Computing Satterthwaite degrees of freedom for glmer models

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1 Introduction

The following is based on the SAS documentation at `https://documentation.sas.com/doc/en/statug/15.2/statug_glimmix_details39.htm`. Also critical was the paper "Fitting Linear Mixed-Effects Models Using lme4" by Bates et al (Journal of Statistical Software, 67(1), 1-48, 2015).

2 Methods

2.1 lmer models

2.1.1 The 1 degree of freedom case

Consider the t-statistic

$$t = \frac{\ell'\hat{\beta}}{\sqrt{\ell' C \ell}} \tag{1}$$

where iℓ is a $p \times 1$ matrix that defines the contrast of interest, $\hat{\beta}$ is the fixed effects coefficient vector estimate (of length p), and C is the variance-covariance matrix of the fixed effects. C depends on σ and the random effects parameters θ (of length q) We can compute the Satterthwaite degrees of freedom for this statistic as

$$\nu = \frac{2(\ell' C \ell)^2}{g' A g} \tag{2}$$

where A is the variance-covariance matrix of the random effects parameter estimates, $\hat{\theta}$, and g is the first derivative of $\ell' C \ell$ with respect to θ, evaluated at $\hat{\theta}$.

Given a lmer fitted object, rslt, we can extract

$$\begin{aligned}
\hat{\theta} &= \text{getME(rslt,"theta")} \\
\hat{\beta} &= \text{fixef(rslt)} \\
C &= \text{vcov(rslt)} \\
A &= \text{MASS::ginv(rslt@optinfo\$derivs\$Hessian/2)[1:q,1:q]}
\end{aligned}$$

where q is the number of random effects parameters (not including σ). Note that this code gives the so-called "theta" formulation of the random effects parameters (instead of the variance formulation). The difficulty is getting g (or equivalently, $dC/d\theta$). to do this we use the grad function (important: use right-sided derivatives) from the numDeriv package by calling

```
g = grad(func_lmer,theta,obj=rslt,lvec=lvec)}
```

The key function, `func_lmer`, returns an updated value of $\ell' C \ell$ for a given value of theta. It does this by extracting information from the fitted object (rslt) and recomputing C as a function of θ, following information provided in the Bates et al article (particularly critical was equation 54 and section 3.6). See the code for `func_lmer`.

2.1.2 The multi-degree of freedom case

Let ℓ be an $r \times p$ matrix defining the r contrasts of interest so that the hypothesis of intertest is

$$Ho : L\hat{\beta} = 0 \tag{3}$$

Define the F statistic as

$$F = \hat{\beta}' L' (LCL')^{-1} L\hat{\beta}$$

F is distributed as according to an F-distribution with r and df degrees of freedom where df is computed as follows.

Compute the spectral decomposition (R function eiger) of $LCL' = U'DU$. Define b_j as the j'th row of UL (an $r \times p$ matrix). Then compute g_j, the gradient of $b_j C b_j'$ with respect to θ (so g_j is a vector of length q). We use the jacobian function from the numDeriv package (important: use right-sided derivatives), along with the `func_lmer` function to compute g_j.

Next, compute

$$\nu_j = \frac{2(D_j)^2}{g_j' A g_j}$$

for $j = 1 \ldots r$, and let

$$E = \sum_{j=1}^{r} \frac{\nu_j}{\nu_j - 2} I(\nu_j > 2)$$

Then

$$df = \frac{E}{E - r}$$

if $E > r$ and 0 otherwise.

3 Discussion

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