Center for Statistics and the Social Sciences Math Camp 2022

Lecture 3 Part 1: Derivatives

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Outline for today

Functions and limits

Motivation for derivatives

Finding the derivative of a function

Second and third derivatives

Application: Optimization (finding minima/maxima)

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Continuous & Piecewise Functions

A **continuous** function behaves without break or interruption. If you can follow the entire function curve with your pencil without picking it up, the function is continuous. Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

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A **piecewise** function is defined in pieces

- May or may not be continuous
- Example: absolute value $f(x) = |x|$ is continuous:

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

- This function has a jump and is not continuous:

$$f(x) = \begin{cases} 0 & \text{if } x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

Limits

Often we are interested in what a function does as it approaches a certain value. This behavior is called the **limit**.

The limit of $f(x)$ as x approaches a is L:

$$lim_{x \to a} f(x) = L$$

It may be that a is not in the domain of $f(x)$ but we can still find the limit by seeing what value $f(x)$ is approaching as x gets very close to a.

Example:

$$\lim_{x \to 3} x^2 = 9$$

$$\lim_{x \to \infty} e^{-x} = 0$$

Limits

Often limits are different depending on the direction from which you approach a. The limit 'from above' is approaching from the right ($x \downarrow a$) and the limit 'from below' ($x \uparrow a$) is approaching from the left.

Example: graph $f(x)$ and find $lim_{x \uparrow 2} f(x)$ and $lim_{x \downarrow 2} f(x)$.

$$f(x) = \begin{cases} 0 & \text{if } x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

Limits

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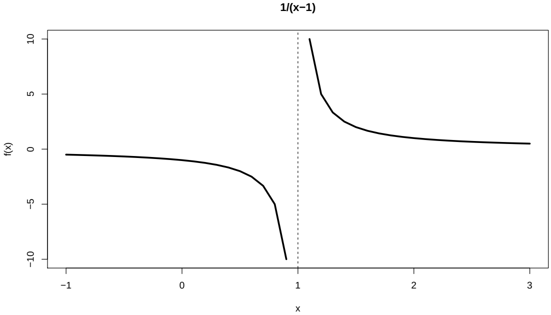
If $f(x) = \frac{1}{x-1}$, we have $lim_{x \downarrow 1} \frac{1}{x-1} = \infty$ and $lim_{x \uparrow 1} \frac{1}{x-1} = -\infty$:



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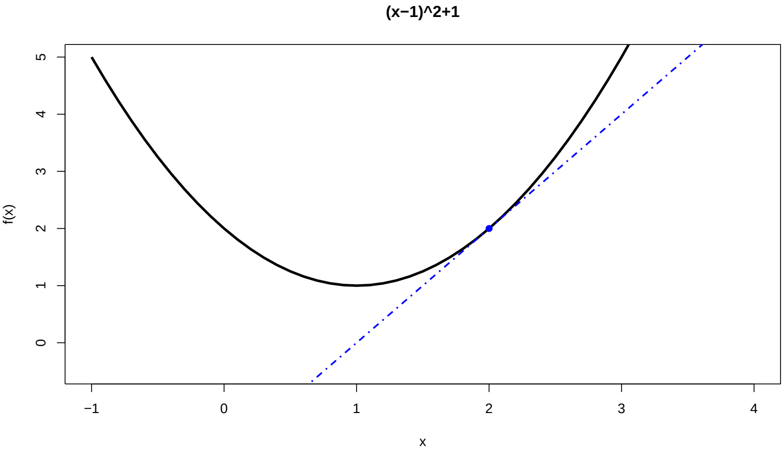
Motivation for derivatives

- The slope of a line tells us how steep the line is
- Can we generalize this idea of steepness for other functions?
- For a line, the slope is the same everywhere
- The slope of a curve as $\Delta y/\Delta x$ isn't well-defined because if you choose different points you get a different answer
- We want some kind of "local" measure of how steep the curve is **at** a point

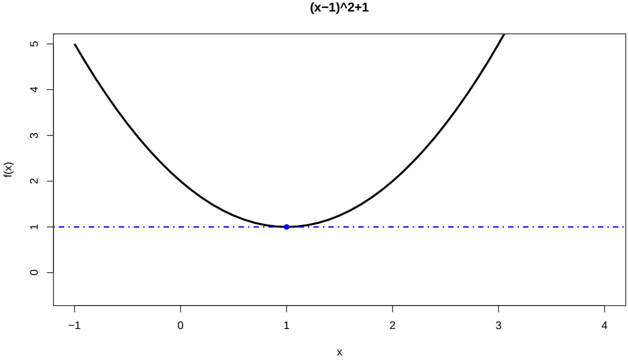
What is the derivative?

The **derivative** of a function can be thought of as the **slope of the line tangent to (touching) the curve**, $f(x)$, at the point x.



A major application: optimization

In statistics we are often interested in derivatives to help us find the values that maximize (or minimize) functions. We will be particularly interested in the values x such that the derivative is zero.



More on this later!

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Finding the Derivative

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- change in distance relative to a change in time
- marginal revenue - change in amount of money from item sales relative to change in demand for the items

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- marginal revenue - change in amount of money from item sales relative to change in demand for the items

Think of it as finding the **slope** of a function at specific point. To find the average rate of change over an interval $[a, b]$, we look at the change in $f(x)$ over the length of the interval.

$$\frac{f(b) - f(a)}{b - a}$$

Finding the derivative using limits

If we want to find the rate of change at a value x, we find the average rate of change over a very small interval (usually of length δ).

$$\frac{f(x+\delta)-f(x)}{\delta}$$

We look at what happens when δ becomes very very small, i.e. when the interval essentially just becomes the point x.

The derivative of $f(x)$ at x is then:

$$lim_{\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta}$$

It is denoted by $\frac{d}{dx}f(x)$ or $f'(x)$.

Differentiation rules

Derivative of a constant:

$$f(x) = a; \;\; f'(x) = 0$$

Derivative of a power:

$$f(x) = ax^n; \;\; f'(x) = n \cdot a \cdot x^{n-1}$$

Derivative of an exponential:

$$f(x) = e^x; \;\; f'(x) = e^x$$

Find the derivatives of these functions:

- $f(x) = x^4$
- $g(x) = 3x^7 - 2$
- $h(x) = 3e^x$

Differentiation rules

logs and trigonometric functions

Derivative of an Logarithmic Function:

$$f(x) = log(x); \;\; f'(x) = 1/x$$

Derivative of a Trigonometric Functions:

$$f(x) = sin(x); \;\; f'(x) = cos(x) \;\;\; \& \;\;\; f(x) = cos(x); \;\; f'(x) = -sin(x)$$

Derivative of a Sum of Functions:

$$f(x) = g(x) + h(x); \;\; f'(x) = g'(x) + h'(x)$$

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Derivative of a Sum of Functions:

$$f(x) = g(x) + h(x); \;\; f'(x) = g'(x) + h'(x)$$

Find the derivatives of these functions:

- $f(x) = 2log(x)$
- $g(x) = sin(x) + e^x$
- $h(x) = 3x^2 + 4x$

Differentiation rules

Product Rule

Derivative of the product of two functions:

$$f(x) = g(x) \cdot h(x); \;\; f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$$

Examples:

- $f(x) = x^2 e^x$
- $f(x) = 3x \log(x)$
- $f(x) = 3x \log(x) + 2x$

Differentiation rules

Quotient Rule

Derivative of the ratio (quotient) of two functions:

$$f(x) = \frac{g(x)}{h(x)}; \;\; f'(x) = \frac{g'(x) \cdot h(x) - h'(x) \cdot g(x)}{h(x)^2}$$

Examples:

- $f(x) = \frac{x^2}{e^x}$

- $f(x) = \frac{3x}{log(x)}$

Differentiation rules

Chain Rule: super useful!

Derivative of a function within a function:

$$f(x) = g(h(x)); \;\; f'(x) = g'(h(x)) \cdot h'(x)$$

Examples:

- $f(x) = e^{3x}$
- $f(x) = \log(1 - x)$
- $f(x) = (2x + 2)^2$

Differentiation rules

Combining rules

We can combine multiple rules:

$$f(x) = 3x(2x+1)^4$$

will require the product rule and the chain rule, where $g(x) = 3x, k(x) = 2x+1$, and $h(k) = k^4$.

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Second and third derivatives

- **Second derivative**: derivative of the derivative
  - Tells you how quickly the first derivative is changing
  - Written $f''(x)$ or $\frac{d^2}{dx^2}f(x)$
- **Third derivative**: derivative of the second derivative
- and so on...

Example:

Relationship between f and f''

The steepness of f affects the magnitude of f'':

The orientation of f affects the sign of f'':

Second and third derivatives

Another example of taking second and third derivatives:

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Critical Values

Optimization: finding the maximum or minimum of a function

- **maximum**: where a function stops increasing and starts to decrease
- **minimum**: where a function stops decreasing and starts increasing
- One application: maximum likelihood estimation (statistics)

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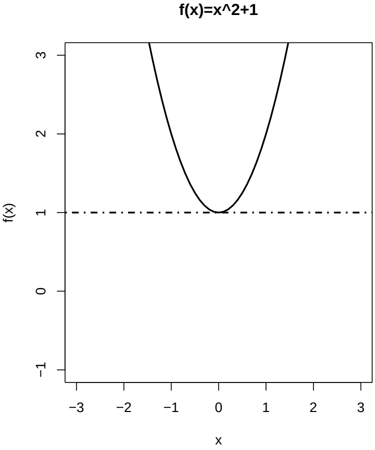
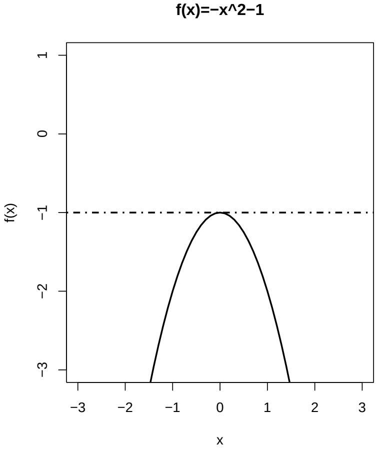
A **critical value** occurs when the behavior of a function changes.

- Occurs where the derivative is equal to zero (goes from $+$ to - or from - to $+$)

Critical Values

Left: $f(x) = x^2 + 1, f'(x) = 2x \implies f'(x) = 0$ when $x = 0$
Right: $f(x) = -x^2 - 1, f'(x) = -2x \implies f'(x) = 0$ when $x = 0$





Critical Values

We can use the first derivative to find the critical point by setting it equal to zero and then solving for x, the root. The goal is to find x such that $f'(x) = 0$.

However, as seen on the previous slide, the derivative is zero for maximums **and** minimums. How do we tell the difference?

Critical Values

Answer: using the second derivative!
For the max, f' is decreasing ($+$ to -), so f'' will be negative.
For the min, f' is increasing (- to $+$), so f'' will be positive.

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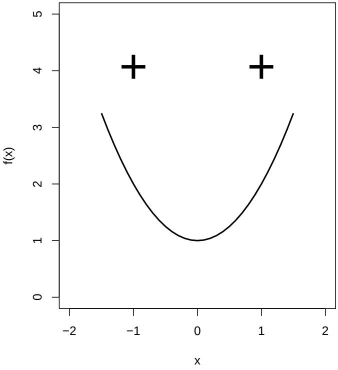
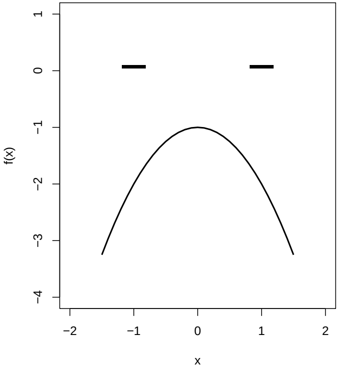
For the min, f' is increasing (- to +), so f'' will be positive.

One way to remember:





Critical Values

Steps

1. Compute $f'(x)$ and $f''(x)$
2. Find the values of x for which $f'(x) = 0$
   - These are the critical points
3. Check the sign of $f''(x)$ at those critical points
   - $f''(x)$ positive $\implies$ minimum at that x value
   - $f''(x)$ negative $\implies$ maximum at that x value
   - $f''(x)$ zero $\implies$ saddle point at that x value

Critical points: an example

Let's return to some examples we saw previously:
$f(x) = x^2$

$f(x) = -x^2$

Critical Values

Another example

$$f(x) = x^3$$

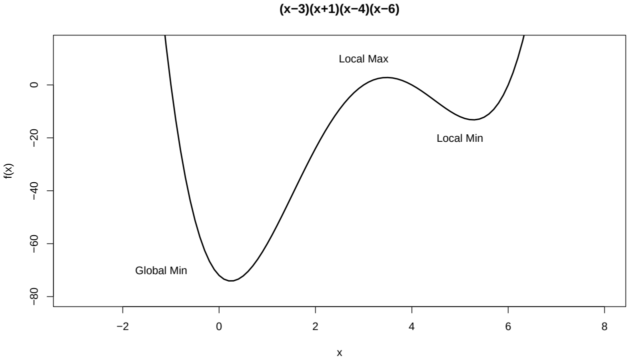
$$f(x) = x^3 + x$$

Critical Values

Global vs. Local

Some functions have more than one maximum or minimum.

We call the largest maximum (or the lowest minimum) the **global** optimum. All others are referred to as **local** optima.



Critical Values

Global vs. Local

Often we want to find the global maximum or minimum

- If we can find all the optima, then we can compare the values to find the largest/smallest one
- In practice, with complicated functions and with more than one or two variables, it is often not feasible to find all the optima, and the one you find might only be a local optimum