_2|H)\mu(H)$$

$$\begin{aligned}1 - \chi_2 &\leq (1 + \frac{1}{\gamma_1}) x(n_1, n_2|H)\mu(H) + \frac{1+\gamma_2}{\gamma_2 - \varepsilon} x(y_1, n_2|H)\mu(H) \\ &\leq \frac{1+\gamma_2}{\gamma_2 - \varepsilon}\mu(H) + (1 + \frac{1}{\gamma_1} - \frac{1+\gamma_2}{\gamma_2 - \varepsilon}) x(n_1, n_2|H)\mu(H)\end{aligned}$$

Therefore, the feasible set of (χ_1, χ_2) must be a subset of the convex hull of Q_1, Q_2, and Q_6.

C.5 Reduction to Pure Recommendation

To complete the derivation, we need to show that it is without loss of generality to focus on pure-action recommendations. For notational convenience, denote by $\psi(\alpha, \omega) := x(\alpha|\omega)\mu(\omega)$ the joint distribution over (α, ω) induced by an outcome x.

We first point out that when the information of a player is private, it is without loss to recommend pure actions to that player. Specifically, let $R \subseteq I$ be the set of players whose information is private—that is, $R = \{i : (i, j) \notin N, \forall j \in I, j \neq i\}$.[18] We have:

Lemma 8. *Fix any feasible outcome x^*. There exists an obedient recommendation x such that (i) players in R are recommended pure actions and (ii) x and x^* induce the same joint distribution over (a, ω) as x^*.*

Proof. Fix any obedient recommendation $x^*(\alpha|\omega)$, where α is a mixed action profile. We define a new recommendation x where players in R are recommended pure actions. Let

$$x(a_R, \alpha_{-R}|\omega) := \sum_{\alpha_R \in \text{supp}_R x^*} \alpha_R(a_R) x^*(\alpha_R, \alpha_{-R}|\omega),$$

where a_R and a_{-R} denote the action profile of players in R and $I \backslash R$, respectively. By definition, x induces the same joint distribution over (a, ω). Next we show x is still an obedient recommendation.

Since x^* is obedient, it holds that for every player i, for all α_{N_i}, a_i, a_i' such that $\alpha_i(a_i) > 0$:[19]

$$\begin{aligned} &\sum_{\alpha_{-N_i}, \omega} \sum_{a_{-i}} u_i(a_i, a_{-i}, \omega) \alpha_{-i}(a_{-i}) \psi^*(\alpha_{N_i}, \alpha_{-N_i}, \omega) \\ \geq &\sum_{\alpha_{-N_i}, \omega} \sum_{a_{-i}} u_i(a_i', a_{-i}, \omega) \alpha_{-i}(a_{-i}) \psi^*(\alpha_{N_i}, \alpha_{-N_i}, \omega) \end{aligned} \tag{C.29}$$

For player $i \notin R$, we have $R \subset -N_i$, so in (C.29) we can first sum over α_R, and then over a_R, to get for all α_{N_i}, a_i, a_i' such that $\alpha_i(a_i) > 0$:

$$\begin{aligned} &\sum_{\alpha_{-N_i \backslash R}, \omega} \sum_{a_{-i}} u_i(a_i, a_{-i}, \omega) \alpha_{-i \backslash R}(a_{-i \backslash R}) \psi(\alpha_{N_i}, \alpha_{-N_i \backslash R}, a_R, \omega) \\ \geq &\sum_{\alpha_{-N_i \backslash R}, \omega} \sum_{a_{-i}} u_i(a_i', a_{-i}, \omega) \alpha_{-i \backslash R}(a_{-i \backslash R}) \psi(\alpha_{N_i}, \alpha_{-N_i \backslash R}, a_R, \omega), \end{aligned}$$

which implies that players not in R are obedient. For all $i \in R$, we can sum (C.29) over $\alpha_i \in \text{supp}_i \psi^*$ to get:

$$\begin{aligned} &\sum_{\alpha_i} \sum_{\alpha_{-N_i}, \omega} \sum_{a_{-i}} \alpha_i(a_i) u_i(a_i, a_{-i}, \omega) \alpha_{-i}(a_{-i}) \psi^*(\alpha_{N_i}, \alpha_{-N_i}, \omega) \\ \geq &\sum_{\alpha_i} \sum_{\alpha_{-N_i}, \omega} \sum_{a_{-i}} \alpha_i(a_i) u_i(a_i', a_{-i}, \omega) \alpha_{-i}(a_{-i}) \psi^*(\alpha_{N_i}, \alpha_{-N_i}, \omega) \end{aligned}$$

which can be written as:

$$\begin{aligned} &\sum_{\alpha_{-N_i \backslash R}, \omega} \sum_{a_{-i}} u_i(a_i, a_{-i}, \omega) \alpha_{-R}(a_{-R}) \psi(\alpha_{N_i \backslash i}, \alpha_{-N_i \backslash R}, a_R, \omega) \\ \geq &\sum_{\alpha_{-N_i \backslash R}, \omega} \sum_{a_{-i}} u_i(a_i', a_{-i}, \omega) \alpha_{-R}(a_{-R}) \psi(\alpha_{N_i \backslash i}, \alpha_{-N_i \backslash R}, a_R, \omega). \end{aligned}$$

This implies player i is obedient under x. □

[18] Recall that when $S = I$, $N^S = N$. Therefore, feasibility is determined by N.

[19] For brevity, we write $\sum_{\alpha_{N_i} \in \text{supp} N_i \psi^*}$ as $\sum_{\alpha_{N_i}}$ in what follows.

An immediate and well-known consequence of Lemma 8 is that, since $N = \varnothing$ implies $R = I$, it is without loss to focus on pure recommendations in this case. In the rest of this section, we shall prove a similar result for the other networks. Before that, we first make a simple observation whose proof is immediate, hence omitted: Convex combinations of obedient recommendations are obedient.

Remark 1. *Suppose $\{\psi^*_\lambda(\alpha, \omega)\}_{\lambda\in\Lambda}$ is a finite family of obedient distributions for some network and q is a probability distribution on Λ. Then*

$$\psi(\alpha, \omega) := \sum_{\Lambda} \psi^*_\lambda(\alpha, \omega) q(\lambda)$$

is also an obedient distribution under that network.

With Lemma 8 and Remark 1, we are ready to show that, as far as the joint distribution of (a_1, a_2) is concerned, it is without loss to focus on pure recommendations when ε is small. We will show this separately for the partial networks and the complete network.

C.5.1 Partial Networks: $N = \{(i, -i)\}$

By Lemma 8, it is without loss of generality to consider recommendations of the form of $x^*(\alpha_i, a_{-i}|\omega)$. From now on, let's focus on a particular α_i and use $\psi^*_{\alpha_i}$ to denote the conditional distribution over (a_{-i}, ω). Abusing notation, we will use α_i to denote the probability on action y.

We first state a useful lemma:

Lemma 9. *If player $-i$ is obedient under $\psi^*_{\alpha_i}$, then $\psi^*_{\alpha_i}$ can be decomposed into $\alpha_i\psi(y_i, a_{-i}, \omega) + (1 - \alpha_i)\psi(n_i, a_{-i}, \omega)$ such that ψ is a well-defined joint distribution that preserves the marginals of ω and (a_i, a_{-i}), and player $-i$ is obedient after both (y_i, a_{-i}) and (n_i, a_{-i}).*

Proof. (Case 1: $a_{-i} = y$). Let's first focus on the case $a_{-i} = y$. The fact that y_{-i} is obedient under $\psi^*_{\alpha_i}$ means

$$\frac{\psi^*_{\alpha_i}(y_{-i}, H)}{\psi^*_{\alpha_i}(y_{-i}, B)} \leq \frac{\gamma_{-i} - \alpha_i\varepsilon}{1 + \alpha_i\varepsilon}. \tag{C.30}$$

We want to find ψ_{α_i} such that:

$$\frac{\psi_{\alpha_i}(y_i, y_{-i}, H)}{\psi_{\alpha_i}(y_i, y_{-i}, B)} \leq \frac{\gamma_{-i} - \varepsilon}{1 + \varepsilon}$$
$$\frac{\psi_{\alpha_i}(n_i, y_{-i}, H)}{\psi_{\alpha_i}(n_i, y_{-i}, B)} \leq \gamma_{-i}$$

and ψ_{α_i} is a well-defined joint distribution that preserves the marginals over ω and (a_i, a_{-i}). To do this, we first define ψ_{α_i} without changing the likelihood ratio of the ω's:

$$\psi_{\alpha_i}(y_i, y_{-i}, \omega) := \alpha_i\psi^*_{\alpha_i}(y_{-i}, \omega), \psi_{\alpha_i}(n_i, y_{-i}, \omega) := (1 - \alpha_i)\psi^*_{\alpha_i}(y_{-i}, \omega).$$

If player $-i$ is already obedient, there is nothing to prove.

Therefore, suppose not—that is, after (y_i, y_{-i}) player $-i$ finds it suboptimal to choose y. In other words:

$$\frac{\psi^*_{\alpha_i}(y_{-i}, H)}{\psi^*_{\alpha_i}(y_{-i}, B)} > \frac{\gamma_{-i} - \varepsilon}{1 + \varepsilon}.$$

Now we choose $0 < \Delta \leq \alpha_i \psi^*_{\alpha_i}(y_{-i}, H)$ such that

$$\frac{\psi_{\alpha_i}(y_i, y_{-i}, H)}{\psi_{\alpha_i}(y_i, y_{-i}, B)} := \frac{\alpha_i \psi^*_{\alpha_i}(y_{-i}, H) - \Delta}{\alpha_i \psi^*_{\alpha_i}(y_{-i}, B) + \Delta} = \frac{\gamma_{-i} - \varepsilon}{1 + \varepsilon}. \tag{C.31}$$

Intuitively, we compensate player $-i$ such that after receiving (y_i, y_{-i}) she is indifferent between y and n. Our last piece is to show

$$\frac{\psi_{\alpha_i}(n_i, y_{-i}, H)}{\psi_{\alpha_i}(n_i, y_{-i}, B)} := \frac{(1 - \alpha_i)\psi^*_{\alpha_i}(y_{-i}, H) + \Delta}{(1 - \alpha_i)\psi^*_{\alpha_i}(y_{-i}, B) - \Delta} \leq \gamma_{-i} \tag{C.32}$$

Note that since we are using the same Δ, the marginals of ω and (a_i, a_{-i}) are preserved.

From (C.31) we know that:

$$(\gamma_{-i} + 1)\Delta = (1 + \varepsilon)\alpha_i \psi^*_{\alpha_i}(y_{-i}, H) - (\gamma_{-i} - \varepsilon)\alpha_i \psi^*_{\alpha_i}(y_{-i}, B)$$

Plugging this into (C.32), we can derive that our desired inequality is exactly (C.30), which finishes the last piece.

We still need to check $(1 - \alpha_i)\psi^*_{\alpha_i}(y_{-i}, B) - \Delta \geq 0$. By (C.30), we know

$$(\gamma_{-i} + 1)\Delta \leq ((1 + \varepsilon)\alpha_i \frac{\gamma_{-i} - \alpha_i \varepsilon}{1 + \alpha_i \varepsilon} - (\gamma_{-i} - \varepsilon)\alpha_i)\psi^*_{\alpha_i}(y_{-i}, B) = \frac{\alpha_i \varepsilon(\gamma_{-i} + 1)(1 - \alpha_i)}{1 + \alpha_i \varepsilon}\psi^*_{\alpha_i}(y_{-i}, B),$$

which simplifies to:

$$\Delta \leq (1 - \alpha_i)\frac{\alpha_i \varepsilon}{1 + \alpha_i \varepsilon}\psi^*_{\alpha_i}(y_{-i}, B).$$

This implies $\Delta \leq (1 - \alpha_i)\psi^*_{\alpha_i}(y_{-i}, B)$ as $\varepsilon > 0$.

(Case 2: $a_{-i} = n$). The argument for this case is almost the same as before with all inequalities reversed. The fact that n_{-i} is obedient under $\psi^*_{\alpha_i}$ means:

$$\frac{\psi^*_{\alpha_i}(n_{-i}, H)}{\psi^*_{\alpha_i}(n_{-i}, B)} \geq \frac{\gamma_{-i} - \alpha_i \varepsilon}{1 + \alpha_i \varepsilon}. \tag{C.33}$$

We want to find ψ_{α_i} such that:

$$\frac{\psi_{\alpha_i}(y_i, n_{-i}, H)}{\psi_{\alpha_i}(y_i, n_{-i}, B)} \geq \frac{\gamma_{-i} - \varepsilon}{1 + \varepsilon}$$
$$\frac{\psi_{\alpha_i}(n_i, n_{-i}, H)}{\psi_{\alpha_i}(n_i, n_{-i}, B)} \geq \gamma_{-i}$$

and ψ_{α_i} is a well-defined joint distribution that preserves the marginals over ω and (a_i, a_{-i}).

To do this, we first decompose ψ_{α_i} without changing the likelihood ratio of the ω's:

$$\psi_{\alpha_i}(y_i, n_{-i}, \omega) := \alpha_i \psi^*_{\alpha_i}(n_{-i}, \omega), \psi_{\alpha_i}(n_i, n_{-i}, \omega) := (1 - \alpha_i)\psi^*_{\alpha_i}(n_{-i}, \omega).$$

If player $-i$ is already obedient, then we are done. Suppose not, then it must be the case that after (n_i, n_{-i}) player $-i$ finds it suboptimal to choose n. In other words:

$$\frac{\psi^*_{\alpha_i}(n_{-i}, H)}{\psi^*_{\alpha_i}(n_{-i}, B)} < \gamma_{-i}.$$

Now we choose $0 < \Delta$ such that:

$$\frac{\psi_{\alpha_i}(n_i, n_{-i}, H)}{\psi_{\alpha_i}(n_i, n_{-i}, B)} := \frac{(1-\alpha_i)\psi^*_{\alpha_i}(y_{-i}, H) + \Delta}{(1-\alpha_i)\psi^*_{\alpha_i}(y_{-i}, B) - \Delta} = \gamma_{-i}. \tag{C.34}$$

Intuitively, we compensate player $-i$ such that after receiving (n_i, n_{-i}) she is indifferent between y and n. Our last piece is to show:

$$\frac{\psi_{\alpha_i}(y_i, n_{-i}, H)}{\psi_{\alpha_i}(y_i, n_{-i}, B)} := \frac{\alpha_i\psi^*_{\alpha_i}(n_{-i}, H) - \Delta}{\alpha_i\psi^*_{\alpha_i}(n_{-i}, B) + \Delta} \geq \frac{\gamma_{-i} - \varepsilon}{1 + \varepsilon} \tag{C.35}$$

Note since we are using the same Δ, we are preserving the marginals of ω and (a_i, a_{-i}).

From (C.34) we know that:

$$(\gamma_{-i} + 1)\Delta = \gamma_{-i}(1-\alpha_i)\psi^*_{\alpha_i}(n_{-i}, B) - (1-\alpha_i)\psi^*_{\alpha_i}(n_{-i}, H)$$

Plugging this into (C.35), we can derive that our desired inequality is exactly (C.33), which finishes the last piece.

We still need to check $\alpha_i\psi^*_{\alpha_i}(y_{-i}, H) - \Delta \geq 0$. By (C.33), we know:

$$(\gamma_{-i} + 1)\Delta \leq (\gamma_{-i}(1-\alpha_i)\frac{1+\alpha_i\varepsilon}{\gamma_{-i} - \alpha_i\varepsilon} - (1-\alpha_i))\psi^*_{\alpha_i}(n_{-i}, H) = \frac{\alpha_i\varepsilon(\gamma_{-i}+1)(1-\alpha_i)}{\gamma_{-i} - \alpha_i\varepsilon}\psi^*_{\alpha_i}(n_{-i}, H),$$

which simplifies to:

$$\Delta \leq \frac{(1-\alpha_i)\varepsilon}{\gamma_{-i} + (1-\alpha_i)\varepsilon - \varepsilon}\alpha_i\psi^*_{\alpha_i}(y_{-i}, H).$$

This implies $\Delta \leq \alpha_i\psi^*_{\alpha_i}(y_{-i}, H)$ as $0 < \varepsilon \leq \gamma_{-i}$. □

Lemma 9 states that we can concentrate on each $\psi^*_{\alpha_i}$ and find the corresponding ψ_{α_i} that is obedient and preserves the marginals over (a_i, a_{-i}) and ω. We still need to check that player i is obedient after both recommendations y_i and n_i. This is easy to see. In the construction of ψ_{α_i} (equation C.31), we have decreased the likelihood of H and increased the likelihood of B when the recommendation is y_i, for both $a_{-i} = y$ and $a_{-i} = n$. Meanwhile, when receiving y_i, the likelihood of y_{-i} and n_{-i} has not been changed, so player i's obedience constraint is relaxed. Similar arguments hold for n_i. Therefore, player i is still obedient under ψ_{α_i}.

Finally, after doing this for every α_i, we have decomposed $\psi^*_{\alpha_i}$ to the corresponding ψ_{α_i} and each of these are obedient. Now we can apply Remark 1 by choosing $\Lambda = \{\alpha_i : \alpha_i \in \text{supp}\psi^*\}$ and $q(\lambda) = \psi^*(\alpha_i)$. The resulting aggregated distribution is obedient and replicates the marginal distribution over ω and (a_i, a_{-i}).

C.5.2 Complete Network: $N = \{(1, 2), (2, 1)\}$

In this case, information is public. Consider any mixed recommendation $x^*(\alpha_1, \alpha_2|\omega)$. First, we argue that when ε is small, player 1 and 2 cannot use mixed strategies simultaneously. When player 1 uses strategy α_1, player 2 is indifferent between y and n when:

$$\frac{\psi^*(\alpha_1, \alpha_2, H)}{\psi^*(\alpha_1, \alpha_2, B)} = \frac{\gamma_2 - \alpha_1\varepsilon}{1 + \alpha_1\varepsilon};$$

Similarly, when player 2 uses strategy α_2, player 1 is indifferent between y and n when:

$$\frac{\psi^*(\alpha_1, \alpha_2, H)}{\psi^*(\alpha_1, \alpha_2, B)} = \frac{\gamma_1 - \alpha_2 \varepsilon}{1 + \alpha_2 \varepsilon}.$$

Therefore, when

$$\gamma_2 < \frac{\gamma_1 - \varepsilon}{1 + \varepsilon} \iff \varepsilon < \frac{\gamma_1 - \gamma_2}{1 + \gamma_2}$$

we know player 1 and 2 cannot use strategies simultaneously. In particular, when $0 < \alpha_1 < 1$, we must have $a_2 = n$, and when $0 < \alpha_2 < 1$, we must have $a_1 = y$.

For recommendation $x^*(\alpha_1, n_2|\omega)$, if the designer instead uses pure recommendation

$$x(y_1, n_2|\omega) = \alpha_1 x^*(\alpha_1, n_2|\omega), x(n_1, n_2|\omega) = (1 - \alpha_1) x^*(\alpha_1, n_2|\omega),$$

player 1 will be obedient because player 2's strategy and the likelihood of ω are unchanged. For player 2, since

$$\frac{\psi(a_1, n_2, H)}{\psi(a_1, n_2, B)} = \gamma_1 > \frac{\gamma_1 - \varepsilon}{1 + \varepsilon} > \gamma_2 \geq \frac{\gamma_2 - \alpha_1 \varepsilon}{1 + \alpha_1 \varepsilon}$$

for all $\alpha_1 \in [0, 1]$, player 2 will also be obedient given $\varepsilon < \frac{\gamma_1 - \gamma_2}{1 + \gamma_2}$.

Similarly, for recommendation $x^*(y_1, \alpha_2|\omega)$, if the designer instead uses pure recommendation

$$x(y_1, y_2|\omega) = \alpha_2 x^*(y_1, \alpha_2|\omega), x(y_1, n_2|\omega) = (1 - \alpha_2) x^*(y_1, \alpha_2|\omega)$$

player 2 will be obedient because player 1's strategy and the likelihood of ω are unchanged. For player 1, since

$$\frac{\psi(y_1, a_2, H)}{\psi(y_1, a_2, B)} = \frac{\gamma_2 - \varepsilon}{1 + \varepsilon} < \gamma_2 < \frac{\gamma_1 - \varepsilon}{1 + \varepsilon} \leq \frac{\gamma_1 - \alpha_2 \varepsilon}{1 + \alpha_2 \varepsilon}$$

for all $\alpha_2 \in [0, 1]$, player 1 will also be obedient given $\varepsilon < \frac{\gamma_1 - \gamma_2}{1 + \gamma_2}$.

Finally, after doing this for every (α_1, α_2), we have decomposed $\psi^*(\alpha_1, \alpha_2, \cdot)$ to corresponding $\psi(\alpha_1, \alpha_2, \cdot)$ and each of these are obedient. Now we can apply Remark 1 by choosing $\Lambda = \{(\alpha_1, \alpha_2) : (\alpha_1, \alpha_2) \in \text{supp}\psi^*\}$ and $q(\lambda) = \psi^*(\alpha_1, \alpha_2)$. The resulting aggregated distribution is obedient and replicates the joint distribution of (a_1, a_2, ω).

D Online Appendix

D.1 Disconnected Network–Seed Systems

Our analysis extends to settings where the network–seed system is not connected. Formally, let $\hat{I} = \{i \in I : N_i \cap S \neq \varnothing\}$ and suppose $I \setminus \hat{I} \neq \varnothing$. For $i \notin \hat{I}$, $N_i \cap S = \varnothing$ and so $|T_i| = 1$ for every initial *and* final information structure. Under the assumption that the chosen π_S is common knowledge to all players, the question is how to characterize the feasible outcomes.

The equivalence result in Theorem 1 goes through almost unchanged. The definition of S-expansion is adapted by requiring that the link (i, j) be added to N if and only if $N_i \cap S \neq \varnothing$ and $N_i \cap S \subseteq N_j$. The same argument behind Theorem 1 then implies that $X(G, N, S) = X(G, N^S, \hat{I})$ for all G.

With regard to the characterization of Theorem 2, it is clear that robust obedience is necessary for an outcome to be feasible for (G, N, S). However, sufficiency has to take into account that with isolated players feasible outcomes have additional statistical properties: If $i \notin \hat{I}$, then α_i cannot depend on the state nor on the action of any other player. Indeed, we can write condition (1) as

$$x(\alpha_1, \ldots, \alpha_I|\omega) = \left[\prod_{i \notin \hat{I}} \mathbb{I}\{\sigma_i(t_i) = \alpha_i\}\right] \sum_{t \in T} \pi(t|\omega) \prod_{i \in \hat{I}} \mathbb{I}\{\sigma_i(t_{N_i}) = \alpha_i\}.$$

There are two ways to proceed. One is to combine robust obedience as in Definition 4 with some constraint on x that captures these statistical properties; the other is to modify the notion of feasible outcomes and obedience, which may offer additional insights.

For the first way to be useful, the additional constraint on outcomes should be linear, so that together with obedience we continue to have a linear problem. One example of such a constraint is as follows: For any $\alpha_{-\hat{I}}, \alpha'_{-\hat{I}} \in \operatorname{supp} x_{-\hat{I}}$,

$$\sum_{\substack{i \notin \hat{I} \\ \omega \in \Omega \\ \alpha_{\hat{I}} \in \operatorname{supp}_{\hat{I}} x}} \sum_{a_i \in A_i} (\alpha'_i(a_i) - \alpha_i(a_i))^2 x(\alpha'_{-\hat{I}}, \alpha_{\hat{I}}|\omega)\mu(\omega) = 0.$$

Clearly, this holds if and only if, for every $i \notin \hat{I}$ and $\alpha_i, \alpha'_i \in \operatorname{supp} x_i$, we have $\alpha_i(a_i) = \alpha'_i(a_i)$ for all $a_i \in A_i$. But this means that x "recommends" to every isolated player the same action independently of the state and the recommendations to others.

For the second approach, we can describe outcomes as a profile of mixed actions for the isolated players and a recommendation mechanism restricted to the non-isolated players. That is, a feasible outcome for (G, N, S) is a pair $(\alpha_{-\hat{I}}, x_{\hat{I}})$, where $x_{\hat{I}}$ satisfies

$$x_{\hat{I}}(\alpha_{\hat{I}}|\omega) = \sum_{t \in T} \pi(t|\omega) \prod_{i \in \hat{I}} \mathbb{I}\{\sigma_i(t_{N_i}) = \alpha_i\}$$

for some initial information structure (T, π), the information structure (T', π') that is induced by (T, π) given N, and an equilibrium $(\alpha_{-\hat{I}}, \sigma_{\hat{I}}) \in \mathrm{BNE}(G, (T', \pi'))$. We can then say that $(\alpha_{-\hat{I}}, x_{\hat{I}})$

is spillover-robust obedient for given G and N if for all $i \in \hat{I}$, $\alpha_{N_i} \in \operatorname{supp}_{N_i} x$, and $a_i \in A_i$,

$$\sum_{\substack{\omega \in \Omega \\ \alpha_{-N_i} \in \operatorname{supp}_{-N_i} x}} \Big(u_i(\alpha_i, \alpha_{\hat{I}\setminus\{i\}}, \alpha_{-\hat{I}}; \omega) - u_i(a_i, \alpha_{\hat{I}\setminus\{i\}}, \alpha_{-\hat{I}}; \omega) \Big) x_{\hat{I}}(\alpha_i, \alpha_{\hat{I}\setminus\{i\}} | \omega) \mu(\omega) \geq 0, \tag{D.1}$$

and for all $i \notin \hat{I}$ and $a_i \in A_i$,

$$\sum_{\substack{\omega \in \Omega \\ \alpha_{\hat{I}} \in \operatorname{supp}_{\hat{I}} x}} \Big(u_i(\alpha_i, \alpha_{\hat{I}}, \alpha_{-\hat{I}\setminus\{i\}}; \omega) - u_i(a_i, \alpha_{\hat{I}}, \alpha_{-\hat{I}\setminus\{i\}}; \omega) \Big) x_{\hat{I}}(\alpha_{\hat{I}} | \omega) \mu(\omega) \geq 0.$$

In words, each non-isolated player finds it optimal to follow the recommendations from $x_{\hat{I}}$, conditional on knowing her sources' recommendations (by spillovers) and the isolated players' behavior (by correct equilibrium conjectures) and on any inference about the other non-isolated players' behavior and the state (through $x_{\hat{I}}$). Each isolated player finds it optimal to implement her mixed action, conditional on knowing the other isolated players' behavior (by correct equilibrium conjectures) and the joint distribution of actions and states induced by $x_{\hat{I}}$. This formulation of obedience shows that now feasibility involves a fixed-point condition between $x_{\hat{I}}$ and the isolated players' behavior.

This second approach also helps to extend the comparative statics results of Section 4.1. Fix I and consider the systems (N, S) and (N', S') that lead to the same set of non-isolated players $\hat{I}$. With regard to Lemma 1, given $\alpha_{-\hat{I}}$ equation (D.1) implies that if (N, S) is *less* connected than (N', S'), then the set of feasible outcomes among the players in $\hat{I}$ under (N, S) contains that under (N', S') for every G. Thus, overall $X(G, N', S') \subseteq X(G, N, S)$. The converse also holds (by the same argument as the baseline case) if we require the ranking of feasible sets to hold for all G.

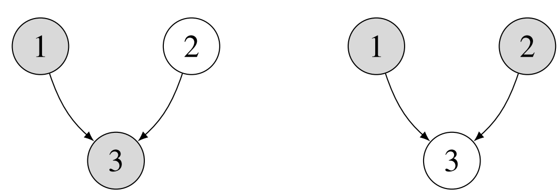

Figure 6: Influence and Disconnected Information Systems

Similarly, we can extend the analysis of Section 4.2 to settings where (N, S) is not connected. Fix N, S', and S''. Define $I' = \{i \in I : N_i \cap S' \neq \emptyset\}$ and $I'' = \{i \in I : N_i \cap S'' \neq \emptyset\}$ and assume that either $I' \subsetneq I$ or $I'' \subsetneq I$. There are two cases worth considering. Suppose that $I' = I''$. Then, S' is more influential than S'' if and only if $(N, S'') \trianglerighteq (N, S')$. This is a direct consequence of what we said in the previous paragraph and can be seen as an extension of Corollary 2. Alternatively, suppose that $N^{S'} = N^{S''}$. Then, S' is more influential than S'' if $I' \supseteq I''$. This latter case cannot arise when the system (N, S) is connected (which implies $I' = I'' = I$) and it is therefore conceptually distinct. We provide an example of this in Figure 6. In such example, let $S' = \{1,2\}$ and $S'' = \{1,3\}$. Note that $I' = I$ and $I'' = S''$. Thus, $I' \supseteq I''$. Moreover, $N^{S'} = N^{S''} = N$. Therefore, we have $X(G, N, S') = X(G, N^{S'}, S') = X(G, N^{S''}, I) \supseteq X(G, N, S'')$ for any G. We conclude that S' is more influential than S''.

D.2 More-connected Systems and Information Aggregation

We briefly explain the relationship between more-connected systems and their ability to aggregate information. Fix $i \in I$ and $(T, \pi) \in \mathcal{P}$. Let $\Delta^{(T,\pi)}(\Omega \times A_i)$ be the set of distributions such that

$$Pr(\omega, a_i) = \sum_{t \in T} \zeta(a_i|t_i)\pi_I(t_i, t_{-i}|\omega)\mu(\omega), \quad (\omega, a_i) \in \Omega \times A_i,$$

for some $\zeta : T_i \to \Delta(A_i)$. In the spirit of Blackwell (1951), we say that (T, π) is more informative than (T', π') for player i if $\Delta^{(T',\pi')}(\Omega \times A_i) \subseteq \Delta^{(T,\pi)}(\Omega \times A_i)$. For every N, denote by $f_N(T, \pi) \in \mathcal{P}_{(N,I)}$ the information structured induced by (T, π) under N. We say that (N, I) aggregates information better than (N', I) if, for all (T, π), $f_N(T, \pi)$ is more informative than $f_{N'}(T, \pi)$ for every player. We have the following characterization.

Lemma 10. *(N, I) aggregates information better than (N', I) if and only if $(N, I) \trianglerighteq (N', I)$.*

Proof. **Part 1 ($\Leftarrow$):** Suppose $(N, I) \trianglerighteq (N', I)$ and so $N_i \supseteq N'_i$, for every i. We want to show that, for every (T, π), $\Delta^{f_{N'}(T,\pi)}(\Omega \times A_i) \subseteq \Delta^{f_N(T,\pi)}(\Omega \times A_i)$ for all i. Fix a payer i and (T, π). Denote $(\hat{T}, \hat{\pi}) = f_N(T, \pi)$ and $(\hat{T}', \hat{\pi}') = f_{N'}(T, \pi)$. Fix $y \in \Delta^{(\hat{T}',\hat{\pi}')}(\Omega \times A_i)$. We will show that $y \in \Delta^{(\hat{T},\hat{\pi})}(\Omega \times A_i)$. Since $y \in \Delta^{(\hat{T}',\hat{\pi}')}(\Omega \times A_i)$, there exists $\gamma' : \hat{T}'_i \to \Delta(A_i)$ (where $\hat{T}'_i = \times_{j \in N'_i} T_j$) such that

$$y(\omega, a_i) = \sum_t \gamma'(a_i|t_{N'_i})\pi(t|\omega)\mu(\omega), \quad (\omega, a_i) \in \Omega \times A_i.$$

Now define $\gamma : \hat{T}_i \to \Delta(A_i)$ (where $\hat{T}_i = \times_{j \in N_i} T_j$) from γ' by letting

$$\gamma(a_i|t_{N'_i}, t_{N_i \setminus N'_i}) = \gamma'(a_i|t_{N'_i})$$

for all $t_{N_i \setminus N'_i}$. That is, γ depends only on the components in N_i that also belong to N'_i and in the same way that γ' depends on them. This is well defined because $N_i \supseteq N'_i$. Clearly, γ leads to the distribution y induced by γ', which therefore belongs to $\Delta^{(\hat{T},\hat{\pi})}(\Omega \times A_i)$.

Part 2 ($\Rightarrow$): We prove this by contrapositive. The only relevant case to consider is that $(N, I) \not\trianglerighteq (N', I)$ and $(N, I) \ntrianglelefteq (N', I)$. This implies that for some i, there exists a k such that $k \in N'_i$ and $k \notin N_i$, and for some j (possibly $i = j$), there exists m such that $m \in N_j$ and $m \notin N'_j$. It follows that there exists a player i_k such that $i_k \neq k$ and there is a direct link from k to i_k in N' but not in N, and there exists a player i_m such that $i_m \neq m$ there is a direct link from m to i_m in N but not in N'.

First, let (T, π) be an information structure such that k receives a fully informative signal and all other players get uninformative signals. Then, under $f_{N'}(T, \pi)$ player i_k gets full information, while under $f_N(T,\pi)$ he still gets no information. Therefore, (N,I) does not aggregate more information than (N',I). Now, let (T',π') be an information structure such that m gets full information and all other players get fully uninformative signals. Then, under $f_N(T',\pi')$ player i_m gets full information, while under $f_{N'}(T',\pi')$ he still gets no information. Therefore, (N',I) does not aggregate information better than (N,I). $\square$

D.3 Omitted Proofs

Proof of Lemma 2

Step 1. Fix an information structure (T,π). Denote by $(T',\pi') \in \mathcal{P}_{(N,S)}$ the information structure induced by (T,π) under N. Let σ be the designer-preferred equilibrium in game $(G,(T',\pi'))$. For every i, every t_{N_i} determines a non-empty subset of optimal actions:

$$A_i(t_{N_i}) = \arg\max_{a_i \in A_i} \mathbb{E}_{\pi,\sigma}\Big(u_i(a_i, a_{-i}, \omega) \;\Big|\; t_{N_i}\Big).$$

Since A_i is finite, every $((T,\pi),\sigma)$ can determine at most finitely many subsets $A_i(t_{N_i})$ for every player i. This requires no more than $|2^{A_i}|$ signals for player i. Therefore, every (π_I,σ) can determine at most finitely many profiles of optimal-action sets of the form $A(t) = \times_i A_i(t_{N_i})$. We conclude that if we are interested in only such profiles, it is enough to consider information structures that satisfy $|T_i| = |2^{A_i}|$ for every i.

Step 2. We now need to transition from profiles of optimal-action sets to distributions over pure-action profiles, which is what ultimately matters for the designer. To this end, we use Theorem 2. Recall that each recommendation profile α can be interpreted, first of all, as a signal realization from the information structure x. Step 1 shows that, if we are interested only in spanning the profiles of optimal-action sets, it is enough to consider xs with finite support. But this may not be enough for the entire set of feasible outcomes intended as joint distributions between actions and states that satisfy obedience.

Suppose that x is a feasible outcome, hence it satisfies obedience. That is, for every i, $\alpha_{N_i} \in \mathrm{supp}_{N_i} x$, and $a_i, a_i' \in A_i$,

$$\sum_{\omega,\alpha_{-N_i}} \Big(\sum_{a_{-i}} \Big(u_i(a_i, a_{-i}, \omega) - u_i(a_i', a_{-i}, \omega)\Big)\alpha_{N_i}(a_{N_i})\alpha_{-N_i}(a_{-N_i})\Big) x(\alpha_{N_i}, \alpha_{-N_i}|\omega)\mu(\omega) \geq 0,$$

where $\alpha_{N_i}(a_{N_i}) = (\alpha_j(a_j))_{j\in N_i}$ and $\alpha_{-N_i}(a_{-N_i}) = (\alpha_j(a_j))_{j\notin N_i}$. We want to construct an alternative x' that is also feasible and induces the same joint distribution between pure-action profiles and states as does x.

From step 1, we know that we can identify finitely many profiles of sets $A^x(\alpha) = \times_{i\in N} A_i^x(\alpha_{N_i})$, where we treat each α as a signal realization from x. Let $\mathcal{A}^x$ be the finite collection of such profiles determined by x. In particular, we know that $|\mathcal{A}^x| \leq \prod_{i\in N} |2^{A_i}|$ independently of x. For every ω, construct x' as follows. For every $A^x \in \mathcal{A}^x$, define

$$\alpha^{A^x,\omega}(a) = \sum_{\alpha \in A^x} \alpha(a) \frac{x(\alpha|\omega)}{\sum_{\alpha' \in A^x} x(\alpha'|\omega)}, \quad a \in A.$$

This is the average mixed-action profile in state ω, conditional on α belonging to A^x. Given this, for every $\alpha^{A^x,\omega}$ so identified, let

$$x'(\alpha^{A^x,\omega}|\omega) = \sum_{\alpha \in A^x} x(\alpha|\omega), \quad \omega \in \Omega.$$

It is immediate to see that x and x' induce the same joint distribution over pure-action profiles for every state: For every a and ω,

$$\begin{aligned}
\sum_{\alpha' \in \operatorname{supp} x'} \alpha'(a)x'(\alpha'|\omega) &= \sum_{A^x \in \mathcal{A}^x} \alpha^{A^x,\omega}(a)x'(\alpha^{A^x,\omega}|\omega) \\
&= \sum_{A^x \in \mathcal{A}^x} \left[\sum_{\alpha \in A^x} \alpha(a) \frac{x(\alpha|\omega)}{\sum_{\alpha' \in A^x} x(\alpha'|\omega)} \right] \sum_{\hat{\alpha} \in A^x} x(\hat{\alpha}|\omega) \\
&= \sum_{A^x \in \mathcal{A}^x} \left[\sum_{\alpha \in A^x} \alpha(a)x(\alpha|\omega) \right] = \sum_{\alpha \in \operatorname{supp} x} \alpha(a)x(\alpha|\omega).
\end{aligned}$$

Let's now consider obedience. If we can show that x' also satisfies obedience, we are done. Fix any player i, any $\alpha'_{N_i} \in \operatorname{supp}_{N_i} x'$, and $a_i, a'_i \in A_i$. Note that α'_{N_i} must equal $\alpha^{A^x,\omega}_{N_i}$ for some some A^x and ω. Let $\mathcal{A}^x(\alpha'_{N_i})$ contain all the profiles A^x that are compatible with α'_{N_i}, i.e., that satisfy $\alpha^{A^x,\omega}_{N_i} = \alpha'_{N_i}$. Letting $\Delta u_i(a_i, a'_i; a_{-i}, \omega) = u_i(a_i, a_{-i}, \omega) - u_i(a'_i, a_{-i}, \omega)$, we have

$$\begin{aligned}
&\sum_{\omega, \alpha'_{-N_i}} \left\{ \sum_{a_{-i}} \Delta u_i(a_i, a'_i; a_{-i}, \omega)\alpha'_{N_i}(a_{N_i})\alpha'_{-N_i}(a_{-N_i}) \right\} x'(\alpha'_{N_i}, \alpha'_{-N_i}|\omega)\mu(\omega) \\
&= \sum_{\omega, A^x \in \mathcal{A}^x(\alpha'_{N_i})} \left\{ \sum_{a_{-i}} \Delta u_i(a_i, a'_i; a_{-i}, \omega)\alpha^{A^x,\omega}_{N_i}(a_{N_i})\alpha^{A^x,\omega}_{-N_i}(a_{-N_i}) \right\} x'(\alpha^{A^x,\omega}_{N_i}, \alpha^{A^x,\omega}_{-N_i}|\omega)\mu(\omega) \\
&= \sum_{\omega, A^x \in \mathcal{A}^x(\alpha'_{N_i})} \left\{ \sum_{a_{-i}} \Delta u_i(a_i, a'_i; a_{-i}, \omega) \sum_{\alpha \in A^x} \alpha_{N_i}(a_{N_i})\alpha_{-N_i}(a_{-N_i}) \frac{x(\alpha_{N_i}, \alpha_{-N_i}|\omega)}{\sum_{\alpha' \in A^x} x(\alpha'|\omega)} \right\} \times \\
&\quad \times \sum_{\alpha \in A^x} x(\alpha_{N_i}, \alpha_{-N_i}|\omega)\mu(\omega) \\
&= \sum_{\omega, A^x \in \mathcal{A}^x(\alpha'_{N_i})} \sum_{\alpha \in A^x} \left\{ \sum_{a_{-i}} \Delta u_i(a_i, a'_i; a_{-i}, \omega)\alpha_{N_i}(a_{N_i})\alpha_{-N_i}(a_{-N_i})x(\alpha_{N_i}, \alpha_{-N_i}|\omega) \right\} \mu(\omega) \\
&= \sum_{A^x \in \mathcal{A}^x(\alpha'_{N_i})} \sum_{\alpha \in A^x} \left\{ \sum_{\omega, a_{-i}} \Delta u_i(a_i, a'_i; a_{-i}, \omega)\alpha_{N_i}(a_{N_i})\alpha_{-N_i}(a_{-N_i})x(\alpha_{N_i}, \alpha_{-N_i}|\omega)\mu(\omega) \right\}.
\end{aligned}$$

Now, recall that for every $\alpha \in A^x$, we have that the set of optimal actions for player i conditional on α_{N_i} is the same. Since x satisfies obedience for player i, her α_i assigns positive probability only to actions that are optimal conditional on α_{N_i}. Therefore, the entire sum must be non-negative. This shows that x' satisfies obedience for player i and every $\alpha'_{N_i} \in \operatorname{supp}_{N_i} x'$. By the same argument, x' satisfies obedience for all players. $\square$

Proof of Lemma 3 Since $V^*(G,N,I) = \sup_{x\in X(G,N,I)} \sum_{\omega,\alpha} v(\alpha,\omega)x(\alpha|\omega)\mu(\omega)$, for every $n \geq 1$ there exists x^n that satisfies

$$V^*(G,N,I) \geq \sum_{\omega,\alpha} v(\alpha,\omega)x^n(\alpha|\omega)\mu(\omega) \geq V^*(G,N,I) - \frac{1}{n}.$$

Moreover, by Lemma 2 we can choose x^n so as to satisfy $|\text{supp}_i x| \leq |2^{A_i}|$ for every i. Let $K_i = |2^{A_i}|$ for every i. To every such x^n there correspond finite subsets $A_i^n \subset \Delta(A_i)$ such that $|A_i^n| = K_i$ for all i, which define a grid in $\times_{i\in I}\Delta(A_i)$ over which we can restrict the construction of x^n itself. (Note that the support of x^n may not use the entire grid, but it is without loss to allow for these extra elements that receive zero probability). Thus, for every i and $k_i = 1,\ldots,K_i$, there is a sequence $\alpha_i^{k_i,n} \in \Delta(A_i)$ where $\alpha_i^{k_i,n} \in A_i^n$ is an element of the grid of player i with (fixed) K_i elements to construct x^n. Also, for each ω and every $(k_1,\ldots,k_I)$ where $k_i = 1,\ldots,K_i$ for every i, we have a sequence of elements $x^n(\alpha_1^{k_1,n},\ldots,\alpha_I^{k_I,n},\omega) \in [0,1]$. Since all these sequences belong to a compact space, each has a converging subsequence. Moreover, since we have finitely many sequences because each K_i is fixed and finite, there exists an overall subsequence of indexes $\tilde{n}$ such that the following holds:

$$\lim_{\tilde{n}\to\infty} \alpha_i^{k_i,\tilde{n}} = \hat{\alpha}_i^{k_i} \in \Delta(A_i), \quad k_i = 1,\ldots,K_i, i \in I;$$

$$\lim_{\tilde{n}\to\infty} x^{\tilde{n}}(\alpha_1^{k_1,\tilde{n}},\ldots,\alpha_I^{k_I,\tilde{n}}|\omega) = \hat{x}(\hat{\alpha}_1^{k_1},\ldots,\hat{\alpha}_I^{k_I}|\omega), \quad k_i = 1,\ldots,K_i, \omega \in \Omega.$$

Since $x^{\tilde{n}} \in X(G,N,I)$ for all $\tilde{n}$ by assumption, it is easy to see that $\hat{x} \in X(G,N,I)$ by continuity of the linear constraints that define $X(G,N,I)$. Finally, we have that

$$\begin{aligned} V^*(G,N,I) \geq \sum_{\omega,\hat{\alpha}} v(\hat{\alpha},\omega)\hat{x}(\hat{\alpha}|\omega)\mu(\omega) &= \lim_{\tilde{n}\to\infty} \sum_{\omega,\alpha^{\tilde{n}}} v(\alpha^{\tilde{n}},\omega)x^{\tilde{n}}(\alpha^{\tilde{n}}|\omega)\mu(\omega) \\ &\geq \lim_{\tilde{n}\to\infty} \left(V^*(G,N,I) - \frac{1}{\tilde{n}}\right) = V^*(G,N,I). \end{aligned}$$

Therefore, $\sum_{\omega,\hat{\alpha}} v(\hat{\alpha},\omega)\hat{x}(\hat{\alpha}|\omega)\mu(\omega) = V^*(G,N,I)$, which proves the result. □

D.4 Examples of General Spillovers

Truthful Belief Announcement. Suppose at each round of communication, every player truthfully reports to all her neighbors her current Bayesian posterior over $\Omega \times T$, namely, about the state and all players' initial signal realizations. We include the signal realizations in the reports because learning about others' information matters for predicting their actions in the final game. Assume that the number of communication rounds K is at least as large as the shortest path between the two most distant players in N.

This communication process fits into the general model outlined in Section 5. Since T is finite, at every round each player can have at most finitely many different posteriors. In the first round, each i reports a degenerate posterior about her private signal t_i to her direct followers in ${}_i\bar{N}$. In the next round, all i's direct followers report a degenerate posterior about t_i to *their* direct followers. Continuing this way, all players for which i is an information source in N will hear a degenerate report of t_i within a number of rounds that cannot exceed the diameter of N. After these rounds, all players' posteriors will be degenerate about their sources' signals and will stop evolving. Thus, this process of truthful belief announcement can be a foundation for our baseline assumption of deterministic spillovers.

Lemma 11. *Fix (T, π) and a signal realization t. For every i and $h_i^K \in \mathcal{H}_i$ consistent with (t, π), $Pr_\pi(\omega, \hat{t}|h_i^K) = Pr_\pi(\omega, \hat{t}|t_{N_i})$ for all $(\omega, \hat{t}) \in \Omega \times T$.*

Proof. Define $N_i^0 = \{i\}$ and $N_i^n = \cup_{j \in N_i^{n-1}} \bar{N}_j$ for $n = 1, \ldots, I$. Note that $N_i^I = N_i$. Fix a signal realization $\bar{t}$ and the corresponding unique $h^K(\bar{t}) = h^K$. For every player i,

$$\xi_i(h_i^0)(\omega, t) = \Pr_\pi(\omega, t|\bar{t}_i) = \Pr_\pi(\omega, t|\bar{t}_{N_i^0}), \quad (\omega, t) \in \Omega \times T.$$

Note that

$$\sum_{\omega, t_{-i}} \Pr_\pi(\omega, t|\bar{t}_i) = \begin{cases} 1 & \text{if } t_i = \bar{t}_i \\ 0 & \text{otherwise.} \end{cases}$$

Fix $n \geq 1$. Given h_i^n, suppose that for every player j,

$$\xi_j(h_j^{n-1})(\omega, t) = \Pr_\pi(\omega, t|\bar{t}_{N_j^{n-1}}), \quad (\omega, t) \in \Omega \times T.$$

Note that

$$\sum_{\omega, t_{-j}} \Pr_\pi(\omega, t|\bar{t}_{N_j^{n-1}}) = \begin{cases} 1 & \text{if } t_{N_j^{n-1}} = \bar{t}_{N_j^{n-1}} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\xi_i(h_i^n)(\omega, t) = \Pr_\pi(\omega, t|\bar{t}_{N_i^n}), \quad (\omega, t) \in \Omega \times T.$$

Since this is true for every i, by induction we have that

$$\xi_i(h_i^K)(\omega, t) = \Pr_\pi(\omega, t|\bar{t}_{N_i^I}) = \Pr_\pi(\omega, t|\bar{t}_{N_i}), \quad (\omega, t) \in \Omega \times T,$$

thus concluding the proof. □

Random Networks as a Form of General Spillovers. Section A.1 claimed that we can think of random networks distributed according to φ as a spillover process ξ^φ over the fixed network $N^\varphi = \cup_{N \in \Phi} N$. Such ξ^φ is constructed as follows. Let the number of communication rounds K be at least as large as the diameter of N^φ. For each $i \in I$, the strategy ξ_i^φ depends on the realization of $\nu_i = (N_i, {}_i\bar{N})$ according to φ as a randomization device. Given n_i, at each round of the communication phase, player i truthfully reports her current Bayesian posterior about $\Omega \times T$ to every $j \in {}_i\bar{N}$—as in the truthful-belief-announcement process described above—and a constant, history-independent, message to every $j \in {}_i\bar{N}^\varphi \setminus {}_i\bar{N}$. Such a ξ^φ induces a process that converges in a finite number of rounds, bounded above by the diameter of N^φ.