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Optimal sharing of surgical costs in the presence of queues*

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Abstract. We deal with a cost-allocation problem arising from sharing a medical service in the presence of queues. We use a standard queuing theory model in a context of several medical procedures, a certain demand for treatment and a maximum average waiting-time guaranteed by the government. We show that the sharing of an operating-theatre to treat patients of different medical disciplines leads to a cost reduction. We then compute an optimal fee per procedure for the use of the operating theatre, based on the Shapley value. Afterwards, considering the recovery time, we characterize the conditions under which the co-operation among treatments has a positive impact on the post-operative costs. Finally, a numerical example, constructed on the basis of real data, is provided to highlight the main features of our model.

Key words: Surgical waiting-lists, Queueing theory, Cost-sharing game

JEL classification: C44, C71, H51

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1 Introduction

The widespread access to public health care in Western European countries is placing such stress on the system that optimal allocation of resources is becoming a major management problem. Citizens are particularly sensitive to some of the phenomena related to health services. One of such phenomena is the persistence of waiting-lists for surgical treatment.

Waiting-lists for elective surgery can be modelled by using queueing theory (for an overview on queueing theory see Gross and Harris (1997), Hillier and Lieberman (1995) and Kleinrock (1975)). This queueing system has some rather peculiar characteristics: (1) There are two sources for the formation of waiting-lists: the capacity of the operating-theatre and the bed-capacity of the hospital; (2) Several different medical procedures share both servers; (3) Each of these medical procedures has its own rate of arrival; (4) Not all medical procedures are considered equally urgent, so that the average waiting-time that is politically considered as adequate, differs among procedures.[1]

In managing such a situation, cost-allocation problems arise. As different procedures may share both the operating-theatre and the hospital beds, we must design a cost-allocation rule for the sharing of joint costs. This, in fact, is the main purpose of this paper. To construct a cost-allocation rule, we use a game theoretical perspective and design a cost-allocation game. In the first part of the paper, we concentrate on the costs associated with the operating-theatre. We construct a game by confronting two situations: one in which each medical procedure has its own operating-theatre, and another in which there is just one theatre for all the operations. Under the assumption that costs are linear in the service time, irrespective of the speciality, we show that sharing the operating-theatre to treat patients from the different specialities, leads to cost reductions. We then construct a cost-sharing game that turns out to be the sum of an additive game plus an "airport game" and, given its characteristics, we offer a cost-sharing rule that recommends the Shapley value allocation of the cost-sharing game. Cost-allocation rules of the same nature have been applied to other economic environments (see, for instance, Littlechild and Thompson (1977) and Fragnelli et al. (2000)).

We then turn to the post-operative costs. Considering the beds as servers, we also model the hospitalization stage as a queueing system. Again, under the assumption that costs associated to beds are constant over the different specialities, we can compute the number of beds required to guarantee the service in different scenarios. The analytical unsolvability of the model, nonetheless, prevents us from arriving at any general results. In spite of that, general conditions that guarantee that co-operation in the operating-theatre stage leads to a reduction in the expected number of beds are obtained.

We finally analyse a numerical example with real data to illustrate the actual computation of the optimal tariff, at the two stages of the surgical process.

[1] Some European public health administrations set different waiting-time guarantees across specialities. See, for instance, Hanning and Wimblad Spånberg (2000) for Sweden, or the Programa Avance INSALUD in Spain (Spanish Ministry of Health and Consumption. Press note: 22_{nd} February 2001).

Most of the literature on hospital waiting-lists has focused on the demand side. This paper, on the contrary, must be considered among the scarce supply-side branch of the literature. For a general survey on waiting-lists, see Cullis et al. (2000).

Joskow (1980) and Worthington (1987), use a queueing model to characterize the hospital-bed supply. They consider the formation of queues at the beds stage. We, instead, consider that the queue is formed at the operating-theatre stage. Thus, when a patient leaves the queue and enters the operating-theatre, we put a small upper bound to the probability that there is not an available hospital bed.

The rest of the paper is organized as follows: Section 2 presents the model. Section 3 studies the operating-theatre costs. Section 4 computes the optimal cost-sharing. Section 5 introduces the recovery time in the model. Section 6 provides a numerical example. Finally, Section 7 offers some concluding remarks.

2 The Model

We consider the basic queueing process: customers requiring a service are generated over time by an input source. These customers arrive to the system and join a queue. At different moments, a customer is selected to receive the service by means of a queue discipline. The service mechanism then provides the service and the customer leaves the system. In our problem, the customers are patients who require elective surgery and the mechanism of the service is a hospital. There are two sorts of *servers*: the operating-theatre and the hospital beds. Any individual entering the system should first go through the operating-theatre, and once released from this server, a bed should be made available to him. The patient only leaves the system when he has been discharged from the hospital.

Let us consider a situation with n kinds of medical procedures, all of which are of an elective nature, $N = \{1, 2, \ldots.., n\}$.

The patients' arrivals at the medical system follow a Poisson process. Let $\lambda_i \in \mathbb{R}_{++}$ be the expected number of arrivals per unit of time, from the i_{th} medical procedure, $i \in N$.

The work that an arriving patient brings to the operating-theatre, equals the time of service he requires. This service time follows a random process, as an operation may either require less time or unexpectedly become more complicated, and hence, require extra time. Thus, we assume that the service time of the i_{th} medical procedure follows an exponential distribution with a mean of $\frac{1}{\mu_i}$, $\mu_i \in \mathbb{R}_{++}$, where $\frac{1}{\mu_i}$ represents the fraction of the total working time of the server employed on one patient. We require $\lambda_i < \mu_i \; \forall i \in N$, otherwise the queue would "explode" and the system would break down.

In order to make the model analytically tractable, we also make some further standard assumptions. First, the number of potential patients is assumed to be infinite. Thus, the number of individuals in the queue does not affect the amount of potential entrants. Secondly, we put no restrictions on the length of the queue. Thirdly, the queue discipline imposed is that of "first come, first served", i.e., patients are chosen to receive the service according their order of arrival. Fourth, arrivals and departures from the system behave as a "birth and death process". Fifth, patients' arrivals are independent events across specialities. Finally, the analysis is performed considering that the steady state of the system has been reached.

As a policy measure, a maximum average waiting-time guarantee t_i, $i \in N$ is set by the government for each procedure. W.l.o.g., let $t_1 \geq t_2 \geq \ldots \geq t_n \equiv T$.

3 Operating-theatre Costs

We first analyze the operating-theatre system. We consider two alternative scenarios. In the first one, each surgical procedure has its own operating-theatre for treating its patients. Let W_i be the expected waiting-time of an individual of type i in the system. As such, it is well known that:

$$W_i = \frac{1}{\mu_i - \lambda_i}. \tag{1}$$

In the second scenario, the different medical procedures share a single operating-theatre. Analogously, if W stands for the expected waiting-time of an individual in the system, independently of his type, we have:

$$W = \frac{1}{\mu - \sum_{i=1}^{n} \lambda_i}, \tag{2}$$

where μ is the capacity of the jointly used operating-theatre.

We assume that the operating-theatres costs are linear in the amount of patients treated, irrespective of their speciality. This limits our analysis to specialities with similar surgical requirements. Nonetheless, it is easily fulfilled for many elective procedures, where surgeons' and theatre staff costs, consumables and equipment are standard.[2]

Next, we study the costs of fulfilling the government's target under the two scenarios.

The overall costs arising from n different operating-theatres are merely the sum of the individual costs. The explicit form of the individual costs is as follows:

$$C_i = k\mu_i(t_i), \text{with } k \in \mathbb{R}_{++}. \tag{3}$$

We must set μ_i (the potential amount of patients for medical procedure i that can be treated per unit of time) in order to guarantee the corresponding legal maximum waiting-time (t_i). To obtain $W_i = t_i$, we must set $\mu_i(t_i) = \frac{1}{t_i} + \lambda_i$. Hence, the costs per medical procedure are increasing in the mean rate of arrival (λ_i), and decreasing in the legal average-time (t_i). The two features are reasonable: the more patients that arrive, and the lower the average-time we can keep them waiting, the higher the cost will be.

[2] On top of this, a report by the Audit Commission (2003a) estimates that staffing costs are the main body of the total theatre costs, and are strongly related to operating hours. They argue, moreover, that a speciality mix is not likely to be a big determinant of the staff costs per operating hour.

The overall costs of keeping n operating-theatres open are, therefore:

$$C^N = \sum\nolimits_{i=1}^{n} C_i = k\left(\sum\nolimits_{i=1}^{n} \frac{1}{t_i} + \sum\nolimits_{i=1}^{n} \lambda_i\right). \tag{3}$$

If there is a single operating-theatre, we can guarantee the legal average-time t_i for all the procedures by simply doing so for the most urgent one. Thus, the total costs will be given by:

$$C^1 = k\mu(T),$$

where $\mu(T)$ is the potential number of patients, coming from any medical procedure, who can be treated per unit of time, and T is the lowest value that the maximum average-time guarantee has across treatments, as defined in Section 2.

To obtain $W = T$, we must set $\mu(T) = \frac{1}{T} + \sum_{i=1}^{n} \lambda_i$. Therefore, the overall costs are:

$$C^1 = k\left(\frac{1}{T} + \sum\nolimits_{i=1}^{n} \lambda_i\right). \tag{4}$$

By comparing (3) and (4) the following proposition follows straightforward.

Proposition 1 *Sharing the operating-theatre leads to cost reduction.*

Proposition 1 states that there are savings if the different surgical procedures share a single operating-theatre. When each medical procedure maintains its own server, it has to suffer not only a cost proportional to the expected number of patients demanding surgical care ($k\lambda_i$), but also a fixed cost, depending on its maximum average waiting-time guarantee ($\frac{1}{t_i}$). This fixed cost corresponds to the additional capacity the procedures have to maintain to cover themselves from the risks derived from the randomness of the system. Under co-operation among the specialities, they can maintain the necessary additional capacity by jointly supporting the fixed extra cost of the most urgent procedure. This is similar to "risk spreading". As we can compensate for the results among treatments, we can cover the demand within the legal average-time with less installed capacity, i.e., at a lower cost.

Our interest now is to distribute the benefits derived from working together among the different specialities. We will model this problem as a cost-sharing co-operative game.

4 Optimal Operating-Theatre Cost-Sharing

We shall now construct a cost-sharing game in order to allocate the operating-theatre costs to the different medical procedures. The *players* are the different surgical procedures, $N = \{1, ..., n\}$. The cost-sharing game $c^o : 2^N \to \mathbb{R}$, assigns to any non-empty coalition $S \subset N$, the minimum cost $c^o(S)$, under which the legal time is fulfilled *for all the surgical procedures in* S, and $c^o(\emptyset) = 0$. According to Proposition 1, $c^o(S)$ is simply the cost required to maintain a single operating-theatre shared by all the medical disciplines included in the set S, namely,

$$c^o(S) = k\left(\frac{1}{T_S} + \sum\nolimits_{i \in S} \lambda_i\right),$$

where $T_S = \min\{t_i : i \in S\}$ is the average-time guarantee established for the most urgent procedure in the set S. This cost function can be divided into two parts: $c_1^o(S) = k \sum_{i \in S} \lambda_i$, proportional to the number of patients within S, and $c_2^o(S) = \left(\frac{k}{T_S}\right)$, related to the shortest average-time guarantee in S. We may interpret $c_1^o(S)$ as a "variable" cost and $c_2^o(S)$ as a "fixed" cost for coalition S. Our cost-sharing game, therefore, is the sum of two other games: c_1^o , which is a linear one, and c_2^o, which is an "airport game".

Given the characteristics of c^o, we shall adopt the recommendation of the Shapley value as a way of distributing the operating-theatre costs among the different surgical procedures. Our tariff, hence, has the properties of efficiency, linearity, symmetry and fairness (See Shapley (1953), Tijs and Driesen (1986) and Young (1994)). As a result, our optimal tariff can be computed as the sum of the Shapley value of the two component games, and belongs to the Core of c^o.

As for c_1^o, it happens that $Sh_i(c_1^o) = k\lambda_i$, for all $i \in N$. As for c_2^o, we may apply the Baker (1965) and Thompson (1971) cost-allocation rule, which gives us the Shapley value allocation (see Littlechild and Owen (1973)). By this rule, each procedure contributes equally to the cost of maintaining an operating-theatre open for the medical treatment with the least priority; the contribution of the procedure with the least priority level $(Sh_1(c_2^o))$ is then computed. All of the remaining procedures also contribute equally to the additional cost of keeping the theatre open for the next treatment in the finite order. This way, the second procedure's contribution $(Sh_2(c_2^o))$ is computed, and so on.

Consequently, if in a certain period of time we receive a set of patients M, where $M = M_1 \cup \cdots \cup M_n$ and M_i stands for the set of patients from procedure i, $m_i = \#M_i$, we have the following result:

Proposition 2 *An optimal schedule of fees for any user $j \in M$ of the operating-theatre $\left(\phi_j^*(c^o)\right)$ is given by:*

$$\phi_j^*(c^o) = \frac{1}{m_1}\left[Sh_1(c_1^o) + Sh_1(c_2^o)\right] = \frac{k}{m_1}\left[\lambda_1 + \frac{1}{nt_1}\right] \text{ if } j \in M_1$$

$$\phi_j^*(c^o) = \frac{1}{m_i}\left[Sh_i(c_1^o) + Sh_i(c_2^o)\right]$$

$$= \frac{k}{m_i}\left[\lambda_i + \frac{1}{nt_1} + \sum\nolimits_{h=1}^{i-1} \frac{1}{n-h}\left(\frac{1}{t_{h+1}} - \frac{1}{t_h}\right)\right] \text{ if } j \in M_i, i \geq 2.$$

5 Post-operative Costs

As the availability of beds at a hospital also affects the possibilities of fulfilling a certain legal average-time, we should introduce a second *set of servers* in the system: the beds for recovering patients.

We consider the recovery costs to be linear in the number of beds.[3] The analysis, hence, is reduced to the computation of the amount of beds required for the adequate functioning of the hospital. We compute this amount by using Erlang's C formula. If patients arrive at beds at a rate λ, their stay in the server is exponentially distributed with a mean of d, and there are b servers in the system, the probability that a patient will have to wait for a bed is:

$$\Pr(\text{queueing}) = \Pr(\mathcal{N} \geq b) = \frac{\left(\frac{(b\rho)^b}{b!}\right)\left(\frac{1}{1-\rho}\right)}{\left[\sum_{k=0}^{b} \frac{(b\rho)^k}{k!} + \left(\frac{(b\rho)^b}{b!}\right)\left(\frac{1}{1-\rho}\right)\right]}, \tag{5}$$

where $\mathcal{N}$ denotes the number of patients in the system and $\rho = \frac{\lambda d}{b} < 1$ is the necessary and sufficient condition for convergence to the steady-state.

As the number of beds needed, b, cannot be analytically derived from (5), no general results on the impact of co-operation in the recovery stage can be obtained. Nonetheless, we can obtain explicit results when we study the impact of sharing the operating-theatre on the *average* number of beds. To do so, let d_i denote the average number of units of time that a patient of type i, $i \in N$ spends recovering in hospital. Once again, we consider the service time to be exponentially distributed.

We take the operating-theatre rate capacity (μ) as the relevant measure of the number of patients requiring recovery treatment. By doing so, we are implicitly assuming that operating-theatres are fully utilized, i.e., there is always a strictly positive stock of patients demanding surgical attention.

The expected number of beds required when the procedures share the operating-theatre and when they do not ($\bar{b}_N$ and $\sum_{i=1}^{n} \bar{b}_i$ respectively) are:[4]

$$\sum_{i=1}^{n} \bar{b}_i = \sum_{i=1}^{n} \frac{d_i}{t_i} + \sum_{i=1}^{n} \lambda_i d_i. \tag{6}$$

$$\bar{b}_N = \sum_{i=1}^{n} p_i d_i = \left(\frac{1}{T} + \sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} d_i\right), \tag{7}$$

where p_i denotes the proportion of type i patients treated in a given time, expressed as a fraction of the operating-theatre total capacity, i.e., $p_i = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} \mu(T)$.

Let us call $\Lambda = \sum_{j=1}^{n} \lambda_j$. In comparing these scenarios we obtain the following result:

Proposition 3 *Sharing the operating-theatre reduces the average post-operative costs if and only if:*

$$\sum_{i=1}^{n} \lambda_i d_i < \Lambda T \sum_{i=1}^{n} \frac{d_i}{t_i}$$

[3] The Audit Commission (2003b) claims that patients should be placed in the first available bed on an appropriate ward. It is quite reasonable, hence, to assume similar bed-cost for procedures in the same ward, as wards cater for different specialities which allows clinical skills and experience to be matched to the needs of patients.

[4] Independence between the capacity of the operating-theatre and the length of the recovery time in hospital is implicitly assumed throughout this section.

Proof. We can rewrite (7) as $\bar{b}_N = \frac{1}{T}\sum_{i=1}^{n}\frac{\lambda_i}{\Lambda}d_i + \sum_{i=1}^{n}\lambda_i d_i$.

Using the expression above and equation (6), it is easy to see that:

$$\sum_{i=1}^{n}\bar{b}_i - \bar{b}_N = \sum_{i=1}^{n}\frac{d_i}{t_i} + \sum_{i=1}^{n}\lambda_i d_i - \left(\frac{1}{T}\sum_{i=1}^{n}\frac{\lambda_i}{\Lambda}d_i + \sum_{i=1}^{n}\lambda_i d_i\right).$$

Thus, $\bar{b}_N < \sum_{i=1}^{n}\bar{b}_i$ if and only if $\sum_{i=1}^{n}\lambda_i d_i < \Lambda T\sum_{i=1}^{n}\frac{d_i}{t_i}$.

This completes the proof. □

Proposition 3 states that sharing an operating-theatre does not always lead to lower average costs in the second stage of the process. The smaller the ratio $\frac{T}{t_i}$ is for each medical treatment, the more difficult it is for the condition in Proposition 3 to be fulfilled.

We shall now provide two sufficient conditions that ensure savings in the expected post-operative costs from sharing the operating-theatre, and which have a clearer intuition. The following one is an immediate consequence of Proposition 3:

Corollary 1 *If $\Lambda T > \lambda_i t_i$, $\forall i = 1, 2, ..., n$, sharing the operating-theatre reduces the average post-operative costs.*

The conditions expressed in Corollary 1 imply that each speciality uses, in expected terms, a fraction of the total capacity which is smaller than the fraction it would have to set if it worked alone. This is enough to guarantee a reduction in the average number of beds, when the different specialities share the operating-theatre.

Proposition 4 *If $d_n > \frac{\sum_{i=1}^{n-1}\lambda_i d_i}{\Lambda - \lambda_n}$, sharing the operating-theatre reduces the average post-operative costs.*

Proof. By some careful but straightforward computations one can see that:

$$\text{If } d_n > \frac{\sum_{i=1}^{n-1}\lambda_i d_i}{\Lambda - \lambda_n}, \text{ then } d_n\left(\frac{\Lambda - \lambda_n}{\Lambda}\right) > \sum_{i=1}^{n-1} d_i\left(\frac{\lambda_i}{\Lambda} - \frac{t_n}{t_i}\right).$$

Thus, it can be shown that $\frac{1}{T}\sum_{i=1}^{n}\lambda_i d_i < \Lambda\sum_{i=1}^{n}\frac{d_i}{t_i}$.

That is, the hypothesis of Proposition 3 holds, which completes the proof. □

Proposition 4 states that if the medical procedure with the highest priority has a longer expected recovery time than the average recovery time of the rest, by co-operating in the operating-theatre the expected amount of required beds is reduced.

The lack of generality of the results on the required number of beds prevents us from computing explicitly the optimal cost-sharing scheme of the post-operative costs. In the following section, we will study a particular problem in which the recovery costs are computed when there is co-operation in the management of beds and when there is not. This way we may choose the most efficient situation, and directly compute the Shapley value of the second stage cost-sharing problem.

6 A Numerical Example

In this section, we provide a numerical example to illustrate the main features of our analysis. The basis of our example is real data obtained from a small hospital, on the expected number of arrivals of patients and their expected recovery time. To set the legal average-time of the procedures, we took the actual time spent by the patients in the waiting-lists of these medical disciplines.

We analyze six medical procedures ($n = 6$), all of an elective nature and with a very short recovery time. The data is summarized in Table 1.

We start by computing the optimal tariff for the operating-theatre costs. In this example, $T = \frac{1}{3}$. The Shapley value of c_1^o and of c^o, as well as the patients' fees are given in Table 2, omitting k to improve the readability of the table.

In Table 2 we see how $Sh_i(c_1^o)$ is increasing in the level of priority of the discipline. $Sh_i(c^o)$, however, does not respect this ranking, as it is also affected by c_2^o. $\phi_j^*(c^o)$ is the fee that each patient j of discipline i, $i \in N$ should pay. $\phi_j^*(c^o)$ is computed following Proposition 2, where m_i denotes the proportion of type i patients treated in a given time, expressed as a fraction of the operating-theatre total capacity, i.e., $m_i = \frac{\lambda_i}{\Lambda}\mu(T)$.

We move now to the analysis of the recovery period. The condition in Proposition 4 (and thus Proposition 3) is fulfilled. Consequently, there is a saving in the expected number of beds when the specialities share the operating-theatre. Then, assuming that the operating-theatre is shared, we analyze the effects on the recovery costs of sharing the second set of servers. To do so, we compute the required number of beds under two different scenarios: (S1) If the procedures share the operating-theatre but do not share the beds; and (S2) If they share both theatre and beds.

By means of Equation (7), we compute the average number of beds in both scenarios. This amount, however, does not guarantee that every patient has a bed when leaving the operating-theatre. Using Erlang's C formula, we compute the minimum number of beds that ensure, with a probability of .9, that there is a bed available when it is needed.

Table 1. Monthly numbers on arrivals, legal average-times and recovery times

	Knee Replacements (r)	Cataract Surgery (c)	Hysterectomies (hy)	Arthroscopies (a)	Inguinal Hernias (h)	Varicose Veins (v)
l_i	12	129	19	39	33	15
t_1	4	2	1	½	1/3	1/3
d_i	0.266	0.043	0.243	0.083	0.074	0.083

Table 2. Sharing of the operating-theatre costs

	Knee Replacements (r)	Cataract Surgery (c)	Hysterectomies (hy)	Arthroscopies (a)	Inguinal Hernias (h)	Varicose Veins (v)
$Sh_i(c_1{}^o)$	1/24	11/120	13/60	11/20	21/20	21/20
$Sh_i(c^o)$	12.041	129.091	19.216	39.55	34.05	16.05
$\phi_j^*(c^o)$	0.99138	0.9887	0.99923	1.00193	1.0194	1.0572

Table 3. Number of beds in scenarios 1 and 2

$\bar{b}_N^{s1}$	b_N^{s1}	$\bar{b}_N^{s2}$	b_N^{s2}
20.50	38.79	20.50	27.23

Table 4. Sharing of the post-operative costs

	Knee Replacements (r)	Cataract Surgery (c)	Hysterectomies (hy)	Arthroscopies (a)	Inguinal Hernias (h)	Varicose Veins (v)
$Sh_i(c^b)$	3.8401	7.3502	6.1996	4.488	3.4734	1.9147
$\phi_j^*(c^b)$	0.317	0.057	0.3263	0.1151	0.1052	0.1276

In (S1), the demand for beds from each speciality is as follows: $b_r^{S1} = 6.26$, $b_c^{S1} = 9.43$, $b_{hy}^{S1} = 8.21$, $b_a^{S1} = 6.34$, $b_h^{S1} = 5.20$ and $b_v^{S1} = 3.35$. In (S2), we compute the amount of beds by considering the rate of patients' arrivals as the sum of the expected arrivals across procedures, and the recovery time as the expected length of stay in hospital for the treatments, weighted by the proportion of patients demanding beds in each procedure.

Table 3 shows the expected number of beds required, and the amount of beds that ensure that an arriving patient does not have to wait ($\bar{b}_N^{S1}$, b_N^{S1} and $\bar{b}_N^{S2}$, b_N^{S2} respectively).

When the servers in the recovery stage are not shared, the presence of randomness implies that, to ensure a negligible probability of waiting, the number of beds has to be almost doubled. This extra amount, however, is reduced when the procedures share the beds. By sharing the second set of servers, therefore, the extra capacity is reduced by 50% . The same waiting probability can be achieved by fixing only 7 extra beds in (S2), instead of the 18 in (S1). Moreover, co-operation in the management of beds yields a nearly 30% reduction in the total need of beds (27.23 instead of 38.79).

Thus, a second cost-sharing game, $c^b : 2^N \to \mathbb{R}$, appears. It assigns to any non-empty coalition $S \subset N$, the minimum bed-cost $c^b(S)$, required for *all the surgical procedures in S* , and $c^b(\emptyset) = 0$. We distribute, then, the recovery costs among the specialities by directly computing the Shapley value, $Sh_i(c^b)$, that, in this case, turns out to be in the Core of c^b. We also compute the optimal fee per patient at the recovery stage, $\phi_j^*(c^b)$. Table 4 presents the results.

As it can be seen, the cost-share assigned to each procedure is increasing in the number of beds that it would require if the servers were not shared. By comparing these cost-shares with the ones computed before for (S1), namely b_i^{S1}, we see how the savings range from the 22% reduction for cataract surgery, to the 43% savings that varicose veins achieve. We also observe that the most demanding discipline, in terms of beds required (cataracts), is the one that benefits the least from co-operation and, conversely, varicose veins (whose necessity of beds is the least) enjoys the greatest fraction of the savings.

The total fee per patient j would be directly computed from tables 2 and 4, as $\phi_j^*(c) = \phi_j^*(c^o) + \phi_j^*(c^b)$.

7 Conclusions

In this paper, we have modeled the problem of the waiting-lists for surgical treatment, making use of queueing theory.

We have made the simplifying assumptions of considering an exponential distribution of the time between two subsequent arrivals and service times. The other extreme would be to assume arrivals and service times that are always constant. Worthington (1987) has suggested that the realistic distribution is often somewhere in between.

We are aware that many hospitals are perpetually in transient conditions because of non-homogeneity in arrivals and servers rates. Almost all of the existing results of queueing theory are, however, obtained for equilibrium conditions. Consequently, our model can be useful in pointing out directions for improving the system, and the results obtained should be viewed as approximate indicators of the real performance of the system.

We have concentrated on the costs that operations generate, considering that the higher the resources spent by the hospital, the shorter its resulting waiting-list. In this respect, two main assumptions have been made. First, we have considered that the costs of the servers (either operating-theatre or hospital-beds) are proportional to the amount of patients treated, irrespective of the speciality they belong to. Second, we have ruled out the possibility that a server that handles all the types of patients could be more expensive to construct (or to maintain). These assumptions are crucial to obtain our results and limit the analysis to the case of specialities with similar surgical and recovery requirements. Most low-risk elective surgery, however, fulfills them. The study of alternative models that deal with more general cost-structures is left for further research.

The numerical example provided illustrates the main features of our model. Even though there are no general results for the beds cost-sharing game, the optimal tariff can be easily computed in the example. This fact also highlights the possibilities of directly addressing cost-sharing games of this type, under different unit-costs per procedure.

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