On a Topological Erdős Similarity Problem

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *On a Topological Erdős Similarity Problem* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Often in data work, one may ask which patterns are possible to find, given a certain data set, or hypothetical relationship. Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets? A set is called universal in another set, when every subset of the larger set contains some scaled and translated copy of original. Paul Erdős proposed a conjecture that no infinite set, is universal in the the collection of sets with positive measure. This paper explores an analogous problem in a topological setting. Instead of sets with positive measure we investigate the collection of dense G-delta sets. Any finite or countable set is found to be topologically universal. Any set containing an interval cannot be topologically universal. We also have the new result that any Cantor sets is not topologically universal. Cantor sets, which contains no interval and are uncountably infinite, are not topologically universal.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

Often in data work, one may ask, "Which patterns are possible to find, given a certain data set, or hypothetical relationship?" Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets?

If we more deeply consider patterns which infinitely repeat scaled copies of itself, this turns out to be a self-similar set. This is very closely related to the concept of fractals as well as a core area of study within dynamical systems. Some fractals can be generated using recursive functions. In dynamical systems some iterated functions may have sets of invariant points. These collections of invariant points may themselves be self-similar. In this respect it is valuable to explore which patterns are found everywhere because they give insight into some key facets within dynamical systems as well as the nature of patterns.

Given a specific set of points, we can formalize this notion of a scaled copy of a pattern as an *affine transformation.* We will formally define this in the next section.

Exploring this notion a little more deeply, we can begin to investigate which patterns appear everywhere. Informally, a set is called *universal* in another collection, when every subset of the collection contains some scaled and translated copy of original pattern.

Paul Erdős proposed a conjecture that no infinite set, is measure-universal in the collection of sets with positive measure. We will explore an analogous problem in a topological setting. This also answers a question posed by Svetic[11], "Is it true that for every uncountably infinite set, E, of real numbers, there exists $S \subset [0, 1]$ of full measure that does not contain an affine copy of E?" Additionally this result can be used to prove some results in higher dimensions.

1.1 An Erdős Self-Similarity Conjecture in Measure Space

There is a long standing conjecture from Paul Erdős on universal sets. Informally the conjecture states there is no infinite set that is universal in the real number line. This is a conjecture about which types of patterns can exist within another sets of numbers.

We can now start by defining *affine transformations* or *affine copies.*

Definition 1.1 (Affine copy)**.** An <u>affine copy</u> of a set $A \subset \mathbb{R}$ is a scaled and translated set A' such that for some $\lambda \neq 0, \lambda \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$A' = \{\lambda a + t : a \in A\}.$$

In this instance, it is a scaled and then translated copy of the set is still one dimensional and therefore need not be a shape. This gives us some flexibility when addressing different sets. This definition is used to define measure universal.

Definition 1.2 (Measure Universal)**.** A set E is called <u>measure universal</u> in X if for every subset $S \subseteq X$, with positive measure, $\mu(S) > 0$, there exist an affine copy of E such that $t + \lambda E \subseteq S$, for some $\lambda \neq 0$ and $\lambda, t \in \mathbb{R}$.

Now we can formally, state Erdős' conjecture as follows.

Conjecture 1.1 (The Erdős Similarity Conjecture)**.** *Let $E \subseteq \mathbb{R}$ be an infinite set of real numbers. Prove that there is a set of real numbers S of positive measure which does not contain an affine copy of E.*

Paul Erdős originally posed this question back in 1974 by building off of the work of Steinhaus. Steinhaus[10] first posed that finite sets are universal in sets with positive measure. In the time since Erdős first posed this conjecture, there has been some progress.

Falconer [4] made a substantial progress by showing slowly decaying sequences are not measure universal. Bourgain [3] expanded on this by showing some faster decaying sequences are also not measure universal. In particular he demonstrated that the sum-set of any three sets, cannot be measure universal. Most recently Kolountzakis [5] demonstrated using probabilistic arguments to demonstrate that certain set with large gaps cannot be measure universal.

Currently it is still an open question whether or not sequences that decay at the rate of 2^{-n} are measure universal.

In this paper we take this idea of measure universality and put it into a topological context. We show in theorem 3.2 a Cantor set with positive Newhouse thickness is not topologically universal.

1.2 An Analogous Theorem in a Topological Setting

In a non-rigorous exploration of the real number line, one might assume that patterns which appear everywhere should have positive measure. However density and measure are not intrinsically linked. Indeed it is possible to have uncountable dense sets with measure zero and to have sets with full measure that are nowhere dense.

Borrowing the concept of G_δ sets from topology, we can explore an extension of the Erdős similarity problem. Instead of exploring sets with positive measure we can explore dense G_δ sets. This is the countable intersection of open sets that are also dense. We will use these dense G_δ sets to define topological-universal.

Definition 1.3 (Topological-Universal)**.** A set E is called Topological-universal in $\mathbb{R}$ if for every dense G_δ subset $S \subseteq \mathbb{R}$, there exist an affine copy of E such that

$t + \lambda E \subseteq S$, for some $\lambda \neq 0$ and $t \in \mathbb{R}$.

As stated above, a dense G_δ set can have measure zero. This also means that there is not a direct relationship between topological universal and measure universal. We make two observations on this fact: a set with an interior cannot have be topologically universal, by the Baire Category Theorem all countable sets are topologically universal.

This motivates the question, "Is an uncountable, nowhere dense set, with an empty interior topologically universal?" Cantor sets for example, have an empty interior and are nowhere dense. This will be our main focus of this thesis.

In chapter 2 we review background definitions and theorems in measure theory and topology. We also give several examples to demonstrate some of the nuances of these facts. In chapter 3 we define Newhouse Thickness, prove the Gap Lemma and prove our main result in Theorem 3.2.

Theorem. *Let J be a Cantor set with positive Newhouse thickness. Then J is not topologically universal.*

Using this result we can partially answer a question of Sevtic[11] with Cantor set that have positive Newhouse thickness and by extension expand on the original Erdő's similarity conjecture for some uncountable nowhere dense set.

We will also extend these results into $\mathbb{R}^d$ by defining the projective Newhouse thickness thereby proving that a large class of nowhere dense Cantor sets in high dimension are not topologically universal. Finally in chapter 4 we present some partial results for Cantor sets with zero Newhouse thickness and conclude with some open questions and remarks.

Chapter 2

Measure & Topology

2.1 Some Measure Theoery

First we want to begin with background definitions and theorems. Erdös' problem specifically deals with infinite set, and affine copies found in measurable set of sets. In our problem, rather than dealing with measurable sets, we will instead use the set of dense G_δ sets.

Underpinning the nuances of this problem, measure theoretic size, and topological size, are not the same. From an intuitive sense of the number line one might think when you are scattered throughout an interval, you would have measure, except in special cases. Similarly one might think that if you have measure, then you would be scattered everywhere. However both of these instances fail when you add in rigorous arguments. Indeed it is possible to construct a set that is no-where dense and has positive measure. It is also possible to construct an uncountable set that is dense and has measure. zero. In other words topological size (density) is not the same thing as measure theoretic size.

First we will review some measure theory. In order to define *measure* and *measurable sets* we first need to define σ-Algebra.

Definition 2.1 (σ-Algebra)**.** Let X be some set and 2^X be the set of subsets of X. Let $\Sigma \subseteq 2^X$. We call Σ a $\sigma-$algebra over X if it satisfies the following three conditions:

1. $\emptyset \in \Sigma$
2. If $E \in \Sigma$, then $X \setminus E \in \Sigma$.
3. If $E_1, E_2, \cdots \in \Sigma$ is a sequence of subsets, then $\bigcup_{k=1}^{\infty} E_k \in \Sigma$.

In this instance we describe set of sets in terms of intersection and union. This allows us to generate an algebraically closed collection of sets.

In the Erdős conjecture, the measure is specifically referencing *Lebesgue Measure*. First we define *Lebesgue outer measure* which is defined on all sets. Then to define *Lebesgue Measure* we will restrict the universe to the appropriate sigma algebra of measurable sets. For details about measure theory, please refer to [2] and [9].

Definition 2.2 (Lebesgue Outer Measure [2])**.** For any interval $I = [a, b]$ (or $I = (a, b)$) in the set $\mathbb{R}$ of real numbers, let $\ell(I) = b - a$ denote its length. For any subset $E \subseteq \mathbb{R}$, the *Lebesgue outer measure* $\lambda^*(E)$ is defined as an inf :

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k\right\}.$$

Finally we define Lebbesgue measurable sets, and Lebesgue measure.

Definition 2.3 (Lebesgue Measurable Sets [9])**.** A subset $E \subset \mathbb{R}$ is *Lebesgue measurable* if for any $\epsilon > 0$, there exists some open subset $\mathcal{O} \subset \mathbb{R}$ such that $E \subset \mathcal{O}$ and

$$\lambda^*(E - \mathcal{O}) < \epsilon.$$

The collection of all Lebesgue measurable sets form a σ-algebra of $\mathbb{R}$.

Definition 2.4 (Lebesgue Measure[9])**.** The Lebesgue measure is the outer measurable defined on the σ-algebra from the Lebesgue measurable sets. That is to say: If E is Lebesgue measurable then we define the *Lebesgue measure* as

$$\lambda(E) = \lambda^*(E).$$

Notice that not all sets are necessarily Lebesgue measurable, such as Vitali sets. We do however make one conjecture in chapter 4 noting that all measurable sets with positive measure contain a non-measurable set.

2.2 Topological and The Baire Category Theorem

Now we will review some Topology that is relevant to our main theorem. We need to define *dense*. From there we will build into G_δ sets. There are many equivalent definitions of dense. We will use the following definition so that we can continue to develop the intuition around intervals and interiors.

Definition 2.5 (Dense)**.** A set S is called dense in X if for every $x \in X$, every neighborhood U of x intersects A.

In a similar fashion to the definitions of *countable* and *uncountable*, the opposite of *dense* is *nowhere dense*.

Definition 2.6 (Nowhere Dense)**.** Let X be a topological space. A subset $B \subseteq X$ of a topological space is called *nowhere dense* in X if its closure has an empty interior. That is to say, B is *nowhere dense* in X if for each open set $U \subseteq X$, $B \cap U$ is not dense in U.

This allows us to now explore the differences between density and measure. As stated earlier, topological size and measure theoretic size are not necessarily related. What do we mean by topologically Large? Uncountable and dense. Similarly what do we mean by measure theoretically large? Non-zero measure. It is helpful to define the opposite of topologically large, namely meager sets.

Definition 2.7 (Meager)**.** A subset $C \subseteq X$ of a topological space is called *meager* in X if it is the countable union of nowhere-dense subsets of X.

Next we will define dense G_δ sets, as well as some useful examples.

Definition 2.8 (G-Delta Set)**.** A G_δ set is the countable intersection of open sets. Namely, let $O_i \subset X$ for $i \in \mathbb{N}$ be a collection of open sets of X. Then $\bigcap_{n=1}^{\infty} O_i$, is a G_δ set.

Example 2.1. The irrational numbers are a G_δ set. Consider the following construction of the set of irrational numbers:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Notice that each $\mathbb{R} \setminus \{q\} = (-\infty, q) \cup (q, \infty)$ is an open subset of $\mathbb{R}$. Furthermore, rational numbers are countable. Therefore the intersection of these sets are a G_δ set. Moreover, in this instance it is a dense G_δ set. We will study these objects further.

Lastly we remark that there is an analogous set which is the countable union of closed sets.

Definition 2.9 (F_σ Set)**.** An F_σ set is the countable union of closed sets. This is equivalent to the compliment of a G_δ set is an F_σ set.

A key theorem that links analysis to set theory is the Baire Category Theorem. This also establishes a link to understanding certain types of topological sets.

Theorem 2.1 (Baire Category Theorem[6])**.** *Let X be a complete space. Then any countable intersection of open dense sets is dense.*

Within the study of measure theory it can sometimes be unclear if a set is dense in another set. For example consider the following set:

$$\mathbb{R}^2 \setminus \{(x, y) : y = mx + b, \text{where } m, b \in \mathbb{Q}.\}$$

Notice that this can also be written as

$$\bigcap_{m,b\in\mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = mx + b\},$$

which is the plane, but removing all lines with rational coefficients, and rational intercepts. Each plane with one line removed, $\mathbb{R}^2\setminus\{(x, y) : y = mx+b, \text{ for some } m, b \in \mathbb{Q}\}$ is an open dense set. Then by the Baire Category Theorem, the countable intersection is dense. Moreover this same idea holds for removing polynomials with rational coefficients. We can write this as the countable intersection

$$\bigcap_{n\in\mathbb{N}} \bigcap_{a_k\in\mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = \sum_{k=0}^{n} a_k x^k\}.$$

Example 2.2 (A dense $\boldsymbol{G_\delta}$ set of measure zero.)**.** First consider an enumeration of the rationals q_n. For each $\epsilon > 0$, let

$$I_\epsilon = \bigcup_{n=1}^{\infty} (q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n}).$$

Clearly I_ϵ is open because it is the union of open intervals and it is dense because of the rational numbers. The Lebesgue measure of each set is at most 2ϵ. Now consider

$$G = \bigcap_{k=1}^{\infty} I_{1/k}.$$

By the Baire Category Theorem, this set is dense, but its measure is less than that of any $|I_{1/k}| = 2/k$, so it has measure zero.

Finally we remark that any countable set is topologically universal. By the Baire category theorem, we can take a sequence of numbers from the dense G_δ set and multiply every element of the countable set, by that collection. This parses the countable set into the G_δ set.

We also remark that any set that contains an interval is not topologically universal. A set that contains an interval has an interior. However a there is a dense G_δ set that has zero measure and therefore contains no intervals.

2.3 Affine Copies and the Self Similarity Property

One of the more popularly known results from math is the the Mandelbrot set fractal. In some senses fractals can be very general objects that are considered to have fractional dimensions. This definition can be very general but by that same token, may not always capture some of the inherent geometry of some fractal type objects. Some objects with fractional dimensions have a self-similar property, and some self-similar objects have fractional dimension. For the scope of this paper, we will not be discussing dimension. However we will investigate the notion of self-similarity.

Even one dimensional objects can have this self-similar property. Take for example the middle Third Cantor set. The middle third Cantor set is defined by recursively removing the open middle third interval of the previous remaining closed intervals. Explicitly this can be constructed using countable intersection.

Example 2.3 (The Middle Third Cantor Set)**.**

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

This set in particular exhibits this self-similar property because each level is a scaled copy of the entire object. The following figure shows the first seven intervals removed.

Figure 2.1: The first seven iterations of the middle third Cantor Set.

Cantor sets are self-similar. In a general sense, a set that is expressible in terms of a finite number of different contraction mappings is self similar.

Definition 2.10 (Contraction[4])**.** Let $0 < r < 1$ and $D \subseteq \mathbb{R}^d$ be a closed subset. .

A <u>contraction</u> $f : D \to D$ is a function such that for all $x, y \in D$

$$|f(x) - f(y)| < r|x - y|.$$

From here we take a finite set of contractions $\{f_1, \ldots, f_n\}$.

Definition 2.11 (Iterated Function System[4])**.** An iterated function system is a finite set of contraction mappings on a complete metric space, X. Symbolically, we write this as, for some $N \in \mathbb{N}$,

$$\{f_i : X \to X | i = 1, 2, \ldots, N\},$$

Definition 2.12 (Self-Similar Set)**.** A set $F \subseteq X$ is self-similar if it is the invariant set of an iterated function system:

$$F = \bigcup_{i=1}^{N} f_i(F).$$

We conclude this section with the Cantor middle third set.

Example 2.4. The Cantor set is self-similar. Consider the closed interval $[0, 1]$. Now we consider the following two contractions, each with the contraction ratio 1/3:

$$f_1(x) = \frac{1}{3}x \quad \text{and} \quad f_2(x) = \frac{x+2}{3}.$$

Each level of the cantor set is another iteration of these two functions. In this case the two functions are the iterated function system and the set of invariant points is the Cantor set.

2.4 Cantor Sets and Other Measure Theoretic and Topological Examples

We begin this section with a special F_σ set. The Cantor set is defined by taking the interval $[0,1]$ and then iterate by removing the open interval containing the middle third, from the previous level. As such it is the countable intersection of closed sets. Formally this can be written as follows.

Definition 2.13 (Cantor Set)**.** The Cantor set $\mathcal{C}$, written as the successive removal of each middle third removed from the previous level is

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

As an F_σ set defined on a closed interval with iteratively removed open intervals we see the middle third Cantor set can also be described with the following three properties.

Definition 2.14 (Cantor Set)**.** *Cantor sets* are compact, perfect sets, and totally disconnected sets.

As a quick reminder to the definitions:

Definition 2.15 (Perfect)**.** A *perfect* set is a closed set that contains no isolated points.

Definition 2.16 (Totally Disconnected)**.** A set is *totally disconnected* if the only connected components are single points.

An equivalent formulation of the Cantor set, is the decimal expansion of all numbers in $[0, 1]$ in base 3, omitting any representation with a 1. This can be a useful tool for thinking through some examples and counter-examples.

Example 2.5 (Decimal Expansion Cantor Set)**.**

$$\mathcal{C} = \{x \in [0, 1] : \ x \text{ has a ternary expansion containing no 1's.}\}$$

Here we notice that although $1/3 \in \mathcal{C}$ can be written as 0.1 using the ternary expansion, it also has another representation as $1/3 = 0.0\overline{2}$ This would be the included representation in the Cantor set. We take a moment to acknowledge that numbers may not have unique representations, where one may be excluded but the other included.

Earlier we defined nowhere dense. Here we see that the Cantor set is an example of a nowhere dense set.

Claim 2.1. The Cantor set is *nowhere dense.*

Proof. Let $\mathcal{C}$ be the middle third Cantor set. Notice that $[0, 1] \setminus \mathcal{C}$ is a set of open

Figure 2.2: The open middle intervals are recursively removed. This removes the interior of the set, while still leaving the original set closed.

intervals:

$$[0,1] \setminus \mathcal{C} = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

Therefore $\mathcal{C}$ is the countable intersection of closed intervals, and itself is closed.

Notice given some radius r, there exists some number t, such that $0 < t < r$ and t has a 1 in its ternary expansion. So if we consider any point $c \in \mathcal{C}$, then an open ball of radius r centered at c then $B_r(c)$ is $(c-r, c+r)$ in ternary, necessarily contains a number containing a 1. Therefore $B_r(c) \nsubseteq \mathcal{C}$, and $\mathcal{C}$ has an empty interior. Finally we conclude because $\mathcal{C}$ is closed and has an empty interior, $\mathcal{C}$ is nowhere dense. ■

Beyond the middle third Cantor set, we can generalize these in a few different ways. Changing the middle interval, having a few different interval widths, series of interval widths.

Example 2.6 (A measure theoretically large set is not necessarily topologically large.)**.** The Smith-Volterra-Cantor set is formed in a similar manor to the middle-third Cantor set. Starting with the closed interval, remove the middle fourth recursively. At each level, 2^{n-1} intervals of length $1/4^n$ are removed. The total length of these removed intervals are

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = \frac{1}{2}.$$

Therefore the Smith-Volterra-Cantor set has measure $1 - 1/2 = 1/2$. This closed set still has an empty interior and is therefore nowhere dense but still has positive measure.

Example 2.7 (A dense, uncountable set of measure zero)**.** As stated above, the middle third Cantor set has measure 0. Let $\mathcal{C}_q$ denote a cantor set translated by a rational number $q \in \mathbb{Q}$. This gluing of sets is dense but still measure zero. This is formed from closed sets.

$$\left| \bigcup_{q \in \mathbb{Q}} \mathcal{C}_q \right| \leq \sum_{q \in \mathbb{Q}} |\mathcal{C}_q| = 0.$$

Chapter 3

An Erdős Similarity Problem in a Topological Setting

3.1 Positive Newhouse Thickness

Cantor sets contain important invariant structures such as Hausdorff dimension, thickness, and denseness. We will investigate thickness and denseness. First we will define the gaps and bounded gaps of Cantor sets in order to construct and define Newhouse thickness.

Definition 3.1 (Gap)**.** Let K be some Cantor set. A *gap* of K is a connected components of $\mathbb{R} \setminus K$.

Informally, the gaps are the intervals surrounding the points of the Cantor set. Some of the lengths of these intervals are bounded some are not. In the example of the middle third Cantor set, the unbounded gaps would be $(-\infty, 0)$ and $(1, \infty)$.

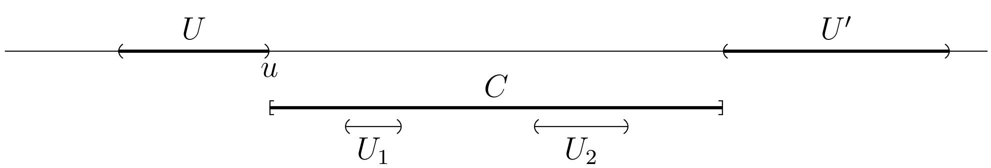
Definition 3.2 (Bounded Gap)**.** Let K be a Cantor set. A *bounded gap* is a bounded connected component of $\mathbb{R} \setminus K$.

Using these two notions we will define the *bridge* of C of Cantor set K.

Definition 3.3 (Bridge)**.** [8] Let K be some cantor set and U be a bounded gap of K with boundary point u. The *bridge* C of K at u is the maximal interval in $\mathbb{R}$ such that:

- u is a boundary point of C
- C contains no point of a gap U' whose length $\ell(U') \geq \ell(U)$..

For clarity the picture below shows that there may be smaller bounded gaps contained in C.



We use this notion to define the *Newhouse Thickness*. Intuitively the thickness of a Cantor set can be thought of as a the infimimum of ratios between the bounded gaps and the bridges.

Definition 3.4 (Newhouse Thickness [8])**.** The *Newhouse Thickness or thickness of K at u* is defined as

$$\tau(K, u) = \frac{\ell(C)}{\ell(U)}.$$

Moreover for $\mathcal{U} = \{\text{set of all boundary points of bounded gaps}\}$, the thickness of the entire Cantor set is

$$\tau(K) = \inf_{u \in \mathcal{U}} \tau(K, u) = \inf_{u \in \mathcal{U}} \frac{\ell(C)}{\ell(U)}.$$

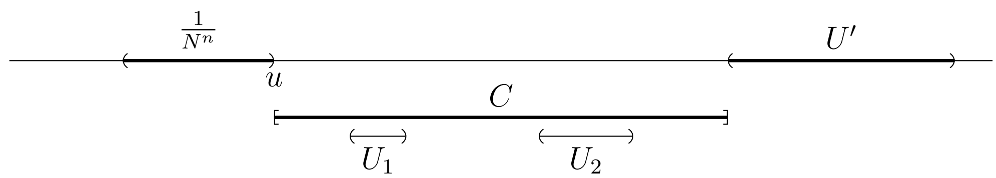
This helps us develop a language to talk about the relative sizes of Cantor sets that is slightly separated from the notion of measure. Importantly it also helps us for a sufficient condition for our main theorem.

Here we will calculate a few examples of Newhouse Thickness. Recall the middle third Cantor set.

Example 3.1 (Newhouse Thickness of the Middle-third Cantor Set)**.** Let K be the middle third cantor set. Then the Newhouse thickness is the infimum of the ratio between gaps and bridges. Here we notice that every bounded gap is one third the previous bridge. Therefore the Newhouse thickness of the set is

$$\tau(K) = \inf_{u \in K} \frac{\ell(C)}{\ell(U)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1.$$

Example 3.2 (Newhouse Thickness of the N-digit Cantor Set)**.** Let K be an N-digit Cantor set. Each gap at the n-th level is of has length N^{-n}. We take a moment to note that the location of the gap matters because it affects the thickness. If we assume that for $2 \leq j \leq n-1$, the j-th digit is removed then we end up with the following cantor set.



With this picture in mind lets take the infimum of the ratios. The thickness of the N-digit expansion Cantor set is

$$\tau(K) = \inf_{u \in K} \frac{\ell(C)}{\ell(U)} = \min\{j, N - j - 1\}$$

3.2 The Gap Lemma

The Gap lemma was originally introduced by Newhouse in his paper "The Abundance of Wild Hyperbolic Sets and Non-Smooth Stable Sets for Diffeomorphism," [7]. This lemma proves to be useful in dynamical systems. The proof of the lemma below is borrowed from Palis and Takens "Hyperbolicity & Sensitive Chaotic Dynamics at Homoclinic Bifurcations." Since their proof was also brief, we expand their proof with more details below. There has also been additional work by Astels, where in his dissertation [1] he details the structure of the intersections.

Lemma 3.1 (The Gap Lemma[8])**.** *Let $K_1, K_2, \subset \mathbb{R}$ be Cantor sets with thickness τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$, K_1 is not contained the gap of K_2, K_2 is not contained in the gap of K_1 then $K_1 \cap K_2 \neq \emptyset$.*

Proof. Let K_1, K_2 be two Cantor sets with thickness τ_1, τ_2 respectively such that K_1

is not contained in the gap of K_2 and K_2 is not contained in the gap of K_1.

Assume for the sake of contradiction that $K_1 \cap K_2 = \emptyset$. Consider the bounded gaps $U_1 \subset K_1^c$ and $U_2 \subset K_2^c$. We call (U_1, U_2) a gap-pair if U_1 contains exactly one boundary point of U_2 and U_2 contains exactly one point of U_1.

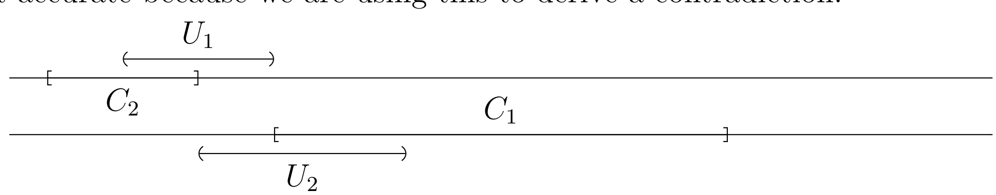
We want to use this technique to demonstrate that $K_1 \cap K_2 \neq \emptyset$. Let C_j^l, C_j^r denote the bridges of K_j for $j = 1, 2$. Returning to our original assumptions, $\tau_1 \cdot \tau_2 > 1$ and therefore

$$\frac{\ell(C_1)}{\ell(U_2)} \cdot \frac{\ell(C_2)}{\ell(U_1)} > 1.$$

The gap of K_1 where $\ell(U_1') < \ell(U_1)$ and (U_1', U_2) is the gap pair we need.

From our construction, the right endpoint of U_2 is in C_1^r or the left endpoint of U_1 is in C_2^l or both. By assumption we know that K_1, K_2 are not contained in the other's gaps. Therefore there exists some gap-pair (U_1, U_2).

A quick picture can help show where these pieces are located: This picture is not accurate because we are using this to derive a contradiction.



Claim 3.1. If $\tau_1 \tau_2 > 1$ then from the interval U_1 (or for that matter U_2) we can construct another sub-interval U_1' such that $\ell(U_1') < \ell(U_1)$ (or similarly U_2' such that $\ell(U_2') < \ell(U_2)$).

Notice that (U_1', U_2) is still a gap-pair, as is (U_1, U_2').

Using this construction we can create a sequence of gap-pairs $(U_1^{(i)}, U_2^{(j)})$. Notice that the sum is finite

$$\sum_{i}^{\infty} \ell(U_1^{(i)}) < \infty,$$

and therefore $\ell(U_1^{(i)}) \to 0$ as $i \to \infty$. From this we construction we have a sequence of gap-pairs such that as $i \to \infty$, $\ell(U_1^{(i)}) \to 0$ and similarly as $j \to \infty$, $\ell(U_2^{(j)}) \to 0$.

Without loss of generality, we can form a subsequence and use the same indexing for the gap pairs, $(U_1^{(i)}, U_2^{(i)})$.

This sequence of gap pairs have a non-empty intersection for all $i \in \mathbb{N}$.

Notice that by picking a sequence of points, $q_i \in U_1^{(i)}$ this forms a convergent subsequence $q_{i_k} \to q$. Notice that $U_1^{(i)}$ is not fully contained in the gap of K_2. Moreover because these intervals are compact and nested, we know that q which is contained in each $U_1^{(i)}$ is therefore in K_2.

Because this construction is symmetric, the same argument applies to $q_i \in U_2^{(i)}$ and so $q \in K_2$. Therefore the Cantor sets share at least one point, $q \in K_1 \cap K_2$.

$\blacksquare$

3.3 A Cantor set with positive Newhouse Thickness is not Topologically Universal

Recall the definition, we say that a set E is *topologically universal* in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t, \lambda \in \mathbb{R}$ and $\lambda \neq 0$. From earlier we have our question, "Is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? " We are now ready to prove our main theorem in the introduction.

Theorem 3.2. *Let J be a cantor set with positive Newhouse thickness. Then J is not topologically universal in the collection of dense G_δ sets.*

Proof. Suppose we have some Cantor set J with Newhouse thickness $\tau(J) > 0$. Without loss of generality, we can assume the convex hull of J $[0, 1]$. Consider Cantor sets K defined by contraction ratio $1/N$ and digits $\{0, 1, ..., N-1\} \setminus \{(N-1)/2\}$ and N is odd. By a simple calculation, $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(J)\tau(K) > 1$.

Using the Cantor set K Define X such that

$$X = \bigcup_{n \in \mathbb{Z}} \bigcup_{\ell \in \mathbb{Z}} N^n(K + \ell), \tag{3.1}$$

creating a dense F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, by the Baire Category Theorem, X^c is a dense G_δ. We now show that X^c contains no affine copy of J.

Suppose we have some affine copy, $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{3.2}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell+1)N^n]. \tag{3.3}$$

Let

$$C_1 = N^n(K+\ell) \text{ and } C_2 = t + \lambda J.$$

The convex hull of C_1, is $[\ell N^n, (\ell+1)N^n]$. So By our choice of t, we know that C_2 is not in the unbounded gap of C_1 and vice versa.

Now we will check the construction of our Cantor sets such that each is not contained in the bounded gaps of the other. For C_1 its largest corresponding open gap interval is $|O_1| = N^{n-1}$ and its largest corresponding closed interval is $|I_1| = N^n$. For C_2 and is corresponding intervals, we find that $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$ and $|I_2| = |\lambda|$ where O_J is the largest open gap interval in J. Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|.$$

Therefore C_1 is not in the gaps of C_2 and C_2 is not fully contained in the gaps of C_1. By our choice of K, the Newhouse thickness of our sets, $\tau(C_1))\tau(C_2) \geq 1$, because Newhouse thickness is scale invariant. Therefore the the Gap Lemma implies $C_1 \cap C_2$ is non-empty and C_2 cannot be in the constructed G_δ set. Therefore we conclude J is not topologically universal in the collection of dense G_δ sets.

■

We also mention that this partially solves a question posed by Svetic[11] in 2000, "Is it true that for every uncountably infinite set, E, of real numbers, there exists $S \subset [0,1]$ of full measure that does not contain an affine copy of E?" From earlier, any uncountable set that contains an interval is not topologically universal. We now solve his question for Cantor set with positive Newhouse thickness.

Theorem 3.3. *Let E be a Cantor set with positive Newhouse thickness. Then there exists a set of full measure in $[0,1]$ that does not contain an affine copy of E.*

Proof. Let E be a Cantor set of positive Newhouse thickness. Consider the G_δ set we constructed in the theorem 3.2 and intersect it with the interval $[0,1]$. By construction, the intersection does not contain an affine copy of E. Note that the complement of the G_δ set defined in (3.1) is a countable union of scaled copies of Cantor sets of Lebesgue measure zero. Hence, it has Lebesgue measure zero, by the countable subadditivity of Lebesgue measure. Therefore the dense G_δ set has full measure and it did not contain any affine copies of E. ■

3.4 Generalizing into Higher Dimensions

Consider a compact set J in $\mathbb{R}^d$. An affine copy of J is $\mathbb{R}^d$ is the set

$$t + \delta O(J)$$

where $t \in \mathbb{R}^d$, $\delta \neq 0$ and O is an orthogonal transformation. We say that J is universal is the collection of dense G_δ sets if any dense G_δ set contains an affine copy of J.

Theorem 3.4. *If $X \subset \mathbb{R}^d$ contains a path connected component, then X is not universal in the set of dense G_δ sets of $\mathbb{R}^d$.*

Proof. Remove the plane

$$\mathbb{R}^d \setminus \bigcup_{i=1}^{d} \bigcup_{r \in \mathbb{Q}} \{X_i = r\}.$$

This is clearly a dense G_δ set. Consider any affine copy of X must contain a path L. The projection of L onto the coordinate axes will be non-degenerate on some interval for at least one of the axes. Call this the i-th axis. This interval will contain a rational number r. Therefore L will intersect with the coordinate plane, $X_i = r$ In other words this dense G_δ set omits at least one point and cannot contain affine copy of X. ■

We can still consider sets that contain no connected component. Cantor dust contains no connected component. We introduce the notion of *projective Newhouse Thickness*, and use it to generalize our one-dimensional results into $\mathbb{R}^d$.

Definition 3.5 (Positive Projective Newhouse Thickness)**.** We say a J set has *positive projective Newhouse thickness* if for all $\mathcal{O} \in O(d)$

$$\tau(P_x \mathcal{O}(J)) > 0$$

where P_x is the orthogonal projection to the x-axis and $O(d)$ is the orthogonal group consisting of all orthogonal transformations in $\mathbb{R}^d$.

We note that the projection some Cantor dust set maybe an interval. If that is the case we say the Newhouse Thickness of the projection is equal to infinity.

Using our one dimensional result, we can generalize

Theorem 3.5. *Suppose $J \subset \mathbb{R}^d$ has positive projective Newhouse thickness. Then J is not topologically-universal in $\mathbb{R}^d$.*

Proof. Suppose we have a set J in $\mathbb{R}^d$ such that it has a positive projective Newhouse thickness. In Theorem 3.2, we can actually see that for all Cantor sets J' with Newhouse thickness at least ϵ, there exists the dense G_δ set omits all affine copies of J'. Hence, we take $\epsilon = 1/n$, there exists G_n, a dense G_δ set in $\mathbb{R}^1$ such that G_n omits all affine copies of all cantor set J', with

$$\tau(J') \geq \frac{1}{n}.$$

Now, from these G_n we construct

$$G = \bigcap_{n=1}^{\infty} G_n \underbrace{\times \cdots \times}_{\text{d-times}} G_n.$$

Each $G_n \underbrace{\times \cdots \times}_{\text{d-times}} G_n$ is a dense G_δ set in $\mathbb{R}^d$ and therefore G is also a dense G_δ set by the Baire Category theorem.

Now all that remains is to prove that G has no affine copy of J. Assume to the contrary that G contains an affine copy of J and denote it by $t + \lambda\mathcal{O}J$. Note that the assumption implies that

$$\tau(P_x(t) + \lambda P_x\mathcal{O}(J)) > 0.$$

Hence, there exists some n so that the above thickness is at least $\frac{1}{n}$. By our construction of G_n, G_n has no affine copy of $P_x\mathcal{O}(J)$. But if $t + \lambda\mathcal{O}J \subset G = \bigcap_{n=1}^{\infty} G_n \times \cdots \times G_n$, then $t + \lambda\mathcal{O}(J) \subset G_n \times \cdots \times G_n$ and it implies that the projection $P_x(t) + \lambda P_x\mathcal{O}(J) \subset G_n$, which is a contradiction. Therefore our assumption is false, and G does not contain an affine copy of J.

$\blacksquare$

Chapter 4

Remarks and Open Questions

Finally we conclude by reviewing a few remarks and positing a few more questions on our results.

4.1 Zero Newhouse Thickness

Can a Cantor sets with zero Newhouse thickness can be universal? We first provide an example for which two Cantor sets with zero Newhouse thickness can still have arithmetic sum equal to an interval, showing that the converse of the Newhouse thickness theorem is not true.

Example 4.1. Let $N_1, N_2, \cdots \in \mathbb{N}_{\geq 2}$. Consider the following construction of a Cantor set using a decomposition of the unit intervals.

$$\begin{aligned}
[0,1] =& \frac{1}{N_1}\{0,1,\ldots,N_1-1\} + \left[0, \frac{1}{N_1}\right] \\
=& \frac{1}{N_1}\{0,1,\ldots,N_1-1\} + \frac{1}{N_1 N_2}\{0,1,\ldots,N_2-1\} + \left[0, \frac{1}{N_1 N_2}\right] \\
=& ... \\
=& \frac{1}{N_1}\{0,1,\ldots,N_1-1\} + \frac{1}{N_1 N_2}\{0,1,\ldots,N_2-1\} + \cdots + \frac{1}{N_1 \cdots N_n}\{0,\ldots,N_n\} + \ldots
\end{aligned}$$

From here we can define the two cantor sets K_1, K_2 where K_1 constitutes the odd indices sets in the above summands and K_2 has the even one. This gives the following constructions for the two Cantor sets:

$$K_1 = \frac{1}{N_1}\{0,1,\ldots,N_1-1\} + \cdots + \frac{1}{N_1 \cdots N_{2n+1}}\{0,\ldots,N_{2n+1}-1\} + \ldots$$

$$K_2 = \frac{1}{N_1 N_2}\{0,1,\ldots,N_2-1\} + \cdots + \frac{1}{N_1 \cdots N_{2n}}\{0,\ldots,N_{2n}-1\} + \ldots$$

From this construction we see that $K_1 + K_2 = [0,1]$ is the interval but from the definition of Newhouse thickness,

$$\tau(K_1) = \inf\left\{\frac{1}{N_1-1}, \frac{1}{N_3-1}, \ldots\right\} = 0$$

$$\tau(K_2) = \inf\left\{\frac{1}{N_2-1}, \frac{1}{N_4-1}, \ldots\right\} = 0.$$

Therefore we have created an interval from two sets with Newhouse thickness 0 if we have $\lim_{n\to\infty} N_n = \infty$.

We will conclude by ask the following questions. Recall I_J denotes the smallest closed interval containing the Cantor set J and O_J denotes the largest open interval in $I_J \setminus J$.

1. Given a Cantor set J, does there exist some K such that $J + K = I_J + I_K$?
2. If we assume that there exists K such that $J + K = I_j + I_K$, can we prove that J is not universal?
3. (rescaling condition) If we assume that $|\lambda_1 I_J| \geq |\lambda_2 O_K|, |\lambda_2 I_K| \geq |\lambda_1 O_J|$ and $J + K = I_J + I_K$, then $\lambda_1 J + \lambda_2 K = \lambda_1 I_J + \lambda_2 I_k$.

We also notice that to solve the second question, we notice that $J + K = I_J + I_K$ implies that

$$(J + a) + (K + b) = (I_J + a) + (I_K + b) \text{ and } bJ + bK = bI_J + bI_K.$$

We can always translate and rescale J, K so that $I_J = [0, a]$ and $I_K = [0, 1]$. Moreover, the following lemma is important.

Lemma 4.1. *Suppose that the Cantor sets J and K satisfies $J + K = I_j + I_K$. Then $|I_J| \geq |O_K|$ and $|I_K| \geq |O_J|$.*

The lemma also said that the condition $|\lambda_1 I_J| \geq |\lambda_2 O_k|, |\lambda_2 I_k| \geq |\lambda_1 O_J|$ is necessary in the rescaling condition.

Proposition 4.2. *Let J be a Cantor set such that $J+K = I_J + I_K$ where $I_J = [0,a]$ and $I_K = [0,1]$. Suppose that the rescaling condition (3) holds. Then J is not universal in the collection of dense G_δ.*

Proof. The proof is similar to the proof in Theorem 3.2. With K given in the assumption. We can assume that $|I_J| > |O_K|$. Suppose that $|I_J| = |O_K|$. Since $|O_J| < 1$, we can choose ϵ such that $(1-\epsilon) > |O_J|$. Then we consider $K' = (1-\epsilon)K$ and we will have $|I_J| > (1-\epsilon)|O_K|$. In this case, by the rescaling condition, $J + K' = I_J + I_{K'}$ and we have another K' such that $|I_J| > |O_{K'}|$.

As now we have $|I_J| > |O_K|$, we can find $0 < \rho < 1$ such that $\rho|I_J| > |O_K|$. We now define

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} \rho^n (K + \ell).$$

Then X^c is a dense G_δ set. Suppose that we have an affine copy $t + \lambda J$, we would like to claim that $t + \lambda J$ intersects non-trivially with $\rho^n(K+\ell)$ for some $n, \ell \in \mathbb{Z}$, which will complete the proof of the theorem.

To justify the claim, we let $0 < \rho < 1$ take the unique n such that

$$|\lambda| \in [\rho^{n+1}, \rho^n) \tag{4.1}$$

and the unique $\ell \in \mathbb{Z}$ such that

$$t \in (\ell\rho^n, (\ell+1)\rho^n]. \tag{4.2}$$

Then we consider the arithmetic sum $\rho^n K - \lambda J$. We now check the assumption in the rescaling condition with $\lambda_1 = \rho^n$ and $\lambda_2 = -\lambda$. Indeed,

$$|\lambda_2 I_J| \geq \rho^{n+1}|I_J| = |\lambda_1|(\rho|I_J|) \geq |\lambda_1 O_K|$$

by our choice of ρ. On the other hand,

$$|\lambda_1 I_K| \geq |\lambda| \geq |\lambda_2 O_J|$$

since $|I_K| \geq |O_J|$ by Lemma 4.1. Hence, using the rescaling condition,

$$\rho^n K - \lambda J = \rho^n I_K - \lambda I_J.$$

If $\lambda > 0$, then we have

$$\rho^n(K+\ell) - \lambda J = [\rho^n \ell - \lambda a, \rho^n(1+\ell)]$$

which contains t by (4.2). Similarly, if $\lambda < 0$, then

$$\rho^n(K+\ell) - \lambda J = [\rho^n \ell, \rho^n(\ell+1) - \lambda a].$$

It also contains t by (4.2). The proof is now complete. ■

We now conclude with several open questions.

Conjecture 4.1. *For any cantor set, does there exist another such that $C_1 + C_2$ adds up to an interval?*

Does the rescaling condition hold for Cantor sets with zero thickness?

Our proof inherently relies on appropriately selecting a scalar and translation that corresponds to a regular (or fairly regular) Cantor set. In this instance we have to find pick the appropriate λ, t based off of a set of associated intervals that are not uniform. Our proof relies on using the regularity to specify where the intersection is.

Do all self-similar sets have positive projective Newhouse thickness? If we consider some of the more commonly known fractals, are they not topologically universal?

These proofs also rely on the Newhouse thickness of Cantor sets but absent this property, we also offer another conjecture.

Conjecture 4.2. *Any Cantor set on $\mathbb{R}^d$ is not topologically universal.*

Finally, if all sets with positive measure contain a non-measurable set, is there a non-measurable set that is universal in sets with positive measure?

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