On a Topological Erdős Similarity Problem

A thesis presented to the faculty of
San Francisco State University
In partial fulfilment of
The Requirements for
The Degree

Master of Arts
In
Mathematics

by

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San Francisco, California

May 2022

CERTIFICATION OF APPROVAL

I certify that I have read *On a Topological Erdős Similarity Problem* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Often in data work, one may ask which patterns are possible to find, given a certain data set, or hypothetical relationship. Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets? A set is called universal in another set, when every subset of the larger set contains some scaled and translated copy of original. Paul Erdos proposed a conjecture that no infinite set, is universal in the the collection of sets with positive measure. This paper explores an analogous problem in a topological setting. Instead of sets with positive measure we investigate the collection of dense G-delta sets. Any finite or countable set is found to be topologically universal. Any set containing an interval cannot be topologically universal. We also have the new result that any Cantor sets is not topologically universal. Cantor sets, which contains no interval and are uncountably infinite, are not topologically universal in the collection of dense G-delta sets.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

ACKNOWLEDGMENTS

I want to take a moment to list a few people who have shaped my pursuit of math. First, thank you Dr. Chun-Kit Lai. I remember my first class in analysis with you. I have learned so much from you and have grown from working with you. You have been a true teacher, mentor, and friend.

I also want to name several other professors. Dr. Arek Goetz, my time at SFSU started with your online calculus course and a recommendation for the Math program. It is so fitting that you are also teaching me my last course at SFSU. Dr. Emily Clader, you have taught me about the subtleties of math, the subtleties of explanation, and how they are not the same thing! You have also done this while encourage a greater diversity in perspectives and collaboration. You have also pointed out my excessive use of commas. Dr. Dusty Ross, your bonus problems sparked my interest in topology and you encouraged me to take my first topology course with Dr. Clader.

Thank you Mom and Dad for supporting me. Mom, you sat with me at the dinner table, where we did puzzles about shapes, volumes, and space. At the same time your fearlessness, your drive for your own career, and your love of our family has inspired my own path for learning. Dad,

you taught me about sizes of infinity and the power of listening to and understanding others. Both of you continue to push the limits of my imagination.

Patrick O'Melveney, I am so glad we have become friends through such unlikely circumstances. You have helped me better understand math and myself.

Finally, thank you to my wife. Kim, you tell it like it is. You help me see things as they are. You gave me the courage to go to graduate school and the support to stay in graduate school. For that I am more grounded and supported.

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Chapter 1

Introduction

Often in data work, one may ask, "Which patterns are possible to find, given a certain data set, or hypothetical relationship?" Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets?

If we more deeply consider patterns which infinitely repeat scaled copies of itself, this turns out to be a self-similar set. This is very closely related to the concept of fractals as well as a core area of study within dynamical systems. Some fractals can be generated using recursive functions. In dynamical systems some iterated functions may have sets of invariant points. These collections of invariant points may themselves be self-similar. In this respect it is valuable to explore which patterns are found everywhere because they give insight into some key facets within dynamical systems as well as the nature of patterns.

Given a specific set of points, we can formalize this notion of a scaled copy of a pattern as an *affine transformation* or *affine copy.* We will formally define this in the next section

Exploring this notion a little more deeply, we can begin to investigate which patterns appear everywhere. Informally, a set is called *universal* in another collection, when every subset of the collection contains some scaled and translated copy of original pattern.

Paul Erdős proposed a conjecture that no infinite set, is measure-universal in the collection of sets with positive measure. We will explore an analogous problem in a topological setting. This also answers a question posed by Svetic[9], "Is it true that for every uncoutably infinite set, E, of real numbers, there exists $S \subset [0, 1]$ of full measure that does not contain an affine copy of E? Additionally this result can be used to prove some results in higher dimensions.

1.1 An Erdős Self-Similarity Conjecture in Measure Space

There is a long standing conjecture from Paul Erdős on universal sets. Informally the conjecture states that there is no infinite set that is universal in the real number line. This is a conjecture about which types of patterns can exist within another sets of numbers.

We can now start by defining *affine transformations* or *affine copies.*

Definition 1.1 (Affine copy)**.** An affine copy of a set $A \subset \mathbb{R}$ is a scaled and translated set A' such that for some $\lambda \neq 0, \lambda \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$A' = \{\lambda a + t : a \in A\}.$$

In this instance, it is a scaled and then translated copy of the set is still one dimensional and therefore need not be a "shape". This gives us some flexibility when addressing different sets. This definition is used to define measure universal.

Definition 1.2 (Measure Universal)**.** A set E is called <u>measure universal</u> in X if for every subset $S \subseteq X$, with positive measure, $\mu(S) > 0$, there exist an affine copy of E such that $t + \lambda E \subseteq S$, for some $\lambda \neq 0$ and $t \in \mathbb{R}$.

Now we can formally, state Erdős' conjecture as follows.

Conjecture 1.1 (The Erdős Self-Similarity Conjecture)**.** *Let $E \subseteq \mathbb{R}$ be an infinite set of real numbers. Prove that there is a set of real numbers S of positive measure which does not contain an affine copy of E.*

Paul Erdős originally posed this question back in 1974 by building off of the work of Steinhaus. Steinhaus[8] first posed that finite sets are universal in sets with positive measure. After Erdős build on this with his conjecture, there has been some progress.

Later Falconer [4] made substatioal progress by showing slowly decaying sequences are not measure universal. Bourgain [2] expanded on this by showing some faster decaying sequences are also not measure universal. In particular he demonstrated that the sumset of any three sets, cannot be measure universal. Most recently Kolountzakis [5] demonstrated using probabilistic arguments to demonstrate that certain set with large gaps cannot be measure universal.

Currently it is still an open question whether or not sequences that decay at the rate of 2^{-n} are measure universal.

In this paper we take this idea of measure universality and put it into a topological context. We show in theorem 3.2 no cantor set with positive Newhouse thickness is universal in the set of dense G_δ sets.

1.2 An Analogous Theorem in a Topological Setting

In a non-rigorous exploration of the real numberline, one might assume that patterns which appear everywhere should have positive measure. However density and measure are not intrinsically linked. Indeed it is possible to have uncountable dense sets with measure zero and to have sets with full measure that are nowhere dense.

Borrowing the conept of G_δ sets from topology, we can explore an extension of the Erdős similarity problem. Instead of exploring sets with positive measure we can explore dense G_δ sets. This is the countable intersection of open sets that are also dense. We will use these dense G_δ sets to define topological-universal.

Definition 1.3 (Topological-Universal)**.** A set E is called <u>Topological-universal</u> in

$\mathbb{R}$ if for every dense G_δ subset $S \subseteq \mathbb{R}$, there exist an affine copy of E such that $t + \lambda E \subseteq S$, for some $\lambda \neq 0$ and $t \in \mathbb{R}$.

As stated above, a dense G_δ set can have measure zero. This also means that there is not a direct relationship between topological universal and measure universal. We make two observations on this fact: a set with an interior cannot have be topologically universal, by the Baire Category Theorem all countable sets are topologically universal.

This motivates the quesion, Is a nowhere dense set, with an empty interior topologically universal? Cantor sets for example, have an empty interior and are nowhere dense.

In chapter 2 we review background definitions and theorems in measure theory and topology. We also give several examples to demonstrate some of the nuances of these facts. In chapter 3 we define Newhouse Thickness, prove the Gap Lemma and prove our main result theorem 3.2.

Theorem. *Let J be a cantor set with positive Newhouse thickness. Then J is not topologically universal in the collection of dense G_δ sets.*

These results can also be extrapolated into $\mathbb{R}^d$ by defininining projective Newhouse thickness. Finally in chapter 4 we conclude with some open questions and remarks.

Chapter 2

Measure & Topology

2.1 Some Measure Theory

First we want to begin with background definitions and theorems. Erdös' problem specifically deals with infinite set, and affine copies found in measurable set of sets. In our problem, rather than dealing with measurable sets, we will instead use the set of dense G_δ sets.

Underpinning the nuances of this problem, measure theoretic size, and topological size, are not the same. From an intuitive sense of the number line one might think when you are scattered throughout an interval, you would have measure, except in special cases. Similarly one might think that if you have measure, then you would be scattered everywhere. However both of these instances fail when you add in rigorous arguments. Indeed it is possible to construct a set that is no-where dense and has positive measure. It is also possible to construct an uncountable set that is dense and has measure. zero. In other words topological size (density) is not the same thing as measure theoretic size.

First we will review some measure theory. In order to define *measure* and *measurable sets* we first need to define σ-Algebra.

Definition 2.1 (σ-Algebra)**.** Let X be some set and 2^X be the set of subsets of X. Let $\Sigma \subseteq 2^X$. We call Σ a $\sigma-$algebra over X if it satisfies the following three conditions:

1. $\emptyset \in \Sigma$

2. If $E \in \Sigma$, then $X \setminus E \in \Sigma$.

3. If $E_1, E_2, \cdots \in \Sigma$ is a sequence of subsets, then $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

In this instance we describe set of sets in terms of inersection and union. This allows us to generate an algebraically closed colleciton of sets.

In the Erdős conjecture, the measure is specifically referencing *Lebesgue Measure.* First we define *Lebesgue outer measure* which is defined on all sets. Then to define *Lebesuge Measure* we will restrict the universe to the appropraite sigma algebra of measurable sets

Definition 2.2 (Lebesgue Outer Measure[1])**.** For any interval $I = [a, b]$ (or $I = (a, b)$) in the set $\mathbb{R}$ of real numbers, let $\ell(I) = b - a$ denote its length. For any subset $E \subseteq \mathbb{R}$, the *Lebesgue outer measure* $\lambda^*(E)$ is defined as an inf :

$$\lambda^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

Finally we define Lebesgue Measure.

Definition 2.3 (Lebesgue Measure[3])**.** A subset $E \subset \mathbb{R}$ is *Lebesgue measurable* if for any $\epsilon > 0$, there exists some open subset $\mathcal{O} \subset \mathbb{R}$ such that $E \subset \mathcal{O}$ and

$$\lambda^*(E - \mathcal{O}) < \epsilon.$$

If E is Lebesgue measurable then we define the *Lebesgue measure* as

$$\lambda(E) = \lambda^*(E).$$

Notice that not all sets are necessarily Lebesgue measurable, such as Vitali sets. We will not explore non-measurable sets because it falls outside of the scope of our current theorems.

2.2 Topological and The Baire Category Theorem

Now we will review some Topology that is relevant to our main theorem. We need to define *dense*. From there we will build into G_δ sets. There are many equivalent definitions of dense. We will use the following definition so that we can continue to develop the intuion around intervals and interiors.

Definition 2.4 (Dense)**.** A set S is called dense in X if for every $x \in X$, every neighborhood U of x intersects A.

In a similar fashion to the definitions of *countable* and *uncountable*, the oppsite of *dense* is *nowhere dense*.

Definition 2.5 (Nowhere Dense)**.** Let X be a topological space. A subset $B \subseteq X$ of a topological space is called *nowhere dense* in X if its closure has an empty interior. That is to say, B is *nowhere dense* in X if for each open set $U \subseteq X$, $B \cap U$ is not dense in U.

This allows us to now explore the differences between density and measure. As stated earlier, topological size and measure theoretic size are not necessarily related. What do we mean by topologically Large? Uncountable and dense. Similarly what do we mean by measure theoretically large? Non-zero measure. It is helpful to define the opposite of topologically large, namely meager sets.

Definition 2.6 (Meager)**.** A subset $C \subseteq X$ of a topological space is called *meager* in X if it is the countable union of nowhere-dense subsets of X.

Next we will define dense G_δ sets, as well as some useful examples.

Definition 2.7 (G-Delta Set)**.** A G_δ set is the countable intersection of open sets. Namely, let $O_i \subset X$ for $i \in \mathbb{N}$ be a collection of open sets of X. Then $\bigcap_{n=1}^{\infty} O_i$, is a G_δ set.

Example 2.1. The irrational numbers are a G_δ set. Consider the following construction of the set of irrational numbers:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Notice that each $\mathbb{R} \setminus \{q\} = (-\infty, q) \cup (q, \infty)$ is an open subset of $\mathbb{R}$. Furthermore, rational numbers are countable. Therefore the intersection of these sets are a G_δ set. Moreover, in this instance it is a dense G_δ set. We will study these objects further.

Lastly we remark that there is an analogous set which is the countable union of closed sets.

Definition 2.8 (F_σ Set)**.** An F_σ set is the countable union of closed sets. This is equivalent to the compliment of a G-delta is an F-sigma set.

A key theorem that links analysis to set theory is the Baire Category Theorem. This also establishes a link to understanding certain types of topological sets.

Theorem 2.1 (Baire Category Theorem[6])**.** *Let X be a complete space. Then any countable intersection of open dense sets is dense.*

Within the study of measure theory it can sometimes be unclear if a set is dense in another set. For example consider the following set:

$$\mathbb{R}^2 \setminus \{(x, y) : y = mx + b, \text{where } m, b \in \mathbb{Q}.\}$$

Notice that this can also be written as

$$\bigcap_{m,b\in\mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = mx + b\},$$

which is the plane, but removing all lines with rational coefficients, and rational intercepts. Each plane with one line removed, $\mathbb{R}^2 \setminus \{(x, y) : y = mx+b, \text{ for some } m, b \in \mathbb{Q}\}$ is an open dense set. Then by the Baire Category Theorem, the countable intersection is dense. Moreover this same idea holds for removing polynomials with rational coefficients. We can write this as the countable intersection

$$\bigcap_{n\in\mathbb{N}} \bigcap_{a_k\in\mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = \sum_{k=0}^{n} a_k x^k\}.$$

Example 2.2 (A dense $\boldsymbol{G_\delta}$ set of measure zero.)**.** First consider an enumeration of the rationals q_n. For each $\epsilon > 0$, let $I_\epsilon = \bigcup_{n=1}^{\infty} (q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n})$. Clearly I_ϵ is an open because of the union of open intervals and dense because of the rational numbers. The Lebesgue measure of each set is at most 2ϵ. Now consider $G = \bigcap_{k=1}^{\infty} I_{1/k}$. By the Baire Category Theorem, this set is dense, but its measure is less than that of any $|I_{1/k}| = 2/k$, so it has measure zero.

Finally we remark that any countable set is topologically universal. By the baire category theorem, we can take a sequence of numbers from the dense G_δ set and multiply every element of the countable set, by that collection. This parses the countable set into the G_δ set.

We also remark that any set that contains an interval is not topologically universal. A set that contains an interval has an interior. However a there is a dense G_δ set that has zero measure and therefore contains no intervals.

2.3 Affine Copies and the Self Similarity Property

One of the more popularly known results from math is the the Mandelbrot set fractal. In some senses fractals can be very general objects that are considered to have fractional dimensions. This definition can be very general but by that same token, may not always capture some of the inherent geometry of some fractal type objects. Some objects with fractional dimensions have a self-similar property, and some self-similar objects have fractional dimension. For the scope of this paper, we will not be discussing dimension. However we will investigate the notion of self-similarity.

Even one dimensional objects can have this self-similar property. Take for example the middle Third Cantor set. The middle third Cantor set is defined by recursively removing the open middle third interval of the previous remaining closed intervals. Explicitly this can be constructed using countable intersection.

Example 2.3 (The Middle Third Cantor Set)**.**

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

This set in particular exhibits this self-similar property because each level is a scaled copy of the entire object. The following figure shows the first seven intervals removed.

Figure 2.1: The first seven iterations of the middle third Cantor Set.

Cantor sets are self-similar. In a general sense, a set that is expressable in terms of a finite number of different contraction mappings is self similar.

Definition 2.9 (Contraction[4])**.** Let $0 < r < 1$ and $D \subseteq \mathbb{R}^d$ be a closed subset. . A contraction $f : D \to D$ is a funciton such taht for all $x, y \in D$

$$|f(x) - f(y)| < r|x - y|.$$

From here we take a finite set of contractions $\{f_1, \ldots, f_n\}$.

Definition 2.10 (Iterated Function System[4])**.** An iterated function system is a finite set of contraction mappings on a complete metric space. Symbolically, we write this as, for some $N \in \mathbb{N}$,

$$\{f_i : X \to X | i = 1, 2, \ldots, N\},$$

Definition 2.11 (Self-Similar Set)**.** A set $F \subseteq D$ is self-similar if it is the invariant set of an iterated function system:

$$F = \bigcup_{i=1}^{N} f_i(F).$$

We conclude this section with the Cantor middle third set.

Example 2.4. The Cantor set is self-similar. Consider the closed interval $[0, 1]$. Now we consider the following two contractions, each with the contraction ratio 1/3:

$$f_1(x) = \frac{1}{3}x \quad \text{and} \quad f_2(x) = \frac{x+2}{3}.$$

Each level of the cantor set is another iteration of these two functions. In this case the two functions are the iterated funciton system and the set of invariant points is the Cantor set.

2.4 Cantor Sets and Other Measure Theoretic and Topological Examples

We begin this section with a special F_σ set. The Cantor set is defined by taking the interval $[0,1]$ and then iteratively removing the open interval containing the middle third, from the previous level. As such it is the countable intersection of closed sets. Formally this can be written as follows.

Definition 2.12 (Cantor Set)**.** The Cantor set $\mathcal{C}$, written as the successive removal of each middle third removed from the previous level is

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

As an F_σ set defined on a closed interval with iteratively removed open intervals we see the middle third Cantor set can also be described with the following three properties.

Definition 2.13 (Cantor Set)**.** *Cantor sets* are compact, perfect sets, and totally disconnected sets.

As a quick reminder to the defitions:

Definition 2.14 (Perfect)**.** A *perfect* set is a closed set that contains no isolated points.

Definition 2.15 (Totally Disconnected)**.** A set is *totally disconnected* if the only connected components are single points.

An equivalent formulation of the Cantor set, is the decimal expansion of all numbers in $[0,1]$ in base 3, omitting any representation with a 1. This can be a useful tool for thinking through some examples and counter-examples.

Example 2.5 (Decimal Expansion Cantor Set)**.**

$$\mathcal{C} = \{x \in [0,1] : \ x \text{ has a ternary expansion containing no 1's.}\}$$

Here we notice that although $1/3 \in \mathcal{C}$ can be written as 0.1 using the trinary expansion, it also has another representation as $1/3 = 0.0\overline{2}$ This would be the included representation in the Cantor set. We take a moment to acknowledge that numbers may not have unique representations, where one may be excluded but the other included.

Earlier we defined nowhere dense. Here we see that the Cantor set is an example of a nowhere dense set.

Claim 2.1. The Cantor set is *nowhere dense*.

Proof. Let $\mathcal{C}$ be the middle third Cantor set. Notice that $[0,1] \setminus \mathcal{C}$ is a set of open intervals:

$$[0,1] \setminus \mathcal{C} = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

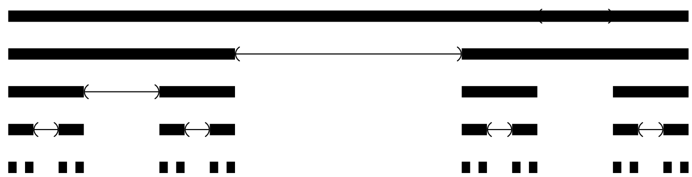


Figure 2.2: The open middle intervals are recursively removed. This removes the interior of the set, while still leaving the original set closed.

Therefore $\mathcal{C}$ is the countable intersection of closed intervals, and itself is closed.

Notice given some radius r, there exists some number t, such that $0 < t < r$ and t has a 1 in its ternary expansion. So if we consider any point $c \in \mathcal{C}$, then an open ball of radius r centered at c then $B_r(c)$ is $(c-r, c+r)$ in ternary, necessarily contains a number containing a 1. Therefore $B_r(c) \not\subseteq \mathcal{C}$, and $\mathcal{C}$ has an empty interior. Finally we conclude because $\mathcal{C}$ is closed and has an empty interior, $\mathcal{C}$ is nowhere dense. ∎

Beyond the middle third Cantor set, we can generalize these in a few different ways. Changing the middle interval, having a few different interval widths, series of interval widths.

Example 2.6. A measure theoretically large set is not necessarily topologically large. The Smith-Volterra-Cantor set is formed in a similar manor to the middle-third Cantor set. Starting with the closed interval, remove the middle fourth recursively. At each level, 2^{n-1} intervals of length $1/4^n$ are removed. The total length of these removed intervals are

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = \frac{1}{2}.$$

Therefore the Smith-Volterra-Cantor set has measure $1 - 1/2 = 1/2$. This closed set still has an empty interior and is therefore nowhere dense but still has positive measure.

Example 2.7 (A dense, uncountable set of measure zero)**.** As stated above, the middle third Cantor set has measure 0. Let $\mathcal{C}_q$ denote a cantor set translated by a rational number $q \in \mathbb{Q}$. This gluing of sets is dense but still measure zero. This is formed from closed sets.

$$\left| \bigcup_{q \in \mathbb{Q}} \mathcal{C}_q \right| \leq \sum_{q \in \mathbb{Q}} |\mathcal{C}_q| = 0.$$

Example 2.8 (A nowhere dense, uncountable set of positive measure.)**.** The Smith-Volterra-Cantor set is formed in a similar manor to the middle-third Cantor set. Starting with the closed interval, remove the middle fourth recursively. At each level, 2^{n-1} intervals of length $1/4^n$ are removed. The total length of these removed intervals are

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = \frac{1}{2}.$$

Therefore the Smith-Volterra-Cantor set has measure $1 - 1/2 = 1/2$.

Chapter 3

An Erdős Similarity Problem in a Topological Setting

3.1 Positive Newhouse Thickness

Cantor sets contain important invariant structures such as Hausdorff dimension, thickness, and denseness. We will investigate thickness and denseness. First we will define the gaps and bounded gaps of Cantor sets in order to construct and define Newhouse thickness.

Definition 3.1 (Gap)**.** Let K be some Cantor set. A *gap* of K is a connected components of $\mathbb{R} \setminus K$.

Informally, the gaps are the intervals surrounding the points of the Cantor set. Some of the lengths of these intervals are bounded some are not. In the example of the middle third Cantor set, the unbounded gaps would be $(-\infty, 0)$ and $(1, \infty)$.

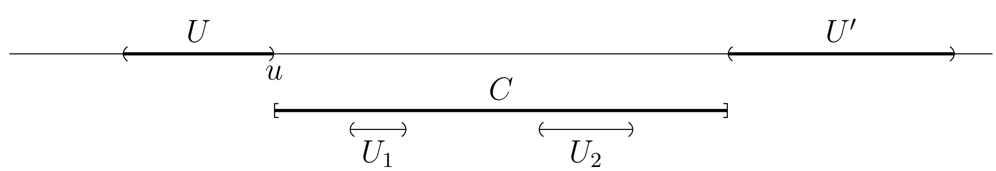
Definition 3.2 (Bounded Gap)**.** Let K be a Cantor set. A *bounded gap* is a bounded connected component of $\mathbb{R} \setminus K$.

Using these two notions we will define the *bridge* of C of Cantor set K.

Definition 3.3 (Bridge)**.** [7] Let K be some cantor set and U be a bounded gap of K with boundary point u. The *bridge* C of K at u is the maximal interval in $\mathbb{R}$ such that:

- u is a boundary point of C
- C contains no point of a gap U' whose length $\ell(U') \geq \ell(U)$..

For clarity the picture below shows that there may be smaller bounded gaps contained in C.



We use this notion to define the *Newhouse Thickness*. Intuitively the thickness of a Cantor set can be thought of as a the infimimum of ratios between the bounded gaps and the bridges.

Definition 3.4 (Newhouse Thickness [7])**.** The *Newhouse Thickness or thickness of K at u* is defined as

$$\tau(K, u) = \frac{\ell(C)}{\ell(U)}.$$

Moreover for $\mathcal{U} = \{\text{set of all boundary points of bounded gaps}\}$, the thickness of the entire Cantor set is

$$\tau(K) = \inf_{u \in \mathcal{U}} \tau(K, u) = \inf_{u \in \mathcal{U}} \frac{\ell(C)}{\ell(U)}$$

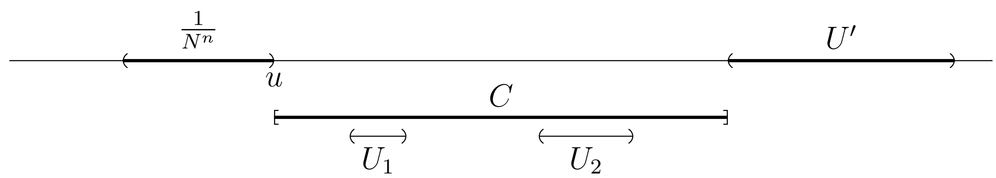
This helps us develop a language to talk about the relative sizes of Cantor sets that is slightly separated from the notion of measure. Importantly it also helps us for a sufficient condition for our main theorem.

Here we will calculate a few examples of Newhouse Thickness. Recall the middle third Cantor set.

Example 3.1 (Newhouse Thickness of the Middle-third Cantor Set)**.** Let K be the middle third cantor set. Then the Newhouse thickness is the infimum of the ratio between gaps and bridges. Here we notice that every bounded gap is one third the previous bridge. Therefore the Newhouse thickness of the set is

$$\tau(K) = \inf_{u \in K} \frac{\ell(C)}{\ell(U)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1.$$

Example 3.2 (Newhouse Thickness of the N-digit Cantor Set)**.** Let K be an N-digit Cantor set. Each gap at the n-th level is of has length N^{-n}. We take a moment to note that the location of the gap matters because it affects the thickness. If we assume that for $2 \leq j \leq n-1$, the j-th digit is removed then we end up with the following cantor set.



With this picture in mind lets take the infimum of the ratios. The thickness of the N-digit expansion Cantor set is

$$\tau(K) = \inf_{u \in K} \frac{\ell(C)}{\ell(U)} = \min\{j, N - j - 1\}$$

3.2 The Gap Lemma

Lemma 3.1 (The Gap Lemma[7]). *Let $K_1, K_2, \subset \mathbb{R}$ be Cantor sets with thickness τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$, K_1 is not contained the gap of K_2, K_2 is not contained in the gap of K_1 then $K_1 \cap K_2 \neq \emptyset$.*

Proof. Let K_1, K_2 be two Cantor sets with thickness τ_1, τ_2 respectively such that K_1 is not contained in the gap of K_2 and K_2 is not contained in the gap of K_1.

Assume for the sake of contradiction that $K_1 \cap K_2 = \emptyset$. Consider the bounded gaps $U_1 \subset K_1^c$ and $U_2 \subset K_2^c$. We call (U_1, U_2) a gap-pair if U_1 contains exactly one boundary point of U_2 and U_2 contains exactly one point of U_1.

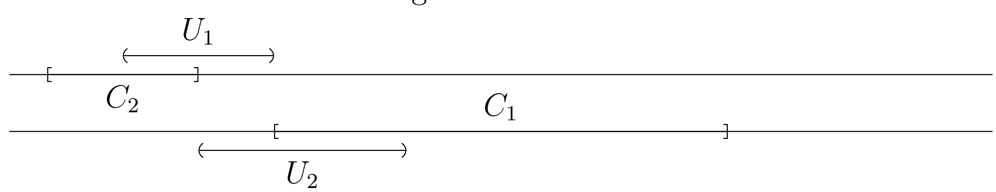
We want to use this technique to demonstrate that $K_1 \cap K_2 \neq \emptyset$. Let C_j^l, C_j^r denote the bridges of K_j for $j = 1, 2$. Returning to our original assumptions, $\tau_1 \cdot \tau_2 > 1$ and therefore

$$\frac{\ell(C_1)}{\ell(U_2)} \cdot \frac{\ell(C_2)}{\ell(U_1)} > 1.$$

The gap of K_1 where $\ell(U_1') < \ell(U_1)$ and (U_1', U_2) is the gap pair we need.

From our construction, the right endpoint of U_2 is in C_1^r or the left endpoint of U_1 is in C_2^l or both. By assumption we know that K_1, K_2 are not contained in the other's gaps. Therefore there exists some gap-pair (U_1, U_2).

A quick picture can help show where these pieces are located: This picture is not accurate because we are using this to derive a contradiction.



Claim 3.1. If $\tau_1\tau_2 > 1$ then from the interval U_1 (or for that matter U_2) we can construct another sub-interval U_1' such that $\ell(U_1') < \ell(U_1)$ (or similarly U_2' such that $\ell(U_2') < \ell(U_2)$).

Notice that (U_1', U_2) is still a gap-pair, as is (U_1, U_2').

Using this construction we can create a sequence of gap-pairs $(U_1^{(i)}, U_2^{(j)})$. Notice that the sum is finite

$$\sum_i^\infty \ell(U_1^{(i)}) < \infty,$$

and therefore $\ell(U_1^{(i)}) \to 0$ as $i \to \infty$. From this we construction we have a sequence of gap-pairs such that as $i \to \infty$, $\ell(U_1^{(i)}) \to 0$ and similarly as $j \to \infty$, $\ell(U_2^{(j)}) \to 0$.

Without loss of generality, we can form a subsequence and use the same indexing for the gap pairs, $(U_1^{(i)}, U_2^{(i)})$.

This sequence of gap pairs have a non-empty intersection for all $i \in \mathbb{N}$.

Notice that by picking a sequence of points, $q_i \in U_1^{(i)}$ this forms a convergent subsequence $q_{i_k} \to q$. Notice that $U_1^{(i)}$ is not fully contained in the gap of K_2. Moreover because these intervals are compact and nested, we know that q which is contained in each $U_1^{(i)}$ is therefore in K_2.

Because this construction is symmetric, the same argument applies to $q_i \in U_2^{(i)}$ and so $q \in K_2$. Therefore the Cantor sets share at least one point, $q \in K_1 \cap K_2$.

■

3.3 A Cantor set with positive Newhouse Thickness is not Topologically Universal

Recall the definition, We say that a set E is *topologically universal* in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdős conjecture in a topological setting.

Theorem 3.2. *Let J be a cantor set with positive Newhouse thickness. Then J is* *not topologically universal in the collection of dense G_δ sets.*

Proof. Suppose we have some Cantor set J with Newhouse thickness $\tau(J) > 0$. Without loss of generality, we can assume the convex hull of J $[0,1]$. Consider Cantor sets K defined by contraction ratio $1/N$ and digits $\{0,1,...,N-1\} \setminus \{(N-1)/2\}$ and N is odd. By a simple calculation, $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(J)\tau(K) > 1$.

Using the Cantor set K Define X such that

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} N^n(K+\ell),$$

creating a dense F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, by the Baire Category Theorem, X^c is a dense G_δ. We now show that X^c contains no affine copy of J.

Suppose we have some affine copy, $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{3.1}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell+1)N^n]. \tag{3.2}$$

Let

$$C_1 = N^n(K + \ell) \text{ and } C_2 = t + \lambda J.$$

The convex hull of C_1, is $[\ell N^n, (\ell + 1)N^n]$. So By our choice of t, we know that C_2 is not in the unbounded gap of C_1 and vice versa.

Now we will check the construction of our Cantor sets such that each is not contained in the bounded gaps of the other. For C_1 its largest corresponding open gap interval is $|O_1| = N^{n-1}$ and its largest corresponding closed interval is $|I_1| = N^n$. For C_2 and is corresponding intervals, we find that $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$ and $|I_2| = |\lambda|$ where O_J is the largest open gap interval in J. Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|.$$

Therefore C_1 is not in the gaps of C_2 and C_2 is not fully contained in the gaps of C_1. By our choice of K, the Newhouse thickness of our sets, $\tau(C_1))\tau(C_2) \geq 1$, because Newhouse thickness is scale invariant. Therefore the the Gap Lemma implies $C_1 \cap C_2$ is non-empty and C_2 cannot be in the constructed G_δ set. Therefore we conclude J is not topologically universal in the collection of dense G_δ sets.

$\blacksquare$

We also mention that this at least partially solves a quetsion posed by Svetic[9] in 2000, "Is it true that for every uncoutably infinite set, E, of real numbers, there exists $S \subset [0,1]$ of full measure that does not contain an affine copy of E?" From earlier, any uncountable set that contains an interval is not topologically universal.

Corollary 3.3. *Let J be a cantor set. Then there exists a set $S \subset [0,1]$ of full measure that does not contain an affine copy of E.*

Proof. Let J be a Cantor set. Then by theorem 3.2, there exists a dense G_δ that does not contain an affine copy of J.

Finally by taking a countable disjoint union of translations of $\bigcup_k G_k$ we can construct a set of arbitraty size such that $\sum \ell(G_k) = 1$. ■

3.4 Generalizing into Higher Dimensions

Consider a compact set J in $\mathbb{R}^d$. An affine copy of J is $\mathbb{R}^d$ is the set

$$t + \delta O(J)$$

where $t \in \mathbb{R}^d$, $\delta \neq 0$ and O is an orthogonal transformation. We say that J is universal is the collection of dense G_δ sets if any dense G_δ set contains an affine copy of J.

Theorem 3.4. *If $X \subset \mathbb{R}^d$ contains a path connected component, then X is not universal in the set of dense G_δ sets of $\mathbb{R}^d$.*

Proof. Remove the plane

$$\mathbb{R}^d \setminus \bigcup_{i=1}^{d} \bigcup_{r \in \mathbb{Q}} \{X_i = r\}.$$

Consider some affine copy of X with a path L. Then X' will contain an affine copy of the path L'. The projection of L' onto the coordinate axises will be non-degenerate on some interval for at least one of the axises. Call this the i-th axis. This interval will contain a rational number r. Therefore L' will intersect with the coordinate plane, $X_i = r$ In other words this dense G-delta set omits at least one point and cannot contain affine copy of X. ■

We can still consider sets that contain no connected component. Cantor dust contains no connected component. We introduce the notion of *projective Newhouse Thickness*, and use it to generalize our one-dimensional results into $\mathbb{R}^d$.

Definition 3.5 (Positive Projective Newhouse Thickness)**.** We say a J set has *positive projective Newhouse thickness* if for all $O \in O(d)$

$$\tau(P_x O(J)) > 0$$

where P_x is the orthogonal projection to the x-axis and $O(d)$ is the orthogonal group consisting of all orthogonal transformations in R^d.

We note that the projection some Cantor dust set maybe an interval. If that is the case we say the Newhouse Thickness of the projection is equal to infinity.

Using our one dimensional result, we can generalize

Theorem 3.5. *Let $J \subset \mathbb{R}^d$ be a compact set such that it has a positive projective Newhouse thickness. Then J is not topologically-universal in $\mathbb{R}^d$.*

Proof. Suppose we have a compact set J in $\mathbb{R}^d$ such that it has a positive projective Newhouse thickness. By Theorem 3.2, there exists a dense G_δ set G_1 in $\mathbb{R}^1$ such that G_1 does not contain any affine copy of $P_x O(J)$. Now, we consider

$$G = G_1 \underbrace{\times \cdots \times}_{\text{d-times}} G_1.$$

Then G is a dense G_δ set in $\mathbb{R}^d$. We claim that there is no affine copy of J in G. To justify the claim. Suppose G contains an affine copy of J such that $t + \delta J \subset G$. Then we take projection and we obtain that

$$P_x(t) + \delta P_x(J) \subset G_1$$

which is a contradiction. Hence, the claim is true and the proof is complete.

■

Chapter 4

Remarks and Open Questions

Finally we conclude by reviewing a few remarks and positing a few more questions on our results.

4.1 Zero Newhouse Thickness

Can a Cantor sets with zero Newhouse thickness can be universal? We first provide an example for which two Cantor sets with zero Newhouse thickness can still have arithmetic sum equal to an interval, showing that the converse of the Newhouse thickness theorem is not true.

Example 4.1. Let $N_1, N_2, \dots \in \mathbb{N}_{\geq 2}$. Consider the following construction of a Cantor set using a decomposition of the unit intervals.

$$\begin{aligned}
[0,1] =& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \left[0, \frac{1}{N_1}\right] \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1 N_2}\{0,1,\dots,N_2-1\} + \left[0, \frac{1}{N_1 N_2}\right] \\
=& ... \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1 N_2}\{0,1,\dots,N_2-1\} + \cdots + \frac{1}{N_1 \cdots N_n}\{0,\dots,N_n\} + \dots
\end{aligned}$$

From here we can define the two cantor sets K_1, K_2 where K_1 constitutes the odd indices sets in the above summands and K_2 has the even one. This gives the following constructions for the two Cantor sets:

$$K_1 = \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \cdots + \frac{1}{N_1 \cdots N_{2n+1}}\{0,\dots,N_{2n+1}-1\} + \dots$$

$$K_2 = \frac{1}{N_1 N_2}\{0,1,\dots,N_2-1\} + \cdots + \frac{1}{N_1 \cdots N_{2n}}\{0,\dots,N_{2n}-1\} + \dots$$

From this construction we see that $K_1 + K_2 = [0,1]$ is the interval but from the definition of Newhouse thickness,

$$\tau(K_1) = \inf\left\{\frac{1}{N_1 - 1}, \frac{1}{N_3 - 1}, \dots\right\} = 0$$

$$\tau(K_2) = \inf\left\{\frac{1}{N_2 - 1}, \frac{1}{N_4 - 1}, \dots\right\} = 0.$$

Therefore we have created an interval from two sets with Newhouse thickness 0 if we have $\lim_{n\to\infty} N_n = \infty$.

We will conclude by ask the following questions. Recall I_J denotes the smallest closed interval containing the Cantor set J and O_J denotes the largest open interval in $I_J \setminus J$.

1. Given a Cantor set J, does there exist some K such that $J + K = I_J + I_K$?

2. If we assume that there exists K such that $J + K = I_j + I_K$, can we prove that J is not universal?

3. (rescaling condition) If we assume that $|\lambda_1 I_J| \geq |\lambda_2 O_K|, |\lambda_2 I_K| \geq |\lambda_1 O_J|$ and $J + K = I_J + I_K$, then $\lambda_1 J + \lambda_2 K = \lambda_1 I_J + \lambda_2 I_k$.

We also notice that to solve the second question, we notice that $J+K = I_J + I_K$ implies that

$$(J + a) + (K + b) = (I_J + a) + (I_K + b) \text{ and } bJ + bK = bI_J + bI_K.$$

We can always translate and rescale J, K so that $I_J = [0, a]$ and $I_K = [0, 1]$. Moreover, the following lemma is important.

Lemma 4.1. *Suppose that the Cantor sets J and K satisfies $J + K = I_j + I_K$. Then $|I_J| \geq |O_K|$ and $|I_K| \geq |O_J|$.*

The lemma also said that the condition $|\lambda_1 I_J| \geq |\lambda_2 O_k|, |\lambda_2 I_k| \geq |\lambda_1 O_J|$ is necessary in the rescaling condition.

Proposition 4.2. *Let J be a Cantor set such that $J+K = I_J + I_K$ where $I_J = [0,a]$ and $I_K = [0,1]$. Suppose that the rescaling condition (3) holds. Then J is not universal in the collection of dense G_δ.*

Proof. The proof is similar to the proof in Theorem 3.2. With K given in the assumption. We can assume that $|I_J| > |O_K|$. Suppose that $|I_J| = |O_K|$. Since $|O_J| < 1$, we can choose ϵ such that $(1-\epsilon) > |O_J|$. Then we consider $K' = (1-\epsilon)K$ and we will have $|I_J| > (1-\epsilon)|O_K|$. In this case, by the rescaling condition, $J + K' = I_J + I_{K'}$ and we have another K' such that $|I_J| > |O_{K'}|$.

As now we have $|I_J| > |O_K|$, we can find $0 < \rho < 1$ such that $\rho|I_J| > |O_K|$. We now define

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} \rho^n (K + \ell).$$

Then X^c is a dense G_δ set. Suppose that we have an affine copy $t + \lambda J$, we would like to claim that $t + \lambda J$ intersects non-trivially with $\rho^n(K+\ell)$ for some $n, \ell \in \mathbb{Z}$, which will complete the proof of the theorem.

To justify the claim, we let $0 < \rho < 1$ take the unique n such that

$$|\lambda| \in [\rho^{n+1}, \rho^n) \tag{4.1}$$

and the unique $\ell \in \mathbb{Z}$ such that

$$t \in (\ell\rho^n, (\ell+1)\rho^n]. \tag{4.2}$$

Then we consider the arithmetic sum $\rho^n K - \lambda J$. We now check the assumption in the rescaling condition with $\lambda_1 = \rho^n$ and $\lambda_2 = -\lambda$. Indeed,

$$|\lambda_2 I_J| \geq \rho^{n+1}|I_J| = |\lambda_1|(\rho|I_J|) \geq |\lambda_1 O_K|$$

by our choice of ρ. On the other hand,

$$|\lambda_1 I_K| \geq |\lambda| \geq |\lambda_2 O_J|$$

since $|I_K| \geq |O_J|$ by Lemma 4.1. Hence, using the rescaling condition,

$$\rho^n K - \lambda J = \rho^n I_K - \lambda I_J.$$

If $\lambda > 0$, then we have

$$\rho^n(K+\ell) - \lambda J = [\rho^n \ell - \lambda a, \rho^n(1+\ell)]$$

which contains t by (4.2). Similarly, if $\lambda < 0$, then

$$\rho^n(K+\ell) - \lambda J = [\rho^n \ell, \rho^n(\ell+1) - \lambda a].$$

It also contains t by (4.2). The proof is now complete. ■

We now conclude with several open questions.

Conjecture 4.1. *For any cantor set, does there exist another such that $C_1 + C_2$ adds up to an interval?*

Does the rescaling condition hold for Cantor sets with zero thickness?

Our proof inherently relies on appropriately selecting a scalar and translation that corresponds to a regular (or fairly regular) Cantor set. In this instance we have to find pick the appropriate λ, t based off of a set of associated intervals that are not uniform. Our proof relies on using the regularity to specify where the intersection is.

Do all self-similar sets have positive projective newhouse thickness?

Finally, if all sets with positive measure contain a non-measurable set, is there a non-meaurable set that is universal in sets with positive measure?

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