a)

$$p(x,0)=\psi(x,0)\cdot\psi*(x,0)$$

$$\int_{-\infty}^{\infty} p(x,0)=1$$

$$\int_{-\infty}^{\infty} \psi(x,0)\cdot\psi*(x,0)=1$$

$$\psi(x,0)=A\cdot e^{-\lambda|x|}$$

note that this is a purely real term.

$$\int_{-\infty}^{\infty} A\cdot e^{-\lambda|x|}\cdot A\cdot e^{-\lambda|x|}=1$$

Rather than integrate the absolute value of x from -∞ to ∞, we can drop the absolute value and double the integral from 0 to ∞, since the curve will be symmetric about the y-axis

$$2\cdot A^2 \int_{0}^{\infty} e^{-2\lambda x}=2\cdot A^2\cdot\frac{e^{-2\lambda x}}{-2\lambda}\Big|_{0}^{\infty}=\frac{A^2}{\lambda}=1$$

$$A=\sqrt{\lambda}$$

b)

given that we know that there is no x-offset, we know that

$$\langle x \rangle = 0$$

however, working out the proof is good practice.

$$p(x,t) = \psi(x,t) \cdot \psi^*(x,t)$$

note here that I have not encountered the complex conjugate of $e^{-i \cdot \omega \cdot t}$

let's stop to figure out what that is

$$C = e^{-i \cdot \omega \cdot t} = \cos(-\omega \cdot t) + i \sin(-\omega \cdot t)$$

$$C^* = \cos(-\omega \cdot t) - i \sin(-\omega \cdot t)$$

If plotted on the complex plane, it could be seen that the angle of the conjugate vector is equal to 2*pi – the angle of the original vector.

Therefore, the complex conjugate of $e^{-i \cdot \omega \cdot t}$ is $e^{i(2 \cdot \pi - [-\omega \cdot t])} = e^{i(2 \cdot \pi + \omega \cdot t)}$

$$p(x,t) = \psi(x,t) \cdot \psi^*(x,t) = (\sqrt{\lambda} \cdot e^{-\lambda \cdot |x|} \cdot e^{-i \cdot \omega \cdot t}) \cdot (\sqrt{\lambda} \cdot e^{-\lambda \cdot |x|} \cdot e^{i(2 \cdot \pi + \omega \cdot t)})$$

$$p(x,t) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot e^{-i \cdot \omega \cdot t + i \cdot 2 \cdot \pi + i \cdot \omega \cdot t} = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot e^{i \cdot 2 \cdot \pi}$$

$$p(x,t) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \quad \text{here, we are using the fact that} \quad e^{i \cdot 2 \cdot \pi} = 1$$

note that this function is independent of time. I'm sort of wondering if I made a mistake here?

$$p(x,t) = p(x)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) \cdot dx = \int_{-\infty}^{\infty} x \cdot \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot dx = \lambda \cdot \int_{0}^{\infty} x \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^{0} x \cdot e^{2 \cdot \lambda \cdot x} \cdot dx$$

We will continue this solution on the next page.

$$\langle x \rangle = \lambda \cdot \int_0^\infty x \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^0 x \cdot e^{2 \cdot \lambda \cdot x} \cdot dx$$

$$u = 2 \lambda x; x = \frac{u}{2 \lambda}; dx = \frac{du}{2 \lambda}$$

$$\langle x \rangle = \lambda \int_0^\infty \frac{u}{2 \lambda} \cdot e^{-u} \cdot \frac{du}{2 \lambda} + \lambda \int_{-\infty}^0 \frac{u}{2 \lambda} \cdot e^u \cdot \frac{du}{2 \lambda} = \frac{1}{4 \lambda} \int_0^\infty u \cdot e^{-u} \cdot du + \frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du$$

we will do integration by parts on each half of the integral separately. For the integral from 0 to ∞:

$$v = u; dw = e^{-u} \cdot du \qquad dv = du; w = -e^{-u}$$

$$\frac{1}{4 \lambda} \int_0^\infty u \cdot e^{-u} \cdot du = \frac{1}{4 \lambda} \int_0^\infty v \cdot dw = \frac{1}{4 \lambda} \cdot v \cdot w \Big|_0^\infty - \frac{1}{4 \lambda} \cdot \int_0^\infty w \cdot dv = \frac{1}{4 \lambda} \cdot u \cdot (-e^{-u}) \Big|_0^\infty - \frac{1}{4 \lambda} \cdot \int_0^\infty -e^{-u} \cdot du$$

$$\frac{1}{4 \lambda} \int_0^\infty u \cdot e^{-u} \cdot du = -\frac{u}{4 \cdot \lambda \cdot e^u} \Big|_0^\infty - \frac{1}{4 \cdot \lambda \cdot e^u} \Big|_0^\infty = -\frac{u+1}{4 \cdot \lambda \cdot e^u} \Big|_0^\infty$$

we'll now have to use L'Hopital's rule to evaluate

$$\frac{1}{4 \lambda} \int_0^\infty u \cdot e^{-u} \cdot du = -\frac{1}{4 \cdot \lambda \cdot e^u} \Big|_0^\infty = \frac{1}{4 \cdot \lambda}$$

For the integral from -∞ to 0:

$$v = u; dw = e^u \cdot du \qquad dv = du; w = e^u$$

$$\frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = \frac{1}{4 \lambda} \int_{-\infty}^0 v \cdot dw = \frac{1}{4 \lambda} \cdot v \cdot w \Big|_{-\infty}^0 - \frac{1}{4 \lambda} \cdot \int_{-\infty}^0 w \cdot dv = \frac{1}{4 \lambda} \cdot u \cdot (e^u) \Big|_{-\infty}^0 - \frac{1}{4 \lambda} \cdot \int_{-\infty}^0 e^u \cdot du$$

$$\frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = \frac{u \cdot e^u}{4 \lambda} \Big|_{-\infty}^0 - \frac{e^u}{4 \lambda} \Big|_{-\infty}^0 = \frac{e^u}{4 \lambda} \cdot (u-1) \Big|_{-\infty}^0 = -\frac{1}{4 \lambda} - \frac{e^u}{4 \lambda} \cdot (u-1) \Big|_{-\infty} = -\frac{1}{4 \lambda}$$

I'm only, like, pretty sure that I did that last part right. I first evaluated the function at 0, and then used a L'Hopital-esque rule, where because $e^{-\infty}$ ends up in the denominator, I differentiated it as if it was in the denominator even though we started with e^u in the numerator... It's not exactly valid but I'm pretty sure I could substitute another variable, say, w=-u, and then evaluate this at positive infinity, and that would be correct.

$$\langle x \rangle = \frac{1}{4 \lambda} \int_0^\infty u \cdot e^{-u} \cdot du + \frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = \frac{1}{4 \lambda} + \left(-\frac{1}{4 \lambda} \right) = 0$$

I got the answer I expected so...I guess I got it right. Maybe.

b)

In deriving $\langle x \rangle$ we found $\quad p(x,t)=p(x)=\lambda\cdot e^{-2\cdot\lambda\cdot|x|}$

$$\langle x^2 \rangle = \lambda\cdot\int_0^\infty x^2\cdot e^{-2\cdot\lambda\cdot x}\cdot dx + \lambda\cdot\int_{-\infty}^0 x^2\cdot e^{2\cdot\lambda\cdot x}\cdot dx$$

$$u=2\lambda x\,;\, x=\frac{u}{2\lambda}\,;\, dx=\frac{du}{2\lambda}$$

$$\langle x^2 \rangle = \lambda\cdot\int_0^\infty \left(\frac{u}{2\cdot\lambda}\right)^2\cdot e^{-u}\cdot\frac{du}{2\cdot\lambda} + \lambda\cdot\int_{-\infty}^0 \left(\frac{u}{2\cdot\lambda}\right)^2\cdot e^{u}\cdot\frac{du}{2\cdot\lambda} = \frac{1}{8\cdot\lambda^2}\cdot\int_0^\infty u^2\cdot e^{-u}\cdot du + \frac{1}{8\cdot\lambda^2}\cdot\int_{-\infty}^0 u^2\cdot e^{u}\cdot du$$

For the integral from 0 to ∞:

$$v=u^2\,;\, dw=e^{-u}\cdot du \qquad dv=2\cdot u\cdot du\,;\, w=-e^{-u}$$

$$\int_0^\infty u^2\cdot e^{-u}\cdot du = \int_0^\infty v\cdot dw = v\cdot w\Big|_0^\infty - \int_0^\infty w\cdot dv = u^2\cdot(-e^{-u})\Big|_0^\infty - \int_0^\infty (-e^{-u})\cdot(2\cdot u\cdot du)$$

$$\int_0^\infty u^2\cdot e^{-u}\cdot du = -\frac{u^2}{e^u}\Big|_0^\infty + 2\int_0^\infty e^{-u}\cdot u\cdot du$$

in the derivation for <x> we found that

$$\frac{1}{4\lambda}\int_0^\infty u\cdot e^{-u}\cdot du = \frac{1}{4\cdot\lambda} \quad \text{which implies that} \quad \int_0^\infty u\cdot e^{-u}\cdot du = 1$$

$$\int_0^\infty u^2\cdot e^{-u}\cdot du = -\frac{u^2}{e^u}\Big|_0^\infty + 2 = 2$$

the integral from -∞ to 0 is evaluated on the next page

For the integral from $-\infty$ to 0:

$$v=u^2 ; dw=e^u \cdot du \qquad dv=2 \cdot u \cdot du ; w=e^u$$

$$\int_{-\infty}^{0} u^2 \cdot e^u \cdot du = \int_{-\infty}^{0} v \cdot dw = v \cdot w \big|_{0}^{\infty} - \int_{-\infty}^{0} w \cdot dv = u^2 \cdot (e^u) \big|_{-\infty}^{0} - \int_{-\infty}^{0} (e^u) \cdot (2 \cdot u \cdot du)$$

$$\int_{-\infty}^{0} u^2 \cdot e^u \cdot du = -\frac{u^2}{e^u} \bigg|_{-\infty}^{0} + 2 \int_{-\infty}^{0} e^u \cdot u \cdot du$$

in the derivation for <x> we found that

$$\frac{1}{4\lambda} \int_{-\infty}^{0} u \cdot e^u \cdot du = -\frac{1}{4\lambda} \quad \text{which implies that} \quad \int_{-\infty}^{0} u \cdot e^u \cdot du = -1$$

$$\int_{-\infty}^{0} u^2 \cdot e^u \cdot du = -u^2 \cdot e^u \big|_{-\infty}^{0} - 2 = -2$$

$$\langle x^2 \rangle = \lambda \cdot \int_{0}^{\infty} x^2 \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^{0} x^2 \cdot e^{2 \cdot \lambda \cdot x} \cdot dx = \lambda \cdot 2 - \lambda \cdot 2 = 0$$

$$\langle x^2 \rangle = 0$$

This is not in keeping with what we've seen in the past for this value – often the average of the square is nonzero even when the average is zero.