

On a Topological Erdős Similarity Problem

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Master of Arts  
In  
Mathematics

by

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## CERTIFICATION OF APPROVAL

I certify that I have read *On a Topological Erdős Similarity Problem* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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# On a Topological Erdős Similarity Problem

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2022

Often in data work, one may ask which patterns are possible to find, given a certain data set, or hypothetical relationship. Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets? A set is called universal in another set, when every subset of the larger set contains some scaled and translated copy of original. Paul Erdos proposed a conjecture that no infinite set, is universal in the the collection of sets with positive measure. This poster explores an analogous problem in a topological setting. Instead of sets with positive measure we investigate the collection of dense G-delta sets. Any finite or countable set is found to be topologically universal. Any set containing an interval cannot be topologically universal. We also have the new result that any Cantor sets is not topologically universal. Cantor sets, which contains no interval and are uncountably infinite, are not topologically universal in the collection of dense G-delta sets.

I certify that the Abstract is a correct representation of the content of this thesis.

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Chair, Thesis Committee

Date

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# Chapter 1

## Introduction

Often in data work, one may ask, "Which patterns are possible to find, given a certain data set, or hypothetical relationship?" Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets?

If we more deeply consider patterns which infinitely repeat scaled copies of itself, this turns out to be a self-similar set. This is very closely related to the concept of fractals as well as a core area of study within dynamical systems. Some fractals can be generated using recursive functions. In dynamical systems some iterated functions may have sets of invariant points. These collections of invariant points may themselves be self-similar. In this respect it is valuable to explore which patterns are found everywhere because they give insight into some key facets within dynamical systems as well as the nature of patterns.

Given a specific set of points, we can formalize this notion of a scaled copy of a



pattern as an *affine transformation* or *affine copy*. We will formally define this in the next section

Exploring this notion a little more deeply, we can begin to investigate which patterns appear everywhere. Informally, a set is called *universal* in another collection, when every subset of the collection contains some scaled and translated copy of original pattern.

Paul Erdős proposed a conjecture that no infinite set, is measure-universal in the collection of sets with positive measure. We will explore an analogous problem in a topological setting. This also answers a question posed by Svetic[9], "Is it true that for every uncountably infinite set,  $E$ , of real numbers, there exists  $S \subset [0, 1]$ ? Additionally this result can be used to prove some results in higher dimensions.

## 1.1 An Erdős Self-Similarity Conjecture in Measure Space

There is a long standing conjecture from Paul Erdős on universal sets. Informally the conjecture states that there is no infinite set that is universal in the real number line. This is a conjecture about which types of patterns can exist within another sets of numbers.

We can now start by defining *affine transformations* or *affine copies*.

**Definition 1.1** (Affine copy). An affine copy of a set  $A \subset \mathbb{R}$  is a scaled and trans-

lated set  $A'$  such that for some  $\lambda \neq 0, \lambda \in \mathbb{R}$  and  $t \in \mathbb{R}$ ,

$$A' = \{\lambda a + t : a \in A\}.$$

In this instance, it is a scaled and then translated copy of the set is still one dimensional and therefore need not be a “shape”. This gives us some flexibility when addressing different sets. This definition is used to define measure universal.

**Definition 1.2** (Measure Universal). A set  $E$  is called measure universal in  $X$  if for every subset  $S \subseteq X$ , with positive measure,  $\mu(S) > 0$ , there exist an affine copy of  $E$  such that  $t + \lambda E \subseteq S$ , for some  $\lambda \neq 0$  and  $t \in \mathbb{R}$ .

Now we can formally, state Erdős’ conjecture as follows.

**Conjecture 1.1** (The Erdős Self-Similarity Conjecture). *Let  $E \subseteq \mathbb{R}$  be an infinite set of real numbers. Prove that there is a set of real numbers  $S$  of positive measure which does not contain an affine copy of  $E$ .*

Paul Erdős originally posed this question back in 1974 by building off of the work of Steinhaus. Steinhaus[7] first posed that finite sets are universal in sets with positive measure. After Erdős build on this with his conjecture, there has been some progress.

Later Falconer [2] made substatioal progress by showing slowly decaying sequences are not measure universal. Bourgain [8] expanded on this by showing some

faster decaying sequences are also not measure universal. In particular he demonstrated that the sumset of any three sets, cannot be measure universal. Most recently Kolountzakis [3] demonstrated using probabilistic arguments to demonstrate that certain set with large gaps cannot be measure universal.

Currently it is still an open question whether or not sequences that decay at the rate of  $2^{-n}$  are measure universal.

In this paper we take this idea of measure universality and put it into a topological context. We show in theorem 3.2 no cantor set with positive Newhouse thickness is universal in the set of dense  $G_\delta$  sets.

## 1.2 An Analogous Theorem in a Topological Setting

In a non-rigorous exploration of the real numberline, one might assume that patterns which appear everywhere should have positive measure. However density and measure are not intrinsically linked. Indeed it is possible to have uncountable dense sets with measure zero and to have sets with full measure that are nowhere dense.

Borrowing the concept of  $G_\delta$  sets from topology, we can explore an extension of the Erdős similarity problem. Instead of exploring sets with positive measure we can explore dense  $G_\delta$  sets. This is the countable intersection of open sets that are also dense. We will use these dense  $G_\delta$  sets to define topological-universal.

**Definition 1.3** (Topological-Universal). A set  $E$  is called Topological-universal in

$\mathbb{R}$  if for every dense  $G_\delta$  subset  $S \subseteq \mathbb{R}$ , there exist an affine copy of  $E$  such that  $t + \lambda E \subseteq S$ , for some  $\lambda \neq 0$  and  $t \in \mathbb{R}$ .

As stated above, a dense  $G_\delta$  set can have measure zero. This also means that there is not a direct relationship between topological universal and measure universal. We make two observations on this fact: a set with an interior cannot have be topologically universal, by the Baire Category Theorem all countable sets are topologically universal.

This motivates the question, Is a nowhere dense set, with an empty interior topologically universal? Cantor sets for example, have an empty interior and are nowhere dense.

In chapter 2 we review background definitions and theorems in measure theory and topology. We also give several examples to demonstrate some of the nuances of these facts. In chapter 3 we define Newhouse Thickness, prove the Gap Lemma and prove our main result theorem 3.2.

**Theorem.** *Let  $J$  be a cantor set with positive Newhouse thickness. Then  $J$  is not topologically universal in the collection of dense  $G_\delta$  sets.*

These results can also be extrapolated into  $\mathbb{R}^d$  by definining projective Newhouse thickness. Finally in chapter 4 we conclude with some open questions and remarks.

## Chapter 2

# Measure & Topology

### 2.1 Some Measure Theory

First we want to begin with background definitions and theorems. Erdős' problem specifically deals with infinite set, and affine copies found in measurable set of sets. In our problem, rather than dealing with measurable sets, we will instead use the set of dense  $G_\delta$  sets.

Underpinning the nuances of this problem, measure theoretic size, and topological size, are not the same. From an intuitive sense of the number line one might think when you are scattered throughout an interval, you would have measure, except in special cases. Similarly one might think that if you have measure, then you would be scattered everywhere. However both of these instances fail when you add in rigorous arguments. Indeed it is possible to construct a set that is no-where dense and has positive measure. It is also possible to construct an uncountable set that

is dense and has measure. zero. In other words topological size (density) is not the same thing as measure theoretic size.

First we will review some measure theory. In order to define *measure* and *measurable sets* we first need to define  $\sigma$ -Algebra.

**Definition 2.1** ( $\sigma$ -Algebra). Let  $X$  be some set and  $2^X$  be the set of subsets of  $X$ . Let  $\Sigma \subseteq 2^X$ . We call  $\Sigma$  a  $\sigma$ -algebra over  $X$  if it satisfies the following three conditions:

1.  $\emptyset \in \Sigma$
2. If  $E \in \Sigma$ , then  $X \setminus E \in \Sigma$ .
3. If  $E_1, E_2, \dots \in \Sigma$  is a sequence of subsets, then  $\bigcup_{k=1}^{\infty} E_k \in \Sigma$ .

In this instance we describe set of sets in terms of intersection and union. This allows us to generate an algebraically closed collection of sets.

In the Erdős conjecture, the measure is specifically referencing *Lebesgue Measure*.

**Definition 2.2** (Lebesgue Measure[6]). For any interval  $I = [a, b]$  (or  $I = (a, b)$ ) in the set  $\mathbb{R}$  of real numbers, let  $\ell(I) = b - a$  denote its length. For any subset  $E \subseteq \mathbb{R}$ , the *Lebesgue outer measure*  $\lambda^*(E)$  is defined as an inf :

$$\lambda^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k \in \mathbb{N}} \text{ is a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

The collection of sets where  $\lambda^*(E)$  is defined forms a  $\sigma$ -algebra. However not all sets are necessarily Lebesgue measurable, such as Vitali sets. We will not explore non-measurable sets because it falls outside of the scope of our current theorems.

We will also need to define *uncountable*. This forms a key distinction for our sets

**Definition 2.3** (Countable). A set  $X$  is called *countable* if for some subset  $N \subset \mathbb{N}$ , that is not necessarily finite, there exists a bijection  $f : X \rightarrow N$ .

**Definition 2.4** (Uncountable). A set  $X$  is called *uncountable*, if it is not *countable*.

## 2.2 Topological and The Baire Category Theorem

Now we will review some Topology that is relevant to our main theorem. We need to define *dense*. From there we will build into  $G_\delta$  sets. There are many equivalent definitions of dense. We will use the following definition so that we can continue to develop the intuition around intervals and interiors.

**Definition 2.5** (Dense). A set  $S$  is called dense in  $X$  if for every  $x \in X$ , every neighborhood  $U$  of  $x$  intersects  $S$ .

In a similar fashion to the definitions of *countable* and *uncountable*, the opposite of *dense* is *nowhere dense*.

**Definition 2.6** (Nowhere Dense). Let  $X$  be a topological space. A subset  $B \subseteq X$  of a topological space is called *nowhere dense* in  $X$  if its closure has an empty interior.

That is to say,  $B$  is *nowhere dense* in  $X$  if for each open set  $U \subseteq X$ ,  $B \cap U$  is not dense in  $U$ .

This allows us to now explore the differences between density and measure. As stated earlier, topological size and measure theoretic size are not necessarily related. What do we mean by topologically Large? Uncountable and dense. Similarly what do we mean by measure theoretically large? Non-zero measure. After reviewing it is helpful to define the opposite of topologically large, namely meager sets.

**Definition 2.7** (Meager). A subset  $C \subseteq X$  of a topological space is called *meager* in  $X$  if it is the countable union of nowhere-dense subsets of  $X$ .

Next we will define dense  $G_\delta$  sets, as well as some useful examples.

**Definition 2.8** (G-Delta Set). A  $G_\delta$  set is the countable intersection of open sets. Namely, let  $O_i \subset X$  for  $i \in \mathbb{N}$  be a collection of open sets of  $X$ . Then  $\bigcap_{n=1}^{\infty} O_i$ , is a  $G_\delta$  set.

**Example 2.1.** The irrational numbers are a  $G_\delta$  set. Consider the following construction of the set of irrational numbers:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Notice that each  $\mathbb{R} \setminus \{q\} = (-\infty, q) \cup (q, \infty)$  is an open subset of  $\mathbb{R}$ . Furthermore, rational numbers are countable. Therefore the intersection of these sets are a  $G_\delta$



set. Moreover, in this instance it is a dense  $G_\delta$  set. We will study these objects further.

Lastly we remark that there is an analogous set which is the countable union of closed sets.

**Definition 2.9** ( $F_\sigma$  Set). An  $F_\sigma$  set is the countable union of closed sets. This is equivalent to the compliment of a G-delta is an F-sigma set.

A key theorem that links analysis to set theory is the Baire Category Theorem. This also establishes a link to understanding certain types of topological sets.

**Theorem 2.1** (Baire Category Theorem[4]). *The countable intersection of open dense sets is dense.*

Within the study of measure theory it can sometimes be unclear if a set is dense in another set. For example consider the following set:

$$\mathbb{R}^2 \setminus \{(x, y) : y = mx + b, \text{ where } m, b \in \mathbb{Q}\}$$

Notice that this can also be written as

$$\bigcap_{m, b \in \mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = mx + b\},$$

which is the plane, but removing all lines with rational coefficients, and rational intercepts. Each plane with one line removed,  $\mathbb{R}^2 \setminus \{(x, y) : y = mx + b, \text{ for some } m, b \in$

$\mathbb{Q}\}$  is an open dense set. Then by the Baire Category Theorem, the countable intersection is dense. Moreover this same idea holds for removing polynomials with rational coefficients. We can write this as the countable intersection

$$\bigcap_{n \in \mathbb{N}} \bigcap_{a_k \in \mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = \sum_{k=0}^n a_k x^k\}.$$

## 2.3 Affine Copies and the Self Similarity Property

One of the more popularly known results from math is the the Mandelbrot set fractal. In some senses fractals can be very general objects that are considered to have fractional dimensions. This definition can be very general but by that same token, may not always capture some of the inherent geometry of some fractal type objects. Some objects with fractional dimensions have a self-similar property, and some self-similar objects have fractional dimension. For the scope of this paper, we will not be discussing dimension. However we will investigate the notion of self-similarity.

Even one dimensional objects can have this self-similar property. Take for example the middle Third Cantor set. The middle third Cantor set is defined by recursively removing the open middle third interval of the previous remaining closed intervals. Explicitly this can be constructed using countable intersection.

**Example 2.2** (The Middle Third Cantor Set).

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

This set in particular exhibits this self-similar property because each level is a scaled copy of the entire object. The following figure shows the first seven intervals removed.



Figure 2.1: The first seven iterations of the middle third Cantor Set.

In a general sense, we can take a set, then dilate and translate a copy of it.

**Definition 2.10** (Dilation). [1] Let  $r > 0$  and  $a \in \mathbb{R}$ . The dilation on  $\mathbb{R}$  with ratio  $r$  and center  $a$  is the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = rx + (1 - r)a.$$

Now we consider the following two dilation functions each with the contraction ratio  $1/3$ :

$$f_1(x) = \frac{1}{3}x \quad \text{and} \quad f_2(x) = \frac{x+2}{3}.$$

We will use these two to demonstrate that the Cantor set is self-similar. This is

an example of the definition of self similar, the collection of invariant points under iterated function systems. In this case the two functions are the iterated function system and the set of invariant points is the Cantor set. Admittedly this is an abstract definition so here is a proposition to work through the specific mechanics.

**Proposition 2.2.** *The Cantor set is self-similar. That is to say the Cantor set satisfies the self-referential equations*

$$\mathcal{C} = f_1[\mathcal{C}] \cup f_2[\mathcal{C}].$$

*Proof.* Recall the definition of the Cantor set, as the iterated removal of the middle third.

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

Here we notice that for the first removal,  $n = 0$ , we are left with the left and right portion of the Cantor set. Specifically the left side is a translated copy of the right side:

$$\begin{aligned}
\left[\frac{2}{3}, 1\right] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right) &= \underbrace{\left\{ \left[0, \frac{1}{3}\right] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right) \right\}}_{\text{the left side of the cantor set}} + \frac{2}{3} \\
&= \frac{1}{3} \underbrace{\left\{ [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right) \right\}}_{\text{the left side of the cantor set}} + \frac{2}{3} \\
&= \frac{1}{3} \left\{ \underbrace{[0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right)}_{\text{the original Cantor set}} \right\} + \frac{2}{3} \\
&= f_2[\mathcal{C}]
\end{aligned}$$

We note that this happens at every level of the cantor set. In otherwords by induction:

$$\mathcal{C}_{n+1} = f_1[\mathcal{C}_n] \cup f_2[\mathcal{C}_n].$$

Notice that by using the above two equations, the Cantor set satisfies the self-referential equations:

$$\mathcal{C} = f_1\mathcal{C} \cup f_2\mathcal{C}.$$

■

**Definition 2.11** (Iterated Function System). An iterated function system is a finite set of contraction mappings on a complete metric space. Symbolically, we write this

as, for some  $N \in \mathbb{N}$ ,

$$\{f_i : X \rightarrow X | i = 1, 2, \dots, N\},$$

**Definition 2.12** (Self-Similar Set). A set  $A$  is self-similar if it is the invariant set of an iterated function system.

The invariant set under this iterated function system is a self similar set.

## 2.4 Cantor Sets and Other Measure Theoretic and Topological Examples

We begin this section with a special  $F_\sigma$  set. The Cantor set is defined by taking the interval  $[0, 1]$  and then iteratively removing the open interval containing the middle third, from the previous level. As such it is the countable intersection of closed sets. Formally this can be written as follows.

**Definition 2.13** (Cantor Set). The Cantor set  $\mathcal{C}$ , written as the successive removal of each middle third removed from the previous level is

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

As an  $F_\sigma$  set defined on a closed interval with iteratively removed open intervals we see the middle third Cantor set can also be described with the following three properties.

**Definition 2.14** (Cantor Set). *Cantor sets* are compact, perfect sets, and totally disconnected sets.

As a quick reminder to the definitions:

**Definition 2.15** (Perfect). A *perfect* set is a closed set that contains no isolated points.

**Definition 2.16** (Totally Disconnected). A set is *totally disconnected* if the only connected components are single points.

An equivalent formulation of the Cantor set, is the decimal expansion of all numbers in  $[0, 1]$  in base 3, omitting any representation with a 1. This can be a useful tool for thinking through some examples and counter-examples.

**Example 2.3** (Decimal Expansion Cantor Set).

$$\mathcal{C} = \{x \in [0, 1] : x \text{ has a ternary expansion containing no 1's.}\}$$

Here we notice that although  $1/3 \in \mathcal{C}$  can be written as 0.1 using the trinary expansion, it also has another representation as  $1/3 = 0.0\bar{2}$ . This would be the included representation in the Cantor set. We take a moment to acknowledge that numbers may not have unique representations, where one may be excluded but the other included.

Earlier we defined nowhere dense. Here we see that the Cantor set is an example of a nowhere dense set.

*Claim 2.1.* The Cantor set is *nowhere dense*.

*Proof.* Let  $\mathcal{C}$  be the middle third Cantor set. Notice that  $[0, 1] \setminus \mathcal{C}$  is a set of open intervals:

$$[0, 1] \setminus \mathcal{C} = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left( \frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

Therefore  $\mathcal{C}$  is the countable intersection of closed intervals, and itself is closed.

Notice given some radius  $r$ , there exists some number  $t$ , such that  $0 < t < r$  and  $t$  has a 1 in its ternary expansion. So if we consider any point  $c \in \mathcal{C}$ , then an open ball of radius  $r$  centered at  $c$  then  $B_r(c)$  is  $(c-r, c+r)$  in ternary, necessarily contains a number containing a 1. Therefore  $B_r(c) \not\subseteq \mathcal{C}$ , and  $\mathcal{C}$  has an empty interior. Finally we conclude because  $\mathcal{C}$  is closed and has an empty interior,  $\mathcal{C}$  is nowhere dense. ■

Beyond the middle third Cantor set, we can generalize these in a few different ways. Changing the middle interval, having a few different interval widths, series of interval widths.

Now we look at our example.

**Example 2.4.** A measure theoretically large set is not necessarily topologically large.

$$\mathcal{C}^* = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \left( \frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}} \right)$$

The total length of each removed interval is the geometric series

$$\sum_{k=0}^{\infty} \frac{2^k}{2^{k+2}} = \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}.$$



Therefore the set has Lebesgue measure

$$\lambda^*(\mathcal{C}^*) = \ell([0, 1]) - \frac{1}{2} = \frac{1}{2}.$$

## Chapter 3

# An Erdős Similarity Problem in a Topological Setting

### 3.1 Positive Newhouse Thickness

Cantor sets contain important invariant structures such as Hausdorff dimension, thickness, and denseness. We will investigate thickness and denseness. First we will define the gaps and bounded gaps of Cantor sets in order to construct and define Newhouse thickness.

**Definition 3.1** (Gap). Let  $K$  be some Cantor set. A *gap* of  $K$  is a connected components of  $\mathbb{R} \setminus K$ .

Informally, the gaps are the intervals surrounding the points of the Cantor set. Some of the lengths of these intervals are bounded some are not. In the example of the middle third Cantor set, the unbounded gaps would be  $(-\infty, 0)$  and  $(1, \infty)$ .

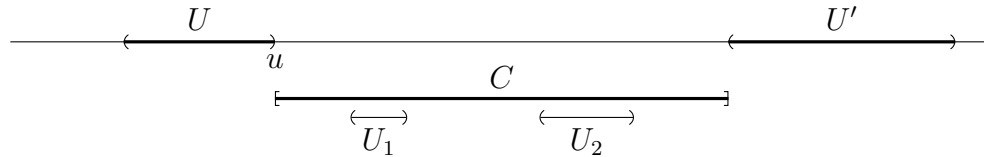
**Definition 3.2** (Bounded Gap). Let  $K$  be a Cantor set. A *bounded gap* is a bounded connected component of  $\mathbb{R} \setminus K$ .

Using these two notions we will define the *bridge* of  $C$  of Cantor set  $K$ .

**Definition 3.3** (Bridge). [5] Let  $K$  be some cantor set and  $U$  be a bounded gap of  $K$  with boundary point  $u$ . The *bridge*  $C$  of  $K$  at  $u$  is the maximal interval in  $\mathbb{R}$  such that:

- $u$  is a boundary point of  $C$
- $C$  contains no point of a gap  $U'$  whose length  $\ell(U') \geq \ell(U)$ .

For clarity the picture below shows that there may be smaller bounded gaps contained in  $C$ .



We use this notion to define the *Newhouse Thickness*. Intuitively the thickness of a Cantor set can be thought of as the infimum of ratios between the bounded gaps and the bridges.

**Definition 3.4** (Newhouse Thickness [5]). The *Newhouse Thickness or thickness* of  $K$  at  $u$  is defined as

$$\tau(K, u) = \frac{\ell(C)}{\ell(U)}.$$

Moreover the thickness of the entire Cantor set is

$$\tau(K) = \inf_{u \in K} \tau(K, u) = \inf .$$

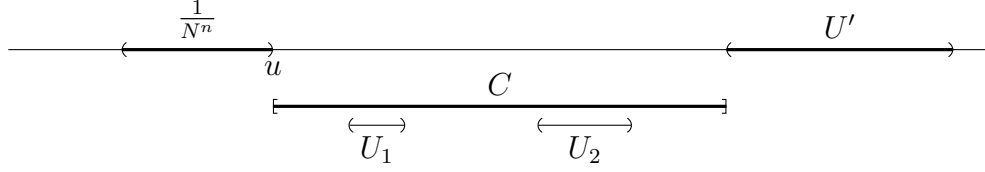
This helps us develop a language to talk about the relative sizes of Cantor sets that is slightly separated from the notion of measure. Importantly it also helps us for a sufficient condition for our main theorem.

Here we will calculate a few examples of Newhouse Thickness. Recall the middle third Cantor set.

**Example 3.1** (Newhouse Thickness of the Middle-third Cantor Set). Let  $K$  be the middle third cantor set. Then the Newhouse thickness is the infimum of the ratio between gaps and bridges. Here we notice that every bounded gap is one third the previous bridge. Therefore the Newhouse thickness of the set is

$$\tau(K) = \inf_{u \in K} \frac{\ell(C)}{\ell(U)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1.$$

**Example 3.2** (Newhouse Thickness of the  $N$ -digit Cantor Set). Let  $K$  be an  $N$ -digit Cantor set. Each gap at the  $n$ -th level is of has length  $N^{-n}$ . We take a moment to note that the location of the gap matters because it affects the thickness. If we assume that for  $2 \leq j \leq n - 1$ , the  $j$ -th digit is removed then we end up with the following cantor set.



With this picture in mind let's take the infimum of the ratios. The thickness of the  $N$ -digit expansion Cantor set is

$$\tau(K) = \inf_{u \in K} \frac{\ell(C)}{\ell(U)} = \min \left\{ \frac{j}{N}, \frac{N-j-1}{N} \right\}$$

### 3.2 The Gap Lemma

**Lemma 3.1** (The Gap Lemma[5]). *Let  $K_1, K_2 \subset \mathbb{R}$  be Cantor sets with thickness  $\tau_1$  and  $\tau_2$ . If  $\tau_1 \cdot \tau_2 > 1$ ,  $K_1$  is not contained in the gap of  $K_2$ ,  $K_2$  is not contained in the gap of  $K_1$  then  $K_1 \cap K_2 \neq \emptyset$ .*

*Proof.* Let  $K_1, K_2$  be two Cantor sets with thickness  $\tau_1, \tau_2$  respectively such that  $K_1$  is not contained in the gap of  $K_2$  and  $K_2$  is not contained in the gap of  $K_1$ .

Assume for the sake of contradiction that  $K_1 \cap K_2 = \emptyset$ . Consider the bounded gaps  $U_1 \subset K_1^c$  and  $U_2 \subset K_2^c$ . We call  $(U_1, U_2)$  a gap-pair if  $U_1$  contains exactly one boundary point of  $U_2$  and  $U_2$  contains exactly one point of  $U_1$ .

We want to use this technique to demonstrate that  $K_1 \cap K_2 \neq \emptyset$ . Let  $C_j^l, C_j^r$  denote the bridges of  $K_j$  for  $j = 1, 2$ . Returning to our original assumptions,  $\tau_1 \cdot \tau_2 > 1$  and

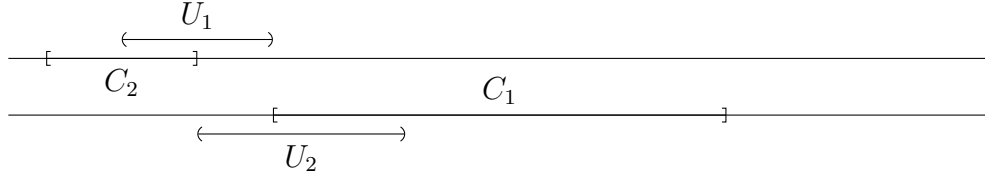
therefore

$$\frac{\ell(C_1)}{\ell(U_2)} \cdot \frac{\ell(C_2)}{\ell(U_1)} > 1.$$

The gap of  $K_1$  where  $\ell(U'_1) < \ell(U_1)$  and  $(U'_1, U_2)$  is the gap pair we need.

From our construction, the right endpoint of  $U_2$  is in  $C_1^r$  or the left endpoint of  $U_1$  is in  $C_2^l$  or both. By assumption we know that  $K_1, K_2$  are not contained in the other's gaps. Therefore there exists some gap-pair  $(U_1, U_2)$ .

A quick picture can help show where these pieces are located: This picture is not accurate because we are using this to derive a contradiction.



*Claim 3.1.* If  $\tau_1\tau_2 > 1$  then from the interval  $U_1$  (or for that matter  $U_2$ ) we can construct another sub-interval  $U'_1$  such that  $\ell(U'_1) < \ell(U_1)$  (or similarly  $U'_2$  such that  $\ell(U'_2) < \ell(U_2)$ ).

Notice that  $(U'_1, U_2)$  is still a gap-pair, as is  $(U_1, U'_2)$ .

Using this construction we can create a sequence of gap-pairs  $(U_1^{(i)}, U_2^{(j)})$ . Notice that the sum is finite

$$\sum_i \ell(U_1^{(i)}) < \infty,$$

and therefore  $\ell(U_1^{(i)}) \rightarrow 0$  as  $i \rightarrow \infty$ . From this construction we have a sequence of gap-pairs such that as  $i \rightarrow \infty$ ,  $\ell(U_1^{(i)}) \rightarrow 0$  and similarly as  $j \rightarrow \infty$ ,  $\ell(U_2^{(j)}) \rightarrow 0$ .

Without loss of generality, we can form a subsequence and use the same indexing for the gap pairs,  $(U_1^{(i)}, U_2^{(i)})$ .

This sequence of gap pairs have a non-empty intersection for all  $i \in \mathbb{N}$ .

Notice that by picking a sequence of points,  $q_i \in U_1^{(i)}$  this forms a convergent subsequence  $q_{i_k} \rightarrow q$ . Notice that  $U_1^{(i)}$  is not fully contained in the gap of  $K_2$ . Moreover because these intervals are compact and nested, we know that  $q$  which is contained in each  $U_1^{(i)}$  is therefore in  $K_2$ .

Because this construction is symmetric, the same argument applies to  $q_i \in U_2^{(i)}$  and so  $q \in K_2$ . Therefore the Cantor sets share at least one point,  $q \in K_1 \cap K_2$ . ■

### 3.3 A Cantor set with positive Newhouse Thickness is not Topologically Universal

Recall the definition, We say that a set  $E$  is *topologically universal* in the collection of dense  $G_\delta$  sets if for all  $G_\delta$  set, we can always find some affine copies of  $E$  inside the set. By an affine copy, we mean sets of the form  $t + \lambda E$  for some  $t \in \mathbb{R}$  and  $\lambda \neq 0$ . A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense  $G_\delta$  sets? This is an exploration of an Erdős conjecture in a topological setting.

**Theorem 3.2.** *Let  $J$  be a cantor set with positive Newhouse thickness. Then  $J$  is*

*not topologically universal in the collection of dense  $G_\delta$  sets.*

*Proof.* Suppose we have some Cantor set  $J$  with Newhouse thickness  $\tau(J) > 0$ . Without loss of generality, we can assume the convex hull of  $J$  is  $[0, 1]$ . Consider Cantor sets  $K$  defined by contraction ratio  $1/N$  and digits  $\{0, 1, \dots, N-1\} \setminus \{(N-1)/2\}$  and  $N$  is odd. By a simple calculation,  $\tau(K) = \frac{N-1}{2}$ . Therefore, we can find a sufficiently large  $N$  so that  $\tau(J)\tau(K) > 1$ .

Using the Cantor set  $K$  Define  $X$  such that

$$X = \bigcup_{n \in \mathbb{Z}} \bigcup_{\ell \in \mathbb{Z}} N^n(K + \ell),$$

creating a dense  $F_\sigma$  set. Now consider  $X^c$ . Because  $K^c$  is open and dense and so is its translated and dilated copies, by the Baire Category Theorem,  $X^c$  is a dense  $G_\delta$ . We now show that  $X^c$  contains no affine copy of  $J$ .

Suppose we have some affine copy,  $t + \lambda J$  where  $t \in \mathbb{R}$  and  $\lambda \neq 0$ . There exists a unique  $n$  such that

$$|\lambda| \in (N^{n-1}, N^n]. \quad (3.1)$$

Similarly there exists a unique  $\ell$  such that

$$t \in (\ell N^n, (\ell + 1)N^n]. \quad (3.2)$$



Let

$$C_1 = N^n(K + \ell) \text{ and } C_2 = t + \lambda J.$$

The convex hull of  $C_1$ , is  $[\ell N^n, (\ell + 1)N^n]$ . So By our choice of  $t$ , we know that  $C_2$  is not in the unbounded gap of  $C_1$  and vice versa.

Now we will check the construction of our Cantor sets such that each is not contained in the bounded gaps of the other. For  $C_1$  its largest corresponding open gap interval is  $|O_1| = N^{n-1}$  and its largest corresponding closed interval is  $|I_1| = N^n$ . For  $C_2$  and its corresponding intervals, we find that  $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$  and  $|I_2| = |\lambda|$  where  $O_J$  is the largest open gap interval in  $J$ . Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|.$$

Therefore  $C_1$  is not in the gaps of  $C_2$  and  $C_2$  is not fully contained in the gaps of  $C_1$ . By our choice of  $K$ , the Newhouse thickness of our sets,  $\tau(C_1)\tau(C_2) \geq 1$ , because Newhouse thickness is scale invariant. Therefore the Gap Lemma implies  $C_1 \cap C_2$  is non-empty and  $C_2$  cannot be in the constructed  $G_\delta$  set. Therefore we conclude  $J$  is not topologically universal in the collection of dense  $G_\delta$  sets.

■

### 3.4 Generalizing into Higher Dimensions

Consider a compact set  $J$  in  $\mathbb{R}^d$ . An affine copy of  $J$  in  $\mathbb{R}^d$  is the set

$$t + \delta O(J)$$

where  $t \in \mathbb{R}^d$ ,  $\delta \neq 0$  and  $O$  is an orthogonal transformation. We say that  $J$  is universal if the collection of dense  $G_\delta$  sets if any dense  $G_\delta$  set contains an affine copy of  $J$ .

**Theorem 3.3.** *If  $X \subset \mathbb{R}^d$  contains a path connected component, then  $X$  is not universal in the set of dense  $G_\delta$  sets of  $\mathbb{R}^d$ .*

*Proof.* Remove the plane

$$\mathbb{R}^d \setminus \bigcup_{i=1}^d \bigcup_{r \in \mathbb{Q}} \{X_i = r\}.$$

Consider some affine copy of  $X$  with a path  $L$ . Then  $X'$  will contain an affine copy of the path  $L'$ . The projection of  $L'$  onto the coordinate axes will be non-degenerate on some interval for at least one of the axes. Call this the  $i$ -th axis. This interval will contain a rational number  $r$ . Therefore  $L'$  will intersect with the coordinate plane,  $X_i = r$ . In other words this dense  $G_\delta$  set omits at least one point and cannot contain affine copy of  $X$ . ■

We can still consider sets that contain no connected component. Cantor dust contains no connected component. We introduce the notion of *projective Newhouse*

*Thickness*, and use it to generalize our one-dimensional results into  $\mathbb{R}^d$ .

**Definition 3.5** (Positive Projective Newhouse Thickness). We say a  $J$  set has *positive projective Newhouse thickness* if for all  $O \in O(d)$

$$\tau(P_x O(J)) > 0$$

where  $P_x$  is the orthogonal projection to the  $x$ -axis and  $O(d)$  is the orthogonal group consisting of all orthogonal transformations in  $\mathbb{R}^d$ .

We note that the projection some Cantor dust set maybe an interval. If that is the case we say the Newhouse Thickness of the projection is equal to infinity.

Using our one dimensional result, we can generalize

**Theorem 3.4.** *Let  $J \subset \mathbb{R}^d$  be a compact set such that it has a positive projective Newhouse thickness. Then  $J$  is not topologically-universal in  $\mathbb{R}^d$ .*

*Proof.* Suppose we have a compact set  $J$  in  $\mathbb{R}^d$  such that it has a positive projective Newhouse thickness. By Theorem 3.2, there exists a dense  $G_\delta$  set  $G_1$  in  $\mathbb{R}^1$  such that  $G_1$  does not contain any affine copy of  $P_x O(J)$ . Now, we consider

$$G = G_1 \underbrace{\times \cdots \times}_{d\text{-times}} G_1.$$

Then  $G$  is a dense  $G_\delta$  set in  $\mathbb{R}^d$ . We claim that there is no affine copy of  $J$  in  $G$ . To justify the claim. Suppose  $G$  contains an affine copy of  $J$  such that  $t + \delta J \subset G$ .

Then we take projection and we obtain that

$$P_x(t) + \delta P_x(J) \subset G_1$$

which is a contradiction. Hence, the claim is true and the proof is complete.

■

## Chapter 4

### Remarks and Open Questions

#### 4.1 Current Research Questions: Zero Newhouse Thickness

This section is devoted to study if Cantor sets with zero Newhouse thickness can be universal. We first provide an example for which two Cantor sets with zero Newhouse thickness can still have arithmetic sum equal to an interval, showing that the converse of the Newhouse thickness theorem is not true.

**Example 4.1.** Let  $N_1, N_2, \dots \in \mathbb{N}_{\geq 2}$ . Consider the following construction of a Cantor set using a decomposition of the unit intervals.

$$\begin{aligned}
[0, 1] &= \frac{1}{N_1} \{0, 1, \dots, N_1 - 1\} + \left[0, \frac{1}{N_1}\right] \\
&= \frac{1}{N_1} \{0, 1, \dots, N_1 - 1\} + \frac{1}{N_1 N_2} \{0, 1, \dots, N_2 - 1\} + \left[0, \frac{1}{N_1 N_2}\right] \\
&= \dots \\
&= \frac{1}{N_1} \{0, 1, \dots, N_1 - 1\} + \frac{1}{N_1 N_2} \{0, 1, \dots, N_2 - 1\} + \dots + \frac{1}{N_1 \dots N_n} \{0, \dots, N_n\} + \dots
\end{aligned}$$

From here we can define the two cantor sets  $K_1, K_2$  where  $K_1$  constitutes the odd indices sets in the above summands and  $K_2$  has the even one. This gives the following constructions for the two Cantor sets:

$$\begin{aligned}
K_1 &= \frac{1}{N_1} \{0, 1, \dots, N_1 - 1\} + \dots + \frac{1}{N_1 \dots N_{2n+1}} \{0, \dots, N_{2n+1} - 1\} + \dots \\
K_2 &= \frac{1}{N_1 N_2} \{0, 1, \dots, N_2 - 1\} + \dots + \frac{1}{N_1 \dots N_{2n}} \{0, \dots, N_{2n} - 1\} + \dots
\end{aligned}$$

From this construction we see that  $K_1 + K_2 = [0, 1]$  is the interval but from the definition of Newhouse thickness,

$$\tau(K_1) = \inf \left\{ \frac{1}{N_1 - 1}, \frac{1}{N_3 - 1}, \dots \right\} = 0$$

$$\tau(K_2) = \inf \left\{ \frac{1}{N_2 - 1}, \frac{1}{N_4 - 1}, \dots \right\} = 0.$$

Therefore we have created an interval from two sets with Newhouse thickness 0 if we have  $\lim_{n \rightarrow \infty} N_n = \infty$ .

We will conclude by ask the following questions. Recall  $I_J$  denotes the smallest closed interval containing the Cantor set  $J$  and  $O_J$  denotes the largest open interval in  $I_J \setminus J$ .

1. Given a Cantor set  $J$ , does there exist some  $K$  such that  $J + K = I_J + I_K$ ?
2. If we assume that there exists  $K$  such that  $J + K = I_J + I_K$ , can we prove that  $J$  is not universal?
3. (rescaling condition) If we assume that  $|\lambda_1 I_J| \geq |\lambda_2 O_K|$ ,  $|\lambda_2 I_K| \geq |\lambda_1 O_J|$  and  $J + K = I_J + I_K$ , then  $\lambda_1 J + \lambda_2 K = \lambda_1 I_J + \lambda_2 I_K$ .

We also notice that to solve the second question, we notice that  $J + K = I_J + I_K$  implies that

$$(J + a) + (K + b) = (I_J + a) + (I_K + b) \text{ and } bJ + bK = bI_J + bI_K.$$

We can always translate and rescale  $J, K$  so that  $I_J = [0, a]$  and  $I_K = [0, 1]$ . Moreover, the following lemma is important.

**Lemma 4.1.** *Suppose that the Cantor sets  $J$  and  $K$  satisfies  $J + K = I_J + I_K$ . Then  $|I_J| \geq |O_K|$  and  $|I_K| \geq |O_J|$ .*

The lemma also said that the condition  $|\lambda_1 I_J| \geq |\lambda_2 O_k|, |\lambda_2 I_k| \geq |\lambda_1 O_J|$  is necessary in the rescaling condition.

**Proposition 4.2.** *Let  $J$  be a Cantor set such that  $J + K = I_J + I_K$  where  $I_J = [0, a]$  and  $I_K = [0, 1]$ . Suppose that the rescaling condition (3) holds. Then  $J$  is not universal in the collection of dense  $G_\delta$ .*

*Proof.* The proof is similar to the proof in Theorem 3.2. With  $K$  given in the assumption. We can assume that  $|I_J| > |O_K|$ . Suppose that  $|I_J| = |O_K|$ . Since  $|O_J| < 1$ , we can choose  $\epsilon$  such that  $(1 - \epsilon) > |O_J|$ . Then we consider  $K' = (1 - \epsilon)K$  and we will have  $|I_J| > (1 - \epsilon)|O_K|$ . In this case, by the rescaling condition,  $J + K' = I_J + I_{K'}$  and we have another  $K'$  such that  $|I_J| > |O_{K'}|$ .

As now we have  $|I_J| > |O_K|$ , we can find  $0 < \rho < 1$  such that  $\rho|I_J| > |O_K|$ . We now define

$$X = \bigcup_{n \in \mathbb{Z}} \bigcup_{\ell \in \mathbb{Z}} \rho^n(K + \ell).$$

Then  $X^c$  is a dense  $G_\delta$  set. Suppose that we have an affine copy  $t + \lambda J$ , we would like to claim that  $t + \lambda J$  intersects non-trivially with  $\rho^n(K + \ell)$  for some  $n, \ell \in \mathbb{Z}$ , which will complete the proof of the theorem.

To justify the claim, we let  $0 < \rho < 1$  take the unique  $n$  such that

$$|\lambda| \in [\rho^{n+1}, \rho^n) \tag{4.1}$$



and the unique  $\ell \in \mathbb{Z}$  such that

$$t \in (\ell\rho^n, (\ell+1)\rho^n]. \quad (4.2)$$

Then we consider the arithmetic sum  $\rho^n K - \lambda J$ . We now check the assumption in the rescaling condition with  $\lambda_1 = \rho^n$  and  $\lambda_2 = -\lambda$ . Indeed,

$$|\lambda_2 I_J| \geq \rho^{n+1} |I_J| = |\lambda_1|(\rho |I_J|) \geq |\lambda_1 O_K|$$

by our choice of  $\rho$ . On the other hand,

$$|\lambda_1 I_K| \geq |\lambda| \geq |\lambda_2 O_J|$$

since  $|I_K| \geq |O_J|$  by Lemma 4.1. Hence, using the rescaling condition,

$$\rho^n K - \lambda J = \rho^n I_K - \lambda I_J.$$

If  $\lambda > 0$ , then we have

$$\rho^n(K + \ell) - \lambda J = [\rho^n \ell - \lambda a, \rho^n(1 + \ell)]$$

which contains  $t$  by (4.2). Similarly, if  $\lambda < 0$ , then

$$\rho^n(K + \ell) - \lambda J = [\rho^n \ell, \rho^n(\ell + 1) - \lambda a].$$

It also contains  $t$  by (4.2). The proof is now complete. ■

We now conclude with several open questions.

**Conjecture 4.1.** *Cantor sets, which are compact, perfect sets, and totally disconnected.*

We also restate the the above two question For any cantor set, does there exist another such that  $C_1 + C_2$  adds up to an interval?

Rescaling condition?

If these two are true then we can settle this in one dimension.

Finally we ask, Do all self-similar sets have positive projective newhouse thickness?

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