

# CSSS Math Camp Lecture 3

Differential Calculus

Authored by: Laina Mercer, PhD

Erin Lipman and Jess Kunke

Department of Statistics

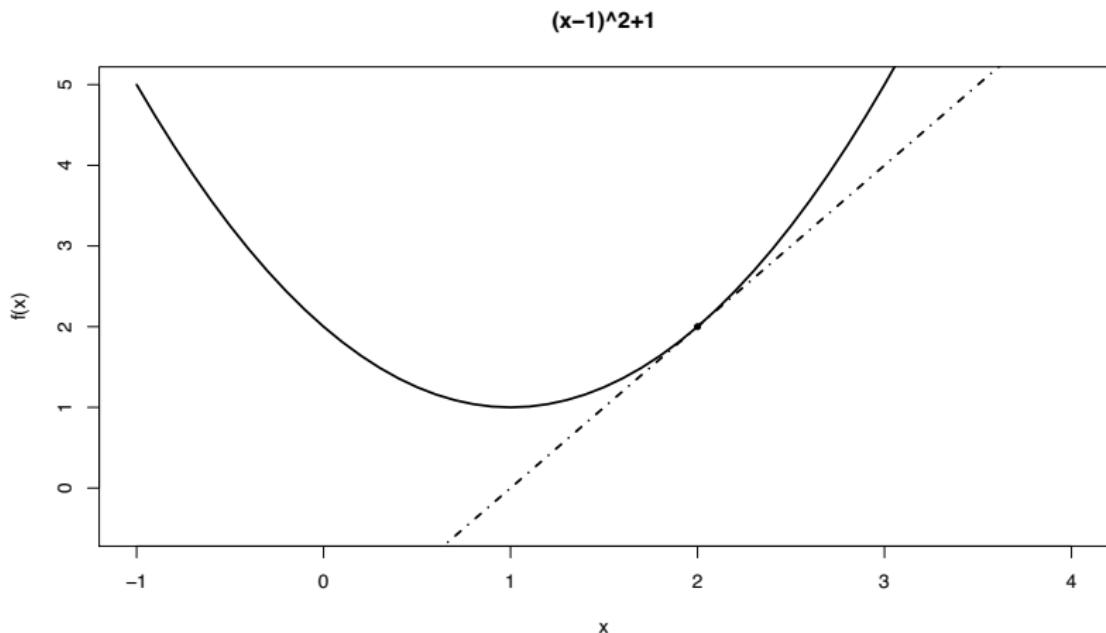
September 13, 2023

# Outline

- Differentiation of functions
    - Defining the derivative
    - Basic differentiation rules
    - Second, third, etc... derivatives
  - Critical points of functions
    - What is a critical point?
    - Maximum, minimum, and using the second derivative to tell the difference
- [conceptual]*  
*[practically]*

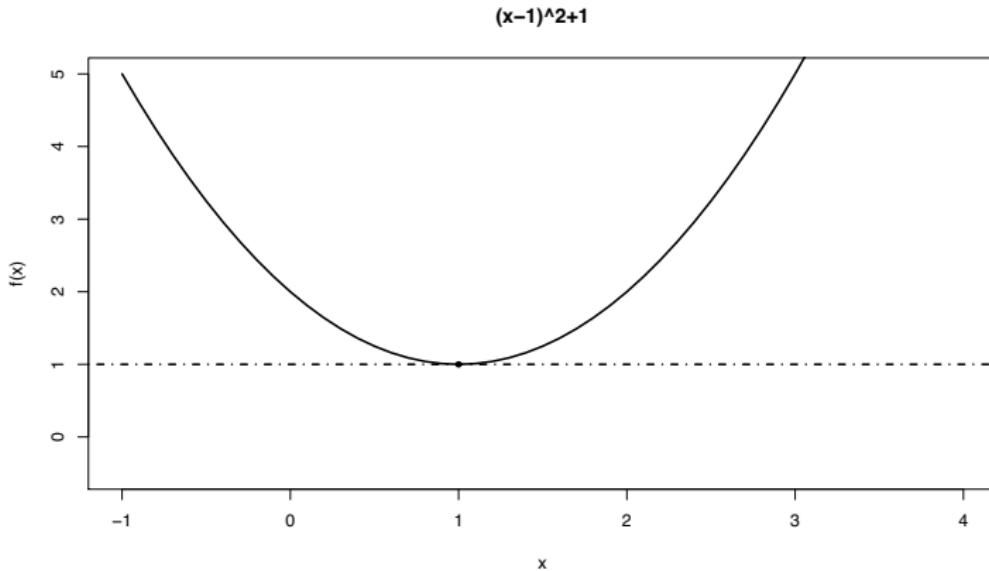
## What is the derivative?

The derivative can be thought of as the slope of the line tangent to  $f(x)$  at the point  $x$ . (Skims the curve, touching only at the point  $x$ ).



## What is the derivative?

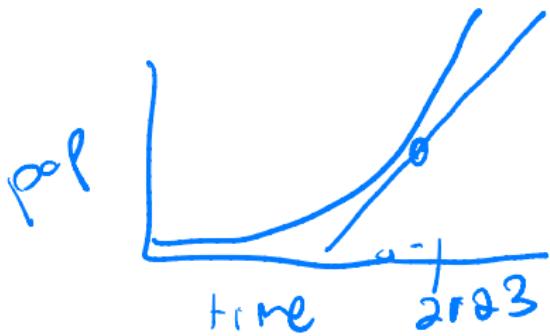
In statistics we are often interested in derivatives to help us find the values that maximize (or minimize) functions. We will be particularly interested in the values  $x$  such that the derivative is zero.



# Finding the Derivative

The derivative of a function  $f(x)$  is the instantaneous rate at which the function is changing at  $x$ . Why would we be interested in finding the derivative of a function?

- growth rate of a population relative to change in time
- change in distance relative to a change in time
- marginal revenue - change in amount of money from item sales relative to change in demand for the items



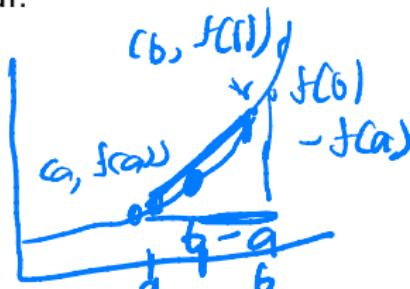
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Think of it as finding the 'slope' of a function at specific point. To find the average rate of change over an interval  $[a, b]$ , we look at the change in  $f(x)$  over the length of the interval.

$$\frac{f(b) - f(a)}{b - a}$$



# Finding the Derivative

So if we want to find the rate of change at a value  $x$ , we find the average rate of change over a very small interval (usually of length  $\delta$ ).

$$\frac{f(x + \delta) - f(x)}{\delta}$$

$(x + \delta) - x = \delta$

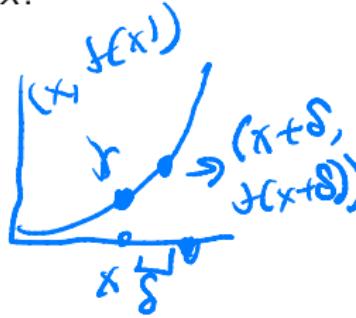
We look at what happens when  $\delta$  becomes very very small, i.e. when the interval essentially just becomes the point  $x$ .

The derivative of  $f(x)$  at  $x$  is then:

$$\lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta}$$

It is denoted by  $\boxed{\frac{d}{dx} f(x)}$  or  $\boxed{f'(x)}$ .

$$\frac{df}{dx}(x)$$



# Differentiation with Limits

Given an  $f(x)$ , how do we find the derivative  $f'(x)$ ?

$$f'(x) = \lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta}$$

to start, let's write out the algebra and then take the limit.

Example:  $f(x) = mx + b$

$$\begin{aligned} f'(x) &= \lim_{\delta \rightarrow 0} \frac{[m(x + \delta) + b] - [mx + b]}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{\cancel{mx} + m\cancel{\delta} + b - \cancel{mx} - \cancel{b}}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{m\cancel{\delta}}{\delta} \\ &= \lim_{\delta \rightarrow 0} m = \boxed{m} \end{aligned}$$

# Differentiation with Limits

Example:  $f(x) = x^2$



$$f'(x) = \lim_{\delta \rightarrow 0} \frac{f(x+\delta) - f(x)}{\delta}$$

$$= \lim_{\delta \rightarrow 0} \frac{(x+\delta)^2 - x^2}{\delta}$$

$$(x+\delta)(x+\delta)$$

$$= x^2 + x\delta + x\delta + \delta^2$$

$$= x^2 + 2x\delta + \delta^2$$

$$= \lim_{\delta \rightarrow 0} \frac{x^2 + 2x\delta + \delta^2 - x^2}{\delta}$$

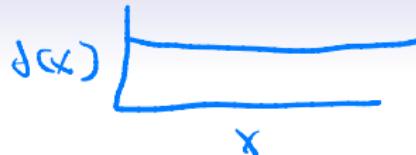
$$= \lim_{\delta \rightarrow 0} \frac{2x\delta + \delta^2}{\delta} = \lim_{\delta \rightarrow 0} 2x + \delta$$

$$= \boxed{2x}$$

# Differentiation Rules

## Special functions

Derivative of a constant:



$$f(x) = a; \quad f'(x) = 0$$

# Differentiation Rules

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Derivative of a power:

$$f(x) = x^n; \quad f'(x) = nx^{n-1}$$

$$\begin{aligned} f(x) &= x^2 \\ f'(x) &= 2x^{2-1} \\ &= 2x^1 \\ &= 2x \end{aligned}$$

# Differentiation Rules

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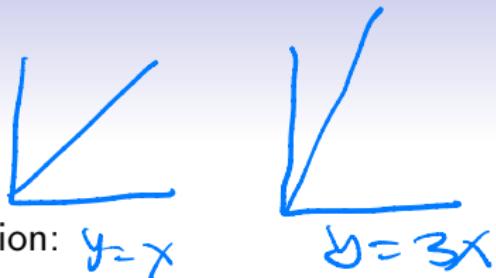
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Derivative of an exponential:

$$f(x) = e^x; \quad f'(x) = e^x$$

# Differentiation rules

Sums and scalar multiples of functions



Derivative of a constant times a function:  $y = x$

$$f(x) = a \cdot g(x); \quad f'(x) = a \cdot g'(x)$$

# Differentiation rules

## Sums and scalar multiples of functions

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Derivative of a sum of Functions:

$$f(x) = g(x) + h(x); \quad f'(x) = g'(x) + h'(x)$$

# Differentiation rules

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Examples:

- $f(x) = 3e^x; f'(x) = 3e^x$
- $f(x) = 2\log(x); f'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$
- $f(x) = 3x^{-2} + 4x; f'(x) = 3 \cdot (-2)x^{-2-1} + 4x^0$   
$$\begin{aligned} &= -6x^{-3} + 4 \\ &= -\frac{6}{x^3} + 4 \end{aligned}$$

# Differentiation rules

## Product Rule

Derivative of the product of two functions:

$$f(x) = g(x) \cdot h(x); \quad f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x)$$

# Differentiation rules

## Product Rule

Derivative of the product of two functions:

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Examples:

$$\bullet f(x) = x^2 \cdot e^x \quad g(x) = x^2 \quad h(x) = e^x$$

$$g'(x) = 2 \cdot x \quad h'(x) = e^x$$

$$\bullet f(x) = 3x \cdot \log(x)$$

$$g(x) = 3x \quad h(x) = \log(x)$$

$$g'(x) = 3 \quad h'(x) = \frac{1}{x}$$

$$f'(x) = 3 \log(x) + \frac{1}{x} \cdot 3x$$

## Differentiation rules

### Quotient Rule

$$\frac{\text{lo } d\text{-hi} - \text{hi } d\text{-lo}}{\text{square the bottom}}$$

Derivative of the division of two functions:

$$f(x) = \frac{g(x)}{h(x)}; \quad f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{h(x)^2}$$

# Differentiation rules

Quotient Rule

$$\frac{a^k}{e^x} = a^{k-x}$$

Derivative of the division of two functions:

$$f(x) = \frac{g(x)}{h(x)}; \quad f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{h(x)^2}$$

Examples:

$$\bullet f(x) = \frac{x^2}{e^x} \quad \frac{e^x(2x) - x^2 e^x}{(e^x)^2} = \frac{e^x x(2-x)}{e^{2x}/e^x}$$

$g(x) = x^2 \quad h(x) = e^x$

$$\bullet f(x) = \frac{3x}{\log(x)} \quad \frac{3 - x}{\log(x)} = \frac{x(2-x)}{e^x}$$

$g'(x) = 3x \quad h'(x) = \log(x)$

and after

$$\frac{g(x) = 3x \quad h(x) = \log(x)}{g(x) = x \quad h(x) = \frac{\log(x)}{3}}$$

# Differentiation rules

## Chain Rule

$$x \xrightarrow{h(y) = 3y} 3x \xrightarrow{g(z) = e^z} e^{3x} \\ g(h(x))$$

Derivative of a function within a function:

$$f(x) = g(h(x)); \quad f'(x) = g'(h(x)) \cdot h'(x)$$

Examples:

$$\bullet f(x) = e^{3x}$$

$$g(z) = e^z \quad h(y) = 3y$$

$$g'(z) = e^z \quad h'(y) = 3$$

$$\bullet f(x) = \log(1 - x)$$

$$g(z) = \log(z) \quad h(y) = 1 - y$$

$$g'(z) = 1/z \quad h'(y) = -1$$

$$\bullet f(x) = (2x + 1)^2$$

$$f'(x) = e^{h(x)} \cdot 3$$

$$= e^{3x} \cdot 3$$

$$= 3e^{3x}$$

$$\boxed{f''(x) = \frac{1}{h(x)} \cdot (-1)}$$

$$= -\frac{1}{1-x} = -\frac{1}{-(1-x)} = \frac{1}{x-1}$$

$$x \xrightarrow{h(y) = 1-y} 1-x \xrightarrow{g(z) = \log z} \log(1-x)$$

$$f(x) = g(b(x))$$

$$f'(x) = g'(b(x)) \cdot b'(x)$$

$$f(x) = (3x+1)^4$$

$$x \xrightarrow{h(y)=3y+1} 3x+1 \xrightarrow{g(z)=z^4} z=3x+1 \\ g'(z)=4(3x+1)^3$$

$$g(z)=z^4 \quad h(y)=3y+1$$

$$\underline{g'(z)=4z^3} \quad h'(y)=3+0$$

$$f'(x) = \underline{g'(h(x)) \cdot h'(x)}$$

$$= 4h(x)^3 \cdot 3$$

$$= 4(3x+1)^3 \cdot 3$$

$$= 12(3x+1)^3$$

# Differentiation rules

## Examples

We can combine many rules, What rules could we combine to find  $f'(x)$  for the following function?

$$f(x) = 3x(2x + 1)^4$$

# Differentiation rules

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$$f(x) = 3x(2x + 1)^4$$

will require the product rule and the chain rule, where  $g(x) = 3x$ ,  $k(x) = 2x + 1$ , and  $h(k) = k^4$ .

$$f(x) = \underbrace{3x}_{g} \cdot \underbrace{(2x+1)^4}_{h}$$

$$x \xrightarrow{k(x)} 2x+1 \xrightarrow{h(z)} (2x+1)^4$$

## Second & Third Derivatives

We can find the second derivative by taking the derivative of the derivative. The third derivative is found by taking the derivative of the second derivative and so on.

The second derivative is the rate of change of the first derivative and can be written as  $f''(x)$  or  $\frac{d^2}{dx^2} f(x)$ .

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Example:

$$\lambda \xrightarrow{y(x) = 4x} \xrightarrow{g(x) = \log(y)} \xrightarrow{\longrightarrow} \log(4x)$$

$$\begin{aligned}f(x) &= \log(4x) \\f'(x) &= \frac{1}{4x} \cdot 4 = \frac{1}{x} = x^{-1} \\f''(x) &= -x^{-2} = -\frac{1}{x^2} \\f'''(x) &= -(-2)x^{-3} = \frac{2}{x^3}\end{aligned}$$

# Differentiation rules

distance, velocity, acceleration

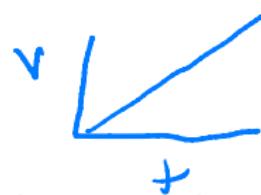
Let's take  $d$ =distance,  $v$ =velocity,  $a$ =acceleration. You may remember from physics, the distance travel after time  $t$

$$d(t) = \frac{a}{2}t^2$$



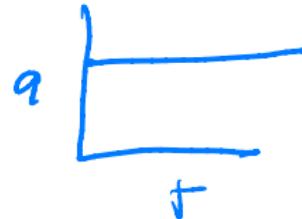
The velocity at any time  $t$  is the instantaneous rate of change of the distance,  $v(t) = d'(t)$ :

$$v(t) = 2 \cdot \frac{a}{2}t = at$$



The acceleration at any time  $t$  is the instantaneous rate of change of the velocity,  $a(t) = v'(t) = d''(t)$ :

$$a(t) = a$$



## Critical Values

Critical value occurs when the behavior of a function changes.

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A *maximum* occurs when a function stops increasing and starts to decrease.

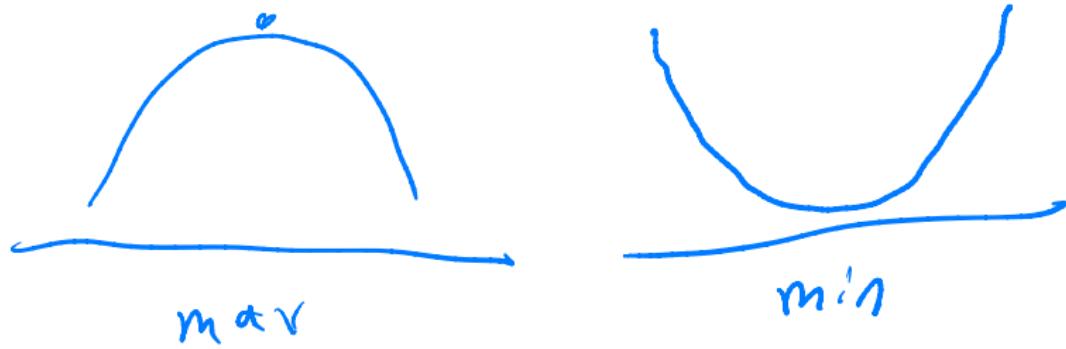


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If we are modeling a population or a behavior, knowing when the maximum or minimum occurs is very useful. In statistics, finding the maximum helps us find values of interest (Maximum Likelihood Estimates).

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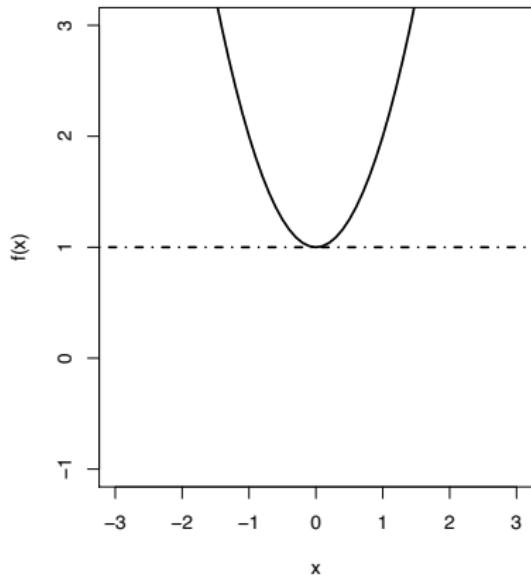
A *minimum* occurs when a function stops decreasing and starts increasing.

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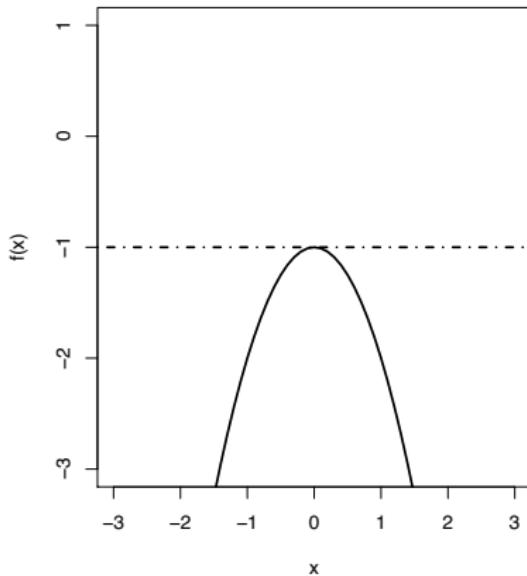
Mathematically, critical points are defined as the points where the derivative is zero. As a function passes through a critical point, the derivative goes from positive to negative (maximum) or negative to positive (minimum).

# Critical Values

$$f(x) = x^2 + 1$$



$$f(x) = -x^2 - 1$$



## Critical Values

We can use the first derivative to find the critical point by setting it equal to zero and then solving for  $x$ , the root. The goal is to find  $x$  such that  $f'(x) = 0$ .

However, as seen on the previous slide, the derivative is zero for maximums **and** minimums. How do we tell the difference?

## Critical Values

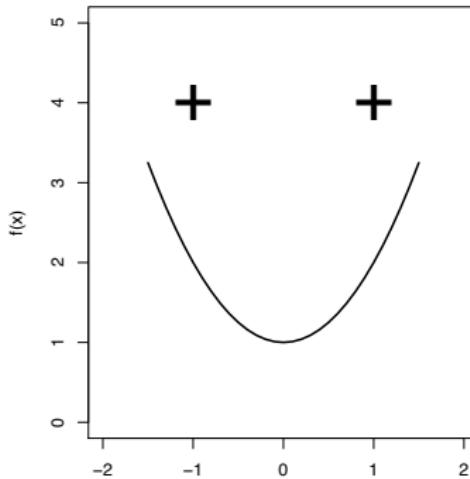
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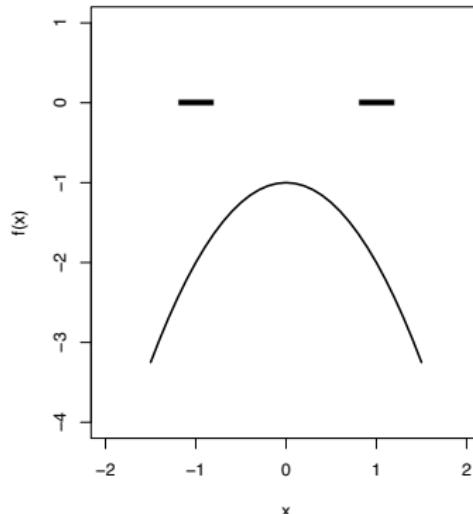
We use the second derivative.

## Critical Values

For the max, the derivative decreases from positive to negative, so the second derivative will be negative. For the min, the derivative increases from negative to positive, so the second derivative will be positive.



deriv -  $\rightarrow$  deriv +  
and deriv is +



# Critical Values

## Examples

$$f(x) = 8x^2 + 4x + 2$$

$$f'(x) = 16x + 4$$

$$0 = 16x + 4$$

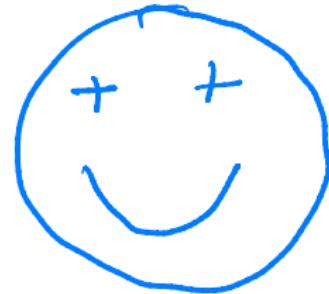
$$-4 = 16x$$

$$x = -\frac{4}{16} = -\frac{1}{4}$$

$$f''(x) = 16 > 0$$



$f(x)$  has a min at  $x = -\frac{1}{4}$



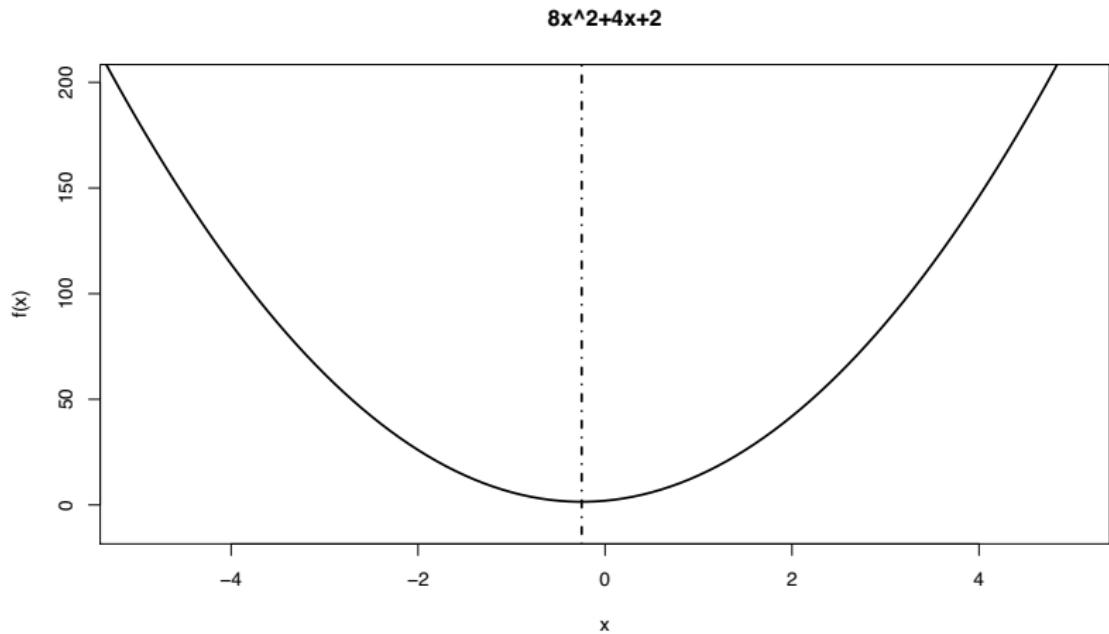
# Critical Values

## Examples

$$f(x) = 8x^2 + 4x + 2$$

# Critical Values

## Examples



# Critical Values

Examples

$$\begin{array}{l} 8 - 3(x^2 + 2x + 4) \\ \parallel = -3x^2 - 7x - 6 \end{array}$$

$$f(x) = 8 - 3(x + 2)^2$$

$$f'(x) = -6(x + 2)$$

$$0 = -6x - 12 \Rightarrow -6x = 12 \Rightarrow x = -2$$

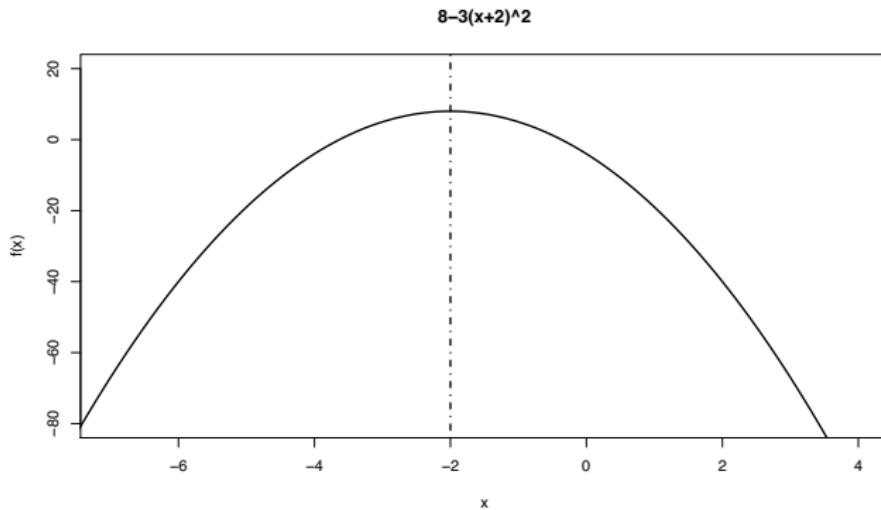
$$f''(x) = -6$$

The critical value is at  $x = -2$  and the second derivative is negative, so it is a maximum.



# Critical Values

## Examples



# Critical Values

## Saddle Points

If  $f''(x) = 0$ , then you have a *saddle point*. This is a critical point where the overall behavior of your function does not change.

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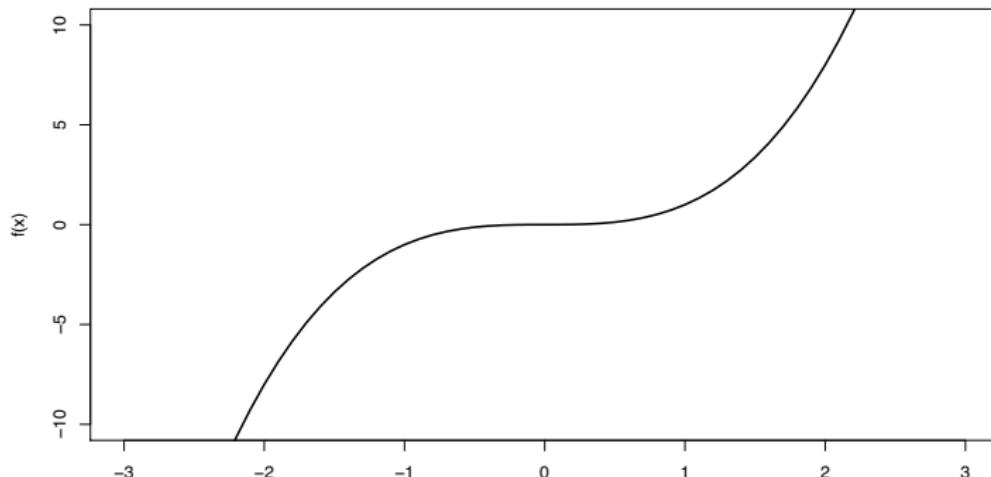
For example:

$f(x) = x^3$ ,  $f'(x) = 3x^2$ ,  $f''(x) = 6x$ . At  $x = 0$ , we have

$$f'(0) = f''(0) = 0.$$

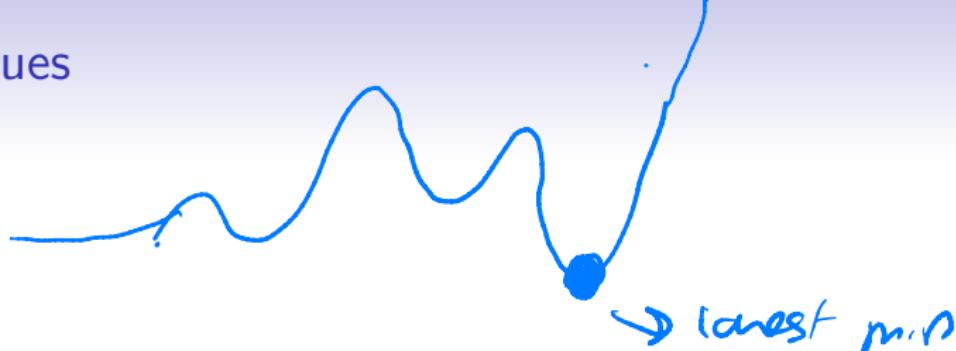
$$f'''(0) = 6 \cdot 0 = 0$$

$x^3$



# Critical Values

Global vs. Local

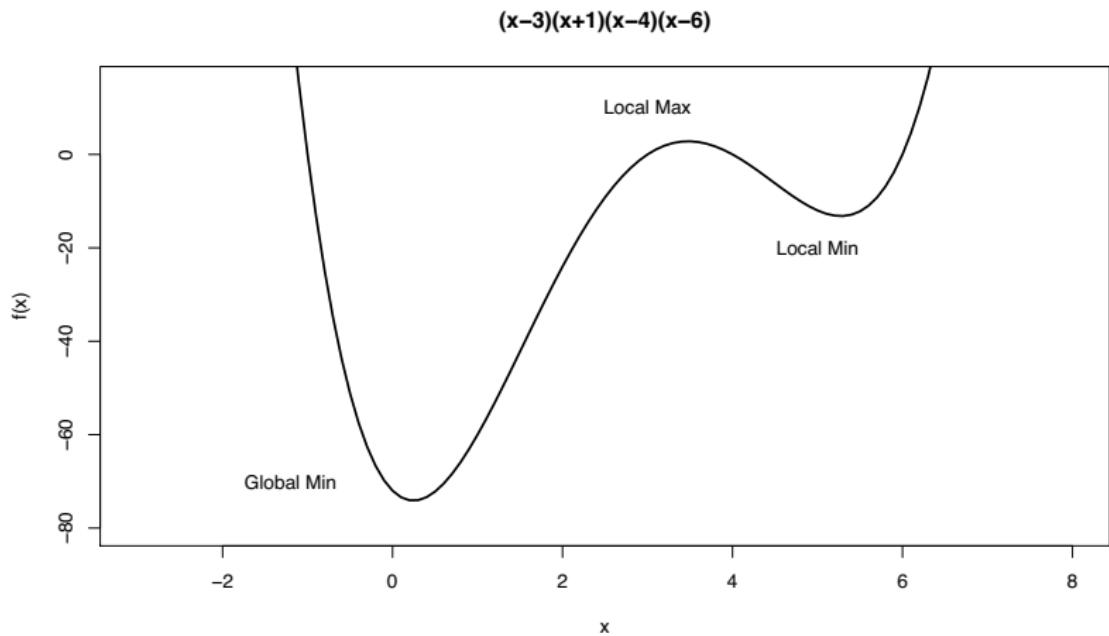


Some functions have more than one maximum or minimum.

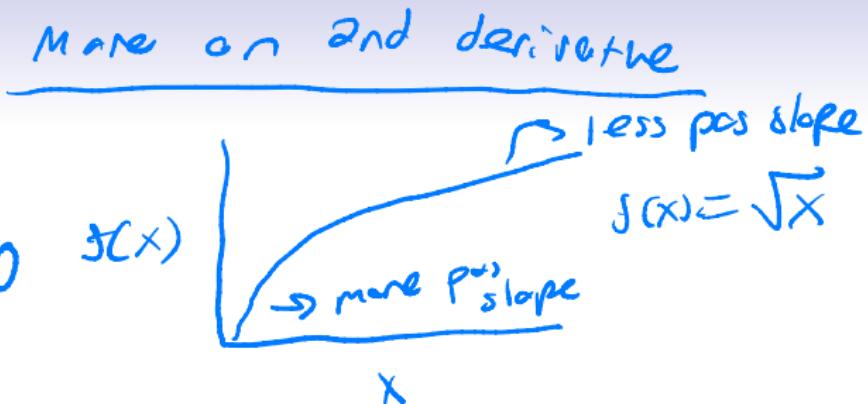
We call the largest maximum or the lowest minimum the *global* critical point. All others are referred to as *local* critical points. When looking, ideally we want to find the global maximum or minimum.

# Critical Values

Global vs. Local



The End  
 $f''(x) < 0$



Questions?



$f'(x)$  : positive

very +  $\rightarrow$  less +

$$\frac{ex}{10}$$

$$\frac{ex}{2}$$

decreasing

$f''(x)$  : negative