Center for Statistics and the Social Sciences Math Camp 2022

Lecture 5: Integrals and probability distributions

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Day 1 math

Outline

Approximating the area under the curve

Computing integrals graphically

Computing integrals algebraically

Some fancy (and useful) tricks

Motivation for integrals in statistics

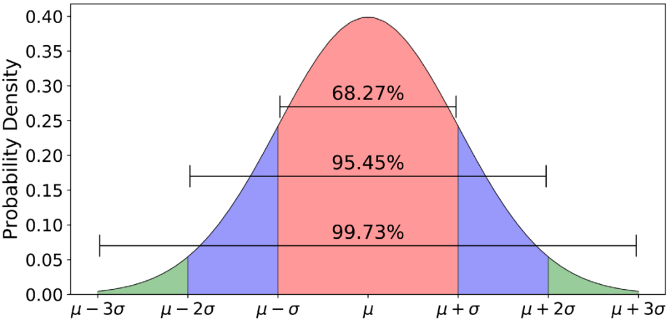

Figure: Standard Normal Density (N(0,1)). Approximately 68% of the probability lies within 1 standard deviation and 95% within 2 standard deviations. The area under the whole curve (from $-\infty$ to ∞) is 1.

How do we compute these areas?

Motivation for integrals in statistics

Integral calculus helps us compute...

- areas under curves
- percentile rankings
- probabilities of events
- expected values and variances

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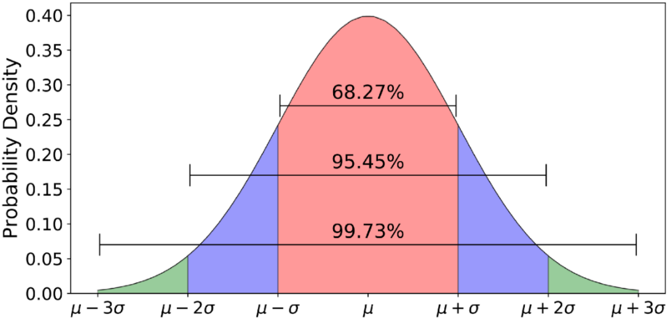
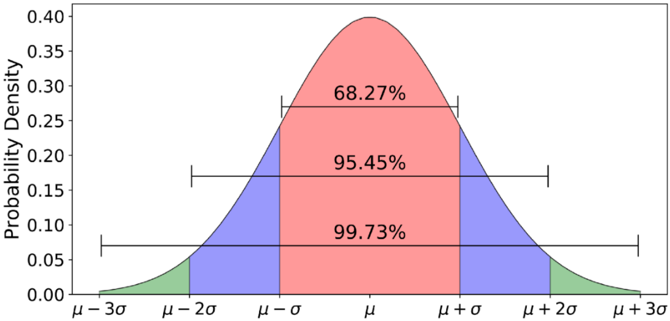
Approximating the area under the curve

What if we wanted to find the area under the curve from -1 to 1?



Approximating the area under the curve

We could approximate with rectangles or trapezoids.



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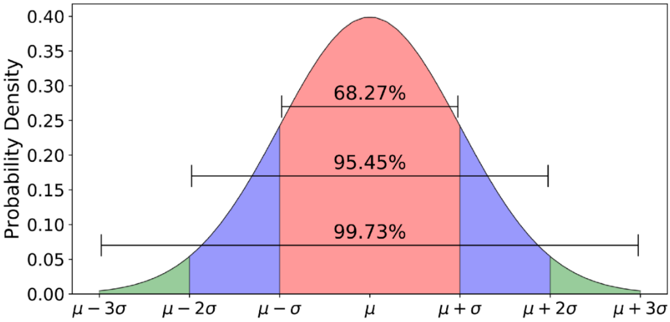

Day 1 math

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Example: birth rate

Example: driving a car

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Integration

The area under a curve is written:

$$\int_a^b f(x)dx$$

This formula is called the **definite integral** of $f(x)$ from a to b.

Here a and b are our endpoints of interest. You can think of the integral as the ‘opposite’ of the derivative.

Integration

More specifically,

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

$F(x)$ is called the **indefinite integral** of $f(x)$. The important relationships between derivatives and integrals are:

$$F'(x) = f(x) \quad \& \quad \int f(x)dx = F(x)$$

What is an integral?

You can think of integrating as looking at a derivative and trying to find the original function.

- $\int 3dx$. What function has a derivative equal to 3?
- $\int 2xdx$. What function has a derivative equal to $2x$?
- $\int e^x dx$. What function has a derivative equal to e^x?

In practice, you don't always have to search for the right function. We have handy shortcuts (rules).

Integration rules

Integrating a constant

$$\int c dx = cx$$

Examples:

- $\int 1 dx$
- $\int 6 dx$
- $\int y dx$

Integration rules

Integrating a power of x

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

Examples:

- $\int x dx$
- $\int \frac{1}{x^2} dx$

Integration rules

Integrating an exponential and logarithmic functions

Exponential:

$$\int e^x dx = e^x$$

(Natural) Logarithm:

$$\int \frac{1}{x} dx = log(x)$$

Integration rules

Basic trigonometric functions

Remember, $\frac{d}{dx}cos(x) = -sin(x)$, thus

$$\int sin(x)dx = -cos(x)$$

and $\frac{d}{dx}sin(x) = cos(x)$, thus

$$\int cos(x)dx = sin(x).$$

Integration rules

Multiple of a function

When you have multiplication by a constant, the constant can just come along for the ride.

$$\int af(x)dx = a \cdot \int f(x)dx = aF(x)$$

Examples:

- $\int 4x^2 dx$
- $\int \frac{3}{x^2} dx$
- $\int \mu y dy$

Integration rules

Sums of functions

$$\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx = F(x) + G(x)$$

Examples:

- $\int 4x + 3x^2 dx = \int 4xdx + \int 3x^2 dx = 4\int xdx + 3\int x^2 dx = 4 \cdot \frac{1}{2}x^2 + 3 \cdot \frac{1}{3}x^3 = 2x^2 + x^3$
- $\int e^x - \frac{2}{x} dx = \int e^x dx - 2\int \frac{1}{x} dx = e^x - 2log(x)$

Finding definite integrals

Often we will be interested in knowing the exact area under the curve $f(x)$, not just the function $F(x)$:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Examples:

- $\int_0^1 x^2 dx$
- $\int_0^\infty e^{-x} dx$
- $\int_2^8 \frac{1}{x} dx$

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u-substitution

Sometimes the function we are integrating is similar to a simpler function with an easy derivative.

For example, $\int \frac{1}{1-x} dx$ is similar to $\int \frac{1}{x} dx$ which we know is $log(x)$. Can we use that?

u-substitution

Similar to the chain rule, we can think about functions within functions.

Let's set $u = 1 - x$. If we differentiate the left with respect to u and the right with respect to x we have $du = -1dx$. Solving for dx we have $dx = -1du$. Now we can substitute these values into our original integral.

$$\int \frac{1}{1-x} dx = \int \frac{1}{u} \cdot (-1) du = -1 \int \frac{1}{u} du$$

u-substitution

Now let's take the integral with respect to u:

$$\int \frac{1}{1-x}dx = -1\int \frac{1}{u}du = -log(u)$$

Then we can plug in the value for $u = 1 - x$:

$$\int \frac{1}{1-x}dx = -1\int \frac{1}{u}du = -log(u) = -log(1-x)$$

Integration by parts

For complicated functions it is often handy to decompose the function into two parts.

$$\int f(x)dx = \int g(x) \cdot h'(x)dx = g(x)h(x) - \int h(x) \cdot g'(x)dx$$

Example:

$$\int xe^x dx$$

$$g(x) = x, \qquad h^{'}(x) = e^x dx$$

$$g^{'}(x) = 1, \qquad h(x) = e^x$$

$$\int xe^x dx = x \cdot e^x - \int e^x \cdot 1dx = xe^x - e^x = e^x(x-1).$$

Demo

Monte Carlo integration! Let's simulate things!