Updating QR Factorization Under Sparse Updates - Fairbanks and Carr

The total number of fill-in entries eliminated is

$$Lazy = \sum_{j=1}^{k}\sum_{t=1}^{j}(i_{j+1} - i_j - t) = \sum_{j=1}^{k} j(i_{j+1} - i_j - \frac{j+1}{2})$$

This is called the lazy elimination strategy because we defer elimination of fill-in entries until the last possible moment. The eager strategy eliminates fill-in as soon as it is created. The total number of fill-in entries removed is

$$Eager = \sum_{j=1}^{k} j(n - i_j) = n\sum_{j=1}^{k} j - \sum_{j=1}^{k} j i_j = \frac{nk(k+1)}{2} - \sum_{j=1}^{k} i_j$$

Both methods must eliminate the original nonzeros which are counted by $\sum_{j=1}^{k} n - i_j$.

Eager method

Suppose that $\Delta A = \sum_{j=1}^{k} A_j$ where each $A_j = xe_{i_j}^T$

$$Q^T A_1 + R = Q_1 R_1$$

$$\vdots$$

$$Q_{k-1}^T A_k + R_{k-1} = Q_k R_k$$

This gives $6(n - i_j)(m - i_j) + 6(n - i_j)(n - i_j)$ flops to perform the rotations and $6n(m - i_j) + 6n(n - i_j)$ flops to update Q Each iteration also requires a single, dense matrix-vector product because we must compute $Q_j^T A_{j+1} + R_j$ We must perform this k times.

Crossover Theory

For a fixed m large enough, the Householder based methods are cubic in n and the sparse updating methods are quadratic in n because we need to accumulate Q. Thus making this simplification we can model the performance ratio as

$$r_e(m,n,k) = \frac{C_m n^3}{n^2 k} = \frac{Cn}{k} \tag{1}$$

The relationship between m and n is accounted for in the dependence of C_m on m. When $r_e = 1$ the two methods will take the same amount of time, and the k for which this occurs is called the break even point.

Results

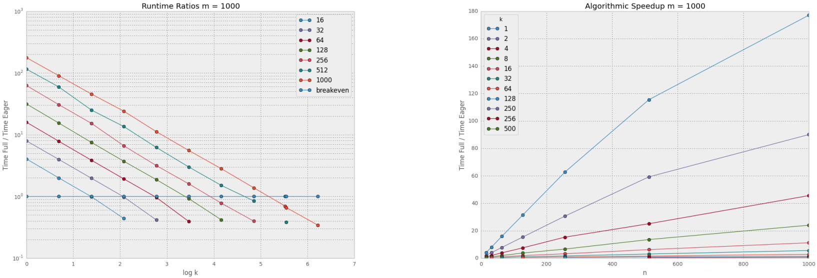

Figure: r_e as a function of k (left) and as a function of n (right)