Center for Statistics and the Social Sciences Math Camp 2022

Lecture 1: Algebra, Functions, & Limits

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Mon 12 Sep 2022

Other resources on campus

- For learning how to use statistical software:
  - CSSS 508 Intro to R for social scientists
  - Workshops
    - Center for Social Science Computation and Research (CSSCR)
    - Center for Studies in Demography and Ecology (CSDE)
    - Workshops offered each quarter, often toward the start of the quarter
    - R, SPSS, GIS, and many other languages and software platforms
    - Introductory sessions as well as sessions on specific skills or packages
  - CSSCR consulting
    - Get help with data wrangling, implementing an analysis in software

Other resources on campus

- For additional math review during the school year
  - CSSS 505 Review of mathematics for social scientists
- For statistical consulting
  - CSSS consulting service
    - Get guidance on model selection and interpretation, research design, best practices, specific data concerns, and more

Introductions

- Name/how you'd like to be addressed
- Program/school/department
- One goal you have for math camp
- One thing you're nervous about (optional)

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

Quadratic equations

Functions and limits

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Numbers and variables

Integers

- Examples: ...,-3,-2,-1,0,1,2,3,...
- Subsets include whole numbers, natural numbers, even numbers

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Real Numbers

- Any number on the number line
- Examples: 2, 3.234, $1/7$, $\sqrt{5}$, π
- The set of real numbers is denoted by $\mathbb{R}$
- "$a \in \mathbb{R}$" means a is in the set of real numbers

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Variables

- Placeholders; can take on different values
- Often represented by letters, e.g. x, y, z

Sums and products

Sums

- Often represented by $\sum$ and summed over some index variable, usually integer-valued
- Example:
$$\sum_{i=1}^{3}(i+1)^2 = (1+1)^2+(2+1)^2+(3+1)^2 = 2^2+3^2+4^2 = 29$$

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Products

- Often represented by $\prod$ and multiplied over some index variable, usually integer-valued
- Example:
$$\prod_{k=0}^{3}(k+1)^2 = (0+1)^2 \times (1+1)^2 \times (2+1)^2 \times (3+1)^2 = ?$$

Order of Operations

Please **E**xcuse **M**y **D**ear **A**unt **S**ally

- **P**arentheses (work from inside out)
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction

Note:

- Multiplication and division are interchangeable
- Addition and subtraction are interchangeable
- When looking at an expression, work from left to right following **PEMDAS**

Order of Operations: Example

A common example: what does each of these equal?

1. $1 + 1/2$

2. $1 + (1/2)$

3. $(1 + 1)/2$

Order of Operations: Examples

1. $\left((1+2)^3\right)^2$

2. $4^3 \cdot 3^2 - 10 + 27/3$

Simplifying variable expressions

Rules:

- Follow PEMDAS
- Combine only like terms (same power of each variable x)

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How can we simplify these expressions?

1. $(x + x)^2 - 2x + 3$
2. $5x + 3x^2 - 2x + 5$
3. $5x + 3xy - 2xy + 5$

Fractions

Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

Examples: $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4}\left(=\frac{1}{2}\right), \frac{8}{1}(=8)$.

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Multiplication: Multiply the numerators; multiply the denominators. Example: $\frac{1}{2} \times \frac{3}{4} =$

Division: Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction. Example: $\frac{1}{2} \div \frac{7}{8} =$

Fractions

Adding & Subtracting

Adding and subtracting requires that **fractions must have the same denominator**. If not, first find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add/subtract the two numerators.

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Examples:

$\frac{1}{7} + \frac{4}{7} =$

$\frac{1}{3} + \frac{1}{4} =$

$\frac{17}{20} - \frac{3}{4} =$

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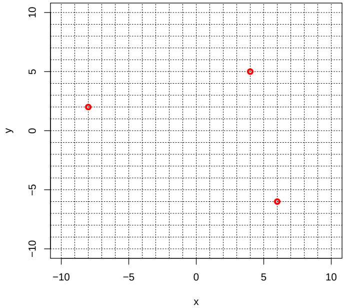
Coordinate plane

- Other names: Cartesian plane, two-dimensional (2D) space, $\mathbb{R}^2$
- The collection of all points (x, y), such that $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$
- **Coordinates** (x, y) provide an "address" for a point in $\mathbb{R}^2$
- The point (0,0) is where the x and y axes intersect and is called the **origin**

Examples: (-8,2),(4,5),(6,-6)



Linear Equations

A line is a collection of points in the plane whose x and y coordinates satisfy a **linear equation**.

Linear Equations

If we have two pairs of points $(x_1, y_1), (x_2, y_2)$, we can find a line between the two points.

A common equation for a line is:

$$y = mx + b$$

where m is the **slope** and b is the **y-intercept**. A line is also a way to define a variable y in terms of another variable x.

Another common form (often used in the regression setting) is

$$y = \beta_0 + \beta_1 x,$$

where β_0 is the **y-intercept** and β_1 is the **slope**. Notice this is really the same equation except that we swapped the order and changed the variable names.

Slopes

The **slope** is the ratio of the difference in the y-values to the difference in the two x-values for any two points on a line. Commonly referred to as **rise** over **run**.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- m measures of the steepness of a line, e.g. how high does the line "rise" in "y-land" when we move one unit to the "right" (toward ∞) in "x"-land.
- The sign of m indicates whether we're going "uphill" $(+)$ or "downhill" (-) when we move to the "right" in "x"-land.

Slopes

Intercepts

The **intercept**, often denoted b, is the value of y when $x = 0$.

- i.e. every line (that isn't a vertical line) has a point $(0, b)$.
- the vertical height where the line crosses the y-axis.

Find the intercept by plugging in one point on the line and the slope into the equation and then solving for the intercept.

$$y_1 = m \cdot x_1 + b \Rightarrow b = y_1 - m \cdot x_1$$

In a simple linear regression setting β_0 can be interpreted as the average value of a dependent variable, y, when the dependent variable x is equal to 0, *if* 0 is a observed or sensible value of your independent variable.

Find the equation of a line using two points

What is the equation of the line that passes through the points $(1, 4)$ and $(2, 1)$? (and why can I say **the** line?)

Solving linear equations algebraically

What if we want to know the value of x when y has a particular value?

- Plug in the values you know
- Do the same thing to both sides of the equation
- Often you undo operations in the reverse order of PEMDAS

Example: Suppose $y = 3x - 2$. What is x when $y = 0$? What is x when $y = 1$?

Solving linear equations graphically

Let's look at our solutions graphically:

Solving Linear Equations

Examples

Your turn:

Say you are at the Garage on Capitol Hill (pre-Covid) and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?

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Solving systems of equations graphically

We often are interested in solving the system of linear equations: finding where the two lines cross/intersect.

We just looked at the case of solving two equations together: one horizontal line and one arbitrary line. What if we have any two lines?

Example: What is the solution to $y = x/2 + 2$ and $y = 3x - 1$?

Solving systems of equations algebraically

Let's try doing that using algebra now. *At the solution*, these two equations give us two different ways to write y in terms of x :

$$y = x/2 + 2 \tag{1}$$

$$y = 3x - 1 \tag{2}$$

So we can set them equal to each other:

Outline for today

Quadratic Equations

Linear equations of x (lines) always take the form $y = mx + b$, where the maximum power of x is 1.

Quadratic equations have the form $y = ax^2 + bx + c$, where $a \neq 0$. Graphically they form parabolas.

Quadratic Equations: Finding roots

For any quadratic equation $f(x) = ax^2 + bx + c$, we find the **root(s)** (values of x such that $f(x) = 0$, or where the function crosses the x-axis) via the **quadratic equation**:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

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$b^2 - 4ac$ is called the **discriminant**. If the discriminant is

- positive, there will be two roots.
- zero, there will be one root.
- negative, there will be no real roots.

Quadratic Equations

Factoring and FOIL

Many quadratic equations can be factored into a more simple form. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

To see that they are equivalent we can FOIL to multiply the two terms on the right hand side of the equation.

- **F**irst: $x \cdot 2x = 2x^2$
- **O**uter: $x \cdot 2 = 2x$
- **I**nner: $-4 \cdot 2x = -8x$
- **L**ast: $-4 \cdot 2 = -8$

Thus, $(x - 4)(2x + 2) = 2x^2 + 2x - 8x - 8 = 2x^2 - 6x - 8$

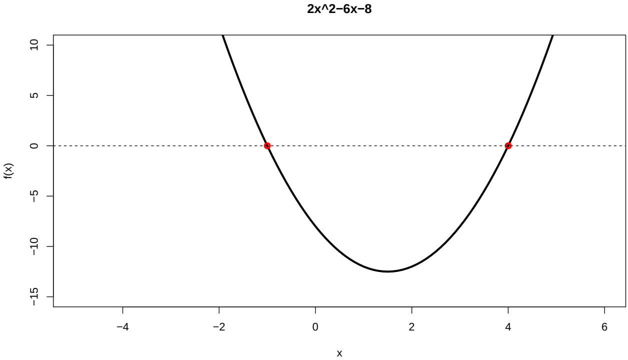
Quadratic Equations

Factoring and FOIL

When your quadratic has been factored you can find the roots by solving each term for zero. For example:

$$2x^2 - 6x - 8 = (x-4)(2x+2)$$

has roots when $x - 4 = 0$ and $2x + 2 = 0$. Thus, the roots are found at $x = -1, 4$.



Quadratic Equations

Factoring and FOIL

Hunting for the FOIL factors can be tricky! Remember the quadratic equation always works!!

- If $b^2 - 4ac$ is a whole number, a fraction, a squared number, then it can be factored into something simple, if not use the quadratic formula.

Quadratic Equations
Factoring and FOIL

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Examples:

- $2x^2 + 4x - 16 \Rightarrow b^2 - 4ac = 4^2 - 4 \cdot 2 \cdot (-16) = 144$; 2 roots; factors
- $3x^2 - 2x + 9 \Rightarrow b^2 - 4ac = (-2)^2 - 4 \cdot 3 \cdot 9 = -104$; no real roots

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We can view linear, quadratic, and many other equations as functions.

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A **function** is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

The **domain** of a function is the set of all possible values that you can plug into the function. The **range** is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- **Domain:**
- **Range:**

Functions

$f(x) = \sqrt{x}$

- **Domain:**
- **Range:**

$f(x) = 1/x$

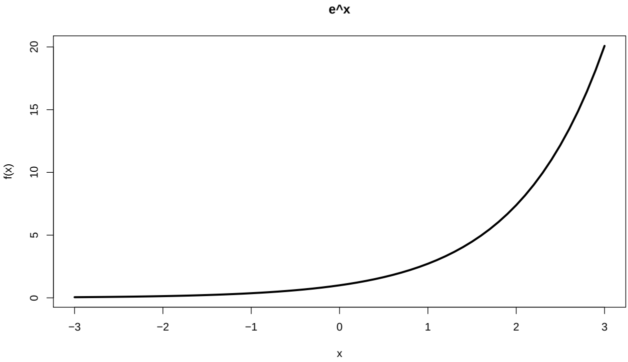
- **Domain:**
- **Range:**

Exponential functions

We'll introduce two new and useful types of functions now

Exponential functions are of the form $f(x) = ae^{bx}$

- Common model for population growth, with $f(x)$ is the population at time x
- Grows more quickly than linear or quadratic functions

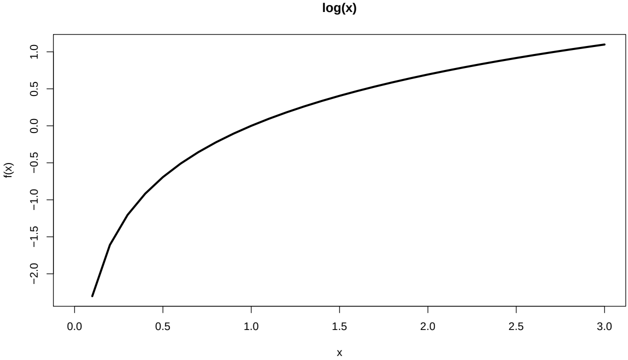


Logarithmic functions

Logarithmic functions are the inverse of exponential functions:

$$f(x) = c + d \cdot log(x)$$

- Can be used to find the time $f(x)$ necessary to reach a certain population x
- Grow more slowly than linear or quadratic functions



Exponents

a^n is 'a to the power of n'. a is multiplied by itself n times. Often a is called the base, n the exponent. Examples: Exponents can be

fractions and/or they can be negative. Examples:

Exponents: useful properties

Logarithms

Logarithms answer the question, what power of this number gives you that number? For example,

$$\log_{10} 100 = ? \iff \text{What power of 10 gives you 100?} \quad \_\_\_$$

$$\log_9 3 = ? \iff \text{What power of } \_\_\_ \text{ gives you } \_\_\_? \quad \_\_\_$$

Logarithms

The three most common bases are 2, 10, and $e \approx 2.718$.

- $\log_e$ is called the natural logarithm and is very common in practice (e.g. exponential growth)
- If no base is specified, often the base is e

Logarithms: three useful properties

1. $\log_c(a \cdot b) = \log_c(a) + \log_c(b)$

2. $\log_c\left(\frac{a}{b}\right) = \log_c(a) - \log_c(b)$

3. $\log_c(a^n) = n \cdot \log_c(a)$

Continuous & Piecewise Functions

A **continuous** function behaves without break or interruption. If you can follow the entire function curve with your pencil without picking it up, the function is continuous. Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

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A **piecewise** function can either have 'jumps' in it or can be made up of different functions for different parts of the domain (possible x-values).

- Example: absolute value $f(x) = |x|$ can be written as

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Limits

Often we are interested in what a function does as it approaches a certain value. This behavior is called the **limit**.

The limit of $f(x)$ as x approaches a is L:

$$lim_{x \to a} f(x) = L$$

It may be that a is not in the domain of $f(x)$ but we can still find the limit by seeing what value $f(x)$ is approaching as x gets very close to a.

Example:

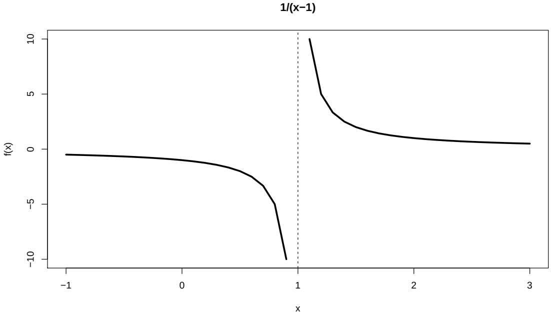
$$\lim_{x \to 3} x^2 = 9$$

$$\lim_{x \to \infty} e^{-x} = 0$$

Limits

Often limits are different depending on the direction from which you approach a. The limit 'from above' is approaching from the right ($x \downarrow a$) and the limit 'from below' ($x \uparrow a$)is approaching from the left.

If $f(x) = \frac{1}{x-1}$, we have $lim_{x \downarrow 1} \frac{1}{x-1} = \infty$ and $lim_{x \uparrow 1} \frac{1}{x-1} = -\infty$:



Limits

Another example (this comes up in probability distributions):

$$f(x) = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases}$$

Graph this and find $lim_{x \uparrow -1}$, $lim_{x \uparrow 2}$, and $lim_{x \downarrow 2}$.