AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING

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CERTIFICATION OF APPROVAL

I certify that I have read *AN ERDOS SIMILARITY PROBLEM IN A TOPOLOGICAL SETTING* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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We say that a set E is *u*niversal in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdös conjecture in a topological setting.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee   Date

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Chapter 1

Introduction

From the perspective of applied math, empirical results are almost always discrete observations yet the relationships that are observed are not necessarily quantized. This means that as we infer mathematical relationships from discrete sets, there can often be a mismatch between our theoretical model, and the true relationships. This notion of embedded relationships is also a core area of study within pure mathematics. Whether observing a pattern and trying to generalize the relationship to a broader context, or deducing a relationship from a different set of assumptions, discrete patters are intrinsically embedded in our universe. Indeed as we examine our world we often notice similarities we want to measure. Using that same measure we want to look for consistency, and should we see something similar to our original observation we would expect a similar measure.

However in mathematics (and life), the rules we construct often lack the subtly to account for nuances. Let us naïvely consider measurement. Suppose we have a string and want to find out its length. After pulling the string taught along a ruler, we might see it is a few centimeters long. Then from this collection of tools we might say that we only need two points and a ruler to be able to describe length. In reality we have only learned of distance.

From that same construction though, we could just as well say, 2 points have no length at all because there is nothing between them. In a sense, both are simultaneously true, because our definition does not address these nuances. This motivates a few new questions: How many points do you need to add in, before we can have a length of string? Can we use this string to measure other things? Can we use collections of points to measure other things? And maybe strangest of all, can collections of points have the same length as a piece of string?

We can now go back to our notion of the universe and ask ourselves these questions again. Suppose we change universes, does our notion of length still exist? In a different universe can we find similar copies of these collections of points?

Key words:

- Dynamical Systems
- Density and Measure are not clearly linked
- Geometry
- Fractals poorly defined
- Self Similarity is more well defined

- Cantor Set
- $\mathbb{R}^n$ Fractals
- Erdös Proposed Conjecture with Measure Space assumptions
- Theorem with Topological Assumptions.
- Open questions

Chapter 2

Measure & Topology

2.1 Topological versus Measure Theoretic Size

First we want to begin with background definitions and theorems, building the supporting context, and motivation for the research. From the perspective of applied math, empirical results are almost always discrete observations, yet the relationships that are observed are not necessarily quantized. This means that as we infer mathematical relationships from discrete sets, there can often be a mismatch between our theoretical model, and the true relationships. This notion of embedded relationships is also a core area of study within pure mathematics. Whether observing a pattern and trying to generalize the relationship to a broader context, or deducing a relationship from a different set of assumptions, discrete patters are intrinsically embedded in our universe.

What do we mean by measure theoretically large? Non-zero Measure.

Topological size is not the same thing as Measure Theoretic size.

A measure theoretically large set is not necessarily topologically large.

Definition 2.1 (Measure)**.** Let X be a set and Σ be a $\sigma-$algebra over X. A function $\mu : \Sigma \to \{\mathbb{R} \cup \infty\}$ is called a measure if it satisfies the following properties:

1. **Non-negativity**: for all $E \in \Sigma$, $\mu(E) \geq 0$.

2. **Null empty set**: $\mu(\emptyset) = 0$.

3. **Countable Additivity** (σ-additivity): For all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

In this instance, a measurable set

Definition 2.2 (Measurable Set)**.** Let (X, Σ) be a measurable space. A set $S \subseteq X$ is a *measurable set* if and only if $S \in \Sigma$.

Note that a non-measurable set is a set that is not in the σ-algebra. This curiously leads to a result that non-measurable sets, also have non-zero measure because zero measure is a measure.

Example 2.1 (The Vitali Set is not Lebesgue-measurable.)**.** Consider the following equivalence relation on $\mathbb{R}$:

$$x \sim y \iff x - y \in \mathbb{Q}.$$

This is the same as the quotient group $\mathbb{R}/\mathbb{Q}$.

Notice for a moment that every single element in $\mathbb{R}/\mathbb{Q}$ intersects with some element of $[0, 1]$ because the $\pmod 1$ operation is subtraction by an integer. Therefore there is a subset of $[0, 1]$ that contains exactly one representative of each element of $\mathbb{R}/\mathbb{Q}$. By the axiom of choice, define the Vitali set V as the set of distinct representatives in $[0, 1]$, one from each equivalence class.

Claim 2.1. The Vitali set V is not Lebesgue measurable.

Proof. Assume for the same of contradiction, that V is measurable. Consider the set of translations of $\{V_q\} \pmod 1$, where for each $q \in \mathbb{Q} \cap [0, 1)$,

$$V_q = \{v + q \pmod 1 : v \in V\}.$$

Because measure is translation invariant, then each V_q is also measurable. Notice that the set $\{V_q\}$ is a countable family of pairwise disjoint sets such that

$$\bigcup_{q \in \mathbb{Q}} V_q = [0, 1].$$

Again because it is countable and pairwise disjoint, this implies that

$$\big|[0,1]\big| = \sum |V_q| = \sum_{k=1}^{\infty} V. \tag{2.1}$$

By assumption V is measurable, however the sum of a constant non-negative number is either 0 or ∞, and $|[0,1]| \neq 0$, and $|[0,1]| \neq \infty$. Therefore our assumption is false and V is not Lebesgue measurable. ■

What do we mean by topologically Large? Uncountable and dense. It is helpful to define the opposite of topologically large, namely meager sets.

Definition 2.3 (Nowhere Dense)**.** Let X be a topological space. A subset $B \subseteq X$ of a topological space is called *nowhere dense* in X if its closure has an empty interior. That is to say, B is *nowhere dense* in X if for each open set $U \subseteq X$, $B \cap U$ is not dense in U.

Definition 2.4 (Meager)**.** A subset $C \subseteq X$ of a topological space is called *meager* in X if it is the countable union of nowhere-dense subsets of X.

Definition 2.5 (G-Delta Set)**.** A G_δ set is the countable intersection of open sets. Namely, let $O_i \subset X$ for $i \in \mathbb{N}$ be a collection of open sets of X. Then $\bigcap_{n=1}^{\infty} O_i$, is a G_δ set.

Example 2.2. The irrational numbers are a G_δ set. Consider the following construction of the set of irrational numbers:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Notice that each $\mathbb{R} \setminus q = (-\infty, q) \cup (q, \infty)$ is an open subset of $\mathbb{R}$. Furthermore, rational numbers are countable. Therefore the intersection of these sets are a G_delta set. Moreover, in this instance it is a dense G_δ set. We will study these objects further.

The compliment of a G-delta is an F-sigma set.

2.2 Cantor Sets

The Cantor set is defined by taking the interval $[0, 1]$ and then iteratively removing the open interval containing the middle third, from the previous level.

Formally this can be written as follows.

Definition 2.6 (Cantor Set)**.** The Cantor set $\mathcal{C}$, written as the successive removal of each middle third removed from the previous level is

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

An equivalent formulation of the Cantor set, is the decimal expansion of all numbers in $[0, 1]$ in base 3, omitting any representation with a 1.

Example 2.3 (Decimal Expansion Cantor Set)**.**

$$\mathcal{C} = \{x \in [0,1] : \ x \text{ has a ternary expansion containing no 1's.}\}$$

Here we notice that although $1/3 \in \mathcal{C}$ can be written as 0.1 using the trinary expansion, it also has another representation as $1/3 = 0.0\overline{2}$ This would be the included representation in the Cantor set. We take a moment to acknowledge that numbers may not have unique representations, where one may be excluded but the other included.

Definition 2.7 (Nowhere dense)**.** A set $A \subseteq X$ is called *nowhere dense* if its closure $\overline{A}$ has an empty interior. Equivalently the set A is *nowhere dense* if A is not not dense in any subset U of X.

Claim 2.2. The Cantor set is *nowhere dense.*

Notice that $[0,1] \setminus \mathcal{C}$ is a set of open intervals:

$$[0,1] \setminus \mathcal{C} = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

Therefore $\mathcal{C}$ is the countable intersection of closed intervals, and itself is closed.

Notice given some radius r, there exists some number t, such that $0 < t < r$ and t has a 1 in its ternary expansion. So if we consider any point $c \in \mathcal{C}$, then an open ball of radius r centered at c then $B_r(c)$ is $(c-r, c+r)$ in ternary, necessarily contains a number containing a 1. Therefore $B_r(c) \not\subseteq \mathcal{C}$, and $\mathcal{C}$ has an empty interior. Finally we conclude because $\mathcal{C}$'s closure has an empty interior, $\mathcal{C}$ is nowhere dense.

2.3 The Baire Category Theorem

A key theorem that links analysis to set theory is the Baire Category Theorem. This also establishes a link to understanding certain types of topological sets.

Theorem 2.1 (Baire Category Theorem)**.** *The countable intersection of open dense sets is dense.*

Within the study of measure theory it can sometimes be unclear if a set is dense in another set. For example consider the following set:

$$\mathbb{R}^2 \setminus \{(x, y) : y = mx + b, \text{where } m, b \in \mathbb{Q}.\}$$

Notice that this can also be written as

$$\bigcap_{m,b\in\mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = mx + b\},$$

which is the plane, but removing all lines with rational coefficients, and rational intercepts.

Chapter 3

An Erdös Conjecture in Measure Theory

3.1 Affine Copies and the Self Similarity Property

In order to define self-similar sets, we first need to define what an affine copy of a set is.

Definition 3.1 (Affine copy)**.** An affine copy of a set A is a scaled and translated set A' such that for some $\lambda \neq 0, \lambda \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$A' = \{\lambda a + t : a \in A\}.$$

In this instance, it is a scaled and then translated copy of the set and need not be a "shape". This gives us some flexibility when addressing different sets. In particular let us examine fractals.

Fractals are geometric objects that have a self-similar property, and a fractional dimension. For the scope of this paper, we will not be discussing dimension, and will instead investigate the notion of self-similarity.

Definition 3.2 (Iterated Function System)**.** An iterated function system is a finite set of contraction mappings on a complete metric space. Symbolically, we write this as, for some $N \in \mathbb{N}$,

$$\{f_i : X \to X | i = 1, 2, \dots, N\},$$

The invariant set under this iterated function system is a self similar set.

Definition 3.3 (Self-Similar Set)**.** A set A is self-similar if it is the invariant set of an iterated function system.

Admittedly this is an abstract definition, so we come back to the Cantor set from earlier.

Claim 3.1*.* The Cantor set is self-similar.

Proof. Recall the definition of the Cantor set, as the iterated removal of the middle third.

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

Here we notice that for the first removal, $n = 0$, we are left with the left and right portion of the Cantor set where ■

3.2 An Erdös Self-Similarity Conjecture in Measure Space

Conjecture 3.1. *There is no infinite universal set.*

Chapter 4

An Erdös Similarity Problem in a Topological Setting

4.1 Positive Newhouse Thickness

Definition 4.1 (Gap)**.** Let K be some Cantor set with an outer hull I, and a sequence of

Definition 4.2 (Newhouse Thickness)**.** Let K be some set

4.2 The Gap Lemma

Lemma 4.1. *The Gap Lemma[3] Let* $K_1, K_2, \subset \mathbb{R}$ *be cCantor sets with thickness* τ_1 *and* τ_2*. If* $\tau_1 \cdot \tau_2 > 1$*, then one of the following three alternatives occurs:* K_1 *is contained the gap of* K_2*;* K_2 *is contained in the gap of* K_1*;* $K_1 \cap K_2 \neq \emptyset$*.*

First we will assume for the sake of contradiction that the opposite is true. These assumptions lead to claim, which under these assumptions must be true. Then that claim leads to a contradiction which means our original assumption must be false.

Proof. Let K_1, K_2 be two Cantor sets with thickness τ_1, τ_2 respectively and assume that K_1 is not contained in the gap of K_2 and K_2 is not contained in the gap of K_1.

Assume that $K_1 \cap K_2 = \emptyset$. Consider the gaps $U_1 \subset K_1^c$ and $U_2 \subset K_2^c$. We call (U_1, U_2) a gap-pair if U_1 contains exactly one boundary point of U_2 and U_2 contains exactly one point of U_1.

By assumption we know that K_1, K_2 are not contained in the other's gaps and therefore there exists some gap-pair (U_1, U_2).

From the interval U_1 (or for that matter U_2) we can construct another sub-interval U_1' such that $l(U_1') < l(U_1)$ (or similarly U_2' such that $l(U_2') < l(U_2)$). Notice that (U_1', U_2) is still a gap-pair, as is (U_1, U_2').

Using this construction we can create a sequence of gap-pairs $(U_1^{(i)}, U_2^{(j)})$. Notice that because it is a summable compact cantor set, the $U_1^{(i)}$ and $U_2^{(j)}$ are compact. Moreover the sums of the lengths

$$\sum_{i}^{\infty} l(U_1^{(i)}) < \infty,$$

and therefore $l((U_1^{(i)} \to 0$ as $i \to \infty$. From this we construction we have a sequence of gap-pairs such that as $i \to \infty$, $l(U_1^{(i)}) \to 0$ and similarly as $j \to \infty$, $l(U_2^{(j)}) \to 0$.

Without loss of generality, we can just use the same indexing the gap pairs, $(U_1^{(i)}, U_2^{(i)})$. If we pick a sequence of points, $q_i \in U_1^{(i)}$ then by the Bolzano Weierstrass theorem, there is a convergent subsequence $q_{i_k} \to q$. Notice that $U_1^{(i)} = (a_i, b_i)$ is not fully contained in the gap of K_2. Moreover because these intervals are compact and nested, we know that $a_{i_k} - b_{i_k} \to 0$ which implies that $q \in K_2$.

Because this construction is symmetric, the same argument applies to $q_i \in U_2^{(i)}$ and so $q_{i_k} \to q$ implies that $q \in K_1 \cap K_2$.

We want to use this technique to demonstrate that $K_1 \cap K_2 \neq \emptyset$. Let C_j^l, C_j^r denote the bridges of K_j for $j = 1, 2$. Returning to our original assumptions, $\tau_1 \cdot \tau_2 > 1$ and therefore

$$\frac{l(C_1)}{l(U_2)} \cdot \frac{l(C_2)}{l(U_1)} > 1.$$

From our construction, the right endpoint of U_2 is in C_1^r or the left endpoint of U_1 is in C_2^l or both. In the case that $q \in U_2$ is the right endpoint, then $q \in K_1$ and $q \in K_2$ and we are done. If $q \notin K_1$, then $q \in U_1'$, the gap of K_1 where $l(U_1') < l(U_1)$ and (U_1', U_2) is the gap pair we need.

■

4.3 A Cantor Set with Positive Newhouse Thickness is not Universal

We say that a set E is *universal* in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdös conjecture in a topological setting.

Theorem 4.2. *Let J be a cantor set with positive Newhouse thickness. Then J is not universal.*

Proof. Suppose we have some Cantor set J with Newhouse thickness $\tau(J) > 0$. Without loss of generality, we can assume the convex hull of J $[0, 1]$. Consider Cantor sets K defined by contraction ratio $1/N$ and digits $\{0, 1, ..., N - 1\} \setminus \{(N - 1)/2\}$ and N is odd. By a simple calculation, $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(J)\tau(K) > 1$.

Using the Cantor set K Define X such that

$$X = \bigcup_{n \in \mathbb{Z}} \bigcup_{\ell \in \mathbb{Z}} N^n(K + \ell),$$

creating a dense F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, by the Baire Category Theorem, X^c is a dense

G_δ. We now show that X^c contains no affine copy of J.

Suppose we have some affine copy, $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{4.1}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell+1)N^n]. \tag{4.2}$$

We claim that this affine copy of J has a non-empty intersection with $N^n(K+\ell)$. This is equivalent to showing that

$$t \in N^n(K+\ell) - \lambda J.$$

For consistent notation with a referenced theorem, let

$$C_1 = N^n(K+\ell) \text{ and } C_2 = -\lambda J.$$

First we check the construction of our Cantor sets. For C_1 its largest corresponding open gap interval is $|O_1| = N^{n-1}$ and its largest corresponding closed interval is $|I_1| = N^n$. For C_2 and is corresponding intervals, we find that $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$ and $|I_2| = |\lambda|$ where O_J is the largest open gap interval in J. Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|$$

as in the condition of Theorem 2.2.1 in [1]. By [1, Theorem 2.2.1][1], given that the Newhouse thickness of our sets, $\tau(K)\tau(J) \geq 1$ then $C_1 + C_2 = I_1 + I_2$. Note that $I_1 = [\ell N^n, (\ell+1)N^n]$, $I_2 = [-\lambda, 0]$ if $\lambda > 0$ and $I_2 = [0, -\lambda]$ if $\lambda < 0$. we find that

$$I_1 + I_2 = [N^n\ell - \lambda, N^n(\ell+1)] \ (\lambda > 0) \text{ and } I_1 + I_2 = [N^n\ell, N^n(\ell+1) - \lambda](\lambda < 0).$$

Then from (2)

$$t \in I_1 + I_2.$$

Therefore the affine copy of the cantor set $t + \lambda J$ has a non-empty intersection with X and J cannot be universal.

$\blacksquare$

It would be interesting to study those Cantor sets with Newhouse thickness zero. We do not know what would happen. However, it seems like if we assume a weaker condition on J.

$(*)$: There exists K such that $J + K = I_J + I_K$, where I_J, I_K are the smallest closed

[1]This might misattribute the theorem. I think Astels '99 Theorem 2.2.1 is actually is quoting Newhouse directly. In particular I think it refers to Newhouse 1979 [2] *The Abundance of Wild Hyperbolic Sets, and Non-smooth Stable Sets for Diffeomorphisms.*

interval containing J and K.

we may be able to show that J cannot be universal for dense G_δ sets.

4.4 Current Research Questions: Zero Newhouse Thickness

This section is devoted to study if Cantor sets with zero Newhouse thickness can be universal. We first provide an example for which two Cantor sets with zero Newhouse thickness can still have arithmetic sum equal to an interval, showing that the converse of theNewhouse thickness theorem us not true.

Example 4.1. Let $N_1, N_2, \cdots \in \mathbb{N}_{\geq 2}$. Consider the following construction of a Cantor set using a decomposition of the unit intervals.

$$\begin{aligned}
[0,1] =& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \left[0, \frac{1}{N_1}\right] \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \left[0, \frac{1}{N_1N_2}\right] \\
=& \ldots \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \cdots + \frac{1}{N_1\cdots N_n}\{0,\dots,N_n\} + \dots
\end{aligned}$$

From here we can define the two cantor sets K_1, K_2 where K_1 constitutes the odd indices sets in the above summands and K_2 has the even one. This gives the following constructions for the two Cantor sets:

$$K_1 = \frac{1}{N_1}\{0, 1, \dots, N_1 - 1\} + \dots + \frac{1}{N_1 \cdots N_{2n+1}}\{0, \dots, N_{2n+1} - 1\} + \dots$$

$$K_2 = \frac{1}{N_1 N_2}\{0, 1, \dots, N_2 - 1\} + \dots + \frac{1}{N_1 \cdots N_{2n}}\{0, \dots, N_{2n} - 1\} + \dots$$

From this construction we see that $K_1 + K_2 = [0, 1]$ is the interval but from the definition of Newhouse thickness,

$$\tau(K_1) = \inf\left\{\frac{1}{N_1 - 1}, \frac{1}{N_3 - 1}, \dots\right\} = 0$$

$$\tau(K_2) = \inf\left\{\frac{1}{N_2 - 1}, \frac{1}{N_4 - 1}, \dots\right\} = 0.$$

Therefore we have created an interval from two sets with Newhouse thickness 0 if we have $\lim_{n\to\infty} N_n = \infty$.

We ask the following questions. Recall I_J denotes the smallest closed interval containing the Cantor set J and O_J denotes the largest open interval in $I_J \setminus J$.

1. Given a Cantor set J, does there exist some K such that $J + K = I_J + I_K$?

2. If we assume that there exists K such that $J + K = I_j + I_K$, can we prove that J is not universal?

3. (rescaling condition) If we assume that $|\lambda_1 I_J| \geq |\lambda_2 O_K|, |\lambda_2 I_K| \geq |\lambda_1 O_J|$ and

$J + K = I_J + I_K$, then $\lambda_1 J + \lambda_2 K = \lambda_1 I_J + \lambda_2 I_k$.

We also notice that to solve the second question, we notice that $J+K = I_J+I_K$ implies that

$$(J + a) + (K + b) = (I_J + a) + (I_K + b) \text{ and } bJ + bK = bI_J + bI_K.$$

We can always translate and rescale J, K so that $I_J = [0, a]$ and $I_K = [0, 1]$. Moreover, the following lemma is important.

Lemma 4.3. *Suppose that the Cantor sets J and K satisfies $J + K = I_j + I_K$. Then $|I_J| \geq |O_K|$ and $|I_K| \geq |O_J|$.*

The lemma also said that the condition $|\lambda_1 I_J| \geq |\lambda_2 O_k|, |\lambda_2 I_k| \geq |\lambda_1 O_J|$ is necessary in the rescaling condition.

Proposition 4.4. *Let J be a Cantor set such that $J+K = I_J+I_K$ where $I_J = [0, a]$ and $I_K = [0, 1]$. Suppose that the rescaling condition (3) holds. Then J is not universal in the collection of dense G_δ.*

Proof. The proof is similar to the proof in Theorem 4.2. With K given in the assumption. We can assume that $|I_J| > |O_K|$. Suppose that $|I_J| = |O_K|$. Since $|O_J| < 1$, we can choose ϵ such that $(1-\epsilon) > |O_J|$. Then we consider $K' = (1-\epsilon)K$ and we will have $|I_J| > (1 - \epsilon)|O_K|$. In this case, by the rescaling condition, $J + K' = I_J + I_{K'}$ and we have another K' such that $|I_J| > |O_{K'}|$.

As now we have $|I_J| > |O_K|$, we can find $0 < \rho < 1$ such that $\rho|I_J| > |O_K|$. We now define

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} \rho^n (K + \ell).$$

Then X^c is a dense G_δ set. Suppose that we have an affine copy $t + \lambda J$, we would like to claim that $t + \lambda J$ intersects non-trivially with $\rho^n(K + \ell)$ for some $n, \ell \in \mathbb{Z}$, which will complete the proof of the theorem.

To justify the claim, we let $0 < \rho < 1$ take the unique n such that

$$|\lambda| \in [\rho^{n+1}, \rho^n) \tag{4.3}$$

and the unique $\ell \in \mathbb{Z}$ such that

$$t \in (\ell\rho^n, (\ell + 1)\rho^n]. \tag{4.4}$$

Then we consider the arithmetic sum $\rho^n K - \lambda J$. We now check the assumption in the rescaling condition with $\lambda_1 = \rho^n$ and $\lambda_2 = -\lambda$. Indeed,

$$|\lambda_2 I_J| \geq \rho^{n+1}|I_J| = |\lambda_1|(\rho|I_J|) \geq |\lambda_1 O_K|$$

by our choice of ρ. On the other hand,

$$|\lambda_1 I_K| \geq |\lambda| \geq |\lambda_2 O_J|$$

since $|I_K| \geq |O_J|$ by Lemma 4.3. Hence, using the rescaling condition,

$$\rho^n K - \lambda J = \rho^n I_K - \lambda I_J.$$

If $\lambda > 0$, then we have

$$\rho^n(K + \ell) - \lambda J = [\rho^n \ell - \lambda a, \rho^n(1 + \ell)]$$

which contains t by (4.4). Similarly, if $\lambda < 0$, then

$$\rho^n(K + \ell) - \lambda J = [\rho^n \ell, \rho^n(\ell + 1) - \lambda a].$$

It also contains t by (4.4). The proof is now complete. ■

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From these questions we have several difficulties associated with each. For the first item it is not always clear which cantor sets can be added to each other. Similarly it is difficult to construct a complementing Cantor set because of the difficulties tracking the notation for the different possible open intervals. There maybe some existing tools. It may also just be messy.

For the second point, our proof inherently relies on appropriately selecting a scalar and translation that corresponds to a regular (or fairly regular) Cantor set. In this instance we have to find pick the appropriate λ, t based off of a set of associated intervals that are not uniform. Our proof relies on using the regularity to specify where the intersection is.

A current tool we are exploring is tracking how scaling and translating the collection of intervals $\{O_j\}_{j\in\mathbb{N}}$ by some appropriate bound M such that we can scale our cantor set by $\frac{1}{M^d}$, and demonstrate an appropriate intersection with X^c.

The last question we discussed for the day focused on how scaling Cantor sets, and scaling intervals are interrelated. With Newhouse thickness, because it relies off of the ratios of $\frac{I_j}{O_{j-1}}$ the scaling factor drops out. Unfortunately if we are considering Cantor sets with Newhouse thickness 0, then there is no corresponding Cantor set with infinite Newhouse thickness. The issue is that from the theorem, the thickness is the product of the two sets so for any finite thickness $0 \cdot \tau(C) = 0$. Therefore Newhouse thickness will not be enough to describe the appropriate construction of the interval. There are a few workarounds that might be possible. In Astels' paper[1] there is a generalized for for countably many cantor sets. Similarly we might be able to find another characterization (measure, dimension etc) of the set, to appropriately find λ and or, another way to combine the two intervals, such that we have a non-empty intersection with X^c.

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