

a)

$$p(x,0) = \psi(x,0) \cdot \psi^*(x,0)$$

$$\int_{-\infty}^{\infty} p(x,0) dx = 1$$

$$\int_{-\infty}^{\infty} \psi(x,0) \cdot \psi^*(x,0) dx = 1$$

$$\psi(x,0) = A \cdot e^{-\lambda|x|}$$

note that this is a purely real term. Therefore, the complex conjugate is the number itself, and the probability density function is just the square of the wave function.

$$\int_{-\infty}^{\infty} A \cdot e^{-\lambda|x|} \cdot A \cdot e^{-\lambda|x|} dx = 1$$

Rather than integrate the absolute value of x from  $-\infty$  to  $\infty$ , we can drop the absolute value and double the integral from 0 to  $\infty$ , since the curve will be symmetric about the y-axis

$$2 \cdot A^2 \int_0^{\infty} e^{-2\lambda x} dx = 2 \cdot A^2 \cdot \left. \frac{e^{-2\lambda x}}{-2\lambda} \right|_0^{\infty} = \frac{A^2}{\lambda} = 1$$

$$A = \sqrt{\lambda}$$

b)

given that we know that there is no x-offset, we know that

$$\langle x \rangle = 0$$

however, working out the proof is good practice.

$$p(x, t) = \psi(x, t) \cdot \psi^*(x, t)$$

note here that I have not encountered the complex conjugate of  $e^{-i \cdot \omega \cdot t}$

let's stop to figure out what that is

$$C = e^{-i \cdot \omega \cdot t} = \cos(-\omega \cdot t) + i \sin(-\omega \cdot t)$$

$$C^* = \cos(-\omega \cdot t) - i \sin(-\omega \cdot t)$$

If plotted on the complex plane, it could be seen that the angle of the conjugate vector is equal to  $2\pi$  – the angle of the original vector. Alternatively, this is just the negative of the original angle (didn't realize this until I looked it up online; however, it gives the same answer which is good – I didn't do anything too dumb).

Therefore, the complex conjugate of  $e^{-i \cdot \omega \cdot t}$  is  $e^{i(2\pi - [-\omega \cdot t])} = e^{i(2\pi + \omega \cdot t)} = e^{i \cdot \omega \cdot t}$   
Another way to understand this is that  $e^{i(2\pi + \omega \cdot t)} = e^{i \cdot 2\pi} \cdot e^{i \cdot \omega \cdot t}$  but  $e^{i \cdot 2\pi} = 1$

$$p(x, t) = \psi(x, t) \cdot \psi^*(x, t) = (\sqrt{\lambda} \cdot e^{-\lambda \cdot |x|} \cdot e^{-i \cdot \omega \cdot t}) \cdot (\sqrt{\lambda} \cdot e^{-\lambda \cdot |x|} \cdot e^{i \cdot \omega \cdot t})$$

$$p(x, t) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot e^{-i \cdot \omega \cdot t + i \cdot 2\pi + i \cdot \omega \cdot t} = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|}$$

$$p(x, t) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|}$$

note that this function is independent of time. I'm sort of wondering if I made a mistake here?

$$p(x, t) = p(x)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) \cdot dx = \int_{-\infty}^{\infty} x \cdot \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot dx = \lambda \cdot \int_0^{\infty} x \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^0 x \cdot e^{2 \cdot \lambda \cdot x} \cdot dx$$

We will continue this solution on the next page.

$$\langle x \rangle = \lambda \cdot \int_0^{\infty} x \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^0 x \cdot e^{2 \cdot \lambda \cdot x} \cdot dx$$

$$u = 2 \lambda x; x = \frac{u}{2 \lambda}; dx = \frac{du}{2 \lambda}$$

$$\langle x \rangle = \lambda \int_0^{\infty} \frac{u}{2 \lambda} \cdot e^{-u} \cdot \frac{du}{2 \lambda} + \lambda \int_{-\infty}^0 \frac{u}{2 \lambda} \cdot e^u \cdot \frac{du}{2 \lambda} = \frac{1}{4 \lambda} \int_0^{\infty} u \cdot e^{-u} \cdot du + \frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du$$

we will do integration by parts on each half of the integral separately. For the integral from 0 to  $\infty$ :

$$v = u; dw = e^{-u} \cdot du \quad dv = du; w = -e^{-u}$$

$$\frac{1}{4 \lambda} \int_0^{\infty} u \cdot e^{-u} \cdot du = \frac{1}{4 \lambda} \int_0^{\infty} v \cdot dw = \frac{1}{4 \lambda} \cdot v \cdot w \Big|_0^{\infty} - \frac{1}{4 \lambda} \cdot \int_0^{\infty} w \cdot dv = \frac{1}{4 \lambda} \cdot u \cdot (-e^{-u}) \Big|_0^{\infty} - \frac{1}{4 \lambda} \cdot \int_0^{\infty} -e^{-u} \cdot du$$

$$\frac{1}{4 \lambda} \int_0^{\infty} u \cdot e^{-u} \cdot du = - \frac{u}{4 \cdot \lambda \cdot e^u} \Big|_0^{\infty} - \frac{1}{4 \cdot \lambda \cdot e^u} \Big|_0^{\infty} = - \frac{u+1}{4 \cdot \lambda \cdot e^u} \Big|_0^{\infty}$$

we'll now have to use L'Hopital's rule to evaluate

$$\frac{1}{4 \lambda} \int_0^{\infty} u \cdot e^{-u} \cdot du = - \frac{1}{4 \cdot \lambda \cdot e^u} \Big|_0^{\infty} = \frac{1}{4 \cdot \lambda}$$

For the integral from  $-\infty$  to 0:

$$v = u; dw = e^u \cdot du \quad dv = du; w = e^u$$

$$\frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = \frac{1}{4 \lambda} \int_{-\infty}^0 v \cdot dw = \frac{1}{4 \lambda} \cdot v \cdot w \Big|_{-\infty}^0 - \frac{1}{4 \lambda} \cdot \int_{-\infty}^0 w \cdot dv = \frac{1}{4 \lambda} \cdot u \cdot (e^u) \Big|_{-\infty}^0 - \frac{1}{4 \lambda} \cdot \int_{-\infty}^0 e^u \cdot du$$

$$\frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = \frac{u \cdot e^u}{4 \lambda} \Big|_{-\infty}^0 - \frac{e^u}{4 \lambda} \Big|_{-\infty}^0 = \frac{e^u}{4 \lambda} \cdot (u-1) \Big|_{-\infty}^0 = -\frac{1}{4 \lambda} - \frac{e^u}{4 \lambda} \cdot (u-1) \Big|_{-\infty}^0 = -\frac{1}{4 \lambda}$$

I'm only, like, pretty sure that I did that last part right. I first evaluated the function at 0, and then used a L'Hopital-esque rule, where because  $e^{-\infty}$  ends up in the denominator, I differentiated it as if it was in the denominator even though we started with  $e^u$  in the numerator... It's not exactly valid but I'm pretty sure I could substitute another variable, say,  $w = -u$ , and then evaluate this at positive infinity, and that would be correct.

$$\langle x \rangle = \frac{1}{4 \lambda} \int_0^{\infty} u \cdot e^{-u} \cdot du + \frac{1}{4 \lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = \frac{1}{4 \lambda} + \left(-\frac{1}{4 \lambda}\right) = 0$$

I got the answer I expected so...I guess I got it right. Maybe.

b)

In deriving  $\langle x \rangle$  we found  $p(x, t) = p(x) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|}$

$$\langle x^2 \rangle = \lambda \cdot \int_0^{\infty} x^2 \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^0 x^2 \cdot e^{2 \cdot \lambda \cdot x} \cdot dx$$

$$u = 2 \lambda x; x = \frac{u}{2 \lambda}; dx = \frac{du}{2 \lambda}$$

$$\langle x^2 \rangle = \lambda \cdot \int_0^{\infty} \left( \frac{u}{2 \cdot \lambda} \right)^2 \cdot e^{-u} \cdot \frac{du}{2 \cdot \lambda} + \lambda \cdot \int_{-\infty}^0 \left( \frac{u}{2 \cdot \lambda} \right)^2 \cdot e^u \cdot \frac{du}{2 \cdot \lambda} = \frac{1}{8 \cdot \lambda^2} \cdot \int_0^{\infty} u^2 \cdot e^{-u} \cdot du + \frac{1}{8 \cdot \lambda^2} \cdot \int_{-\infty}^0 u^2 \cdot e^u \cdot du$$

For the integral from 0 to  $\infty$ :

$$v = u^2; dw = e^{-u} \cdot du \quad dv = 2 \cdot u \cdot du; w = -e^{-u}$$

$$\int_0^{\infty} u^2 \cdot e^{-u} \cdot du = \int_0^{\infty} v \cdot dw = v \cdot w \Big|_0^{\infty} - \int_0^{\infty} w \cdot dv = u^2 \cdot (-e^{-u}) \Big|_0^{\infty} - \int_0^{\infty} (-e^{-u}) \cdot (2 \cdot u \cdot du)$$

$$\int_0^{\infty} u^2 \cdot e^{-u} \cdot du = - \frac{u^2}{e^u} \Big|_0^{\infty} + 2 \int_0^{\infty} e^{-u} \cdot u \cdot du$$

in the derivation for  $\langle x \rangle$  we found that

$$\frac{1}{4 \lambda} \int_0^{\infty} u \cdot e^{-u} \cdot du = \frac{1}{4 \cdot \lambda} \quad \text{which implies that} \quad \int_0^{\infty} u \cdot e^{-u} \cdot du = 1$$

$$\int_0^{\infty} u^2 \cdot e^{-u} \cdot du = - \frac{u^2}{e^u} \Big|_0^{\infty} + 2 = 2$$

the integral from  $-\infty$  to 0 is evaluated on the next page

For the integral from  $-\infty$  to 0:

$$v = u^2; dw = e^u \cdot du \quad dv = 2 \cdot u \cdot du; w = e^u$$

$$\int_{-\infty}^0 u^2 \cdot e^u \cdot du = \int_{-\infty}^0 v \cdot dw = v \cdot w \Big|_{-\infty}^0 - \int_{-\infty}^0 w \cdot dv = u^2 \cdot (e^u) \Big|_{-\infty}^0 - \int_{-\infty}^0 (e^u) \cdot (2 \cdot u \cdot du)$$

$$\int_{-\infty}^0 u^2 \cdot e^u \cdot du = -\frac{u^2}{e^u} \Big|_{-\infty}^0 + 2 \int_{-\infty}^0 e^u \cdot u \cdot du$$

in the derivation for  $\langle x \rangle$  we found that

$$\frac{1}{4\lambda} \int_{-\infty}^0 u \cdot e^u \cdot du = -\frac{1}{4\lambda} \quad \text{which implies that} \quad \int_{-\infty}^0 u \cdot e^u \cdot du = -1$$

$$\int_{-\infty}^0 u^2 \cdot e^u \cdot du = -u^2 \cdot e^u \Big|_{-\infty}^0 - 2 = -2$$

$$\langle x^2 \rangle = \lambda \cdot \int_0^{\infty} x^2 \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^0 x^2 \cdot e^{2 \cdot \lambda \cdot x} \cdot dx = \lambda \cdot 2 - \lambda \cdot 2 = 0$$

$$\langle x^2 \rangle = 0$$

This makes no sense – if this is the case, the standard deviation is 0, and the particle is exactly constrained to  $x=0$  for all time...