Center for Statistics and the Social Sciences Math Camp 2022

Lecture 1: Algebra, Functions, & Limits

Jess Kunke & Erin Lipman

Department of Statistics
University of Washington

Mon 12 Sep 2022

Other resources on campus

- For learning how to use statistical software:
  - CSSS 508 Intro to R for social scientists
  - Workshops
    - Center for Social Science Computation and Research (CSSCR)
    - Center for Studies in Demography and Ecology (CSDE)
    - Workshops offered each quarter, often toward the start of the quarter
    - R, SPSS, GIS, and many other languages and software platforms
    - Introductory sessions as well as sessions on specific skills or packages
  - CSSCR consulting
    - Get help with data wrangling, implementing an analysis in software

Other resources on campus

- For additional math review during the school year
  - CSSS 505 Review of mathematics for social scientists
- For statistical consulting
  - CSSS consulting service
    - Get guidance on model selection and interpretation, research design, best practices, specific data concerns, and more

Introductions

- Name/how you'd like to be addressed
- Program/school/department
- One goal you have for math camp
- One thing you're nervous about (optional)

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

Quadratic equations

Functions and limits

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

Quadratic equations

Functions and limits

Numbers and variables

Integers

- Examples: ...,-3,-2,-1,0,1,2,3,...
- Subsets include whole numbers, natural numbers, even numbers

Numbers and variables

Integers

- Examples: ...,-3,-2,-1,0,1,2,3,...
- Subsets include whole numbers, natural numbers, even numbers

Real Numbers

- Any number on the number line
- Examples: 2, 3.234, $1/7$, $\sqrt{5}$, π
- The set of real numbers is denoted by $\mathbb{R}$
- "$a \in \mathbb{R}$" means a is in the set of real numbers

Numbers and variables

Integers

- Examples: ...,-3,-2,-1,0,1,2,3,...
- Subsets include whole numbers, natural numbers, even numbers

Real Numbers

- Any number on the number line
- Examples: 2, 3.234, $1/7$, $\sqrt{5}$, π
- The set of real numbers is denoted by $\mathbb{R}$
- "$a \in \mathbb{R}$" means a is in the set of real numbers

Variables

- Placeholders; can take on different values
- Often represented by letters, e.g. x, y, z

Sums and products

Sums

- Often represented by $\sum$ and summed over some index variable, usually integer-valued
- Example:
$$\sum_{i=1}^{3}(i+1)^2 = (1+1)^2 + (2+1)^2 + (3+1)^2 = 2^2 + 3^2 + 4^2 = 29$$

Sums and products

Sums

- Often represented by $\sum$ and summed over some index variable, usually integer-valued
- Example:
$$\sum_{i=1}^{3}(i+1)^2 = (1+1)^2+(2+1)^2+(3+1)^2 = 2^2+3^2+4^2 = 29$$

Products

- Often represented by $\prod$ and multiplied over some index variable, usually integer-valued
- Example:
$$\prod_{k=0}^{3}(k+1)^2 = (0+1)^2 \times (1+1)^2 \times (2+1)^2 \times (3+1)^2 = ?$$

Order of Operations

Please **E**xcuse **M**y **D**ear **A**unt **S**ally

- **P**arentheses (work from inside out)
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction

Note:

- Multiplication and division are interchangeable
- Addition and subtraction are interchangeable
- When looking at an expression, work from left to right following **PEMDAS**

Order of Operations: Example

A common example: what does each of these equal?

1. $1 + 1/2$
2. $1 + (1/2)$
3. $(1 + 1)/2$

Order of Operations: Examples

1. $\left((1+2)^3\right)^2$

2. $4^3 \cdot 3^2 - 10 + 27/3$

Simplifying variable expressions

Rules:

- Follow PEMDAS
- Combine only like terms (same power of each variable x)

Simplifying variable expressions

Rules:

- Follow PEMDAS
- Combine only like terms (same power of each variable x)

How can we simplify these expressions?

1. $(x+x)^2 - 2x + 3$
2. $5x + 3x^2 - 2x + 5$
3. $5x + 3xy - 2xy + 5$

Fractions

Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

Examples: $\frac{2}{3}, \frac{16}{4}(= 4), \frac{2}{4}\left(= \frac{1}{2}\right), \frac{8}{1}(= 8)$.

Fractions

Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

Examples: $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4}\left(=\frac{1}{2}\right), \frac{8}{1}(=8)$.

Multiplication: Multiply the numerators; multiply the denominators. Example: $\frac{1}{2} \times \frac{3}{4} =$

Fractions
Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

Examples: $\frac{2}{3}, \frac{16}{4}(= 4), \frac{2}{4}\left(= \frac{1}{2}\right), \frac{8}{1}(= 8)$.

Multiplication: Multiply the numerators; multiply the denominators. Example: $\frac{1}{2} \times \frac{3}{4} =$

Division: Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction. Example: $\frac{1}{2} \div \frac{7}{8} =$

Fractions
Adding & Subtracting

Adding and subtracting requires that **fractions must have the same denominator**. If not, first find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add/subtract the two numerators.

Fractions

Adding & Subtracting

Adding and subtracting requires that **fractions must have the same denominator**. If not, first find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add/subtract the two numerators.

Examples:

$\frac{1}{7} + \frac{4}{7} =$

$\frac{1}{3} + \frac{1}{4} =$

$\frac{17}{20} - \frac{3}{4} =$

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

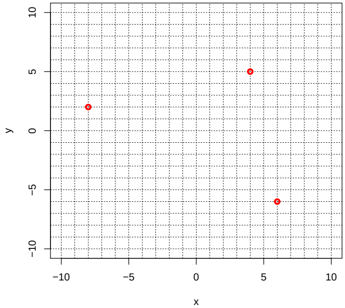
Quadratic equations

Functions and limits

Coordinate plane

- Other names: Cartesian plane, two-dimensional (2D) space, $\mathbb{R}^2$
- The collection of all points (x, y), such that $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$
- **Coordinates** (x, y) provide an "address" for a point in $\mathbb{R}^2$
- The point (0,0) is where the x and y axes intersect and is called the **origin**

Examples: (-8,2),(4,5),(6,-6)



Linear Equations

A line is a collection of points in the plane whose x and y coordinates satisfy a **linear equation**.

Linear Equations

If we have two pairs of points $(x_1, y_1), (x_2, y_2)$, we can find a line between the two points.

A common equation for a line is:

$$y = mx + b$$

where m is the **slope** and b is the **y-intercept**. A line is also a way to define a variable y in terms of another variable x.

Another common form (often used in the regression setting) is

$$y = \beta_0 + \beta_1 x,$$

where β_0 is the **y-intercept** and β_1 is the **slope**. Notice this is really the same equation except that we swapped the order and changed the variable names.

Slopes

The **slope** is the ratio of the difference in the y-values to the difference in the two x-values for any two points on a line. Commonly referred to as **rise** over **run**.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- m measures of the steepness of a line, e.g. how high does the line "rise" in "y-land" when we move one unit to the "right" (toward ∞) in "x"-land.
- The sign of m indicates whether we're going "uphill" $(+)$ or "downhill" (-) when we move to the "right" in "x"-land.

Slopes

Intercepts

The **intercept**, often denoted b, is the value of y when $x = 0$.

- i.e. every line (that isn't a vertical line) has a point $(0, b)$.
- the vertical height where the line crosses the y-axis.

Find the intercept by plugging in one point on the line and the slope into the equation and then solving for the intercept.

$$y_1 = m \cdot x_1 + b \Rightarrow b = y_1 - m \cdot x_1$$

In a simple linear regression setting β_0 can be interpreted as the average value of a dependent variable, y, when the dependent variable x is equal to 0, *if* 0 is a observed or sensible value of your independent variable.

Find the equation of a line using two points

What is the equation of the line that passes through the points $(1, 4)$ and $(2, 1)$? (and why can I say **the** line?)

Solving linear equations algebraically

What if we want to know the value of x when y has a particular value?

- Plug in the values you know
- Do the same thing to both sides of the equation
- Often you undo operations in the reverse order of PEMDAS

Example: Suppose $y = 3x - 2$. What is x when $y = 0$? What is x when $y = 1$?

Solving linear equations graphically

Let's look at our solutions graphically:

Solving Linear Equations

Word problem example

Say you are at the Garage on Capitol Hill (pre-Covid) and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

Quadratic equations

Functions and limits

Solving systems of equations graphically

We often are interested in solving the system of linear equations: finding where the two lines cross/intersect.

We just looked at the case of solving two equations together: one horizontal line and one arbitrary line. What if we have any two lines?

Example: What is the solution to $y = x/2 + 2$ and $y = -x + 5$?

Solving systems of equations algebraically

Let's try doing that using algebra now. *At the solution*, these two equations give us two different ways to write y in terms of x :

$$y = x/2 + 2 \tag{1}$$

$$y = -x + 5 \tag{2}$$

So we can set them equal to each other:

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

Quadratic equations

Functions and limits

Quadratic Equations

Linear equations of x (lines) always take the form $y = mx + b$, where the maximum power of x is 1.

Quadratic equations have the form $y = ax^2 + bx + c$, where $a \neq 0$. Graphically they form parabolas.

Quadratic Equations: Finding roots

For any quadratic equation $y = ax^2 + bx + c$, we can find the **root(s)** (values of x such that $y = 0$, or where the parabola crosses the x-axis) via the **quadratic formula**:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Equations: Finding roots

For any quadratic equation $y = ax^2 + bx + c$, we can find the **root(s)** (values of x such that $y = 0$, or where the parabola crosses the x-axis) via the **quadratic formula**:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Find the roots of the following equations. What do you notice?

- $x^2 + x - 2$
- $x^2 - 4x + 4$
- $x^2 + x + 1$

The discriminant

How can we tell how many roots there will be?

The expression under the square root sign, $b^2 - 4ac$, is called the **discriminant**. If the discriminant is

- positive, there will be two roots.
- zero, there will be one root.
- negative, there will be no real roots.

Quadratic Equations

Factoring and FOIL

Many quadratic equations can be factored into a more simple form. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

To see that they are equivalent we can FOIL to multiply the two terms on the right hand side of the equation.

- **F**irst: $x \cdot 2x = 2x^2$
- **O**uter: $x \cdot 2 = 2x$
- **I**nner: $-4 \cdot 2x = -8x$
- **L**ast: $-4 \cdot 2 = -8$

Thus, $(x - 4)(2x + 2) = 2x^2 + 2x - 8x - 8 = 2x^2 - 6x - 8$

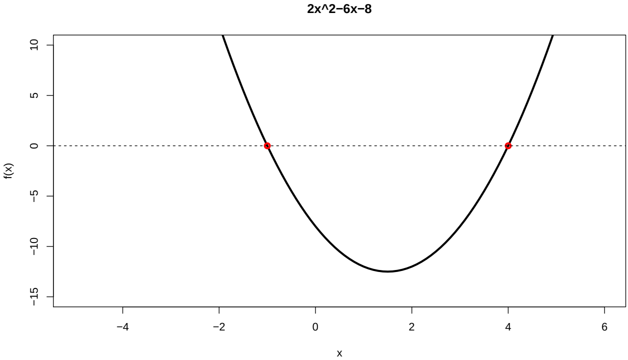
Quadratic Equations

Factoring and FOIL

When your quadratic has been factored you can find the roots by solving each term for zero. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

has roots when $x - 4 = 0$ and $2x + 2 = 0$. Thus, the roots are found at $x = -1, 4$.



Quadratic Equations

Factoring and FOIL

Hunting for the FOIL factors can be tricky! Remember the quadratic equation always works!!

- If $b^2 - 4ac$ is a whole number, a fraction, a squared number, then it can be factored into something simple, if not use the quadratic formula.

Quadratic Equations

Factoring and FOIL

Hunting for the FOIL factors can be tricky! Remember the quadratic equation always works!!

- If $b^2 - 4ac$ is a whole number, a fraction, a squared number, then it can be factored into something simple, if not use the quadratic formula.

Examples:

- $2x^2 + 4x - 16 \Rightarrow b^2 - 4ac = 4^2 - 4 \cdot 2 \cdot (-16) = 144$; 2 roots; factors
- $3x^2 - 2x + 9 \Rightarrow b^2 - 4ac = (-2)^2 - 4 \cdot 3 \cdot 9 = -104$; no real roots

Outline for today

Preliminaries

Lines, equations, and the coordinate plane

Solving systems of equations

Quadratic equations

Functions and limits

Functions

We can view linear, quadratic, and many other equations as functions.

A **function** is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

Functions

We can view linear, quadratic, and many other equations as functions.

A **function** is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

The **domain** of a function is the set of all possible values that you can plug into the function. The **range** is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- **Domain:**
- **Range:**

Functions

$f(x) = \sqrt{x}$

- **Domain:**
- **Range:**

$f(x) = 1/x$

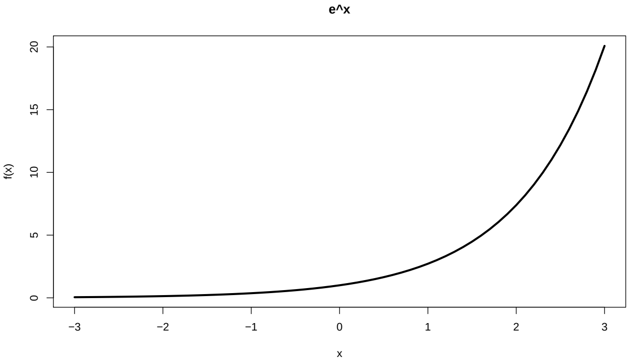
- **Domain:**
- **Range:**

Exponential functions

We'll introduce two new and useful types of functions now

Exponential functions are of the form $f(x) = ae^{bx}$

- Common model for population growth, with $f(x)$ is the population at time x
- Grows more quickly than linear or quadratic functions

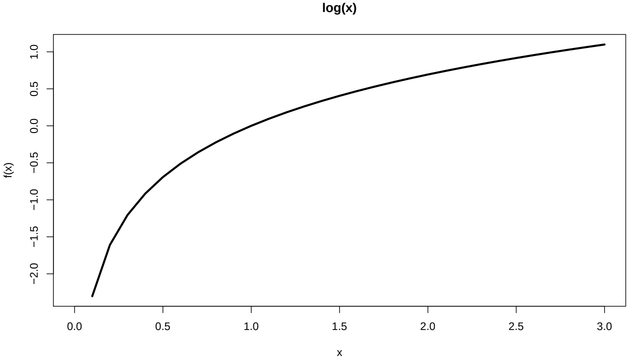


Logarithmic functions

Logarithmic functions are the inverse of exponential functions:

$$f(x) = c + d \cdot log(x)$$

- Can be used to find the time $f(x)$ necessary to reach a certain population x
- Grow more slowly than linear or quadratic functions



Exponents

a^n is 'a to the power of n'. a is multiplied by itself n times. Often a is called the base, n the exponent.

Examples:

Exponents can be fractions and/or they can be negative. We will see examples on the next slide.

Exponents: useful properties

Logarithms

Logarithms answer the question, what power of this number gives you that number? For example,

$$\log_{10} 100 = ? \iff \text{What power of 10 gives you 100?} \quad \_\_\_$$

$$\log_{9} 3 = ? \iff \text{What power of \_\_\_ gives you \_\_\_?} \quad \_\_\_$$

Logarithms

The three most common bases are 2, 10, and $e \approx 2.718$.

- $\log_e$ is called the natural logarithm and is very common in practice (e.g. exponential growth)
- If no base is specified, often the base is e

Logarithms: three useful properties

1. $\log_c(a \cdot b) = \log_c(a) + \log_c(b)$

2. $\log_c\left(\frac{a}{b}\right) = \log_c(a) - \log_c(b)$

3. $\log_c(a^n) = n \cdot \log_c(a)$

Continuous & Piecewise Functions

A **continuous** function behaves without break or interruption. If you can follow the entire function curve with your pencil without picking it up, the function is continuous. Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

Continuous & Piecewise Functions

A **continuous** function behaves without break or interruption. If you can follow the entire function curve with your pencil without picking it up, the function is continuous. Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

A **piecewise** function can either have 'jumps' in it or can be made up of different functions for different parts of the domain (possible x-values).

- Example: absolute value $f(x) = |x|$ can be written as

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Limits

Often we are interested in what a function does as it approaches a certain value. This behavior is called the **limit**.

The limit of $f(x)$ as x approaches a is L:

$$lim_{x \to a} f(x) = L$$

It may be that a is not in the domain of $f(x)$ but we can still find the limit by seeing what value $f(x)$ is approaching as x gets very close to a.

Example:

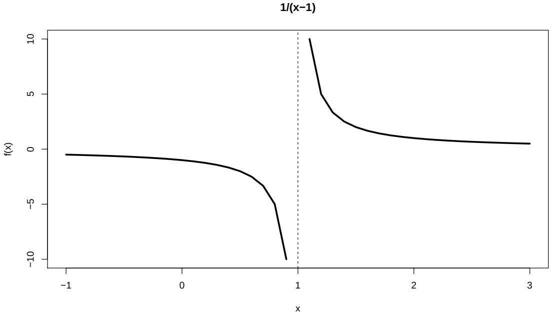
$$\lim_{x \to 3} x^2 = 9$$

$$\lim_{x \to \infty} e^{-x} = 0$$

Limits

Often limits are different depending on the direction from which you approach a. The limit 'from above' is approaching from the right ($x \downarrow a$) and the limit 'from below' ($x \uparrow a$)is approaching from the left.

If $f(x) = \frac{1}{x-1}$, we have $lim_{x \downarrow 1} \frac{1}{x-1} = \infty$ and $lim_{x \uparrow 1} \frac{1}{x-1} = -\infty$:



Limits

Another example (this comes up in probability distributions):

$$f(x) = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases}$$

Graph this and find $lim_{x\uparrow -1}$, $lim_{x\uparrow 2}$, and $lim_{x\downarrow 2}$.