1. Proof by Induction

We start with the following definitions

$$\phi = \frac{1+\sqrt{5}}{2} \tag{1}$$

$$\psi = \frac{1-\sqrt{5}}{2} \tag{2}$$

and are asked to prove that

$$\text{fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}} \tag{3}$$

For a proof by induction we must first prove that the trivial case is true, this is a trivial task. Next we must show that if we have n, that $n+1$ is guaranteed to work. We will show this by establishing that

$$\text{fib}(n+1) = \text{fib}(n) + \text{fib}(n-1) \tag{4}$$

The binomial expansion states that

$$(1+a) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k \tag{5}$$

We rewrite ϕ and ψ accordingly

$$(2\phi)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k \tag{6}$$

$$(2\psi)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-1)^k a^k \tag{7}$$

where $a = \sqrt{5}$, to ease reading.

The difference between ϕ and ψ are the terms that are negative in ψ.

$$\phi^n - \psi^n = \left(\frac{1}{2}\right)^n \sum_{k=1,odd}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k \tag{8}$$

We can then redefine our fib function as

$$\text{fib}(n) = \left(\frac{1}{2}\right)^n \sum_{k=1,odd}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{k-1} \tag{9}$$

Now with the combination identity of

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} \tag{10}$$

It is trivial now to establish that the case of n and $n-1$ will sum to the case for $n+1$. This is a property of the binomial theorem.