CSSS Math Camp Lecture 1b

Functions & Limits

Authored by: Laina Mercer, PhD

Erin Lipman and Jess Kunke

Department of Statistics

September 11, 2023

CSSS Math Camp Lecture 1b

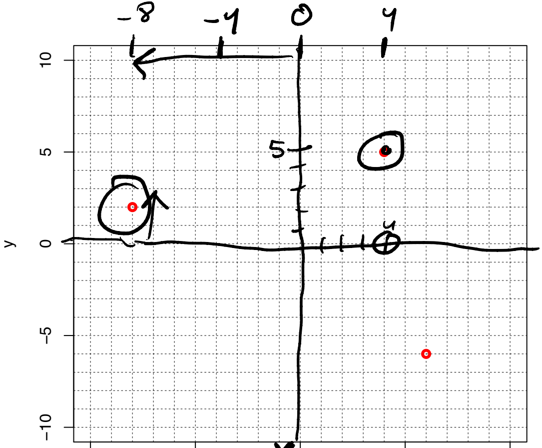
$y = mx + b$

- Linear functions
- Other examples of functions
- Domain and range
- Continuous and piecewise functions
- Limits

Coordinates

A pair of real numbers, written (x, y), can be plotted on a corrdinate plane. The plot has two axes: x (horizontal) and y (vertical).

Examples: (-8,2),(4,5),(6,-6)



Equation of a Line

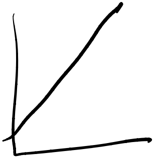
Linear Equations

A line is a set of points, for example the (x, y) pairs satisfying the equation $y = 2x + 1$

$(0, 1)$ $\quad y = 2 \cdot 0 + 1 = 1$

$(1, 3)$ $\quad y = (2 \cdot 1) + 1 = 2 + 1 = 3$

$(5, 11)$ $\quad y = 2 \cdot 5 + 1 = 10 + 1 = 11$

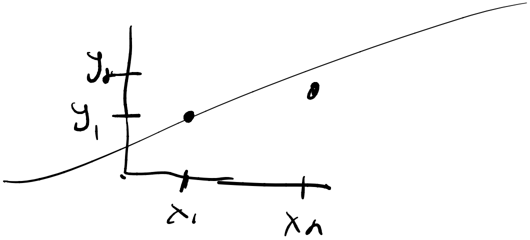


Equation of a Line

Linear Equations

A line is a set of points, for example the (x, y) pairs satisfying the equation $y = 2x + 1$

If we have two points $(x_1, y_1), (x_2, y_2)$, we can find a line between the two points.



Equation of a Line

Linear Equations

A line is a set of points, for example the (x, y) pairs satisfying the equation $y = 2x + 1$

If we have two points $(x_1, y_1), (x_2, y_2)$, we can find a line between the two points.

A common equation for a line is: $\boxed{y = \underline{m}x + \underline{b}}$ where m is the *slope* and b is the *y-intercept*.

$y = 2x + 7$

$y = -2x + 0 = -2x$

Equation of a Line

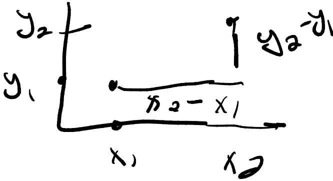
Linear Equations

$$y = \underbrace{m}_{\text{slope}}x + b$$

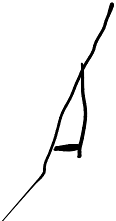
The slope is a measure of the steepness of a line. A line with a slope 5 is steeper than a line with a with slope 2. The slope is the ratio of the difference in the two y-values to the difference in the two x-values. Commonly referred to as *rise* over *run*.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\text{rise}}{\text{run}}$$



$$m = \frac{0}{3} = 0$$



Equation of a Line

Linear Equations

The slope is a measure of the steepness of a line. A line with a slope 5 is steeper than a line with a with slope 2. The slope is the ratio of the difference in the two y-values to the difference in the two x-values. Commonly referred to as *rise* over *run*.

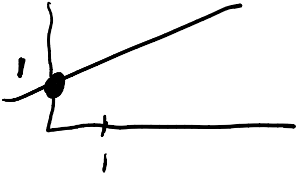
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The intercept is the value of y when $x = 0$. This is the vertical height where the line crosses the y-axis.

$y = mx + b$

slope

intercept



Equation of a Line

Linear Equations

The slope is a measure of the steepness of a line. A line with a slope 5 is steeper than a line with a with slope 2. The slope is the ratio of the difference in the two y-values to the difference in the two x-values. Commonly referred to as *rise* over *run*.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The intercept is the value of y when $x = 0$. This is the vertical height where the line crosses the y-axis.

Once you have the slope, you can find the intercept by plugging in one point and the slope into the equation and then solving for the intercept.

$$b = y_1 - m \cdot x_1$$

$y_1 = m x_1 + b$

Equation of a Line

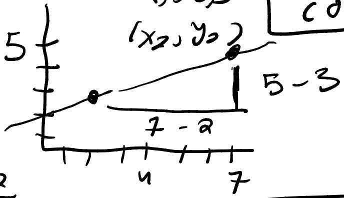
Linear Equations Example

Given the points $(2,3), (7,5)$:

$(x_1, y_1)\ (x_2, y_2)$

(x_1, y_1) (x_2, y_2)

$(1, 3)$
$(0, -2)$



slope: $\frac{3-5}{2-7} = \frac{-2}{-5} = \frac{2}{5}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{7-2} = \frac{2}{5}$$

$$\boxed{y = \frac{2}{5}x + \frac{11}{5}}$$

intercept

$y_1 = m x_1 + b$

$3 = \frac{2}{5} \cdot 2 + b$

$3 = \frac{4}{5} + b$

$3 - \frac{4}{5} = \underbrace{\frac{4}{5} - \frac{4}{5}} + b$

$\frac{3}{1} - \frac{4}{5} = b$

$\frac{3 \times 5}{1 \times 5} - \frac{4}{5} = b$

$\frac{15}{5} - \frac{4}{5} = b$

$b = \frac{11}{5}$

Equation of a Line

Linear Equations Example

Given the points $(2, 3), (7, 5)$:

Slope: $m =$

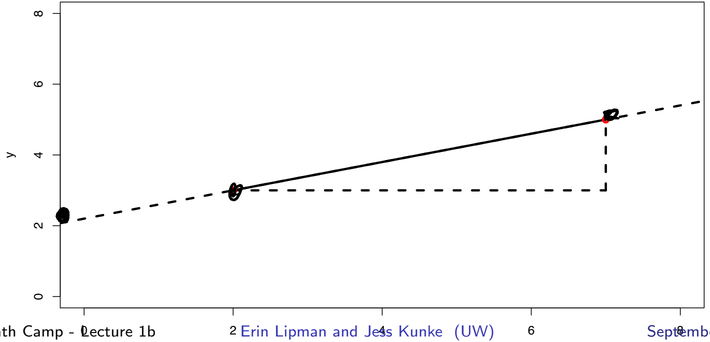
Intercept: $b =$

Equation of the line: $y =$

$$y = \frac{2}{5}x + \frac{11}{5}$$

$$\parallel$$

$$2.2$$



Solving Linear Equations

Often we would like to find the *root* of a linear equation. This is the value of x that maps $f(x)$ to 0 (where the line crosses the x-axis).

$x \longrightarrow \boxed{f} \longrightarrow f(x)$

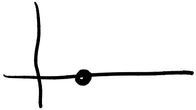
y

$$f(x) = mx + b$$

To find the root we need to solve (find x-intercept) $= m\cancel{x}+b$

$$\begin{aligned} 0 &= m\boxed{x} + b \\ -b &= mx \\ \frac{-b}{m} &= x \end{aligned}$$

solve for x



The value $\boxed{-b/m}$ is the root of $f(x) = mx + b$.

x intercept

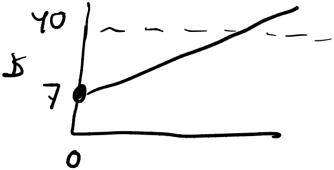
Solving Linear Equations

Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?

Let's take x is hours and $f(x)$ total price.



$$f(x) = \underline{7} + 11x$$

How long can you bowl?

$$
\begin{aligned}
40 &= 11x + 7 \\
-7 & \qquad\quad -7 \\
33 &= 11x \\
\div 11 & \qquad \div 11 \\
3 &= x
\end{aligned}
$$

Solving Linear Equations

Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?
Let's take x is hours and $f(x)$ total price.

$$f(x) = 7 + 11x$$

How long can you bowl?

$$\begin{aligned} 40 &= 11x + 7 \\ 40 - 7 &= 11x \end{aligned}$$

Solving Linear Equations

Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?
Let's take x is hours and $f(x)$ total price.

$$f(x) = 7 + 11x$$

How long can you bowl?

$$\begin{aligned} 40 &= 11x + 7 \\ 40 - 7 &= 11x \\ 33 &= 11x \end{aligned}$$

Solving Linear Equations

Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?
Let's take x is hours and $f(x)$ total price.

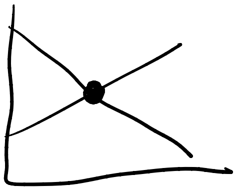
$$f(x) = 7 + 11x$$

How long can you bowl?

$$\begin{aligned} 40 &= 11x + 7 \\ 40 - 7 &= 11x \\ 33 &= 11x \\ 33/11 &= 3 = x \end{aligned}$$

Solving Systems of Linear Equations

We often are interested in finding the values of x and y where two lines cross. This is called solving the system of linear equations. A common example is supply and demand curves.



Solving Systems of Linear Equations

We often are interested in finding the values of x and y where two lines cross. This is called solving the system of linear equations. A common example is supply and demand curves.

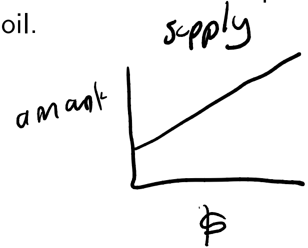
Supply Curve: As the price of oil increases, producers are willing to provide more to the market.

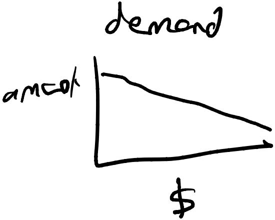
Solving Systems of Linear Equations

We often are interested in finding the values of x and y where two lines cross. This is called solving the system of linear equations. A common example is supply and demand curves.

Supply Curve: As the price of oil increases, producers are willing to provide more to the market.

Demand Curve: As price increases, consumers will demand less oil.



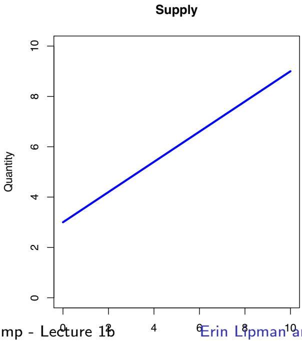


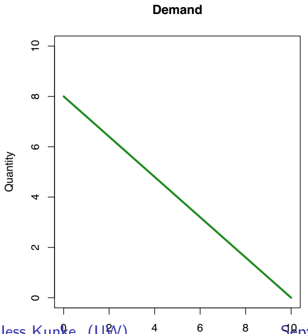
Solving Systems of Linear Equations

We often are interested in finding the values of x and y where two lines cross. This is called solving the system of linear equations. A common example is supply and demand curves.

Supply Curve: As the price of oil increases, producers are willing to provide more to the market.

Demand Curve: As price increases, consumers will demand less oil.

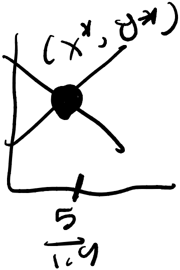




Solving Systems of Linear Equations

There are many ways to approach solving linear equations. We are interested in finding the point (x, y) that falls on both lines. If Supply is $\boxed{y = 3 + 0.6x}$ and Demand is $\boxed{y = 8 - 0.8x}$ we could take the following approach:

$y^* = 3 + 0.6x^*$ $\qquad$ $y^* = 8 - .8x^*$



$$3 + 0.6x = 8 - 0.8x$$

$$-3 \qquad\qquad -3$$

S

$$0.6x = 5 - .8x$$

A

$$1.4x = 5$$

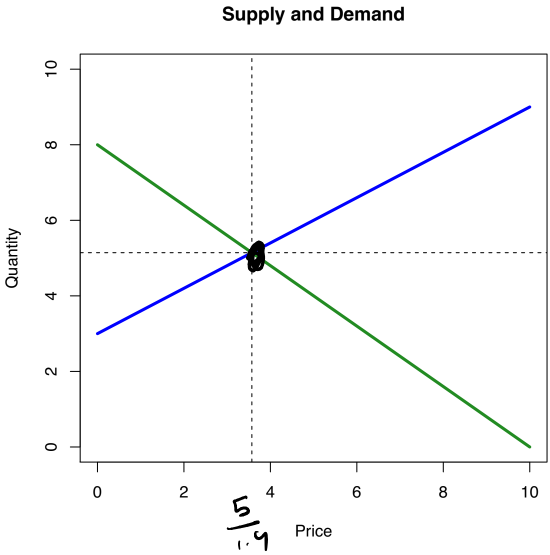
P

$$x = \frac{5}{1.4}$$

The y-value is found using either equation:

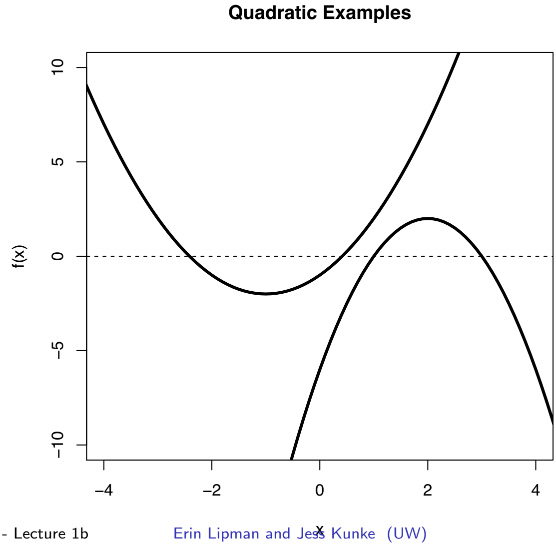
$$y = 3 + .6 \times \frac{5}{1.4}$$

Solving Systems of Linear Equations



Quadratic Equations

A *quadratic* function has the form $\boxed{f(x) = ax^2 + bx + c.}$ The quadratic function is associated with the parabola.



Quadratic Equations

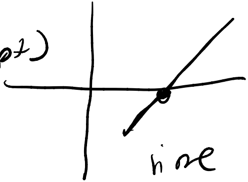
Finding Roots

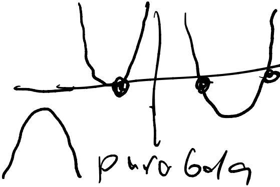
(x intercept)

$f(x) = 0$

solve

$0 = ax^2 + bx + c$





For any quadratic equation $f(x) = ax^2 + bx + c$, we find the root(s) (values of x such that $f(x) = 0$) by using the 'quadratic equation':

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note: Quadratics may have only one root (both roots are the same) or no real root.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

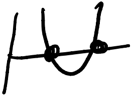
$$f(x) = ax^2 + bx + c$$

Quadratic Equations

Finding Roots

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The value of $\boxed{b^2 - 4ac}$ (called the *discriminant*) tells us how many roots the equation has



- If $b^2 - 4ac$ is positive, there will be two roots.
- If $b^2 - 4ac$ is zero, there will be one root. → $x = \frac{-b}{2a}$
- If $b^2 - 4ac$ is negative, there will be no real roots.

Examples:

$\underbrace{b^2}_{4^2} - 4ac$

- $2x^2 + 4x - 16 \Rightarrow 4^2 - 4 \cdot 2 \cdot (-16) = 144$; 2 roots; factors
- $3x^2 - 2x + 9 \Rightarrow (-2)^2 - 4 \cdot 3 \cdot 9 = -104$; no real roots

$b^2 - 4ac$

ex $x^2 - 2x + 1$

$b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0$

$x = \frac{-(-2)}{2 \cdot 1} = \frac{2}{2} = 1$

Quadratic Equations

Factoring and FOIL

$(x-4)(2x+2)$

Many quadratic equations can be factored into a more simple form. For example:

$x - 4 = 0 \rightarrow x = 4$

$$2x^2 - 6x - 8 = (x-4)(2x+2)$$

$2x + 2 = 0$
$2x = -2$
$x = -1$

To see that they are equivalent we can FOIL.

- **F**irst: $x \cdot 2x = 2x^2$
- **O**uter: $x \cdot 2 = 2x$
- **I**nner: $-4 \cdot 2x = -8x$
- **L**ast: $-4 \cdot 2 = -8$

Thus, $(x-4)(2x+2) = 2x^2 + 2x - 8x - 8 = 2x^2 - 6x - 8$

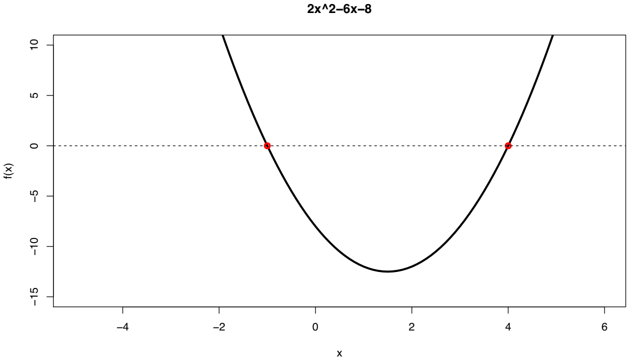
Quadratic Equations

Factoring and FOIL

When your quadratic has been factored you can find the roots by solving each term for zero. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

has roots when $x - 4 = 0$ and $2x + 2 = 0$. Thus, the roots are found at $x = -1, 4$.

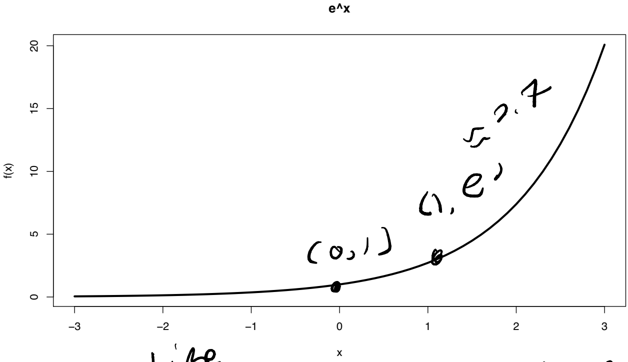


Exponential Functions

e^x $\quad f(x) = 2e^{-x}$ $\quad f(x) = e^{5x}$

Exponential Functions are of the form $f(x) = \boxed{ae^{bx}.}$ Often used as a model for population increase where $f(x)$ is the population at time x.

pop



time

$1 = 1$

$f(x) = e^1$

$= e$

≈ 2.7

$f(x) = e^x$

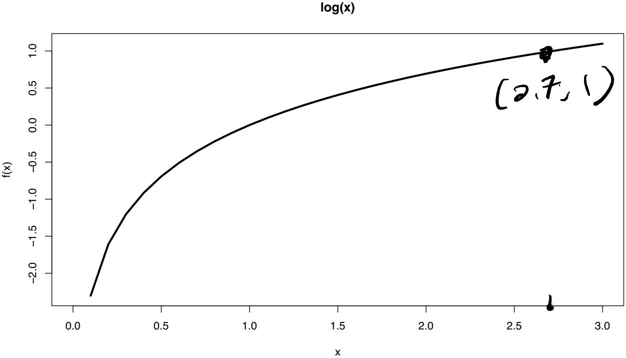
$x = 0$

$f(x) = e^0 = 1$

Logarithmic Functions

Logarithmic Functions, $\boxed{f(x) = c + d \cdot log(x)}$, can be used to find the time $f(x)$ necessary to reach a certain population x. It can be thought of as an 'inverse' of the exponential function.

time



pop

Note: $c = -1/b \cdot log(a)$ and $d = 1/b$ from the previous exponential model.

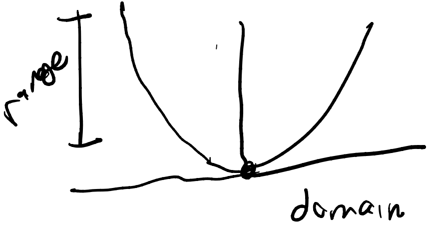
Domain and Range

A *function* is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

The *domain* of a function is the set of all possible values that you can plug into the function. The *range* is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- Domain: All real numbers $\mathbb{R}$



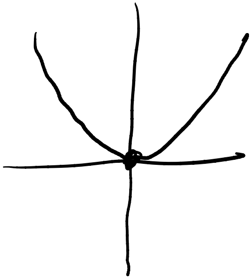
Domain and Range

A *function* is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

The *domain* of a function is the set of all possible values that you can plug into the function. The *range* is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- Domain: all real numbers $\mathbb{R}$
- Range:



Domain and Range

A *function* is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

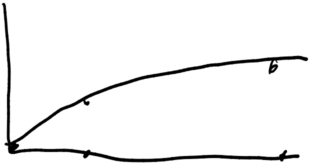
The *domain* of a function is the set of all possible values that you can plug into the function. The *range* is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- Domain: all real numbers $\mathbb{R}$
- Range: zero and all positive real numbers, $\boxed{f(x) \geq 0}$

Domain and Range

Examples



$f(x) = \sqrt{x}$

(0,0) (1,1)

- Domain: all positive #s and 0 $\boxed{x \geq 0}$

Range: all positive numbers and 0

$\boxed{f(x) \geq 0}$

Domain and Range

Examples

$f(x) = \sqrt{x}$

- Domain: zero and all positive real numbers, $x \geq 0$
- Range:

Domain and Range

Examples

ex

$f(x) = \frac{1}{x-1}$

undefined when

$x-1=0$

$\Rightarrow x=1$

$f(x) = \sqrt{x}$

- Domain: zero and all positive real numbers, $x \geq 0$
- Range: zero and all positive real numbers, $x \geq 0$

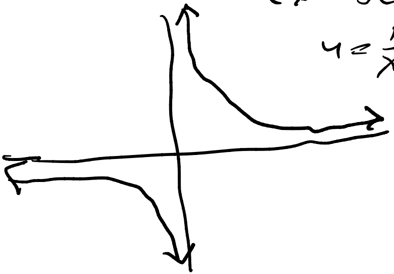
ex $f(x)=4$

$4=\frac{1}{x} \Rightarrow x=\frac{1}{4}$

$f(x) = 1/x$

- Domain:

All real #s
except 0



- Range:

All reall #s
except 0

Domain and Range

Examples

$f(x) = \sqrt{x}$

- Domain: zero and all positive real numbers, $x \geq 0$
- Range: zero and all positive real numbers, $x \geq 0$

$f(x) = 1/x$

- Domain: all real numbers except zero
- Range:

Domain and Range

Examples

$f(x) = \sqrt{x}$

- Domain: zero and all positive real numbers, $x \geq 0$
- Range: zero and all positive real numbers, $x \geq 0$

$f(x) = 1/x$

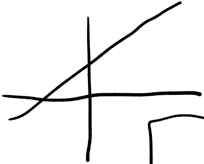
- Domain: all real numbers except zero
- Range: all real numbers except zero

Continuous & Piecewise Functions

A *continuous* function behaves without break or interruption. If you can follow the ENTIRE graph of a function with your pencil without picking it up, the function is continuous.Examples:

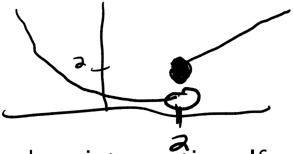
- $f(x) = x^2$

- $f(x) = x + 4$

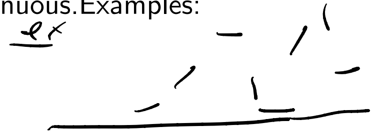


not continuous

Continuous & Piecewise Functions



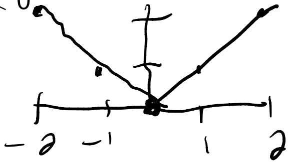
A *continuous* function behaves without break or interruption. If you can follow the ENTIRE graph of a function with your pencil without picking it up, the function is continuous.Examples:

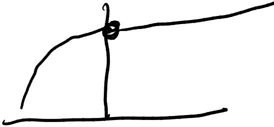


- $f(x) = x^2$
- $f(x) = x + 4$

A *piecewise* functioncan either have 'jumps' in it or can be made up of different functions for different parts of the domain (possible x-values). Example:

- Absolute Value $f(x) = |x|$ can be written as $f(x) = x, x \geq 0$ and $f(x) = -x, x < 0$





Limits

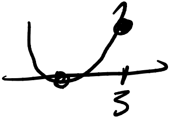
Often we are interested in what a function does as it approaches a certain value. This behavior is called the *limit*.

The limit of $f(x)$ as x approaches a is L:

$$lim_{x \to a} f(x) = L$$

It may be that a is not in the domain of $f(x)$ but we can still find the limit by seeing what value $f(x)$ is approaching as x gets very close to a. Examples:

- $lim_{x \to 3} x^2 = 9$ (3 is in the domain)
- $lim_{x \to \infty} (1 + 1/x)^x = e$



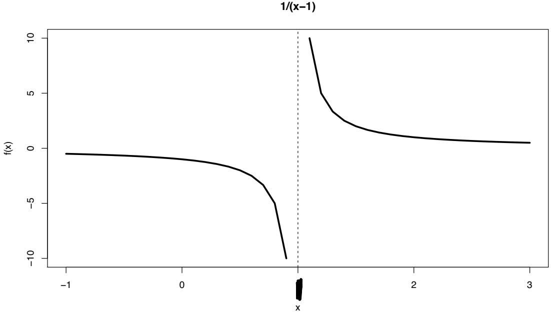
<ins>ex</ins> $\lim_{x \to \infty} x^2 = \infty$

$\lim_{x \to -\infty} x^2 = \infty$

Limits

Often limits are different depending on the direction from which you approach a. The limit 'from above' is approaching from the right ($x \downarrow a$) and the limit 'from below' ($x \uparrow a$)is approaching from the left.

$\lim_{x \to \infty} \frac{1}{x-1} = 0$



If $f(x) = \frac{1}{x-1}$ we have $lim_{x\downarrow 1} \frac{1}{x-1} = \infty$ and $lim_{x\uparrow 1} \frac{1}{x-1} = -\infty$

The End

Questions?

$f(x) = \frac{1}{x}$

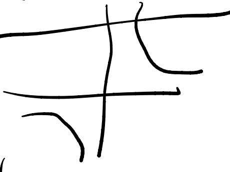
$\lim_{x \to \infty} \frac{1}{x} = ?$

ex

$f(x) = x$

$\lim_{x \to \infty} f(x)$

$f(\infty) = \infty$



$x = 1 \to f(x) = \frac{1}{1} = 1$

$x = 10 \to f(x) = \frac{1}{10} = .1$

$x = 100 \to f(x) = \frac{1}{100} = .01$

$x = 10000 \to f(x) = \frac{1}{10000} = .0001$

$\frac{1}{\text{very big \#}} \approx 0$