Linear and Nonlinear Oscillators

Jeffrey Eldred

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Primes and Dots

x-prime means derivative of x with respect to s, where s is some spatial coordinate.

$$x'(s) = \frac{\partial}{\partial s} x(s)$$

x-dot mean derivative of x with respect to t, where t is some temporal coordinate.

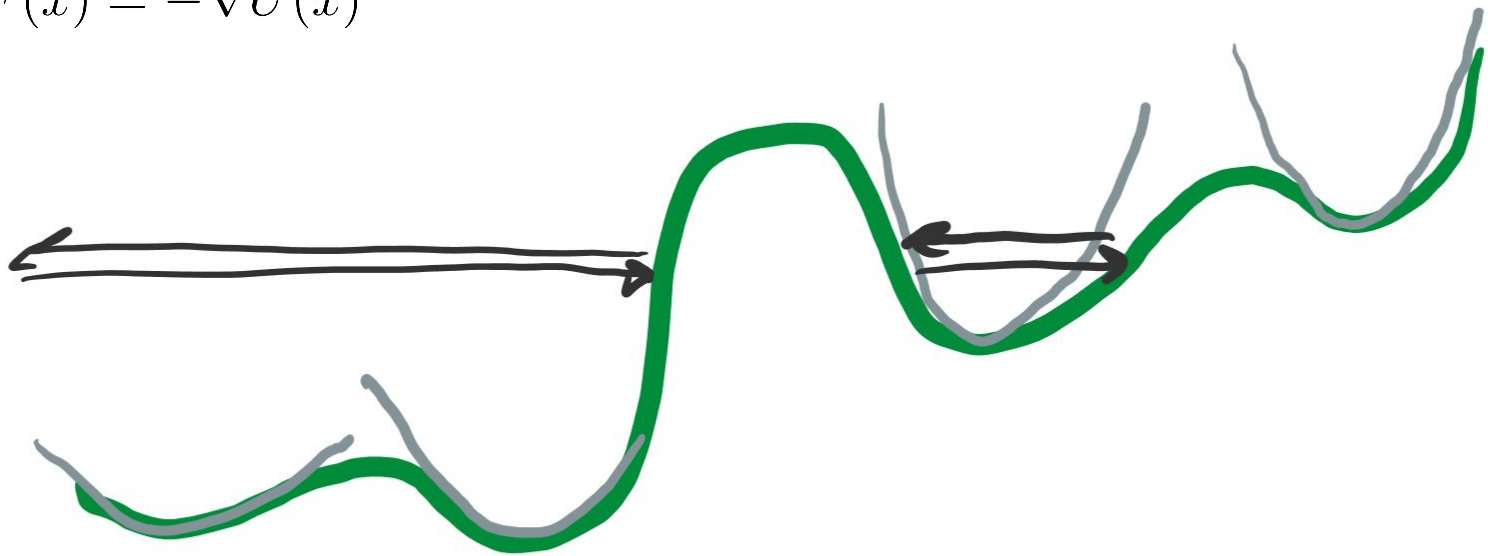
$$\dot{x}(t) = \frac{\partial}{\partial t} x(t)$$

Linearization of a second order differential equation

Taylor Expansion of a general 1D force on a particle

Consider an arbitrary force on a particle: $m\ddot{x} = F(x)$

$$F(x) = -\nabla U(x)$$



Usually, it can be linearized as a damped harmonic oscillator:

$$m\ddot{x} = F(x) \approx F(x_0) + (x - x_0)\frac{\partial F}{\partial x}\bigg|_{x_0} + \frac{1}{2}(x - x_0)^2\frac{\partial^2 F}{\partial x^2}\bigg|_{x_0}$$

Taylor Expansion of a general 1D force on a particle

That is for a conservative field, in which F is a function of the position x, but not a function of the velocity x-dot.

We can include a linear damping (or excitory force) by:

$$m\ddot{x} = F(x, \dot{x}) \approx F(x_0, \dot{x}_0) + (x - x_0)\left.\frac{\partial F}{\partial x}\right|_{x_0, \dot{x}_0} + (\dot{x} - \dot{x}_0)\left.\frac{\partial F}{\partial \dot{x}}\right|_{x_0, \dot{x}_0}$$

$$a\ddot{x} + b\dot{x} + cx = 0$$

This is a linear second-order homogeneous differential equation, otherwise known as the damped oscillator.

How do we solve it? and why can we solve it?

Linear Differential Equation

A differential equation is linear if the following is true

(I) If x_0 is a solution, αx_0 is a solution.
(II) If x_1 and x_2 are solutions, $x_1 + x_2$ is a solution.

Linear Example:

$$x'' = -\omega^2 x$$
$$(\alpha x_1 + \beta x_2)'' = -\omega^2(\alpha x_1 + \beta x_2)$$

Nonlinear Example:

$$x'' = -\omega^2 x^2$$
$$(\alpha x_1 + \beta x_2)'' = -\omega^2 \alpha x_1^2 + \omega^2 \beta x_2^2$$
$$(\alpha x_1 + \beta x_2)'' \neq -\omega^2(\alpha x_1 + \beta x_2)^2$$

Definition of the Exponent e (Euler's Number)

Euler's number e, and natural exponent function exp are defined to be solutions to the first-order linear differential equation.

$$\text{If } \dot{x} = \lambda x,$$

$$\text{Then } x(t) = Ae^{\lambda t} = A \cdot \exp(\lambda t)$$

For someone unknown value of A. Any number other than e would be have some coefficient in front of λ.

And for a second-order linear differential equation, we'd just write:

$$\text{If } \ddot{x} = \lambda^2 x,$$

$$\text{Then } \dot{x} = \lambda x \text{ and } x(t) = Ae^{\lambda t}$$

Now, we can define sine & cosine, as a solution to a differential equation in the same way.

Definition of Sine & Cosine (Geometry)

Definition of Sine & Cosine (Differential Equations)

How do we solve this equation, define functions which are solutions!

$$\text{If } \ddot{x} = -\omega^2 x,$$

$$\text{Then } x(t) = A\cos(\omega t) + B\sin(\omega t),$$
$$\text{where } x(0) = A,\ \dot{x}(0) = B\omega.$$

And so it must be related to that other thing that we defined:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),\ \text{where } i = \sqrt{-1}$$

So we “solved” the second-order linear equation with cosine, sine, and e.

But we haven’t solved anything, have we? We just defined our ignorance!

Yes, except all properties of cosine, sine, and natural exponent are just shorthand representation of statements about these differential equations.

Phase space of Sine & Cosine

Oscillator Trajectory:

$$x(t) = A\cos(\omega t + \phi)$$

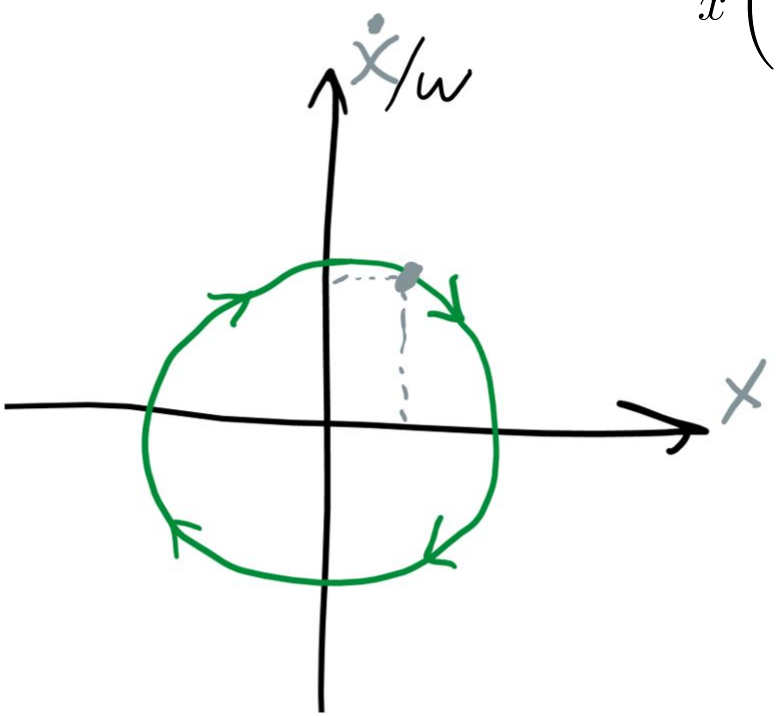
$$\dot{x}(t) = -A\omega\sin(\omega t + \phi)$$

Periodicity:

$$\cos(\theta + 2\pi) = \cos(\theta)$$

$$\sin(\theta + 2\pi) = \sin(\theta)$$

$$x\left(t + \frac{2\pi}{\omega}\right) = x(t)$$



Homogeneous Second-Order Linear Differential Equation

(Solving the Damped Harmonic Oscillator)

Homogeneous Second-Order Linear Diff. Eq.

Damped Harmonic Oscillator:

$$a\ddot{x} + b\dot{x} + cx = 0 \qquad \gamma = \frac{b}{a}, \; \omega_0^2 = \frac{c}{a}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Start with an "ansatz" for the form of the solution: $x_c(t) = Ae^{\lambda t}$

$$\lambda^2 x_c + \gamma\lambda x_c + \omega_0^2 x_c = 0$$

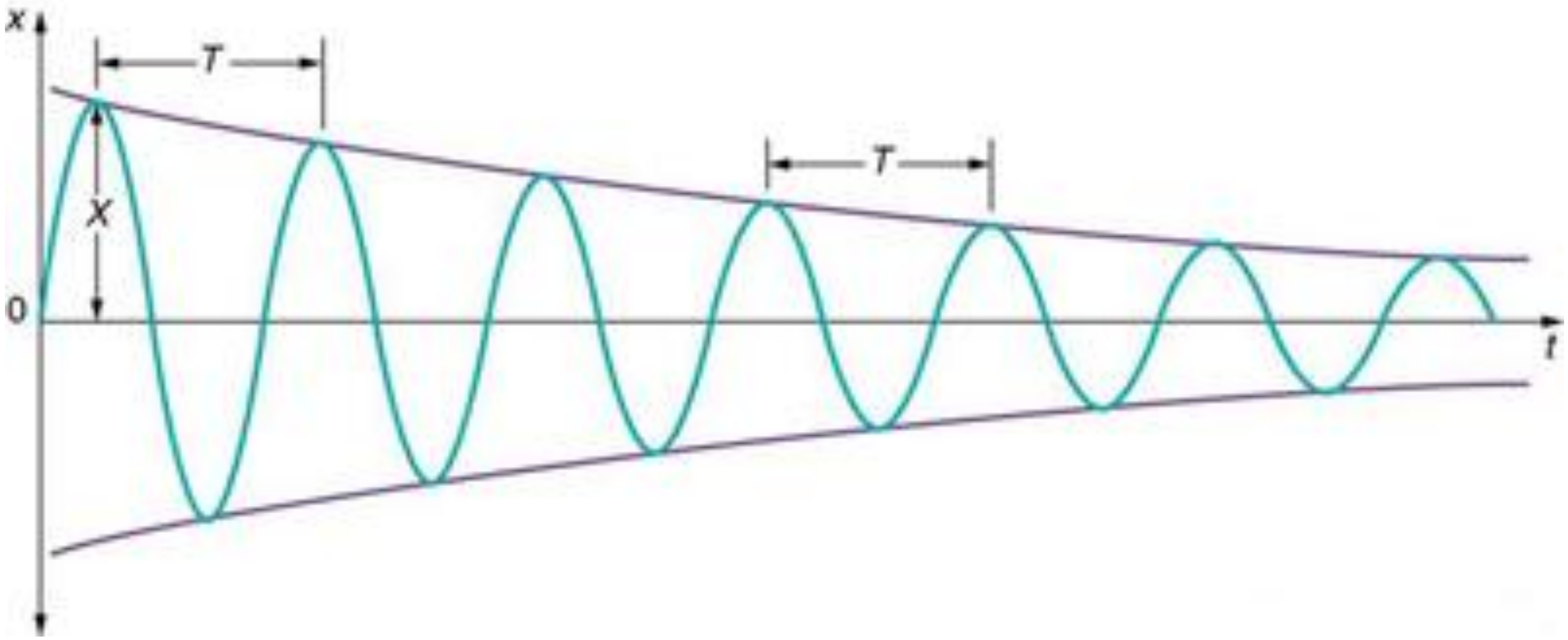
$$\lambda_\pm = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

The general solution is a linear combination of the independent solutions:

$$x(t) = x_{c1}(t) + x_{c2}(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$$

$$x(t) = A_0 e^{-(\gamma/2)t} \cos\left[\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} t + \phi_0\right]$$

Sinesoid with Decaying Amplitude



$$x(t) = A_0 e^{-(\gamma/2)t} \cos\left(\omega_1 t + \phi_0\right)$$

$$\omega_1 = \omega_0 \sqrt{1 - \left(\frac{\gamma}{2\omega_0}\right)^2}$$

Accelerator Example

Consider particle's oscillations in the Low Energy Ring in PEP-II at SLAC.

In the transverse direction, the particle executes "betatron" oscillations with the frequency about 40 times larger than the revolution frequency of 136 kHz. This means

$$\omega_\beta \approx 2\pi \ \times \ 5.4 \text{ MHz}$$

The damping time $2/\gamma$ due to the synchrotron radiation is about 60 ms, which means that γ is about 30 Hz.

Clearly, the ratio $\gamma/\omega_\beta \approx 10^{-6}$ is extremely small for these oscillations, and the damping can be neglected in first approximation.

Stupakov

Fermilab

Inhomogeneous Second-Order Linear Differential Equation

(Adding a time-dependent force)

Example: Constantly Progressing Force

Example:

Damped Harmonic Oscillator with a constantly progressing force:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f(t)$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \epsilon t + \eta$$

Find one solution that matches the differential form of the force:

$$x_p(t) = At + B$$

$$\ddot{x_p} + \gamma\dot{x_p} + \omega_0^2 x_p = \epsilon t + \eta$$

$$0 + \gamma A + \omega_0^2 At + \omega_0^2 B = \epsilon t + \eta$$

$$A = \frac{\epsilon}{\omega_0^2}, \;\; B = \frac{\eta\omega_0^2 - \gamma\epsilon}{\omega_0^4}$$

$$x_p(t) = \left(\frac{\epsilon}{\omega_0^2}\right) t + \left(\frac{\eta\omega_0^2 - \gamma\epsilon}{\omega_0^4}\right)$$

Example: Constantly Progressing Force (cont.)

The general solution is the sum of complimentary solution (the general solution to the homogeneous equation) and the particular solution (a solution to the inhomogeneous equation):

$$\ddot{x}_c + \gamma\dot{x}_c + \omega_0^2 x_c = 0$$

$$\ddot{x}_p + \gamma\dot{x}_p + \omega_0^2 x_p = \epsilon t + \eta$$

$$x_c(t) = A_0 e^{-(\gamma/2)t}\cos\left[\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}\, t + \phi_0\right]$$

$$x_p(t) = \left(\frac{\lambda}{\omega_0^2}\right)t + \left(\frac{\eta\omega_0^2 - \gamma\lambda}{\omega_0^4}\right)$$

$$x(t) = x_c(t) + x_p(t)$$

$$x(t) = A_0 e^{-(\gamma/2)t}\cos\left[\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}\, t + \phi_0\right] + \left(\frac{\epsilon}{\omega_0^2}\right)t + \left(\frac{\eta\omega_0^2 - \gamma\epsilon}{\omega_0^4}\right)$$

Example: Sinesoidal Driving Force

Now there is a drive frequency ω, and a natural (dynamical) frequency ω_0:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$$

Write the equation in complex form

$$\ddot{\xi} + \gamma\dot{\xi} + \omega_0^2\xi = f_0 e^{i\omega t}, \quad \text{where } \Re[\xi(t)] = x(t)$$

$$\xi_p(t) = \xi_0 e^{i\omega t}$$

$$[-\omega^2 + i\omega\gamma + \omega_0^2]\xi_0 e^{i\omega t} = f_0 e^{i\omega t}$$

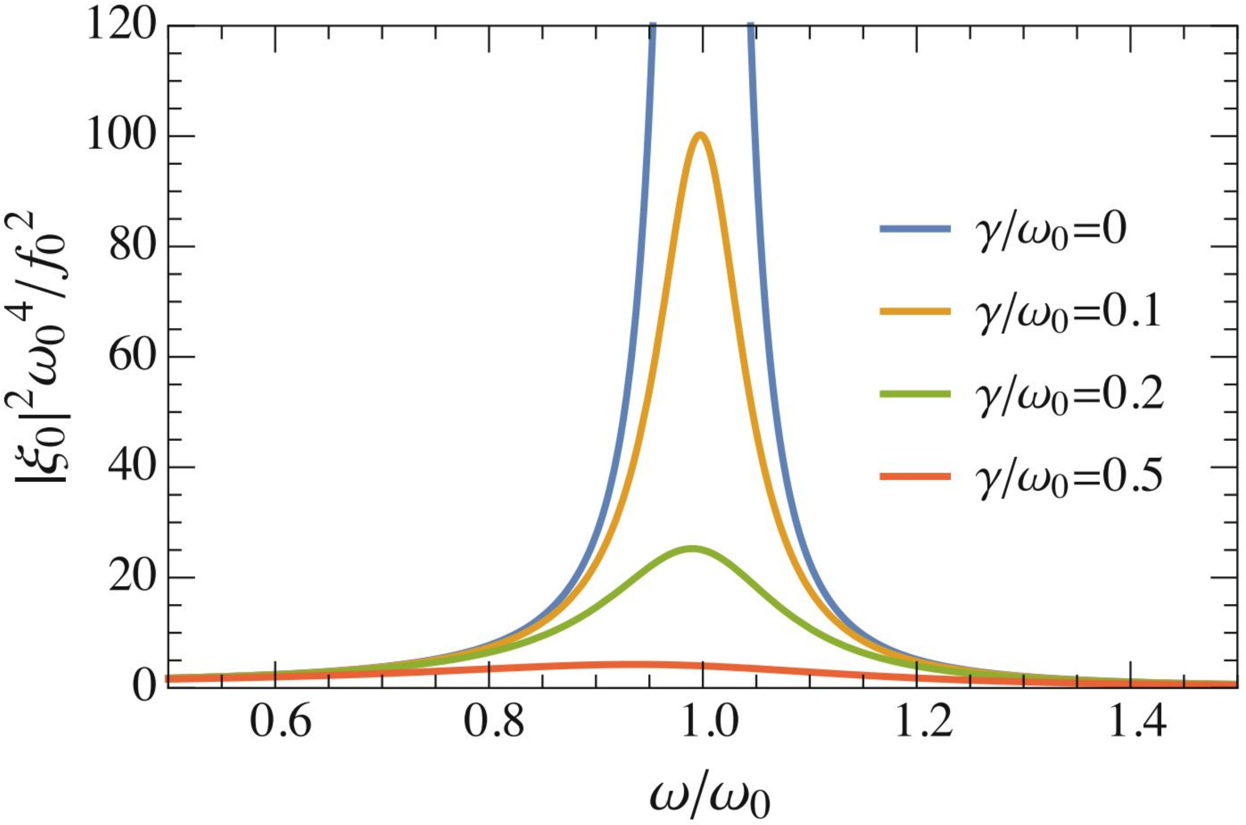
$$|\xi_0|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$x(t) = x_c(t) + x_p(t)$$

$$x(t) = A_0 e^{-(\gamma/2)t}\cos\left[\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}t + \phi_0\right] + |\xi_0|\cos(\omega t + \psi_0)$$

Example: Sinesoidal Driving Force

$$|\xi_0|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$



Stupakov

Example: Sinesoidal Driving Force

That's the amplitude, what about the phase?

$$\ddot{\xi} + \gamma\dot{\xi} + \omega_0^2\xi = f_0 e^{i\omega t}, \quad \text{where } \Re[\xi(t)] = x(t)$$

$$\xi_p(t) = \xi_0 e^{i\omega t}$$

$$[-\omega^2 + i\omega\gamma + \omega_0^2]\xi_0 e^{i\omega t} = f_0 e^{i\omega t}$$

$$[-\omega^2 + i\omega\gamma + \omega_0^2]\left(\Re\xi_0 + i\Im\xi_0\right) = f_0$$

$$(\omega_0^2 - \omega^2)\Re[\xi_0] - \omega\gamma\Im[\xi_0] = f_0$$

$$(\omega_0^2 - \omega^2)\Im[\xi_0] + \omega\gamma\Re[\xi_0] = 0$$

$$\left[1 + \frac{(\omega\gamma)^2}{(\omega_0^2 - \omega^2)^2}\right]\Im[\xi_0] = -\frac{\omega\gamma}{(\omega_0^2 - \omega^2)} f_0$$

$$\left[\frac{(\omega_0^2 - \omega^2) + \omega\gamma}{\omega\gamma}\right]\Re[\xi_0] = \frac{(\omega_0^2 - \omega^2)}{\omega\gamma} f_0$$

Sidebar: Solution without Complex Number

Now there is a drive frequency ω, and a dynamical frequency ω_0:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

Write the equation in complex form

$$x_p(t) = x_0 \cos(\omega t + \psi_0) = x_0[\cos(\omega t)\cos(\psi_0) - \sin(\omega t)\sin(\psi_0)]$$

$$[-\omega^2 \cos(\psi_0) - \omega\gamma \sin(\psi_0) + \omega_0^2 \cos(\psi_0)]x_0 \cos(\omega t) = f_0 \cos(\omega t)$$

$$[\omega^2 \sin(\psi_0) - \omega\gamma \cos(\psi_0) - \omega_0^2 \sin(\psi_0)]x_0 \sin(\omega t) = 0$$

$$\cos(\psi_0)^2 + \sin(\psi_0)^2 = 1$$

$$\sin(\psi_0) = -\omega\gamma \cos(\psi_0)/(\omega_0^2 - \omega^2)$$

$$\cos(\psi_0) = (\omega_0^2 - \omega^2)/[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]^{1/2}$$

$$x_0 = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]^{1/2}}$$

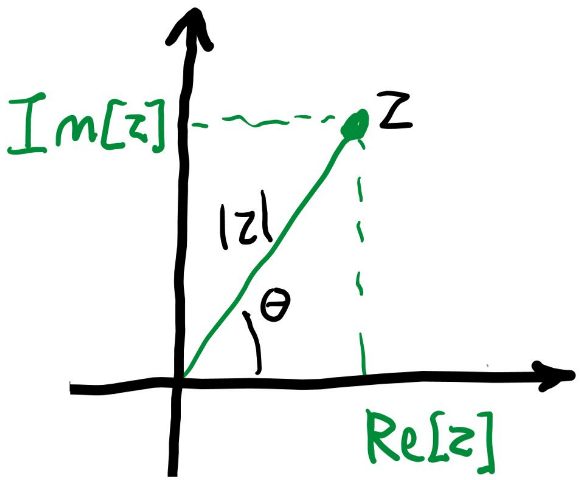
Sidebar: Complex Numbers Refresher

Recall that for a complex number you have: $z = a + ib = \Re[z] + i\Im[z]$

The complex conjugate is defined: $z^* = a - ib = \Re[z] - i\Im[z]$

The complex magnitude |z| and complex argument ϕ can be calculated by:

$$zz^* = |z|^2 = a^2 + b^2$$
$$z = |z|e^{i\phi} = a + ib$$
$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$



Complex fractions can be rewritten: $\dfrac{1}{a + ib} = \dfrac{a - ib}{a^2 + b^2}$

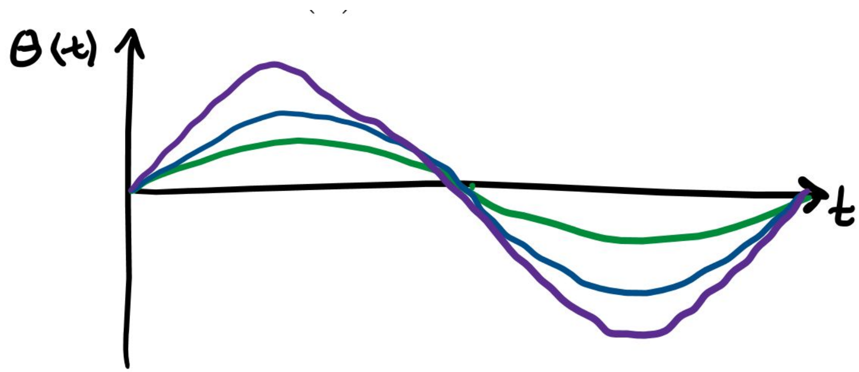
Nonlinear Oscillator

Pendulum as Nonlinear Oscillator

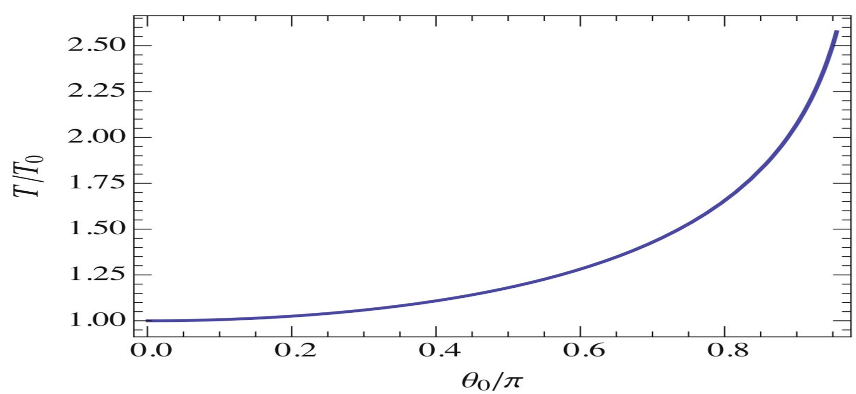
The equation of motion for a simple pendulum:

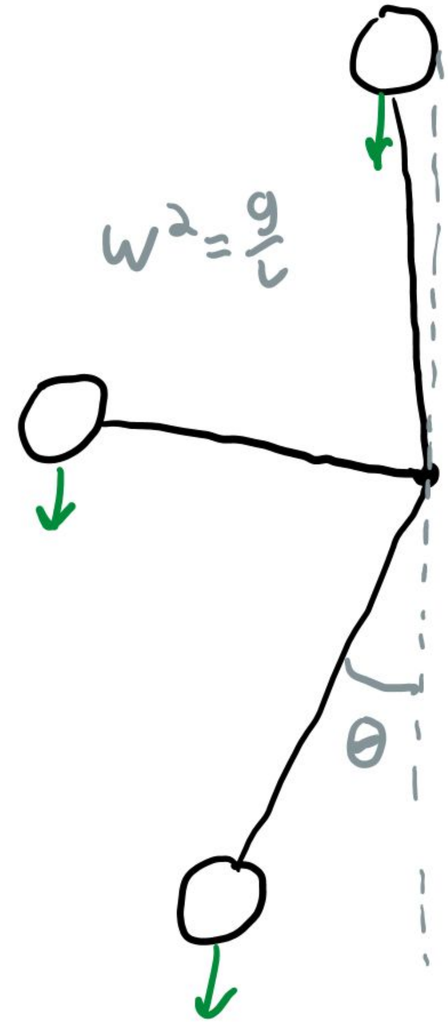
$$\ddot{\theta} = -\omega^2 \sin(\theta) \approx -\omega^2\theta + \omega^2\frac{\theta^3}{6} + \ldots$$

oscillation waveform distortion:



oscillation frequency change:





Perturbation Methods

A differential equation may be difficult (or impossible) to solve explicitly. We need to know the trajectory to calculate the force on the particle, but we need to know the force on the particle to calculate the trajectory!

So one may be able to obtain an approximate solution can be decomposed as a series of simpler solutions:

1. Make a severe approximation of the differential equation that admits a simple solution for the trajectory.
2. Use that simple trajectory to calculate the force.
3. Use that force to calculate a new trajectory.
4. Iterate until the trajectory converges for a certain order of precision.

Pendulum Perturbation

The equation of motion for a simple pendulum:

$$\ddot{\theta} = -\omega^2 \sin(\theta) \approx -\omega^2\theta + \omega^2\frac{\theta^3}{6} + \ldots$$

Start with a simplest solution, small angle θ:

$$\ddot{\theta}_1 = -\omega^2\theta_1$$

$$\theta_1(t) = A\cos(\omega t + \phi)$$

Expand out the RHS, plug in the simplest solution, solve for LHS:

$$\ddot{\theta}_2 = -\omega^2\theta_1 + \omega^2\frac{\theta_1^3}{6}$$

$$\ddot{\theta}_2 = -\omega^2 A\cos(\omega t + \phi) + \omega^2\frac{A^3}{6}\cos^3(\omega t + \phi)$$

$$\ddot{\theta}_2 = -\omega^2 A\cos(\omega t + \phi) + \omega^2\frac{A^3}{6}\left[\frac{3}{4}\cos(\omega t + \phi) + \frac{1}{4}\cos(3\omega t + 3\phi)\right]$$

Pendulum Perturbation (cont.)

$$\theta_1(t) = A\cos(\omega t + \phi)$$

$$\ddot{\theta}_2 = -\omega^2\theta_1 + \omega^2\frac{\theta_1^3}{6}$$

$$\ddot{\theta}_2 = -\omega^2 A\cos(\omega t + \phi) + \omega^2\frac{A^3}{6}\left[\frac{3}{4}\cos(\omega t + \phi) + \frac{1}{4}\cos(3\omega t + 3\phi)\right]$$

$$\ddot{\theta}_2 = -\omega^2 A\left[1 - \frac{A^2}{8}\right]\cos(\omega t + \phi) + \omega^2\frac{A^3}{24}\cos(3\omega t + 3\phi)$$

$$\theta_2(t) = A\left[1 - \frac{A^2}{8}\right]\cos(\omega t + \phi) - \frac{A^3}{216}\cos(3\omega t + 3\phi) \ ??$$

$$\theta_2(t) = A\cos[\omega(1+\sigma)t + \phi] - \frac{A^3}{216}\cos[3\omega(1+\sigma)t + 3\phi]$$

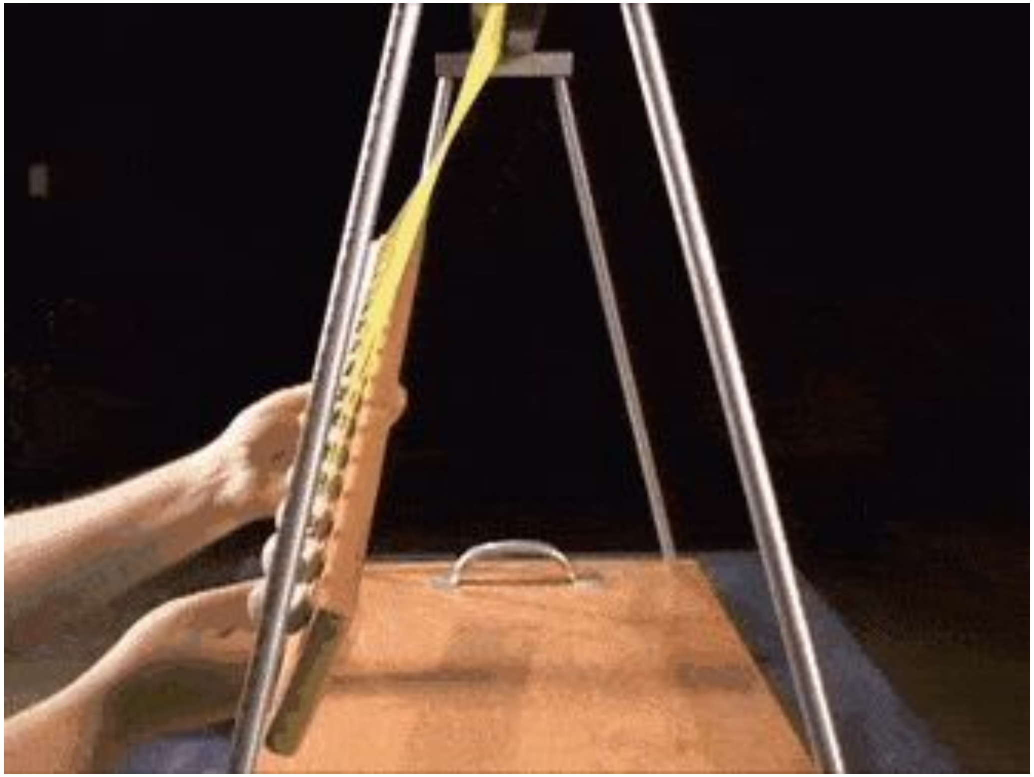
$$-\omega^2(1+\sigma)^2 \approx \omega^2\left[1 - \frac{A^2}{8}\right]$$

$$\sigma \approx -\frac{A^2}{16}$$

The next order iteration will refine σ, adjust 3ω coefficient, include 5ω term.

Pendulum Perturbation (cont.)

Every amplitude has a different frequency.
Pendulums of different lengths will decohere over time.



The video also shows the pendulums re-cohering, that wouldn't be the case with a larger number of pendulums.

Parametric Resonance (advanced topics)

Parametric Resonance

Let the focusing $\omega_0^{\ 2}$ term vary sinesoidally with time, creating terms dependent on x & t:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2(t)x = 0$$

$$\omega_0^2(t) = \Omega^2 - h\Omega^2\cos(\nu t)$$

$$\ddot{x} = -\Omega^2 x - \gamma\dot{x} + h\Omega^2 x\cos(\nu t)$$

Let's start by examining it perturbatively (small x, small h, small γ):

$$x_0(t) \approx A\cos(\Omega t + \phi)$$

$$\ddot{x}_1 \approx -\Omega^2 x_0 - \gamma\dot{x}_0 + h\Omega^2 x_0\cos(\nu t)$$

$$\ddot{x}_1 \approx -\Omega^2 A\cos(\Omega t + \phi) + \Omega\gamma A\sin(\Omega t + \phi)$$
$$+ \frac{1}{2}h\Omega^2 A[\cos(\Omega t + \nu t + \phi) + \cos(\Omega t - \nu t + \phi)]$$

$$x_1(t) \approx A\cos(\Omega t + \phi) - \frac{\gamma}{\Omega}A\sin(\Omega t + \phi)$$
$$+ \frac{1}{2}h\Omega^2 A\left[\frac{\cos(\Omega t + \nu t + \phi)}{(\Omega+\nu)^2} + \frac{\cos(\Omega t - \nu t + \phi)}{(\Omega-\nu)^2}\right]$$

Parametric Resonance (cont.)

Now that we know what terms to look for, let's do the whole thing:

$$\ddot{x} = -\Omega^2 x - \gamma \dot{x} + h\Omega^2 x \cos(\nu t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} A_k \cos[(\Omega + k\nu)t + \phi] + B_k \sin[(\Omega + k\nu)t + \phi]$$

$$x(t) \equiv \sum A_k c_k + B_k s_k$$

$$\sum -(\Omega + k\nu)^2 (A_k c_k + B_k s_k) = \sum - \Omega^2 A_k c_k + \Omega\gamma A_k s_k - \Omega^2 B_k s_k - \Omega\gamma B_k c_k$$

$$+ \frac{1}{2} h\Omega^2 A_k (c_{k+1} + c_{k-1}) + \frac{1}{2} h\Omega^2 B_k (s_{k+1} + s_{k-1})$$

$$\ldots$$

$$A_k = \frac{h\Omega}{4k\nu} \left[\frac{A_{k-1} + A_{k+1}}{1 + (k\nu)/(2\Omega)} \right] + \mathcal{O}\left(\frac{\gamma^2}{\Omega^2}, h\gamma \right)$$

$$A_k \sim h^k A_0, \quad \text{unless } \Omega \approx \frac{k}{2}\nu$$

Parametric Resonance (cont.)

$$\sum -(\Omega + k\nu)^2(A_k c_k + B_k s_k) = \sum - \Omega^2 A_k c_k + \Omega\gamma A_k s_k - \Omega^2 B_k s_k - \Omega\gamma B_k c_k$$
$$+ \frac{1}{2}h\Omega^2 A_k(c_{k+1} + c_{k-1}) + \frac{1}{2}h\Omega^2 B_k(s_{k+1} + s_{k-1})$$

$$-(\Omega + k\nu)^2 A_k c_k = - \Omega^2 A_k c_k - \Omega\gamma B_k c_k + \frac{1}{2}h\Omega^2(A_{k-1} + A_{k+1})c_k$$
$$-(\Omega + k\nu)^2 B_k s_k) = - \Omega^2 B_k s_k + \Omega\gamma A_k s_k + \frac{1}{2}h\Omega^2(B_{k-1} + B_{k+1})s_k$$

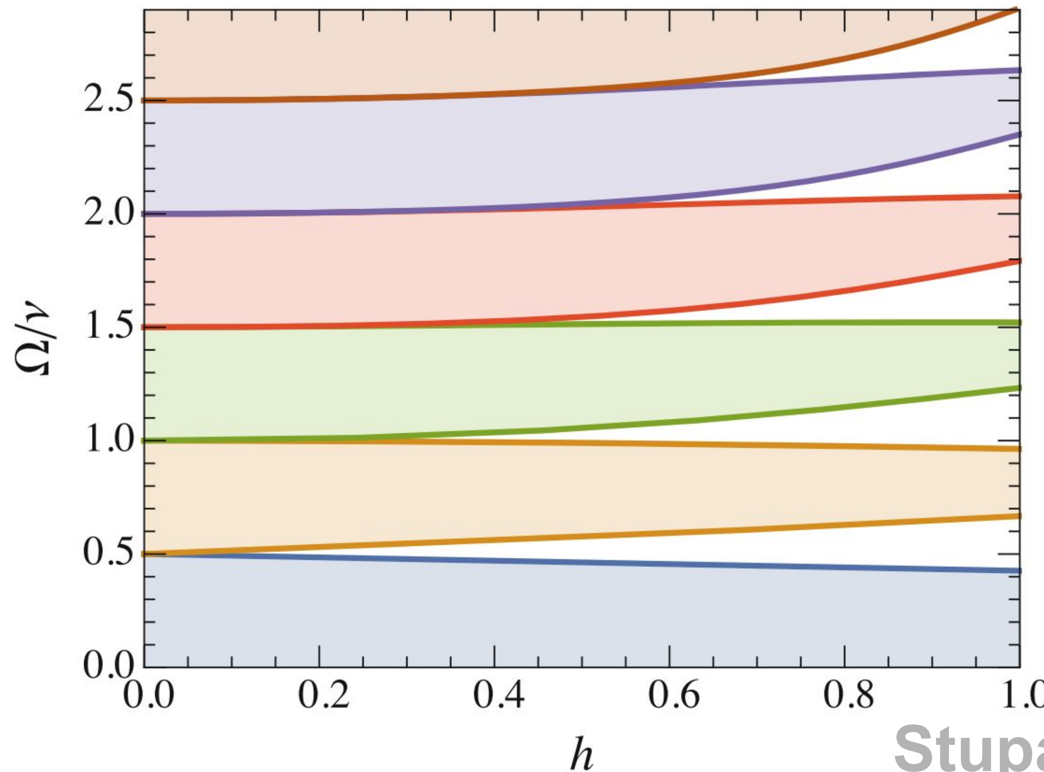
$$A_k = \frac{h\Omega}{4k\nu}\left[\frac{A_{k-1} + A_{k+1}}{1 + (k\nu)/(2\Omega)}\right] + \frac{\gamma}{2k\nu}\left[\frac{B_k}{1 + (k\nu)/(2\Omega)}\right]$$
$$A_k \approx \frac{h\Omega}{4k\nu}\left[\frac{A_{k-1} + A_{k+1}}{1 + (k\nu)/(2\Omega)}\right] + \mathcal{O}\left(\frac{\gamma^2}{k^2\nu^2}, \frac{h\gamma\Omega}{k^2\nu^2}\right)$$
$$A_k \sim h^k A_0, \quad \text{unless } \Omega \approx \frac{k}{2}\nu$$

Parametric Resonance (cont.)

$$A_k \approx \frac{h\Omega}{4k\nu}\left[\frac{A_{k-1}+A_{k+1}}{1+(k\nu)/(2\Omega)}\right]+\mathcal{O}\left(\frac{\gamma^2}{k^2\nu^2},\frac{h\gamma\Omega}{k^2\nu^2}\right)$$

$$A_k \sim h^k A_0, \quad \text{unless } \Omega \approx \frac{k}{2}\nu$$

Fig. 4.2 Stability regions for the Mathieu equation (4.16), (4.17) as functions of amplitude modulation h. The stable regions are shadowed



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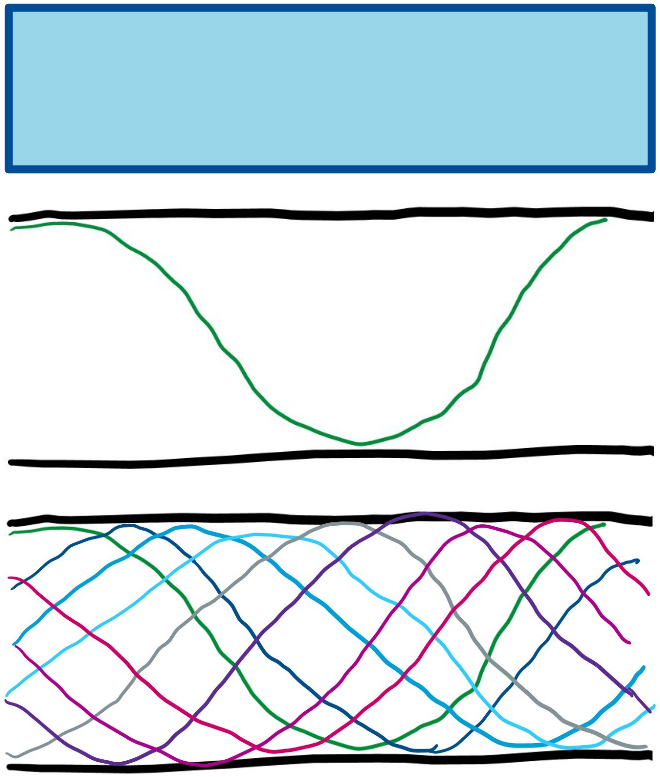
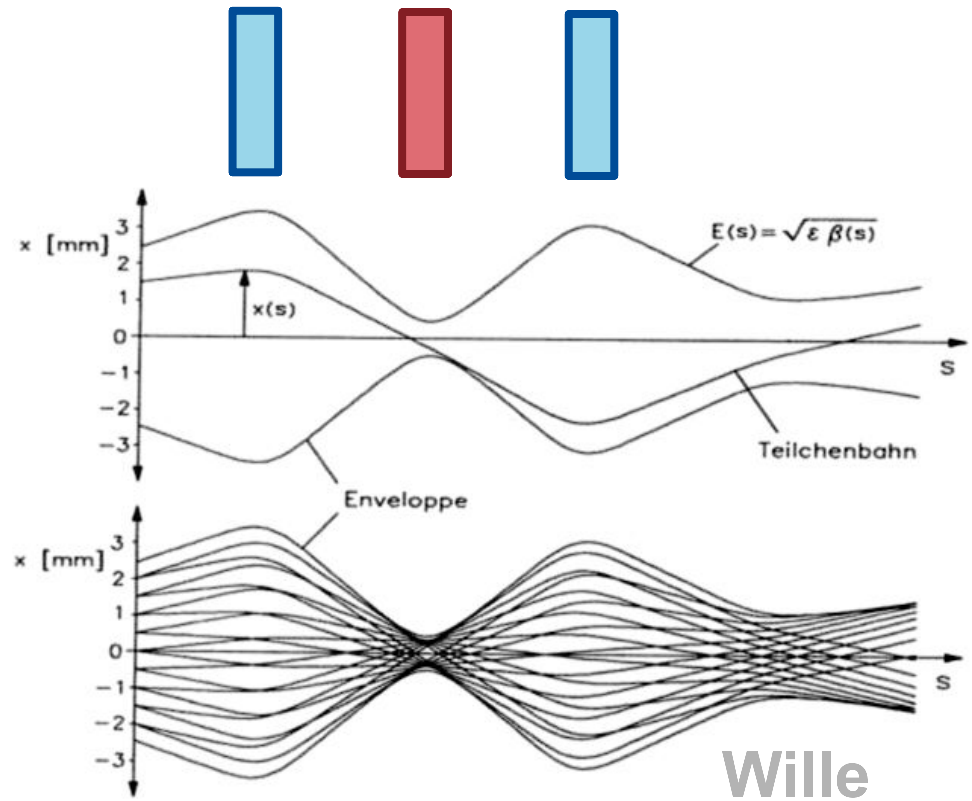
Parametric Resonance in Accelerators

Continuous Focusing:



Discrete focusing elements:



$\Omega \rightarrow \omega_{\beta_x}$ $\qquad$ $\nu \rightarrow \omega_{rev}$

Betatron tune = (number of transverse oscillations)/(revolution), $\dfrac{\Omega}{\nu} \rightarrow Q_x$
should not be half-integer (i.e. k/2) for this reason

Anecdote: Slip-stacking Perturbation

$$\ddot{\phi} = -\omega_s^2[\sin(\phi) + \sin(\phi)\cos(\omega_\phi t) - \cos(\phi)\sin(\omega_\phi t)].$$

Full Perturbative Solution:

$$\phi = \sum_{n=1}^{\infty} A_n \sin(n\omega_\phi t) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}\sin[m(1+\sigma)\omega_s t + n\omega_\phi t + m\psi] + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,-n}\sin[m(1+\sigma)\omega_s t - n\omega_\phi t + m\psi].$$

$B_{m,n}$ are of the order $\rho^m \alpha_s^{-2|n|}$

A_n are of the order $\alpha_s^{-2|n|}$

Lowest Order Non-Trivial Perturbation:

$$\phi = A_1\sin(\omega_\phi t) + A_2\sin(2\omega_\phi t) + \rho\sin[(1+\sigma)\omega_s t + \psi] + B_{1,1}\sin[(1+\sigma)\omega_s t + \omega_\phi t + \psi] + B_{1,-1}\sin[(1+\sigma)\omega_s t - \omega_\phi t + \psi],$$

$$A_1 = -\frac{1}{\alpha_s^2 - 1},$$

$$A_2 = \frac{1}{(2\alpha_s)^2 - 1}\left(\frac{A_1}{2}\right),$$

$$B_{1,\pm1} = \frac{\alpha_s^{-1}}{\alpha_s \pm 2}\left(\frac{\rho}{2}\right),$$

$$\sigma = \frac{3}{4}\alpha_s^{-4}.$$

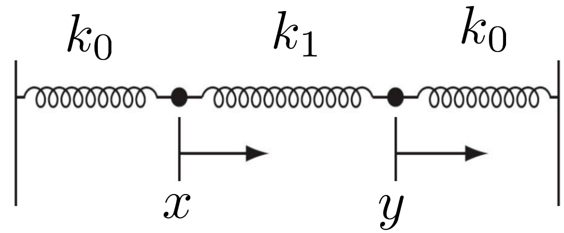
This was the main part of my first publication.

Coupled Oscillators

Coupled Oscillators

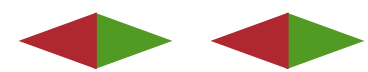
Two oscillators, with some coupling force between them.

$$\ddot{x} = -\omega_0^2 x - \omega_1^2 (x - y)$$
$$\ddot{y} = -\omega_0^2 y + \omega_1^2 (x - y)$$

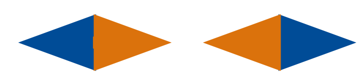


By taking linear combinations, we can find independent modes:

$$\frac{d^2}{dt^2}\left[\frac{x+y}{2}\right] = -\omega_0^2 \left[\frac{x+y}{2}\right]$$



$$\frac{d^2}{dt^2}\left[\frac{x-y}{2}\right] = -(\omega_0^2 + 2\omega_1^2)\left[\frac{x-y}{2}\right] = -\omega_-^2 \left[\frac{x-y}{2}\right]$$



Once we solve the mode, we can then transform back to the original:

$$x(t) = \left[\frac{x+y}{2}\right](t) + \left[\frac{x-y}{2}\right](t) = A_+ \cos\left(\omega_0 t + \phi_0\right) + A_- \cos\left(\omega_- t + \phi_-\right)$$

$$y(t) = \left[\frac{x+y}{2}\right](t) - \left[\frac{x-y}{2}\right](t) = A_+ \cos\left(\omega_0 t + \phi_0\right) - A_- \cos\left(\omega_- t + \phi_-\right)$$

Coupled Oscillators (cont.)

How to obtain these modes? Diagonalization of a matrix representation.

$$\ddot{x} = -\omega_0^2 x - \omega_1^2 (x - y)$$
$$\ddot{y} = -\omega_0^2 y + \omega_1^2 (x - y)$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \text{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Representation

$$\frac{d^2}{dt^2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(\omega_0^2 + \omega_1^2) & \omega_1^2 \\ \omega_1^2 & -(\omega_0^2 + \omega_1^2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalues

$$\det[M - \lambda I_2] = (\omega_0^2 + \omega_1^2 + \lambda)^2 - \omega_1^4 = 0$$
$$(\omega_0^2 + \lambda)(\omega_0^2 + 2\omega_1^2 + \lambda) = 0$$

Eigenvectors

$$\begin{bmatrix} -\omega_1^2 & \omega_1^2 \\ \omega_1^2 & -\omega_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \omega_1^2 & \omega_1^2 \\ \omega_1^2 & \omega_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ \text{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Coupled Oscillators (cont.)

How to obtain these modes? Diagonalization of a matrix representation.

Matrix Representation

$$\frac{d^2}{dt^2}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -(\omega_0^2+\omega_1^2) & \omega_1^2 \\ \omega_1^2 & -(\omega_0^2+\omega_1^2) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvectors

$$\begin{bmatrix} -\omega_1^2 & \omega_1^2 \\ \omega_1^2 & -\omega_1^2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \omega_1^2 & \omega_1^2 \\ \omega_1^2 & \omega_1^2 \end{bmatrix}\begin{bmatrix} 1 \\ \text{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Diagonalization

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \text{-1} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{d^2}{dt^2}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\omega_0^2 & 0 \\ 0 & -(\omega_0^2+2\omega_1^2) \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

Eigenvalues and Eigenvectors: Reminder

The eigenvalues λ and eigenvectors v of a matrix M fulfill:

$$M\vec{v_i} = \lambda_i \vec{v_i}$$

The eigenvalues are given by the n solutions to the equation:

$$\det[M - \lambda I_n] = 0$$

The eigenvector v_i for the ith eigenvalue λ_i is given by solving:

$$[M - \lambda_i I_n]\vec{v_i} = 0$$

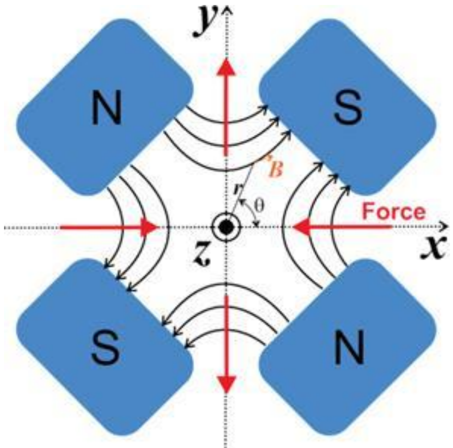
Usually, the eigenvectors are then normalized by:

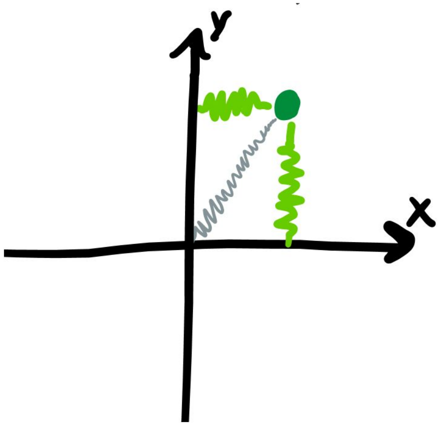
$$\vec{v_i} \to \frac{1}{|v_i|}\vec{v_i}$$

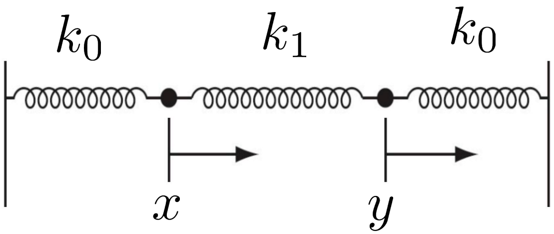
If there are n distinct eigenvalues there will be n orthogonal eigenvectors, but this is not necessarily guaranteed.

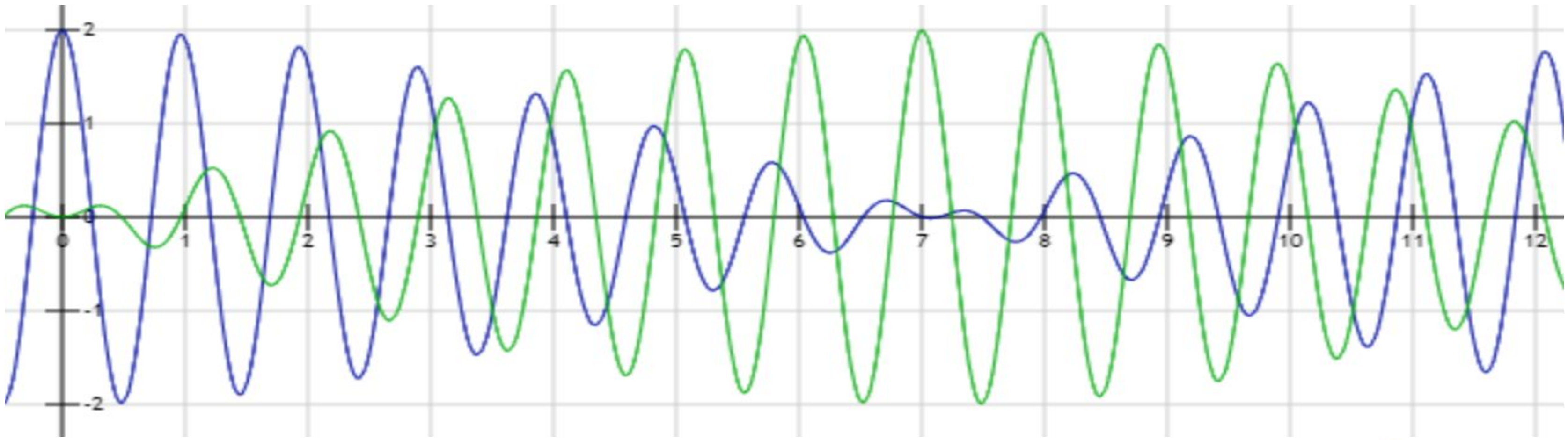
Coupled Oscillators in Accelerators

Most accelerators are formally designed with independent focusing in the horizontal and vertical plane. In practice, there is usually coupling.









Weakly Coupled Oscillators

Two oscillators, with some coupling force between them.

$$x(t) = A_+ \cos(\omega_0 t + \phi_0) + A_- \cos(\omega_- t + \phi_-)$$
$$y(t) = A_+ \cos(\omega_0 t + \phi_0) - A_- \cos(\omega_- t + \phi_-)$$

Let ω_1 be weak:

$$\omega_- = \omega_0 \sqrt{1 + 2\frac{\omega_1^2}{\omega_0^2}} \approx \omega_0 + \omega_1 \frac{\omega_1}{\omega_0} = \omega_0 + \epsilon$$

Now consider beating:

$$A_+ = A_-, \quad \phi_0 = 0, \quad \phi_- = 0$$

$$x(t) = A_0 \left[\cos(\omega_0 t) + \cos(\omega_0 t + \epsilon t)\right] \approx A_0 \left(2 - \frac{\epsilon^2 t^2}{2}\right) \cos(\omega_0 t) - A_0 \epsilon t \sin(\omega_0 t)$$

$$y(t) = A_0 \left[\cos(\omega_0 t) - \cos(\omega_0 t + \epsilon t)\right] \approx A_0 \left(\frac{\epsilon^2 t^2}{2}\right) \cos(\omega_0 t) + A_0 \epsilon t \sin(\omega_0 t)$$

Summary

- Oscillators are an important building block of more complicated systems.

- We learned how to solve damped oscillator equations,
    - which are also called 2nd order linear differential equations

- The homogeneous equation has no external driving force.
    - The solution to the homogeneous equation is called complementary.
    - The complementary solution is always the same.

- The inhomogeneous equation has an external driving force.
    - The solution to the inhomogeneous equation is called particular.
    - The form of the particular solution often matches the driving force.
    - The full solution is the complementary + the particular.

- Using perturbation theory, we can learn more about nonlinear systems by approximating the solution with a series of individual oscillators.

- Using eigendecompositon, we can solve several linearly coupled oscillators by representing them as oscillating modes.