On a Topological Erdos Similarity Problem

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In
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by

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CERTIFICATION OF APPROVAL

I certify that I have read *On a Topological Erdos Similarity Problem* by John P Gallagher and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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On a Topological Erdos Similarity Problem

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Often in data work, one may ask which patterns are possible to find, given a certain data set, or hypothetical relationship. Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets? A set is called universal in another set, when every subset of the larger set contains some scaled and translated copy of original. Paul Erdos proposed a conjecture that no infinite set, is universal in the the collection of sets with positive measure. This poster explores an analogous problem in a topological setting. Instead of sets with positive measure we investigate the collection of dense G-delta sets. Any finite or countable set is found to be topologically universal. Any set containing an interval cannot be topologically universal. We also have the new result that any Cantor sets is not topologically universal. Cantor sets, which contains no interval and are uncountably infinite, are not topologically universal in the collection of dense G-delta sets.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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Chapter 1

Introduction

Often in data work, one may ask, "Which patterns are possible to find, given a certain data set, or hypothetical relationship?" Mathematically a similar problem can be proposed. Which patterns, finite or infinite, exist within another collection of sets? A set is called universal in another set, when every subset of the larger set contains some scaled and translated copy of original. Paul Erdos proposed a conjecture that no infinite set, is universal in the the collection of sets with positive measure. We will explore an analogous problem in a topological setting. Instead of sets with positive measure we investigate the collection of dense G-delta sets. Any finite or countable set is found to be topologically universal. Any set containing an interval cannot be topologically universal. We also have the new result that any Cantor sets is not topologically universal. Cantor sets, which contains no interval and are uncountably infinite, are not topologically universal in the collection of dense G-delta sets.

There is a long standing conjecture from Paul Erdös on universal sets. Informally the conjecture states that there is no infinite set that is universal in the real number line. This is a conjecture about which types of patterns can exist within another sets of numbers. Before we formally state the theorem, we first need to review a few definitions: affine copies, universality, and non-zero measure.

Definition 1.1 (Affine copy)**.** An affine copy of a set A is a scaled and translated set A' such that for some $\lambda \neq 0, \lambda \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$A' = \{\lambda a + t : a \in A\}.$$

In this instance, it is a scaled and then translated copy of the set and need not be a "shape". This gives us some flexibility when addressing different sets. This definition is used to define universal.

Definition 1.2 (Universal)**.** A set E is called universal in X if for every subset $S \subseteq X$, with positive measure, $\mu(S) > 0$, there exist an affine copy of E such that $t + \lambda E \subseteq S$, for some $\lambda \neq 0$ and $t \in \mathbb{R}$.

More formally, we state this conjecture as follows.

Conjecture 1.1 (The Erdös Self-Similarity Conjecture)**.** *Let $E \subseteq \mathbb{R}$ be an infinite set of real numbers. Prove that there is a set of real numbers S of positive measure which does not contain an affine copy of E.*

In this paper we take this idea of universality and put it into a topological context. We show that no cantor set is universal in the set of dense G_δ sets.

If it is countable then baire category works directly. So we need to work with

1.1 Affine Copies and the Self Similarity Property

In order to define self-similar sets, we first need to define what an affine copy of a set is.

Definition 1.3 (Affine copy)**.** An affine copy of a set A is a scaled and translated set A' such that for some $\lambda \neq 0, \lambda \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$A' = \{\lambda a + t : a \in A\}.$$

In this instance, it is a scaled and then translated copy of the set and need not be a "shape". This gives us some flexibility when addressing different sets. In particular let us examine fractals.

Fractals are geometric objects that have a self-similar property, and a fractional dimension. For the scope of this paper, we will not be discussing dimension, and will instead investigate the notion of self-similarity.

Definition 1.4 (Iterated Function System)**.** An iterated function system is a finite set of contraction mappings on a complete metric space. Symbolically, we write this as, for some $N \in \mathbb{N}$,

$$\{f_i : X \to X | i = 1, 2, \ldots, N\},$$

The invariant set under this iterated function system is a self similar set.

Definition 1.5 (Self-Similar Set)**.** A set A is self-similar if it is the invariant set of an iterated function system.

Admittedly this is an abstract definition, so we come back to the Cantor set from earlier.

Claim 1.1. The Cantor set is self-similar.

Proof. Recall the definition of the Cantor set, as the iterated removal of the middle third.

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

Here we notice that for the first removal, $n = 0$, we are left with the left and right portion of the Cantor set where ■

1.2 An Erdös Self-Similarity Conjecture in Measure Space

Conjecture 1.2. *There is no infinite universal set.*

1.3 An Analogous Theorem in a Topological Setting

Chapter 2

Measure & Topology

2.1 Topological versus Measure Theoretic Size

First we want to begin with background definitions and theorems. Erdös' problem specifically deals with infinite set, and affine copies found in measurable set of sets. In our problem, rather than dealing with measurable sets, we will instead use the set of dense G_δ sets.

Underpinning the nuances of this problem, measure theoretic size, and topological size, are not the same. From an intuitive sense of the number line one might think when you are scattered throughout an interval, you would have measure, except in special cases. Similarly one might think that if you have measure, then you would be scattered everywhere. However both of these instances fail when you add in rigorous arguments. Indeed it is possible to be a set that is no-where dense and have positive measure, as well as be dense and have zero measure. In other words topological size (density) is not the same thing as measure theoretic size.

What do we mean by topologically Large? Uncountable and dense. Similarly what do we mean by measure theoretically large? Non-zero measure. It is helpful to define the opposite of topologically large, namely meager sets.

Definition 2.1 (Nowhere Dense)**.** Let X be a topological space. A subset $B \subseteq X$ of a topological space is called *nowhere dense* in X if its closure has an empty interior. That is to say, B is *nowhere dense* in X if for each open set $U \subseteq X$, $B \cap U$ is not dense in U.

Definition 2.2 (Meager)**.** A subset $C \subseteq X$ of a topological space is called *meager* in X if it is the countable union of nowhere-dense subsets of X.

Now we look at our example.

Example 2.1. A measure theoretically large set is not necessarily topologically large.

Consider the interval $[0, 1]$ and for all positive integers $a, n \in \mathbb{N}$ remove the intervals $(\frac{a}{2^n} - \frac{1}{2^{n+1}}, \frac{a}{2^n} + \frac{1}{2^{n+1}})$. Notice that the intervals are a geometric series and for each n add up to at most $\frac{1}{2^{n+1}}$. Therefore the set

$$[0, 1] \setminus \bigcup_{a,n \in \mathbb{N}} \left(\frac{a}{2^n} - \frac{1}{2^{n+1}}, \frac{a}{2^n} + \frac{1}{2^{n+1}} \right),$$

is closed, has an empty interior, and is of positive measure.

Next we will define dense G_δ sets, as well as some useful examples.

Definition 2.3 (G-Delta Set)**.** A G_δ set is the countable intersection of open sets. Namely, let $O_i \subset X$ for $i \in \mathbb{N}$ be a collection of open sets of X. Then $\bigcap_{n=1}^{\infty} O_i$, is a G_δ set.

Example 2.2. The irrational numbers are a G_δ set. Consider the following construction of the set of irrational numbers:

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\}.$$

Notice that each $\mathbb{R} \setminus \{q\} = (-\infty, q) \cup (q, \infty)$ is an open subset of $\mathbb{R}$. Furthermore, rational numbers are countable. Therefore the intersection of these sets are a G_δ set. Moreover, in this instance it is a dense G_δ set. We will study these objects further.

Lastly we remark that there is an analogous set which is the countable union of closed sets.

Definition 2.4 (F_σ Set)**.** An F_σ set is the countable union of closed sets. This is equivalent to the compliment of a G-delta is an F-sigma set.

2.2 Cantor Sets

We begin this section with a special F_σ set. The Cantor set is defined by taking the interval $[0,1]$ and then iteratively removing the open interval containing the middle third, from the previous level. As such it is the countable intersection of closed sets. Formally this can be written as follows.

Definition 2.5 (Cantor Set)**.** The Cantor set $\mathcal{C}$, written as the successive removal of each middle third removed from the previous level is

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

An equivalent formulation of the Cantor set, is the decimal expansion of all numbers in $[0,1]$ in base 3, omitting any representation with a 1. This can be a useful tool for thinking through some examples and counter-examples.

Example 2.3 (Decimal Expansion Cantor Set)**.**

$$\mathcal{C} = \{x \in [0,1] : \ x \text{ has a ternary expansion containing no 1's.}\}$$

Here we notice that although $1/3 \in \mathcal{C}$ can be written as 0.1 using the trinary expansion, it also has another representation as $1/3 = 0.0\overline{2}$ This would be the included representation in the Cantor set. We take a moment to acknowledge that numbers may not have unique representations, where one may be excluded but the other included.

Earlier we defined nowhere dense. Here we see that the Cantor set is an example of a nowhere dense set.

Claim 2.1. The Cantor set is *nowhere dense*.

Proof. Let $\mathcal{C}$ be the middle third Cantor set. Notice that $[0,1] \setminus \mathcal{C}$ is a set of open intervals:

$$[0,1] \setminus \mathcal{C} = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

Therefore $\mathcal{C}$ is the countable intersection of closed intervals, and itself is closed.

Notice given some radius r, there exists some number t, such that $0 < t < r$ and t has a 1 in its ternary expansion. So if we consider any point $c \in \mathcal{C}$, then an open ball of radius r centered at c then $B_r(c)$ is $(c-r, c+r)$ in ternary, necessarily contains a number containing a 1. Therefore $B_r(c) \nsubseteq \mathcal{C}$, and $\mathcal{C}$ has an empty interior. Finally we conclude because $\mathcal{C}$ is closed and has an empty interior, $\mathcal{C}$ is nowhere dense. ■

Beyond the middle third Cantor set, we can generalize these in a few different ways. Changing the middle interval, having a few different interval widths, series of interval widths.

2.3 The Baire Category Theorem

A key theorem that links analysis to set theory is the Baire Category Theorem. This also establishes a link to understanding certain types of topological sets.

Theorem 2.1 (Baire Category Theorem)**.** *The countable intersection of open dense sets is dense.*

Within the study of measure theory it can sometimes be unclear if a set is dense in another set. For example consider the following set:

$$\mathbb{R}^2 \setminus \{(x, y) : y = mx + b, \text{where } m, b \in \mathbb{Q}.\}$$

Notice that this can also be written as

$$\bigcap_{m,b \in \mathbb{Q}} \mathbb{R}^2 \setminus \{(x, y) : y = mx + b\},$$

which is the plane, but removing all lines with rational coefficients, and rational intercepts.

Chapter 3

An Erdös Similarity Problem in a Topological Setting

3.1 Positive Newhouse Thickness

Definition 3.1 (Gap)**.** Let K be some Cantor set with an outer hull I, and a sequence of

Definition 3.2 (Newhouse Thickness)**.** Let K be some set

3.2 The Gap Lemma

Lemma 3.1. *The Gap Lemma[2] Let $K_1, K_2, \subset \mathbb{R}$ be Cantor sets with thickness τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$, then one of the following three alternatives occurs: K_1 is contained the gap of K_2; K_2 is contained in the gap of K_1; $K_1 \cap K_2 \neq \emptyset$.*

First we will assume for the sake of contradiction that the opposite is true. These assumptions lead to claim, which under these assumptions must be true. Then that claim leads to a contradiction which means our original assumption must be false.

Proof. Let K_1, K_2 be two Cantor sets with thickness τ_1, τ_2 respectively and assume that K_1 is not contained in the gap of K_2 and K_2 is not contained in the gap of K_1.

Assume that $K_1 \cap K_2 = \emptyset$. Consider the gaps $U_1 \subset K_1^c$ and $U_2 \subset K_2^c$. We call (U_1, U_2) a gap-pair if U_1 contains exactly one boundary point of U_2 and U_2 contains exactly one point of U_1.

By assumption we know that K_1, K_2 are not contained in the other's gaps and therefore there exists some gap-pair (U_1, U_2).

From the interval U_1 (or for that matter U_2) we can construct another subinterval U_1' such that $l(U_1') < l(U_1)$ (or similarly U_2' such that $l(U_2') < l(U_2)$). Notice that (U_1', U_2) is still a gap-pair, as is (U_1, U_2').

Using this construction we can create a sequence of gap-pairs $(U_1^{(i)}, U_2^{(j)})$. Notice that because it is a summable compact cantor set, the $U_1^{(i)}$ and $U_2^{(j)}$ are compact. Moreover the sums of the lengths

$$\sum_{i}^{\infty} l(U_1^{(i)}) < \infty,$$

and therefore $l((U_1^{(i)} \to 0$ as $i \to \infty$. From this we construction we have a sequence of gap-pairs such that as $i \to \infty$, $l(U_1^{(i)}) \to 0$ and similarly as $j \to \infty$, $l(U_2^{(j)}) \to 0$.

Without loss of generality, we can just use the same indexing the gap pairs, $(U_1^{(i)}, U_2^{(i)})$. If we pick a sequence of points, $q_i \in U_1^{(i)}$ then by the Bolzano Weierstrass theorem, there is a convergent subsequence $q_{i_k} \to q$. Notice that $U_1^{(i)} = (a_i, b_i)$ is not fully contained in the gap of K_2. Moreover because these intervals are compact and nested, we know that $a_{i_k} - b_{i_k} \to 0$ which implies that $q \in K_2$.

Because this construction is symmetric, the same argument applies to $q_i \in U_2^{(i)}$ and so $q_{i_k} \to q$ implies that $q \in K_1 \cap K_2$.

We want to use this technique to demonstrate that $K_1 \cap K_2 \neq \emptyset$. Let C_j^l, C_j^r denote the bridges of K_j for $j = 1, 2$. Returning to our original assumptions, $\tau_1 \cdot \tau_2 > 1$ and therefore

$$\frac{l(C_1)}{l(U_2)} \cdot \frac{l(C_2)}{l(U_1)} > 1.$$

From our construction, the right endpoint of U_2 is in C_1^r or the left endpoint of U_1 is in C_2^l or both. In the case that $q \in U_2$ is the right endpoint, then $q \in K_1$ and $q \in K_2$ and we are done. If $q \notin K_1$, then $q \in U_1'$, the gap of K_1 where $l(U_1') < l(U_1)$ and (U_1', U_2) is the gap pair we need.

■

3.3 Positive Newhouse Thickness

3.4 A Cantor set with positive Newhouse Thickness is not universal

We say that a set E is *universal* in the collection of dense G_δ sets if for all G_δ set, we can always find some affine copies of E inside the set. By an affine copy, we mean sets of the form $t + \lambda E$ for some $t \in \mathbb{R}$ and $\lambda \neq 0$. A natural question we have is that is there a nowhere dense Cantor Set that is universal in the collection of dense G_δ sets? This is an exploration of an Erdös conjecture in a topological setting.

Theorem 3.2. *Let J be a cantor set with positive Newhouse thickness. Then J is not topologically universal in the collection of dense G_δ sets.*

Proof. Suppose we have some Cantor set J with Newhouse thickness $\tau(J) > 0$. Without loss of generality, we can assume the convex hull of J $[0, 1]$. Consider Cantor sets K defined by contraction ratio $1/N$ and digits $\{0, 1, ..., N-1\} \setminus \{(N-1)/2\}$ and N is odd. By a simple calculation, $\tau(K) = \frac{N-1}{2}$. Therefore, we can find a sufficiently large N so that $\tau(J)\tau(K) > 1$.

Using the Cantor set K Define X such that

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} N^n(K + \ell),$$

creating a dense F_σ set. Now consider X^c. Because K^c is open and dense and so is its translated and dilated copies, by the Baire Category Theorem, X^c is a dense G_δ. We now show that X^c contains no affine copy of J.

Suppose we have some affine copy, $t + \lambda J$ where $t \in \mathbb{R}$ and $\lambda \neq 0$. There exists a unique n such that

$$|\lambda| \in (N^{n-1}, N^n]. \tag{3.1}$$

Similarly there exists a unique ℓ such that

$$t \in (\ell N^n, (\ell + 1)N^n]. \tag{3.2}$$

We claim that this affine copy of J has a non-empty intersection with $N^n(K + \ell)$. This is equivalent to showing that

$$t \in N^n(K + \ell) - \lambda J.$$

For consistent notation with a referenced theorem, let

$$C_1 = N^n(K + \ell) \text{ and } C_2 = t + \lambda J.$$

First we check the construction of our Cantor sets. For C_1 its largest corresponding open gap interval is $|O_1| = N^{n-1}$ and its largest corresponding closed interval is $|I_1| = N^n$. For C_2 and is corresponding intervals, we find that $|O_2| = |\lambda| \cdot |O_J| \leq |\lambda|$

and $|I_2| = |\lambda|$ where O_J is the largest open gap interval in J. Therefore by our construction in (1) the following two inequalities hold:

$$|O_1| \leq |I_2| \text{ and } |O_2| \leq |I_1|.$$

Therefore C_1 is not in the gaps of C_2 and C_2 is not fully contained in the gaps of C_1. By our choice of K, the Newhouse thickness of our sets, $\tau(C_1))\tau(C_2) \geq 1$, because Newhouse thickness is scale invariant. Therefore the the Gap Lemma implies $C_1 \cap C_2$ is non-empty and C_2 cannot be in the constructed G_δ set. Therefore we conclude J is not topologically universal in the collection of dense G_δ sets.

■

Consider a compact set J in $\mathbb{R}^d$. An affine copy of J is $\mathbb{R}^2$ is the set

$$t + \delta O(J)$$

where $t \in \mathbb{R}^d$, $\delta \neq 0$ and O is an orthogonal transformation. We say that J is universal is the collection of dense G_δ sets if any dense G_δ set contains an affine copy of J.

Definition 3.3 (Projective Newhouse Thickness)**.** We define *projective Newhouse thickness* of J to be

$$\tau_P(J) = \inf_{O \in O(n)} \tau(P_x O(J))$$

where P_x is the orthogonal projection to the x-axis.

Theorem 3.3. *Let $J \subset \mathbb{R}^d$ be a compact set such that it has a positive projective Newhouse thickness. Then K is not universal in the set of dense G_δ set in $\mathbb{R}^2$.*

Proof. Suppose we have a compact set J in $\mathbb{R}^d$ such that it has a positive projective Newhouse thickness. By Theorem 1, there exists a dense G_δ set G_1 in $\mathbb{R}^1$ such that G_1 does not contain any affine copy of $\operatorname{proj}_x O(J)$. Now, we consider

$$G = G_1 \times G_1.$$

Then G is a dense G_δ set in $\mathbb{R}^d$. We claim that there is no affine copy of J in G. To justify the claim. Suppose G contains an affine copy of J such that $t + \delta J \subset G$. Then we take projection and we obtain that

$$\operatorname{proj}_x(t) + \delta \operatorname{proj}_x(J) \subset G_1$$

which is a contradiction. Hence, the claim is true and the proof is complete.

∎

Chapter 4

Remarks and Open Questions

4.1 Current Research Questions: Zero Newhouse Thickness

This section is devoted to study if Cantor sets with zero Newhouse thickness can be universal. We first provide an example for which two Cantor sets with zero Newhouse thickness can still have arithmetic sum equal to an interval, showing that the converse of the Newhouse thickness theorem is not true.

Example 4.1. Let $N_1, N_2, \dots \in \mathbb{N}_{\geq 2}$. Consider the following construction of a Cantor set using a decomposition of the unit intervals.

$$\begin{aligned}
[0,1] =& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \left[0, \frac{1}{N_1}\right] \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \left[0, \frac{1}{N_1N_2}\right] \\
=& ... \\
=& \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \dots + \frac{1}{N_1\cdots N_n}\{0,\dots,N_n\} + \dots
\end{aligned}$$

From here we can define the two cantor sets K_1, K_2 where K_1 constitutes the odd indices sets in the above summands and K_2 has the even one. This gives the following constructions for the two Cantor sets:

$$K_1 = \frac{1}{N_1}\{0,1,\dots,N_1-1\} + \dots + \frac{1}{N_1\cdots N_{2n+1}}\{0,\dots,N_{2n+1}-1\} + \dots$$

$$K_2 = \frac{1}{N_1N_2}\{0,1,\dots,N_2-1\} + \dots + \frac{1}{N_1\cdots N_{2n}}\{0,\dots,N_{2n}-1\} + \dots$$

From this construction we see that $K_1 + K_2 = [0,1]$ is the interval but from the definition of Newhouse thickness,

$$\tau(K_1) = \inf\left\{\frac{1}{N_1-1}, \frac{1}{N_3-1}, \dots\right\} = 0$$

$$\tau(K_2) = \inf\left\{\frac{1}{N_2-1}, \frac{1}{N_4-1}, \dots\right\} = 0.$$

Therefore we have created an interval from two sets with Newhouse thickness 0 if we have $\lim_{n\to\infty} N_n = \infty$.

We ask the following questions. Recall I_J denotes the smallest closed interval containing the Cantor set J and O_J denotes the largest open interval in $I_J \setminus J$.

1. Given a Cantor set J, does there exist some K such that $J + K = I_J + I_K$?

2. If we assume that there exists K such that $J + K = I_j + I_K$, can we prove that J is not universal?

3. (rescaling condition) If we assume that $|\lambda_1 I_J| \geq |\lambda_2 O_K|, |\lambda_2 I_K| \geq |\lambda_1 O_J|$ and $J + K = I_J + I_K$, then $\lambda_1 J + \lambda_2 K = \lambda_1 I_J + \lambda_2 I_k$.

We also notice that to solve the second question, we notice that $J + K = I_J + I_K$ implies that

$$(J + a) + (K + b) = (I_J + a) + (I_K + b) \text{ and } bJ + bK = bI_J + bI_K.$$

We can always translate and rescale J, K so that $I_J = [0, a]$ and $I_K = [0, 1]$. Moreover, the following lemma is important.

Lemma 4.1. *Suppose that the Cantor sets J and K satisfies $J + K = I_j + I_K$. Then $|I_J| \geq |O_K|$ and $|I_K| \geq |O_J|$.*

The lemma also said that the condition $|\lambda_1 I_J| \geq |\lambda_2 O_k|, |\lambda_2 I_k| \geq |\lambda_1 O_J|$ is necessary in the rescaling condition.

Proposition 4.2. *Let J be a Cantor set such that $J+K = I_J + I_K$ where $I_J = [0,a]$ and $I_K = [0,1]$. Suppose that the rescaling condition (3) holds. Then J is not universal in the collection of dense G_δ.*

Proof. The proof is similar to the proof in Theorem 3.2. With K given in the assumption. We can assume that $|I_J| > |O_K|$. Suppose that $|I_J| = |O_K|$. Since $|O_J| < 1$, we can choose ϵ such that $(1-\epsilon) > |O_J|$. Then we consider $K' = (1-\epsilon)K$ and we will have $|I_J| > (1-\epsilon)|O_K|$. In this case, by the rescaling condition, $J + K' = I_J + I_{K'}$ and we have another K' such that $|I_J| > |O_{K'}|$.

As now we have $|I_J| > |O_K|$, we can find $0 < \rho < 1$ such that $\rho|I_J| > |O_K|$. We now define

$$X = \bigcup_{n\in\mathbb{Z}} \bigcup_{\ell\in\mathbb{Z}} \rho^n (K + \ell).$$

Then X^c is a dense G_δ set. Suppose that we have an affine copy $t + \lambda J$, we would like to claim that $t + \lambda J$ intersects non-trivially with $\rho^n(K+\ell)$ for some $n, \ell \in \mathbb{Z}$, which will complete the proof of the theorem.

To justify the claim, we let $0 < \rho < 1$ take the unique n such that

$$|\lambda| \in [\rho^{n+1}, \rho^n) \tag{4.1}$$

and the unique $\ell \in \mathbb{Z}$ such that

$$t \in (\ell\rho^n, (\ell+1)\rho^n]. \tag{4.2}$$

Then we consider the arithmetic sum $\rho^n K - \lambda J$. We now check the assumption in the rescaling condition with $\lambda_1 = \rho^n$ and $\lambda_2 = -\lambda$. Indeed,

$$|\lambda_2 I_J| \geq \rho^{n+1}|I_J| = |\lambda_1|(\rho|I_J|) \geq |\lambda_1 O_K|$$

by our choice of ρ. On the other hand,

$$|\lambda_1 I_K| \geq |\lambda| \geq |\lambda_2 O_J|$$

since $|I_K| \geq |O_J|$ by Lemma 4.1. Hence, using the rescaling condition,

$$\rho^n K - \lambda J = \rho^n I_K - \lambda I_J.$$

If $\lambda > 0$, then we have

$$\rho^n(K+\ell) - \lambda J = [\rho^n \ell - \lambda a, \rho^n(1+\ell)]$$

which contains t by (4.2). Similarly, if $\lambda < 0$, then

$$\rho^n(K+\ell) - \lambda J = [\rho^n \ell, \rho^n(\ell+1) - \lambda a].$$

It also contains t by (4.2). The proof is now complete. ■

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From these questions we have several difficulties associated with each. For the first item it is not always clear which cantor sets can be added to each other. Similarly it is difficult to construct a complementing Cantor set because of the difficulties tracking the notation for the different possible open intervals. There maybe some existing tools. It may also just be messy.

For the second point, our proof inherently relies on appropriately selecting a scalar and translation that corresponds to a regular (or fairly regular) Cantor set. In this instance we have to find pick the appropriate λ, t based off of a set of associated intervals that are not uniform. Our proof relies on using the regularity to specify where the intersection is.

A current tool we are exploring is tracking how scaling and translating the collection of intervals $\{O_j\}_{j\in\mathbb{N}}$ by some appropriate bound M such that we can scale our cantor set by $\frac{1}{M^d}$, and demonstrate an appropriate intersection with X^c.

The last question we discussed for the day focused on how scaling Cantor sets, and scaling intervals are interrelated. With Newhouse thickness, because it relies off of the ratios of $\frac{I_j}{O_{j-1}}$ the scaling factor drops out. Unfortunately if we are considering Cantor sets with Newhouse thickness 0, then there is no corresponding Cantor set with infinite Newhouse thickness. The issue is that from the theorem, the thickness is the product of the two sets so for any finite thickness $0 \cdot \tau(C) = 0$. Therefore Newhouse thickness will not be enough to describe the appropriate construction of the interval. There are a few workarounds that might be possible. In Astels' paper[1] there is a generalized for for countably many cantor sets. Similarly we might be able to find another characterization (measure, dimension etc) of the set, to appropriately find λ and or, another way to combine the two intervals, such that we have a non-empty intersection with X^c.

Bibliography

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