Introduction to Artificial Gauge Fields for Neutral Atoms

PHY 2203 2019 Seminar Notes

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"To study material systems, theorists create 'spherical cow' models of real materials, whereas in cold atom physics experimentalists can actually make spherical cows."
— Ref. [1]

1 Introduction and Motivation

Disclaimer: if you want a more thorough and less prone-to-error review, see Ref. [2]. I drew the most inspiration from there though I tried to add material from the other papers cited. In my opinion, some details are easier to understand in the original papers (cited in Ref. [2]), though they will almost always require some understanding of second quantisation. Here I avoid second quantisation and focus on two-level systems to keep with the theme of the class.

Artificial (or synthetic) gauge fields are a way of engineering Hamiltonians (in our context, using ultracold atomic systems) so that we can study other systems indirectly. In this way, we view it as a powerful platform for quantum simulation. That is, we seek to "engineer" a Hamiltonian in a cloud of cold atoms such that it looks like the Hamiltonian of another system of interest (note: this is not simulation in the sense as being on a computer; our simulator is a cloud of atoms). It is also interesting in its own right, as simulating magnetic fields with neutral atoms is not an easy task. Thanks to artificial gauge fields, we can indirectly study models used in other fields, from many-body or condensed matter systems to particle physics.

In this seminar I'll briefly outline the key ideas of an artificial gauge field, limiting myself to topics and ideas that we have already covered. In order to avoid going too far outside the scope of the course, I shall not talk too much about fancy applications, which are bountiful. Instead, I'll give one application which I personally found the most interesting and luckily does not require any deep understanding of another field of study.[1] This presentation is by no means complete, of course, but is instead an introduction for students in PHY 2203.

As in the presentation, I will start with perhaps the easiest-to-understand means of constructing an artificial gauge field (rotating the gas), then move on to a more sophisticated technique of adiabatically following dressed states, before concluding with an application: simulating pseudo-spin-1/2 Bosons.

[1] For example, I am *not* going to discuss the extremely rich connection between artificial gauge fields and topological quantum matter, because although a huge field, I feel in order to really appreciate it I would have to talk about topological insulators at some depth. If this stuff interests you, learning about how to use artificial gauge fields for spin-orbit coupling seems to be a good springboard to get started.

Also, just note that while I try to be careful with notation when I first introduce a concept, I tend to get a little sloppy and drop things like hats and vectors – it should be obvious what's what. Sorry!

2 Rotation as Simulation

Perhaps the easiest form of an artificial gauge field to understand is a rotating gas. While most of the interesting applications (at least to my knowledge) do *not* come from this form of simulation, it is a rather satisfying approach insofar that is it very simple (conceptually) and captures the basic essence of what we are trying to do throughout this seminar. An elementary motivation may be taken as noticing a similarity between the Lorentz force for a charged particle in a magnetic field $\vec{F}_{\text{Lorentz}} = q\vec{v} \times \vec{B}$ and the Coriolis force $\vec{F}_{\text{Coriolis}} = 2m\vec{v} \times \vec{\omega}_F$ where $\vec{\omega}_F$ is the rotation vector for the frame. For a more in-depth discussion on the Coriolis force, you can consult most classical mechanics textbooks.[2] A qualitative picture can be seen in figure 1.

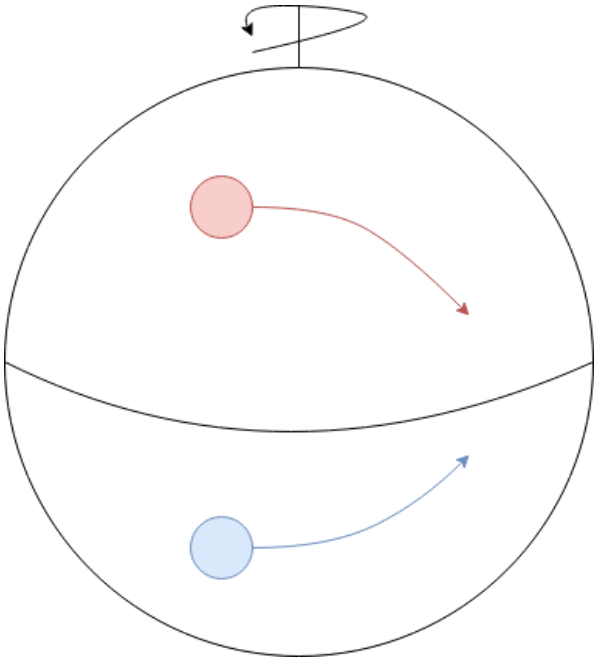

Figure 1: Basic illustration of the Coriolis force. The Earth (depicted as the sphere) is rotating with some angular momentum (depicted with the arrow at the top). The two spheres are collecions of gas particles as they are deflected by the Coriolis force while travelling from an area of higher pressure to lower pressure. This can all be deduced by looking at the form of the equation $\vec{F}_{\text{Coriolis}} = 2m\vec{v} \times \vec{\omega}_F$. Image (and idea) basically taken from http://ww2010.atmos.uiuc.edu/%28Gh%29/guides/mtr/fw/crls.rxml though I recreated the image.

If we consider the Hamiltonian in the rotating frame[3]

$$\hat{H} \to \frac{\hat{p}^2}{2m} + V(\hat{r}) - \omega_F \hat{J}_z \tag{1}$$

where we made the transformation

$$|\psi\rangle \to e^{-i\omega_F \hat{J}_z t/\hbar} |\psi\rangle \tag{2}$$

[2]For example, see Landau and Lifshitz *Mechanics* section 39.
[3]If you wish to see a derivation, see lecture 7.

of the ket to the rotating frame. J_z is our total angular momentum operator.

If we make the not-so-obvious substitution $\hat{\vec{\mathcal{A}}} \cdot \hat{\vec{p}} = m\omega_F \hat{J}_z$ (where this is how we define our newly introduced value $\hat{\vec{\mathcal{A}}}$) then we can "massage" our Hamiltonian to get

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} - \omega_F \hat{J}_z + V(\hat{r}) \\ &= \frac{\hat{p}^2}{2m} - \frac{1}{m}\hat{\vec{p}} \cdot \hat{\vec{\mathcal{A}}} + V(\hat{r}) \\ &= \frac{\hat{p}^2 - 2\hat{\vec{p}} \cdot \hat{\vec{\mathcal{A}}} + \hat{\mathcal{A}}^2}{2m} - \frac{\hat{\mathcal{A}}^2}{2m} + V(\hat{r}) \\ &= \frac{(\hat{p} - \hat{\vec{\mathcal{A}}})^2}{2m} - \frac{\hat{\mathcal{A}}^2}{2m} + V(\hat{r}). \end{aligned} \tag{3}$$

This invites us to think of $\hat{\vec{\mathcal{A}}}$ in this equation as as $q\hat{\vec{A}}$, where $\hat{\vec{A}}$ is the vector potential, as the above expression is then very similar to the Hamiltonian for a charged particle in a magnetic field. This $\hat{\vec{\mathcal{A}}}$ is our gauge field.

To actually use equation 3 as a "simulation" for a charged particle in a magnetic field, we need to get rid of the second term, and do something with V. To start, lets assume the potential V, which just came along for the ride throughout equation 3, somehow restricts us to the xy-plane. With this adjustment, we have

$$\hat{J}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = (\hat{x}\vec{u}_y - \hat{y}\vec{u}_x) \cdot \hat{\vec{p}} \tag{4}$$

where $\vec{u}_x$ and $\vec{u}_y$ are unit vector in the x and y directions, respectively.

With this substitution (and $\mathcal{A} = qA$), we can see the extra term in equation 3 becomes

$$-\frac{q^2A^2}{2m} = \frac{m^2\omega_F^2}{m}(xu_y - yu_x)^2 = m\omega_F^2 r^2. \tag{5}$$

Finally, this gives us an illuminating form in which to write our Hamiltonian,

$$\hat{H} = \frac{(\hat{p} - qA(\hat{r}))^2}{2m} - \frac{1}{2}\omega_F^2 r^2 + V(r) \tag{6}$$

where the second term is exactly the centrifugal potential we expect to see in the rotating frame. Now, we wish to only keep the first term, as this would give us a quantum simulation for a particle in a magnetic field. Thus, we also require $V(r)$ to cancel the centrifugal potential.

While this is possible, it is very difficult in practice ($V(r)$ must not only restrict the particles to a plane, but it must also cancel the centrifugal potential). However, it has been done successfully in experiment. One such (slightly exotic) example discussed in lecture was the emergence of vortices in a rotating Bose-Einstein condensate (Ref. [3]).

3 Simulation by Adiabatically Following Dressed States

Consider a system described by some set of external parameters $\vec{\lambda}(t)$ (which are, in turn, functions of time). Lets then make a closed loop in parameter space, slowly. That is, let

$$\vec{\lambda}(0) \to \vec{\lambda}(t) \to \vec{\lambda}(T) \equiv \vec{\lambda}(0) \tag{7}$$

where T is the period, letting this evolution be sufficiently slow that we can use the adiabatic theorem. Naively, it is tempting to think that the system will be the same at times $t = 0$ and $t = T$ since $\vec{\lambda}$ will

be the same. However, this is not true. An archetypal example from classical mechanics is the Foucault pendulum. In that case, T represents one full day (the suspension point plays the role of the external parameter; it is back to its original spot after a full rotation of the Earth). Instead of being in the same plane of oscillation, though, the pendulum has rotated by an angle that depends on the latitude.

3.1 Berry Connection and Berry Curvature

To start talking about artificial gauge fields by adiabatic following of dressed states, it pays to first learn about the Berry phase and Berry curvature.

Consider a Hamiltonian defined by some set of external parameters $\hat{H}(\lambda)$, and suppose we have a complete basis for the Hilbert space $\{|\psi_n\rangle\}$ where

$$\hat{H}(\lambda)\,|\psi_n(\lambda)\rangle = E_n(\lambda)\,|\psi_n(\lambda)\rangle \tag{8}$$

Now consider the total state $|\Psi\rangle$ and expand it in terms of the eigenstates

$$|\Psi(t)\rangle = \sum_n c_n(t)\,|\psi_n(\lambda(t))\rangle\,. \tag{9}$$

If $|\Psi(t=0)\rangle = c_l(0)\,|\psi_l(\lambda(t=0))\rangle$ then if we evolve sufficiently slowly, $|\Psi(t)\rangle \propto |\psi_l(\lambda(t))\rangle$ for all time by the adiabatic theorem. Call the coefficient $c_l(t)$; i.e. $|\Psi\rangle = c_l(t)\,|\psi_l(\lambda(t))\rangle$. Then

$$\hat{H}\,|\Psi\rangle = c_l E_l\,|\psi_l\rangle \tag{10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Psi\rangle = \dot{c}_l\,|\psi_l\rangle + \dot{\lambda}\,|\nabla\psi_l\rangle\,. \tag{11}$$

Thus, the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\Psi\rangle = \hat{H}\,|\Psi\rangle \tag{12}$$

in this case becomes simply

$$i\hbar\dot{c}_l(t)\,|\psi_l(\lambda(t))\rangle + \dot{\lambda}(t)\left|\vec{\nabla}\psi_l(\lambda(t))\right\rangle = E_l c_l(t)\,|\psi_l(\lambda(t))\rangle \tag{13}$$

which can be simplified by multiplying on the left by $\langle\psi_l(\lambda(t))|$ and isolating for $\dot{c}_l$:

$$i\hbar\dot{c}_l = [E_l(t) - \dot{\lambda}\underbrace{i\hbar\,\langle\psi_l|\nabla\psi_l\rangle}_{\equiv\mathcal{A}_l(\lambda)}]c_l \tag{14}$$

where I have defined the Berry connection $\vec{\mathcal{A}}(\lambda(t))$. Here we also see the emergence of a gauge-dependent quantum ($\mathcal{A}$) coupled in an equation of motion. Indeed, we'll be engineering this quantity in the following sections.

Equation 14 is a simple first-order ordinary differential equation whose answer can readily be deduced by direct integration. Supposing, as before, $\lambda(T) = \lambda(0)$ (making a closed loop in parameter space), then

$$c_l(T) = e^{i\Phi_{\mathrm{dynamical}}(T)}e^{i\Phi_{\mathrm{Berry}}(T)}c_l(0) \tag{15}$$

where $\Phi_{\mathrm{dynamical}}$ is the familiar term that shows up in time-dependent problems

$$\Phi_{\mathrm{dynamical}}(T) = -\frac{1}{\hbar}\int_0^T \mathrm{d}t E_l(t) \tag{16}$$

but the newly introduced Berry (or geometric) phase is

$$\Phi_{\text{Berry}} = \frac{1}{\hbar}\int_0^T \mathrm{d}t\dot{\lambda}\cdot\vec{\mathcal{A}}_l(\lambda). \tag{17}$$

This new phase clearly does not actually depend on time, but instead only the trajectory, because it can be rewritten as a line integral over the closed path in parameter space, which in turn can be rewritten as a surface integral over a surface delimited by the closed loop. That is (using basic vector calculus)

$$\Phi_{\text{Berry}} = \frac{1}{\hbar}\oint \mathrm{d}\lambda\cdot\vec{\mathcal{A}}_l(\lambda) = \frac{1}{\hbar}\iint \mathrm{d}^2S\cdot\boldsymbol{\nabla}\times\vec{\mathcal{A}}_l. \tag{18}$$

Now we can introduce the gauge-independent quantity $\vec{\mathcal{B}}_l \equiv \boldsymbol{\nabla}\times\vec{\mathcal{A}}_l$, called the Berry curvature. This will essentially play the role of our magnetic field.

3.2 General Approach

Consider now an atom interacting with light under the usual rotating wave approximation.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{H}_{\text{int}}(\hat{\vec{r}}) \tag{19}$$

where for a dressed state $|\psi_n\rangle$ we have

$$\hat{H}_{\text{int}}(\hat{\vec{r}})|\psi_n(\vec{r})\rangle = E_n(\vec{r})\,|\psi_n(\vec{r})\rangle\,. \tag{20}$$

We have

$$|\Psi(\vec{r},t)\rangle = \sum_n c_n(\vec{r},t)\,|\psi_n(\vec{r})\rangle \tag{21}$$

as the set of dressed states forms a basis set for the Hilbert space associated with $\hat{H}_{\text{int}}$.

Now lets take $|\Psi(r,t=0)\rangle = |\psi_l(r)\rangle$ and evolve sufficiently slow such that the adiabatic theorem can be applied. So, $c_n(t=0)=0$ for all $n\neq l$, and $|\Psi\rangle$ stays proportional to $|\psi_l\rangle$.

As before, we want to use the Schrödinger equation and massage the result to include a term like $\frac{(p-\mathcal{A}_l)^2}{2m}$. We multiply on the left of the Schrödinger equation (on both sides) with $\langle\psi_l|$. So, on the left-hand side we have

$$\langle\psi_l|\,i\hbar\frac{\partial}{\partial t}\,|\Psi\rangle = i\hbar\,\langle\psi_l|\,(\dot{c}_l\,|\psi_l\rangle) = i\hbar\dot{c}_l \tag{22}$$

while on the right-hand side we have, after a lot of work (recall $p = -i\hbar\vec{\nabla}$ and $\mathcal{A}_l = i\hbar \langle \psi_l | \nabla \psi_l \rangle$)

$$\begin{aligned}
\langle \psi_l | \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + \hat{H}_{\text{int}} \right) |\Psi\rangle &= \langle \psi_l | \left(\frac{p^2}{2m} + \hat{H}_{\text{int}} \right) c_l |\psi_l\rangle \\
&= \frac{1}{2m} \langle \psi_l | \, p^2 (c_l |\psi_l\rangle) + E_l c_l \\
&= \frac{1}{2m} \langle \psi_l | \left(c_l (p^2 |\psi_l\rangle) + 2(pc_l)(p|\psi_l\rangle) + (p^2 c_l)|\psi_l\rangle \right) + E_l c_l \\
&= \frac{1}{2m} \langle \psi_l | \, c_l (p^2 |\psi_l\rangle) + \frac{1}{2m} \langle \psi_l | \, 2(pc_l)(p|\psi_l\rangle) + \frac{1}{2m} \langle \psi_l | \, (p^2 c_l)|\psi_l\rangle + E_l c_l \\
&= \frac{c_l}{2m} \langle \psi_l | \, p^2 |\psi_l\rangle + \frac{2pc_l}{2m} \langle \psi_l | \, p |\psi_l\rangle + \frac{p^2 c_l}{2m} \langle \psi_l | \psi_l \rangle + E_l c_l \\
&= \frac{1}{2m} \left[\langle \psi_l | \, p^2 |\psi_l\rangle + 2p \langle \psi_l | \, p |\psi_l\rangle + p^2 \right] c_l + E_l c_l \\
&= \frac{1}{2m} \left[\sum_n \langle \psi_l | \, p |\psi_n\rangle \langle \psi_n | \, p |\psi_l\rangle - i\hbar 2p \langle \psi_l | \nabla \psi_l \rangle + p^2 \right] c_l + E_l c_l \\
&= \frac{1}{2m} \left[\sum_n \langle \psi_l | \, (-i\hbar) |\nabla \psi_n\rangle \langle \nabla \psi_n | \, (i\hbar) |\psi_l\rangle - 2p \cdot \mathcal{A}_l + p^2 \right] c_l + E_l c_l \\
&= \frac{1}{2m} \left[\hbar^2 \sum_{n \neq l} \langle \psi_l | \nabla \psi_n \rangle \langle \nabla \psi_n | \psi_l \rangle + \langle \psi_l | \, i\hbar |\nabla \psi_l\rangle \langle \psi_l | \, (i\hbar) |\nabla \psi_l\rangle - 2p \cdot \mathcal{A}_l + p^2 \right] c_l + E_l c_l \\
&= \frac{1}{2m} \left[\hbar^2 \sum_{n \neq l} | \langle \nabla \psi_l | \psi_n \rangle |^2 + \mathcal{A}_l^2 - 2p \cdot \mathcal{A}_l + p^2 \right] c_l + E_l c_l \\
&= \left[\frac{(p - \mathcal{A}_l)^2}{2m} + E_l + \frac{\hbar^2}{2m} \sum_{n \neq l} | \langle \nabla \psi_l | \psi_n \rangle |^2 \right] c_l
\end{aligned} \tag{23}$$

Putting these two together we get[4]

$$i\hbar \frac{\partial c_l}{\partial t}(\vec{r}, t) = \left[\frac{(p - \mathcal{A}_l(\vec{r}))^2}{2m} + E_l(\vec{r}) + \frac{\hbar^2}{2m} \sum_{n \neq l} | \langle \nabla \psi_l | \psi_n \rangle |^2 \right] c_l(r, t) \tag{24}$$

where the last term in the brackets can be understand to represent the kinetic energy associated to the motion of the atom as it makes virtual transitions between $|\psi_l\rangle$ and all other dressed states. We see that this has the form that we want, i.e. this is exactly the form of the Schrödinger equation for a particle in a magnetic field with associated vector potential $\mathcal{A}_l/q$ where q is the charge of the particle. The effective/simulated magnetic field comes from the Berry curvature $\mathcal{B} = \nabla \times \mathcal{A}$, which we will want to make sure is nonzero.

3.3 Two-Level Case

Now we specialise the discussion of the previous section to something more tractable and familiar to us: a two-level system. We could use a familiar two-level system like in figure 2, for example. Alternatively, we could use an effective two-level system like in figure 3, which involves a lambda scheme but with the excited/intermediate state off resonance. While not central to this discussion, it will be necessary in the next section, so I introduce it now.

[4] In my presentation, I accidentally cut off the last c_l on the right-hand side.

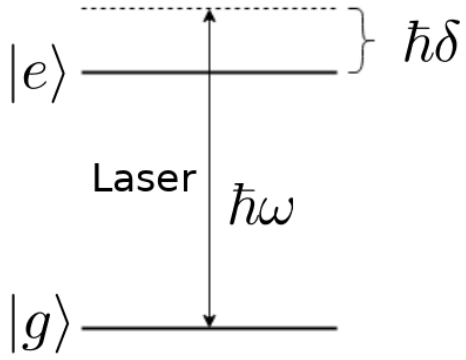

Figure 2: The usual two-level atom approximation, with detuning δ.

For the scheme in figure 3, we have an effective Rabi frequency from a 2-photon process,

$$\Omega_R^{\text{eff}} = \frac{\Omega_R^a \Omega_R^{b*}}{2\delta_e}. \tag{25}$$

For a (brief) discussion of where this comes from, see appendix A. Now either way, we can proceed with some Rabi frequency (the rest of the discussion assumes either the familiar Rabi frequency in the case of figure 2 of this effective Rabi frequency in the case of figure 3).

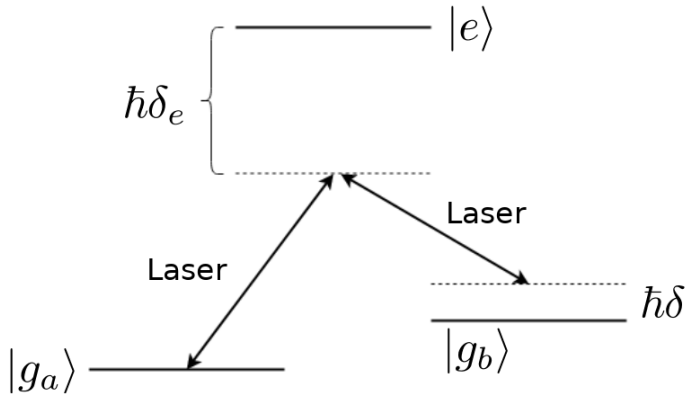

Figure 3: A lambda scheme with the intermediate level off-resonance. This is effectively a two-level system between states $|g_a\rangle$ and $|g_b\rangle$, and will be important in the next section.

In the basis of the two levels between which the atom oscillates (e.g. $\{|g\rangle, |e\rangle\}$), we can write the Hamiltonian $\hat{H}_{\text{int}}$ as a 2 by 2 matrix (see, for example, lecture 15).

$$H_{\text{int}} = \frac{\hbar}{2}\begin{bmatrix} \delta & \Omega_R^* \\ \Omega_R & -\delta \end{bmatrix}. \tag{26}$$

This should look familiar, as it is like the Hamiltonian discussed in question 3 of homework 3.

Make the substitutions $\Omega = \sqrt{\delta^2 + |\Omega_R|^2}$, $\cos\theta = \frac{\delta}{\Omega}$, $\sin\theta = \frac{|\Omega_R|}{\Omega}$, and $\Omega_R = |\Omega_R|e^{i\varphi}$. Then

$$\begin{aligned}\hat{H}_{\text{int}} &= \frac{\hbar}{2}\begin{bmatrix}\Omega\cos\theta & |\Omega_R|e^{-i\varphi}\\ |\Omega_R|e^{i\varphi} & -\cos\theta\end{bmatrix}\\ &= \frac{\hbar}{2}\begin{bmatrix}\Omega\cos\theta & \Omega\sin\theta e^{-i\varphi}\\ \Omega\sin\theta e^{i\varphi} & -\Omega\cos\theta\end{bmatrix}\\ &= \frac{\Omega\hbar}{2}\begin{bmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\end{bmatrix}.\end{aligned} \tag{27}$$

We have $\operatorname{Tr}\hat{H}_{\text{int}} = 0$ so eigenvalues $\lambda_1 + \lambda_2 = 0$. However,

$$\det\hat{H}_{\text{int}} = \frac{\Omega\hbar}{2}(-\cos^2\theta - \sin^2\theta) = -\frac{\Omega\hbar}{2} = \lambda_1\lambda_2 \tag{28}$$

Thus, the eigenvalues are $\pm\Omega\hbar/2$. Using angle-sum formulae it is also clear that the eigenstates are [5]

$$|\psi_+\rangle = \begin{bmatrix}\cos(\theta/2)\\ e^{i\varphi}\sin(\theta/2)\end{bmatrix}, |\psi_-\rangle = \begin{bmatrix}-e^{-i\varphi}\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix}. \tag{29}$$

These eigenstates are our dressed states (eigenstates of $\hat{H}_{\text{int}}$). Next suppose we follow one of these dressed states adiabatically. Now, in order to have this successfully simulate anything meaningful, we need to calculate the Berry curvature. In turn, this requires us to calculate the Berry connection. By the previous subsection, if we have $\mathcal{B} \neq 0$ we are done! For simplicity, suppose we follow $|\psi_+\rangle$. We have Berry connection

$$\begin{aligned}\mathcal{A}_+ &= i\hbar\langle\psi_+|\nabla\psi_+\rangle\\ &= i\hbar\begin{bmatrix}\cos(\theta/2)\\ e^{-i\varphi}\sin(\theta/2)\end{bmatrix}\cdot\begin{bmatrix}-(\nabla\theta/2)\sin(\theta/2)\\ i\nabla\varphi e^{i\varphi}\sin(\theta/2) + e^{i\varphi}(\nabla\theta/2)\cos(\theta/2)\end{bmatrix}\\ &= i\hbar(-\cos(\theta/2)(\nabla\theta/2)\sin(\theta/2) + e^{-i\varphi}\sin(\theta/2)(i\nabla\varphi e^{i\varphi}\sin(\theta/2) + e^{i\varphi}(\nabla\theta/2)\cos(\theta/2)))\\ &= i\hbar(-\cos(\theta/2)\sin(\theta/2)(\nabla\theta/2) + i\nabla\varphi\sin^2(\theta/2) + (\nabla\theta/2)\sin(\theta/2)\cos(\theta/2))\\ &= i\hbar(i\nabla\varphi\sin^2(\theta/2))\\ &= -\hbar\nabla\varphi\frac{1}{2}(1-\cos\theta)\\ &= \frac{\hbar}{2}(\cos\theta - 1)\nabla\varphi.\end{aligned} \tag{30}$$

and corresponding Berry curvature

$$\begin{aligned}\mathcal{B}_+ &= \nabla\times\mathcal{A}_+\\ &= \frac{\hbar}{2}\nabla\cos\theta\times\nabla\varphi\end{aligned} \tag{31}$$

where I have used the identity $\nabla\times(f\vec{F}) = \nabla f\times\vec{F} + f\nabla\times\vec{F}$ where $f = \cos\theta - 1$ and $\vec{F} = \nabla\varphi$ and the second term gives zero since $\nabla\times\nabla f = 0$ for any function f.

This final expression for $\mathcal{B}$ is not zero in general, so we are done, having successfully simulated a magnetic field by adiabatically following dressed states. Of course, this would require that either the gradient of the intensity or the gradient of the detuning be nonvanishing.

[5] There is a typo in my seminar slides. Slide 14, the second eigenstate should have $\sin(\theta/2)$ not $\sin(\varphi/2)$.

3.4 Validity of Adiabaticity

Note: I did not discuss this in my presentation, as it wasn't central to the discussion, but I thought it should be mentioned here at least. Adiabaticity is valid when the angular velocity of the dressed state we are following is much smaller than the generalised Rabi frequency. The energy corresponding to this angular velocity of the dressed state must at least be on the order of the "recoil energy" felt from the photon(s), $\hbar^2k^2/2m$. Thus, in order for the adiabatic theorem to hold, we need

$$\frac{\hbar^2k^2}{2m} \ll \hbar\Omega. \tag{32}$$

4 Application: Spin-1/2 Bosons

Now we look at an interesting application of artificial gauge fields, outside of what we have already been discussing (we generalise away from only simulating magnetic fields). The review Ref. [2] discusses this as spin-orbit coupling, but this coupling only really emerges in an obvious way in the higher dimensional cases (which I do not discuss – they require 3-photon processes). Ref. [4] discusses the same mechanism and calls it (pseudo-)spin-1/2 bosons, which has a more dramatic flair. Regardless, this is a good starting point to study more advanced forms of quantum simulation, especially if we are interesting in topological insulators.

Consider a one-dimensional gas of Bosons interacting with two lasers propagating with equal and opposite wave vectors $\pm k$. That is, the oscillations in figure 3 between the $|g_i\rangle$ states and the intermediate state $|e\rangle$ are driven by photons carrying momenta $\pm\hbar k$. By figure 3 (and appendix A) we have a two-level system. We can think of $|g_a\rangle$ as $|\uparrow\rangle$ and $|g_b\rangle$ as $|\downarrow\rangle$ (preluding our spin-1/2 idea). Now introduce states labelled by the momenta: $|\downarrow, p-\hbar k\rangle$ and $|\uparrow, p+\hbar k\rangle$. This allows us to write the Hamiltonian in this basis,

$$\hat{H}(p) = \begin{bmatrix} \frac{(p-\hbar k)^2}{2m} + \frac{\hbar\delta}{2} & \frac{\hbar\Omega_R}{2} \\ \frac{\hbar\Omega_R}{2} & \frac{(p+\hbar k)^2}{2m} - \frac{\hbar\delta}{2} \end{bmatrix}. \tag{33}$$

The Pauli matrices along with identity form an orthogonal basis for the complex Hilbert space of 2 by 2 matrices. Therefore, we can rewrite this Hamiltonian in terms of Pauli matrices (and identity). It doesn't take a lot of effort to see

$$\hat{H}(p) = \begin{bmatrix} \frac{(p-\hbar k)^2}{2m} & 0 \\ 0 & \frac{(p+\hbar k)^2}{2m} \end{bmatrix} + \frac{\hbar\delta}{2}\sigma_z + \frac{\hbar\Omega_R}{2}\sigma_x \tag{34}$$

but what about the first, diagonal matrix? Insides the brackets, it looks sort of like a σ_z. Indeed, take our gauge field to be $\mathcal{A} \equiv \hbar k\sigma_z$. Then

$$\begin{bmatrix} \frac{(p-\hbar k)^2}{2m} & 0 \\ 0 & \frac{(p+\hbar k)^2}{2m} \end{bmatrix} = \frac{1}{2m}(p-\mathcal{A})^2 \tag{35}$$

and more completely,

$$\hat{H}(p) = \frac{(p-\mathcal{A})^2}{2m} + \frac{\hbar\delta}{2}\hat{\sigma}_z + \frac{\hbar}{2}\Omega_R\hat{\sigma}_x. \tag{36}$$

We also see there the connection with artificial gauge fields, with the emergence of $\mathcal{A}$.

Next, lets take the resonant case, $\delta = 0$, just for simplicity (not really necessary). Also expand the square. Then

$$\hat{H} = \frac{1}{2m}p^2 - \frac{2p\hbar k}{2m}p\sigma_z + \frac{\hbar^2k^2}{2m} + \frac{\hbar\Omega_R}{2}\sigma_x = (\frac{\hbar^2k^2}{2m} + \frac{1}{2m}p^2)\mathbb{1} + \frac{\hbar\Omega_R}{2}\sigma_x - \frac{p\hbar k}{m}\sigma_z. \tag{37}$$

When diagonalising, the $\mathbb{1}$ term just offsets the energy (eigenvalue). Now let $\tilde{H} = \frac{\hbar\Omega_R}{2}\sigma_x - \frac{p\hbar k}{m}\sigma_z$. Then if $|n\rangle$ is an eigenstate with energy E_n then $\tilde{H}|n\rangle = E_n|n\rangle$. Operating again we get $\tilde{H}^2|n\rangle = \tilde{H}E_n|n\rangle = E_n\tilde{H}|n\rangle = E_n^2|n\rangle$. Now, using the fact that the Pauli matrices commute so the cross terms in the square cancel (and the square of a Pauli matrix is identity), we can readily deduce

$$\tilde{H}^2 = (\frac{\hbar\Omega_R}{2})^2 - (\frac{p\hbar k}{m})^2 \tag{38}$$

which tells us

$$E_n = \sqrt{(\frac{\hbar\Omega_R}{2})^2 - (\frac{p\hbar k}{m})^2}. \tag{39}$$

Putting this together with the identity term, we get energy eigenvalues for our original Hamiltonian,

$$E_\pm(p) = \frac{p^2}{2m} + \frac{\hbar^2k^2}{2m} \pm \frac{\hbar}{2}\sqrt{\Omega_R^2 + 4\frac{k^2p^2}{m^2}} \tag{40}$$

which could be a degenerate ground state! A plot of these energy values as a function of $p/\hbar k$ is shown in figure 4.

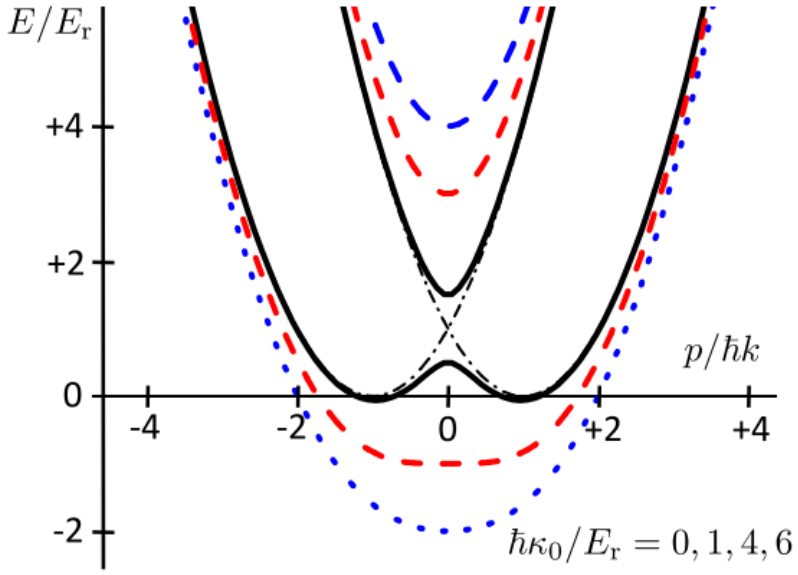

Figure 4: Image taken from Ref. [2]. Ignore the notation, as it is different from ours. The different curves represent the eigenspectrum (E_+ above, E_- below) for different choices of Ω_R. When $\Omega_R < \frac{2\hbar k^2}{m}$, as discussed in the body of the text, we get two degenerate minima and a local maximum at $p = 0$. This is shown in the black curve. As we change Ω_R, we get different curves: the red one is when $\Omega_R = \frac{2\hbar k^2}{m}$ and the blue one when $\Omega_R > \frac{2\hbar k^2}{m}$ which are not interesting for our purposes.

This is unexpected for Bosons, and leads to unique physics at temperatures low enough for Bose-Einstein condensation. But when exactly does a degenerate ground state occur? Well,

$$E_-(p) = \frac{p^2}{2m} + \frac{\hbar^2k^2}{2m} - \frac{\hbar}{2}\sqrt{\Omega_R^2 + 4\frac{k^2p^2}{m^2}} \tag{41}$$

is obviously the ground state. Now find the locations of the extrema by taking the derivative with respect to p and setting it equal to zero:

$$0 = \frac{\partial}{\partial p}E_- = \frac{p}{m} - \frac{\hbar}{4}\left[\Omega_R^2 + 4\frac{k^2p^2}{m^2}\right]^{-1/2} \cdot 8\frac{k^2}{m^2}p \tag{42}$$

which implies

$$\frac{p}{m}\left[\Omega_R^2 + 4\frac{k^2p^2}{m^2}\right]^{1/2} = \frac{2\hbar k^2}{m^2}p. \tag{43}$$

It is clear that $p = 0$ is a solution to this equation, and indeed this is a minima for certain choices of Ω_R as can be seen in figure 4, but this is the case that does not interest us. Instead $p \neq 0$ and lets see if we can find another extremum. This would imply

$$\left[\Omega_R^2 + 4\frac{k^2p^2}{m^2}\right]^{1/2} = 2\hbar\frac{k^2}{m} \tag{44}$$

or

$$\Omega_R^2 + 4\frac{k^2p^2}{m^2} = \frac{4\hbar^2k^4}{m^2} \tag{45}$$

which gives the solution

$$p = \sqrt{\hbar^2k^2 - \frac{m^2\Omega_R^2}{4k^2}}. \tag{46}$$

It is clear that this solution only exists (i.e. is real) when $\Omega_R \leq \frac{2\hbar k^2}{m}$ (technically, we should discard the $=$ case and have only $<$ since this would just give zero again). This, then, is our condition for a degenerate ground state (under this condition $p = 0$ becomes a local maximum).[6] Thus, we have successfully simulated a Bose-Einstein condensate that has degenerate ground state, which is something that does not normally exist in nature – this is an illustration of how powerful artificial gauge fields are.

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[6] Note that this condition implies we do *not* have an adiabatic process! See section 3.4. This technique is fundamentally different from the one in the previous section. If we were to do this adiabatically, we would end up in the usual nondegenerate Bose-Einstein condensate.

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A Two-Photon Rabi Frequency

Understanding the two-photon Rabi frequency is basically just an extension of what we did at the start of the semester when we introduced the Rabi frequency from first order perturbation theory. This time, however, we go up to second order. For a refresher, see either the file `Lec4and5_NotesTDPTandHI.pdf` posted on Piazza, or section 1.2.4 of the course textbook (Grynberg, Aspect, Fabre).[7]

After doing first order perturbation theory with a sinusoidal perturbation, we get (quoted from the aforementioned file)[8]

$$S_{ki}^{(1)}(t) = -\frac{W_{ki}}{2\hbar}\left(\frac{e^{i(\omega_{ki}-\omega)t}-1}{\omega_{ki}-\omega}\right) \tag{47}$$

within the rotating wave approximation. For simplicity/ease of reading I set $t_0 = 0$ and $\varphi = 0$.

If we take mod squared of this, we get the (hopefully familiar) probability of transition

$$P_{i\to k}^{(1)} = T\frac{|W_{ki}|^2}{4}\frac{2\pi}{\hbar}\delta_T(E_k - E_i - \hbar\omega) = T\frac{|W_{ki}|^2}{4}\frac{2\pi}{\hbar}\delta_T(E_k - E_i - \hbar\omega) \tag{48}$$

which we used to motivate our definition of the Rabi frequency:

$$W_{ki} = -dE = \hbar\Omega_R \tag{49}$$

which we can use to rewrite equation 48 (along with plugging in δ_T) to get

$$P_{i\to k}(T) = \left(\frac{\Omega_R T}{2}\right)^2\left(\frac{\sin(\delta T/2)}{\delta T/2}\right)^2. \tag{50}$$

Now, if we recall our second order perturbation theory from undergraduate quantum mechanics (or earlier on in the course), we use an iterative approach to get the next order. So for intermediate state labelled by j (in our case $|e\rangle$) and amplitude S we have

$$i\hbar\dot{S}_{ki}^{(2)} = \sum_j H_{kj}S_{ji}^{(1)}e^{i\omega_{kj}} \tag{51}$$

where $S_{ji}^{(1)}$ represents first order approximation for going from the initial state i to an intermediate state j. In our case, we are only interested in a single intermediate state so we may omit the sum over j and just fix for some j. This gives (note this is the same as the double integral for $S_{ki}^{(2)}$ found in the notes, which was done generally before plugging in a form for the perturbation)

$$S_{ki}^{(2)} = \frac{1}{i\hbar}\int_0^t \mathrm{d}t' H_{kj}(t')e^{i\omega_{kj}t'}S_{ji}^{(1)}(t'). \tag{52}$$

[7] Also perhaps worth mentioning is that the notes/textbook do second-order perturbation theory for a constant perturbation but not a sinusoid.

[8] In this section, I employ the notation that the superscript $^{(i)}$ indicates perturbation order i.

Now we impose a form for the perturbation. We have two fields instead of just one this time, so we have two sinusoids. This time we'll write the perturbation as

$$\hat{H} = \hat{W}^1 \cos(\omega_1 t) + \hat{W}^2 \cos(\omega_2 t) \tag{53}$$

which is the same as what we did for a single-photon process but with two terms denoted by 1 and 2. Note that this also means that we have two terms in the first order S, though it's the sum of two very similar terms (just for the two fields). Furthermore, since we are operating in the rotating wave approximation, we may expand cos and omit some terms. Positive exponentials like $e^{i\omega t}$ correspond to emission of photons while negative corresponds to absorption. In this particular two photon process, we are interested in absorption of the first photon, with energy $\hbar\omega_1$ and emission of the second photon with energy $\hbar\omega_2$, so we can get rid of half the exponentials. Thus, we may simplify our lives a little more by writing

$$\hat{H} = \hat{W}^1 \frac{e^{-i\omega_1 t}}{2} + \hat{W}^2 \frac{e^{i\omega_2 t}}{2} \tag{54}$$

which is really just to avoid the need to write down lots of terms in the full expansion. Now we plug this into equation 52 (remembering that $S^{(1)}$ has two terms), and integrate. Since we have made these simplifying assumptions, we only get 4 terms (much fewer than if we didn't make any approximations), all of which are exponentials.

In particular, we have

$$S^{(1)}_{ji} = \frac{1}{2\hbar}\left(\frac{W^1_{ji}(e^{-i(\omega_1-\omega_{ji})t}-1)}{\omega_1-\omega_{ji}} + \frac{W^2_{ji}(e^{-i(-\omega_2-\omega_{ji})t}-1)}{-\omega_2-\omega_{ji}}\right) \tag{55}$$

and plugging into equation 52 we get all very similar terms, which gives us a mess after integration

$$\begin{aligned} S^{(2)}_{ki} = \frac{1}{4\hbar^2}[&\frac{W^1_{kj}W^1_{ji}}{\omega_1-\omega_{ji}}\frac{e^{i(\omega_0-2\omega_1)t}-1}{\omega_0-2\omega_1} + \frac{W^2_{kj}W^2_{ji}}{-\omega_2-\omega_{ji}}\frac{e^{i(\omega_0+2\omega_2)t}-1}{\omega_0+2\omega_2} \\ &+\frac{W^2_{kj}W^1_{ji}}{\omega_1-\omega_{ji}}\frac{e^{i(\omega_0-\omega_1+\omega_2)t}-1}{\omega_0-\omega_1+\omega_2} + \frac{W^1_{kj}W^2_{ji}}{-\omega_2-\omega_{ji}}\frac{e^{i(\omega_0-\omega_1+\omega_2)t}-1}{\omega_0-\omega_1+\omega_2}] \end{aligned} \tag{56}$$

where $\omega_0 = \omega_k - \omega_i$. Now we can get rid of two terms right off the bat on physical grounds: the first because it corresponds to absorption of two $\hbar\omega_1$ photons, and the second because it corresponds to emittance of two $\hbar\omega_2$ photons. Finally, by the quasi-resonance criterion the third term is much larger than the fourth term (by the denominator).

This give us our glorious result

$$S^{(2)}_{ki} = \frac{1}{4\hbar^2}\frac{W^2_{kj}W^1_{ji}}{\omega_1-\omega_{ji}}\frac{e^{i(\omega_0-\omega_1+\omega_2)t}-1}{\omega_0-\omega_1+\omega_2}. \tag{57}$$

At a glance, this is just another ugly formula. However, the resonance between states i and k come from a resonance condition $\omega_0 = \omega_1 - \omega_2$, meaning our effective detuning is $\delta_{ki} = \omega_0 - (\omega_1 - \omega_2)$, which is of course nothing other than what is found in the denominator and in the exponential above. Furthermore, the other frequency term in the denominator is simply how far off-resonance ω_1 is to the intermediate state, which I called δ_e originally. That is, $\delta_e = \omega_{ji} - \omega_1$. With these considerations, we can take the modulus squared to get the transition probability to second order

$$P^{(2)}_{i\to k}(t) = \left|\frac{1}{4\hbar^2}\frac{W^2_{kj}W^1_{ji}}{\delta_e}\frac{e^{i\delta_{ki}t}-1}{\delta_{ki}}\right|^2. \tag{58}$$

But the last fraction can be rewritten by multiplying and dividing by $e^{-i\delta_{ki}t/2}$, (recall that this is just a phase term so the part we divide by we can just get rid of)

$$\begin{aligned}\left|\frac{e^{i\delta_{ki}t}-1}{\delta_{ki}}\right|^2 &= \left|e^{i\delta_{ki}t/2}\frac{e^{i\delta_{ki}t/2}-e^{-i\delta_{ki}t/2}}{\delta_{ki}}\right|^2 \\ &= \left|\frac{e^{i\delta_{ki}t/2}-e^{-i\delta_{ki}t/2}}{\delta_{ki}}\right|^2 \\ &= \left|\frac{2\sin(\delta_{ki}t/2)}{\delta_{ki}}\right|^2.\end{aligned} \tag{59}$$

Plugging this back in we get[9]

$$\begin{aligned}P^{(2)}_{i\to k}(t) &= \left|\frac{1}{4\hbar^2}\frac{W^2_{kj}W^1_{ji}}{\delta_e}\frac{2\sin(\delta_{ki}t/2)}{\delta_{ki}}\right|^2 \\ &= \left|\frac{t}{4\hbar^2}\frac{W^2_{kj}W^1_{ji}}{\delta_e}\frac{\sin(\delta_{ki}t/2)}{\delta_{ki}t/2}\right|^2 \\ &= \left|\frac{t}{2}\frac{(W^2_{kj}/\hbar)(W^1_{ji}/\hbar)}{2\delta_e}\right|^2\left(\frac{\sin(\delta_{ki}t/2)}{\delta_{ki}t/2}\right)^2\end{aligned} \tag{60}$$

where in the last line I rewrote it to look exactly like equation 50, but with an effective Rabi frequency

$$\frac{(W^2_{kj}/\hbar)(W^1_{ji}/\hbar)}{2\delta_e}. \tag{61}$$

The first term in brackets represents the Rabi frequency for the transition between the final state k and the intermediate state j whereas the other bracketed term is the Rabi frequency for the transition between the initial state i and the intermediate state j. If we were a little more careful, we would have used a complex conjugate since we were really taking the adjoint when we took the emission vs absorption term (since the Perturbation is necessarily Hermitian; the sum of the two exponentials are complex conjugates of each other). Thus, finally, we have (in our original notation)

$$\Omega^{\text{eff}}_R = \frac{\Omega^a_R\Omega^{b*}_R}{2\delta_e} \tag{62}$$

which is what I claimed in the main body of these notes. From here we can see that we effectively have exactly the same dynamics as if we had a first-order process, as the transition probability becomes exactly the same after this substitution. This justifies treating the lambda scheme discussed in figure 3 as a two-level system.

[9]Note that since I set $t_0 = 0$, we have $T \equiv t - t_0 = t$.