AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING

A thesis presented to the faculty of
San Francisco State University
In partial fulfilment of
The Requirements for
The Degree

Master of Arts
In
Mathematics

by

John P Gallagher

San Francisco, California

May 2022

Copyright by
John P Gallagher
2022

CERTIFICATION OF APPROVAL

I certify that I have read *AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Dr. Chun-Kit Lai
Associate Professorof Mathematics

Dr. Emily Clader
Assistant Professorof Mathematics

Dr. Arek Goetz
Professorof Mathematics

AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING

John P Gallagher
San Francisco State University
2022

We say that a set E is *u*niversal in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdös conjecture in a topological setting.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

ACKNOWLEDGMENTS

I want to take a moment to list a few people who have shaped my pursuit of math. Firstly, thank you Mom and Dad for supporting. Mom, you sat with me at the dinner table teaching me

TABLE OF CONTENTS

Chapter 1

Introduction

Key words:

- Dynamical Systems
- Density and Measure are not clearly linked
- Geometry
- Fractals poorly defined
- Self Similarity is more well defined
- Cantor Set
- $\mathbb{R}^n$ Fractals
- Erdös Proposed Conjecture with Measure Space assumptions
- Theorem with Topological Assumptions.

- Open questions

Chapter 2

Measure & Topology

2.1 Topological versus Measure Theoretic Size

Topological size is not the same thing as Measure Theoretic size.

A measure theoretically large set is not necessarily topologically large.

What do we mean by measure theoretically large? Non-zero Measure.

Definition 2.1 (Measure)**.** Let X be a set and Σ be a $\sigma-$algebra over X. A function $\mu : \Sigma \to \{\mathbb{R} \cup \infty\}$ is called a measure if it satisfies the following properties:

1. **Non-negativity**: for all $E \in \Sigma$, $\mu(E) \geq 0$.

2. **Null empty set**: $\mu(\emptyset) = 0$.

3. **Countable Additivity** (σ-additivity): For all countable collections $\{E_k\}_{k=1}^{\infty}$

of pairwise disjoint sets in Σ,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

What do we mean by topologically Large? Uncountable and dense. It is helpful to define the opposite of topologically large, namely meager sets.

Definition 2.2 (Nowhere Dense)**.** Let X be a topological space. A subset $B \subseteq X$ of a topological space is called *nowhere dense* in X if its closure has an empty interior. That is to say, B is *nowhere dense* in X if for each open set $U \subseteq X$, $B \cap U$ is not dense in U.

Definition 2.3 (Meager)**.** A subset $C \subseteq X$ of a topological space is called *meager* in X if it is the countable union of nowhere-dense subsets of X.

Definition 2.4 (G-Delta Set)**.** A G_δ set is the countable intersection of open sets. Namely, let $O_i \subset X$ for $i \in \mathbb{N}$ be a collection of open sets of X. Then $\bigcap_{n=1}^{\infty} O_i$, is a G_δ set.

Example 2.1. The irrational numbers are a G_δ set. Consider the following construction of the set of irrational numbers:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Notice that each $\mathbb{R} \setminus q = (-\infty, q) \cup (q, \infty)$ is an open subset of $\mathbb{R}$. Furthermore, rational numbers are countable. Therefore the intersection of these sets are a G_delta set. Moreover, in this instance it is a dense G_δ set. We will study these objects further.

2.2 Cantor Sets

The Cantor set is defined by taking the interval $[0, 1]$ and then iteratively removing the open interval containing the middle third, from the previous level.

Formally this can be written as follows.

Definition 2.5 (Cantor Set)**.** The Cantor set $\mathcal{C}$, written as the successive removal of each middle third removed from the previous level is

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

Definition 2.6. Generalized Cantor's Set Middle Third Cantor Set

Example 2.2. Decimal Expansion Cantor Set

2.3 The Baire Category Theorem

Theorem 2.1 (Baire Category Theorem)**.** *The countable intersection of open dense sets is dense.*

Example 2.3. The irrational numbers

Chapter 3

An Erdös Conjecture in Measure Theory

3.1 Affine Copies and the Self Similarity Property

In order to define self-similar sets, we first need to define

Definition 3.1. Affine An affine copy of a set A is a set A' such that

$$A' = \{\lambda a + t : a \in A, t \in \mathbb{R}, and \lambda \neq 0\}.$$

3.2 An Erdös Self-Similarity Conjecture in Measure Space

Chapter 4

An Erdös Similarity Problem in a Topological Setting

4.1 The Gap Lemma

Lemma 4.1. *The Gap Lemma[3] Let $K_1, K_2, \subset \mathbb{R}$ be cCantor sets with thickness τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$, then one of the following three alternatives occurs: K_1 is contained the gap of K_2; K_2 is contained in the gap of K_1; $K_1 \cap K_2 \neq \emptyset$.*

4.2 Positive Newhouse Thickness

4.3 A Cantor set with positive Newhouse Thickness is not universal

We say that a set E is *universal* in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdös conjecture in a topological setting.

Theorem 4.2. *Let J be a cantor set with positive Newhouse thickness. Then J is not universal.*

Proof. Suppose we have some Cantor set J with Newhouse thickness $\tau(J) > 0$. Without loss of generality, we can assume the convex hull of J $[0, 1]$. Consider Cantor sets K defined by contraction ratio $1/N$ and digits $\{0, 1, ..., N-1\} \setminus \{(N-1)/2\}$ and N is odd. By a simple calculation, $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(J)\tau(K) > 1$.

Using the Cantor set K Define X such that

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} N^n(K + \ell),$$

creating a dense F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, by the Baire Category Theorem, X^c is a dense G_δ. We now show that X^c contains no affine copy of J.

Suppose we have some affine copy, $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{4.1}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell+1)N^n]. \tag{4.2}$$

We claim that this affine copy of J has a non-empty intersection with $N^n(K+\ell)$. This is equivalent to showing that

$$t \in N^n(K+\ell) - \lambda J.$$

For consistent notation with a referenced theorem, let

$$C_1 = N^n(K+\ell) \text{ and } C_2 = -\lambda J.$$

First we check the construction of our Cantor sets. For C_1 its largest corresponding open gap interval is $|O_1| = N^{n-1}$ and its largest corresponding closed interval is $|I_1| = N^n$. For C_2 and is corresponding intervals, we find that $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$

and $|I_2| = |\lambda|$ where O_J is the largest open gap interval in J. Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|$$

as in the condition of Theorem 2.2.1 in [1]. By [1, Theorem 2.2.1][1], given that the Newhouse thickness of our sets, $\tau(K)\tau(J) \geq 1$ then $C_1 + C_2 = I_1 + I_2$. Note that $I_1 = [\ell N^n, (\ell+1)N^n]$, $I_2 = [-\lambda, 0]$ if $\lambda > 0$ and $I_2 = [0, -\lambda]$ if $\lambda < 0$. we find that

$$I_1 + I_2 = [N^n\ell - \lambda, N^n(\ell+1)] \ (\lambda > 0) \text{ and } I_1 + I_2 = [N^n\ell, N^n(\ell+1) - \lambda](\lambda < 0).$$

Then from (2)

$$t \in I_1 + I_2.$$

Therefore the affine copy of the cantor set $t + \lambda J$ has a non-empty intersection with X and J cannot be universal.

$\square$

It would be interesting to study those Cantor sets with Newhouse thickness zero. We do not know what would happen. However, it seems like if we assume a weaker condition on J.

[1] This might misattribute the theorem. I think Astels '99 Theorem 2.2.1 is actually is quoting Newhouse directly. In particular I think it refers to Newhouse 1979 [2] *The Abundance of Wild Hyperbolic Sets, and Non-smooth Stable Sets for Diffeomorphisms.*

$(*)$: There exists K such that $J+K = I_J + I_K$, where I_J, I_K are the smallest closed interval containing J and K.

we may be able to show that J cannot be universal for dense G_δ sets.

4.4 Current Research Questions: Zero Newhouse Thickness

This section is devoted to study if Cantor sets with zero Newhouse thickness can be universal. We first provide an example for which two Cantor sets with zero Newhouse thickness can still have arithmetic sum equal to an interval, showing that the converse of theNewhouse thickness theorem us not true.

Example 4.1. Let $N_1, N_2, \dots \in \mathbb{N}_{\geq 2}$. Consider the following construction of a Cantor set using a decomposition of the unit intervals.

$$\begin{aligned}
[0,1] =& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \left[0, \frac{1}{N_1}\right] \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \left[0, \frac{1}{N_1N_2}\right] \\
=& \dots \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \cdots + \frac{1}{N_1\cdots N_n}\{0,\dots,N_n\} + \dots
\end{aligned}$$

From here we can define the two cantor sets K_1, K_2 where K_1 constitutes the odd indices sets in the above summands and K_2 has the even one. This gives the following constructions for the two Cantor sets:

$$K_1 = \frac{1}{N_1}\{0, 1, \dots, N_1 - 1\} + \dots + \frac{1}{N_1 \cdots N_{2n+1}}\{0, \dots, N_{2n+1} - 1\} + \dots$$

$$K_2 = \frac{1}{N_1 N_2}\{0, 1, \dots, N_2 - 1\} + \dots + \frac{1}{N_1 \cdots N_{2n}}\{0, \dots, N_{2n} - 1\} + \dots$$

From this construction we see that $K_1 + K_2 = [0, 1]$ is the interval but from the definition of Newhouse thickness,

$$\tau(K_1) = \inf\left\{\frac{1}{N_1 - 1}, \frac{1}{N_3 - 1}, \dots\right\} = 0$$

$$\tau(K_2) = \inf\left\{\frac{1}{N_2 - 1}, \frac{1}{N_4 - 1}, \dots\right\} = 0.$$

Therefore we have created an interval from two sets with Newhouse thickness 0 if we have $\lim_{n\to\infty} N_n = \infty$.

We ask the following questions. Recall I_J denotes the smallest closed interval containing the Cantor set J and O_J denotes the largest open interval in $I_J \setminus J$.

1. Given a Cantor set J, does there exist some K such that $J + K = I_J + I_K$?

2. If we assume that there exists K such that $J + K = I_j + I_K$, can we prove that J is not universal?

3. (rescaling condition) If we assume that $|\lambda_1 I_J| \geq |\lambda_2 O_K|, |\lambda_2 I_K| \geq |\lambda_1 O_J|$ and

$J+K=I_J+I_K$, then $\lambda_1 J+\lambda_2 K=\lambda_1 I_J+\lambda_2 I_k$.

We also notice that to solve the second question, we notice that $J+K=I_J+I_K$ implies that

$$(J+a)+(K+b)=(I_J+a)+(I_K+b) \text{ and } bJ+bK=bI_J+bI_K.$$

We can always translate and rescale J,K so that $I_J=[0,a]$ and $I_K=[0,1]$. Moreover, the following lemma is important.

Lemma 4.3. *Suppose that the Cantor sets J and K satisfies $J+K=I_j+I_K$. Then $|I_J|\geq|O_K|$ and $|I_K|\geq|O_J|$.*

The lemma also said that the condition $|\lambda_1 I_J|\geq|\lambda_2 O_k|, |\lambda_2 I_k|\geq|\lambda_1 O_J|$ is necessary in the rescaling condition.

Proposition 4.4. *Let J be a Cantor set such that $J+K=I_J+I_K$ where $I_J=[0,a]$ and $I_K=[0,1]$. Suppose that the rescaling condition (3) holds. Then J is not universal in the collection of dense G_δ.*

Proof. The proof is similar to the proof in Theorem 4.2. With K given in the assumption. We can assume that $|I_J|>|O_K|$. Suppose that $|I_J|=|O_K|$. Since $|O_J|<1$, we can choose ϵ such that $(1-\epsilon)>|O_J|$. Then we consider $K'=(1-\epsilon)K$ and we will have $|I_J|>(1-\epsilon)|O_K|$. In this case, by the rescaling condition, $J+K'=I_J+I_{K'}$ and we have another K' such that $|I_J|>|O_{K'}|$.

As now we have $|I_J| > |O_K|$, we can find $0 < \rho < 1$ such that $\rho|I_J| > |O_K|$. We now define

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} \rho^n (K + \ell).$$

Then X^c is a dense G_δ set. Suppose that we have an affine copy $t + \lambda J$, we would like to claim that $t + \lambda J$ intersects non-trivially with $\rho^n(K + \ell)$ for some $n, \ell \in \mathbb{Z}$, which will complete the proof of the theorem.

To justify the claim, we let $0 < \rho < 1$ take the unique n such that

$$|\lambda| \in [\rho^{n+1}, \rho^n) \tag{4.3}$$

and the unique $\ell \in \mathbb{Z}$ such that

$$t \in (\ell\rho^n, (\ell + 1)\rho^n]. \tag{4.4}$$

Then we consider the arithmetic sum $\rho^n K - \lambda J$. We now check the assumption in the rescaling condition with $\lambda_1 = \rho^n$ and $\lambda_2 = -\lambda$. Indeed,

$$|\lambda_2 I_J| \geq \rho^{n+1}|I_J| = |\lambda_1|(\rho|I_J|) \geq |\lambda_1 O_K|$$

by our choice of ρ. On the other hand,

$$|\lambda_1 I_K| \geq |\lambda| \geq |\lambda_2 O_J|$$

since $|I_K| \geq |O_J|$ by Lemma 4.3. Hence, using the rescaling condition,

$$\rho^n K - \lambda J = \rho^n I_K - \lambda I_J.$$

If $\lambda > 0$, then we have

$$\rho^n(K + \ell) - \lambda J = [\rho^n \ell - \lambda a, \rho^n(1 + \ell)]$$

which contains t by (4.4). Similarly, if $\lambda < 0$, then

$$\rho^n(K + \ell) - \lambda J = [\rho^n \ell, \rho^n(\ell + 1) - \lambda a].$$

It also contains t by (4.4). The proof is now complete. $\square$

****

From these questions we have several difficulties associated with each. For the first item it is not always clear which cantor sets can be added to each other. Similarly it is difficult to construct a complementing Cantor set because of the difficulties tracking the notation for the different possible open intervals. There maybe some existing tools. It may also just be messy.

For the second point, our proof inherently relies on appropriately selecting a scalar and translation that corresponds to a regular (or fairly regular) Cantor set. In this instance we have to find pick the appropriate λ, t based off of a set of associated intervals that are not uniform. Our proof relies on using the regularity to specify where the intersection is.

A current tool we are exploring is tracking how scaling and translating the collection of intervals $\{O_j\}_{j\in\mathbb{N}}$ by some appropriate bound M such that we can scale our cantor set by $\frac{1}{M^d}$, and demonstrate an appropriate intersection with X^c.

The last question we discussed for the day focused on how scaling Cantor sets, and scaling intervals are interrelated. With Newhouse thickness, because it relies off of the ratios of $\frac{I_j}{O_{j-1}}$ the scaling factor drops out. Unfortunately if we are considering Cantor sets with Newhouse thickness 0, then there is no corresponding Cantor set with infinite Newhouse thickness. The issue is that from the theorem, the thickness is the product of the two sets so for any finite thickness $0 \cdot \tau(C) = 0$. Therefore Newhouse thickness will not be enough to describe the appropriate construction of the interval. There are a few workarounds that might be possible. In Astels' paper[1] there is a generalized for for countably many cantor sets. Similarly we might be able to find another characterization (measure, dimension etc) of the set, to appropriately find λ and or, another way to combine the two intervals, such that we have a non-empty intersection with X^c.

Bibliography

[1] S. Astels. “Thickness measures for Cantor sets”. In: *Electron. Res. Announc. Amer. Math. Soc.* 5 (1999) (1999), pp. 108–111.

[2] Sheldon E. Newhouse. “The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms”. en. In: *Publications Mathématiques de l’IHÉS* 50 (1979), pp. 101–151. URL: `http://www.numdam.org/item/PMIHES_1979___50__101_0/`.

[3] J. Palis and F. Takens. *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations.* 40 West 20th St., New York, NY 10011, USA: Cambridge University Press, 1993.