AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING

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In
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by

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CERTIFICATION OF APPROVAL

I certify that I have read *AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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2022

Your abstract comes here.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

ACKNOWLEDGMENTS

I want to take a moment to list a few people who have shaped my pursuit of math. Firstly, thank you Mom and Dad for supporting. Mom, you sat with me at the dinner table teaching me

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Chapter 1

Introduction

Here's where you text starts. You might (but don't have to) have sections like…

1.1 Introduction to the Introduction

…or subsections…

1.1.1 Introduction to the Introduction to the Introduction

To include text in your document, you just type. You don't have to worry about spacing between sentences; LaTeX does that automatically. To start a new paragraph, just insert an empty line.

Here comes the new paragraph. You can include tables, such as Table 1.1, which can also be turned if necessary, as seen, e.g. in Table 1.2. (I think the latter is automatically on its own page; the former will just be put where LaTeX thinks it fits best.)

d	2	3	4	5	6	7	8	9
bound	3.6	8.5	15.8	25.7	38.3	53.5	71.4	92.0

Table 1.1: Bounds in various dimensions d.

To start a new chapter, you don't need to add a pagebreak…

d	2	3	4	5	6	7	8	9
bound	3.6	8.5	15.8	25.7	38.3	53.5	71.4	92.0

Table 1.2: More bounds in various dimensions d.

Chapter 2

The Real Stuff

. . . LaTeX does that automatically.

2.1 Various Tricks

Here are a few tidbits that came to my mind, in no particular order. . .

I'm sure most people know about `\label` and `\ref`; this feature is enough reason for me to use LaTeX. For example, you might have something like. . .

Theorem 2.1 (Euler)**.** *Leonhard says* $e^{2\pi i} = 1$.

. . . which you can then later (or earlier) reference as Theorem 2.1 and you can point to page 4 on which it appears. If you label equations like

$$e^{2\pi i} = 1\,, \tag{2.1}$$

you can reference them most lazily using (2.1).

There are two ways to define internal macros: `\def` and `\newcommand`. The difference is that the former overwrites any possibly existing command, where as the latter induces a LaTeX complaint if you're redefining an existing command (in which case you should use `\renewcommand` instead). Since I'm lazy, I tend to use `\def`, with one important exception: `\newcommand` allows you to use arguments. For example, the definition

```
\newcommand\floor[1]{\left\lfloor {#1} \right\rfloor}
```

produces a flexible floor function $\left\lfloor \frac{3\pi}{2} \right\rfloor$. And yes, the `[1]` can be replaced by `[n]` if you have use for `n` arguments (which get the placeholders `#1`, `#2`, ..., `#n`).

The previous example reminds me to strongly recommend the use of `\left` and `\right` whenever you use parentheses:

$$(\frac{a}{b} - 2) \text{ just doesn't look good compared with } \left(\frac{a}{b} - 2\right).$$

There are three dashes in LaTeX—one like the one you just saw, one that's used in "Berndt–Zaharescu's Theorem" or "Chapter 7–9," and one that's used in "well-known identity."

One of my favorite packages is `enumerate`:

i) first item

ii) second item

iii) third item

iv) ...

TO DO: I stole the idea of a to-do box from a friend. This is useful when editing a paper and you want to remind your co-author or yourself about something...

2.2 Math

You can put a □ at the end of a math line if a proof happens to end with a math line. The version...

Proof. This follows from

$$e^{2\pi i} = 1\,.$$

□

...wastes space and doesn't look nearly as cool as...

Proof. This follows from

$$e^{2\pi i} = 1\,. \qquad \square$$

Speaking about proofs, sometimes you need a paragraph or two between a theorem and its proof, in which case you can use...

Proof of Theorem 2.1. This follows from

$$e^{2\pi i} = 1\,. \qquad \square$$

You can (and should) define your own math operators with `\operatorname`. For example,

$$\operatorname{lcm}(2,3)=6 \quad \text{looks better than} \quad lcm(2,3)=6\,.$$

Speaking about lcm's, I find it amusing that `\gcd` is a pre-defined operator, whereas `\lcm` is not.

The `\dots` command is relatively smart in LaTeX; e.g., it knows automatically where to put the dots in $\{1,2,\dots,n\}$ as compared to $1+2+\dots+n$. If you ever need to force a certain alignment of the dots, use $\ldots$, $\cdots$, $\vdots$, or $\ddots$.

For math stuff that takes several lines I prefer the environment `align` over `eqnarray`. The difference is minor but I still prefer

$$\begin{aligned} \sum_{j=1}^{k-1} \chi(j) \sin^2 \left(\frac{\pi j}{k} \right) \tan \left(\frac{2\pi j}{k} \right) &= \sqrt{k} \left(-\frac{1}{2} + (\chi(2)-2)\, h(-k) \right) \\ &= \begin{cases} \sqrt{k} \left(-\frac{1}{2} - 3\, h(-k) \right) & \text{if } k \equiv 3 \bmod 8, \\ \sqrt{k} \left(-\frac{1}{2} - h(-k) \right) & \text{if } k \equiv 7 \bmod 8 \end{cases} \end{aligned}$$

over

$$\begin{eqnarray} \sum_{j=1}^{k-1} \chi(j) \sin^2 \left(\frac{\pi j}{k} \right) \tan \left(\frac{2\pi j}{k} \right) &=& \sqrt{k} \left(-\frac{1}{2} + (\chi(2)-2)\, h(-k) \right) \\ &=& \begin{cases} \sqrt{k} \left(-\frac{1}{2} - 3\, h(-k) \right) & \text{if } k \equiv 3 \bmod 8, \\ \sqrt{k} \left(-\frac{1}{2} - h(-k) \right) & \text{if } k \equiv 7 \bmod 8. \end{cases} \end{eqnarray}$$

2.3 Graphics

My favorite way to produce graphics is through a (public-domain) program called `jPicEdt`; it produces LaTeX code that can be read right into the file (using the `\input` command). My second favorite way to go about graphics is through the package `graphicx`. I produce my graphics with a separate program, export them into pdf, and then overlay them with TeX symbols if needed; see Figure 2.1 for an example.

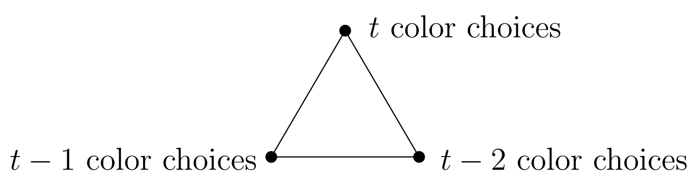

Figure 2.1: Proper t-colorings of K_3.

The overlaying of figures reminds me of two of my other favorite LaTeX commands: `\vspace` and `\hspace`. For example, they allow you to place just about

anywhere.

anything

This can be very useful, e.g., for presentations in which you might move pictures around.

2.4 Bibliographic Stuff

For references like [2, Section 2] I recommend using `bibtex`; it means that you have to do only minimal work, especially when you get the entries from *MathSciNet*. I keep all the references I've ever used in the same file and use this in all my documents...

If you could use an index at the end of your document, use `makeindex`—one of best reasons to use LaTeX if you're writing a book.

2.5 A Cantor set with positive Newhouse Thickness is not universal

We say that a set E is *universal* in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdös conjecture in a topological setting.

Theorem 2.2. *Let J be a cantor set with positive Newhouse thickness. Then J is not universal.*

Proof. Suppose we have some Cantor set J with Newhouse thickness $\tau(J) > 0$. Without loss of generality, we can assume the convex hull of J $[0, 1]$. Consider Cantor sets K defined by contraction ratio $1/N$ and digits $\{0, 1, ..., N-1\} \setminus \{(N-1)/2\}$ and N is odd. By a simple calculation, $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(J)\tau(K) > 1$.

Using the Cantor set K Define X such that

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} N^n(K+\ell),$$

creating a dense F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, by the Baire Category Theorem, X^c is a dense G_δ. We now show that X^c contains no affine copy of J.

Suppose we have some affine copy, $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{2.2}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell+1)N^n]. \tag{2.3}$$

We claim that this affine copy of J has a non-empty intersection with $N^n(K+\ell)$. This is equivalent to showing that

$$t \in N^n(K+\ell) - \lambda J.$$

For consistent notation with a referenced theorem, let

$$C_1 = N^n(K + \ell) \text{ and } C_2 = -\lambda J.$$

First we check the construction of our Cantor sets. For C_1 its largest corresponding open gap interval is $|O_1| = N^{n-1}$ and its largest corresponding closed interval is $|I_1| = N^n$. For C_2 and is corresponding intervals, we find that $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$ and $|I_2| = |\lambda|$ where O_J is the largest open gap interval in J. Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|$$

as in the condition of Theorem 2.2.1 in [1]. By [1, Theorem 2.2.1][1], given that the Newhouse thickness of our sets, $\tau(K)\tau(J) \geq 1$ then $C_1 + C_2 = I_1 + I_2$. Note that $I_1 = [\ell N^n, (\ell + 1)N^n]$, $I_2 = [-\lambda, 0]$ if $\lambda > 0$ and $I_2 = [0, -\lambda]$ if $\lambda < 0$. we find that

$$I_1 + I_2 = [N^n\ell - \lambda, N^n(\ell + 1)]\ (\lambda > 0) \text{ and } I_1 + I_2 = [N^n\ell, N^n(\ell + 1) - \lambda](\lambda < 0).$$

Then from (2)

$$t \in I_1 + I_2.$$

[1]This might misattribute the theorem. I think Astel '99 Theorem 2.2.1 is actually is quoting Newhouse directly. In particular I think it refers to Newhouse 1979 [3] *The Abundance of Wild Hyperbolic Sets, and Non-smooth Stable Sets for Diffeomorphisms.*

Therefore the affine copy of the cantor set $t + \lambda J$ has a non-empty intersection with X and J cannot be universal.

$\square$

It would be interesting to study those Cantor sets with Newhouse thickness zero. We do not know what would happen. However, it seems like if we assume a weaker condition on J.

$(*)$: There exists K such that $J + K = I_J + I_K$, where I_J, I_K are the smallest closed interval containing J and K.

we may be able to show that J cannot be universal for dense G_δ sets.

Appendix A: Triangulations of Polytopes

This creates an appendix, which is not numbered (and therefore has to be added to the table of contents semi-manually).

For code, the `verbatim` environment (maybe in conjunction with `\include`) is helpful:

```
It reproduces text
  exactly
    as
      it
        is
typed.
```

Good luck and enjoy writing!

Bibliography

[1] S. Astels. “Thickness measures for Cantor sets”. In: *Electron. Res. Announc. Amer. Math. Soc.* 5 (1999) (1999), pp. 108–111.

[2] Christos A. Athanasiadis. “Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley”. In: *J. Reine Angew. Math.* 583 (2005), pp. 163–174. ISSN: 0075-4102.

[3] Sheldon E. Newhouse. “The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms”. en. In: *Publications Mathématiques de l’IHÉS* 50 (1979), pp. 101–151. URL: http://www.numdam.org/item/PMIHES_1979__50__101_0/.