CSSS Math Camp Lecture 1a

Review of Math Notation and Algebra

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Introductions

- Name and pronouns
- Program / school / department
- One goal you have for math camp
- One thing you are nervous about for math camp (optional)

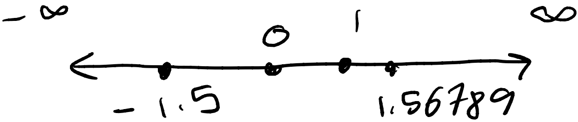
Math camp overview

Monday:	Algebra, Linear equations and systems of equations, Functions and limits
Tuesday:	Matrices
Wednesday:	Derivatives
Thursday:	Introduction to probabability
Friday:	Integrals

Lecture 1a: Review of Math Notation and Algebra

- Math notation
- Fractions
- Rules of exponents, logarithms
- Order of operations

Notation



Real Numbers

- Any number that falls on the continuous line. Often represented by a, b, c, d
- Examples: 2, 3.234, 1/7, $\sqrt{5}$, π = .1415...
- The set of real numbers is denoted by $\mathbb{R}$. Then $a \in \mathbb{R}$ means a is in the set of real numbers.

is in

OR

is an element of

Notation

Real Numbers

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Integers

- Any whole number. Often represented by i, j, k, l
- Examples: ...,-3,-2,-1,0,1,2,3,...

Notation

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Variables

- Can take on different values
- Often represented by x, y, z

Notation

Functions

- Often represented by f, g, h
- Examples: $f(x) = x^2 + 3$, $g(y) = 6y^2 - 2y$, $h(z) = z^3$

$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$

$$f(x) = x^2$$

$$f(3) = 3^2 = 9$$

Notation

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Summations

- Often represented by $\sum$ and summed over some integer
- Example:

$i=1$ $i=2$ $i=3$

$$\sum_{i=1}^{3}(i+1)^2 = (\boxed{1}+1)^2 + (\boxed{2}+1)^2 + (\boxed{3}+1)^2 = 2^2 + 3^2 + 4^2 = 29$$

end at 3

start at 1
(variable is i)

Notation

Functions

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Summations

- Often represented by $\sum$ and summed over some integer
- Example:

$4^2 = 4 \times 4 = 16$

$$\sum_{i=1}^{3}(i+1)^2 = (1+1)^2 + (2+1)^2 + \overbrace{(3+1)^2} = 2^2 + 3^2 + 4^2 = 29$$

$2^2 \quad 3^2 = 3 \times 3 \quad 4 + 9 + 16$

$4 + 9$

Products

- Often represented by $\prod$ and multiplied over some integer

$k = 1$

- Example: $\prod_{k=1}^{3}(y_k + 1)^2 = (y_1 + 1)^2 \times (y_2 + 1)^2 \times (y_3 + 1)^2$

y_1, y_2, y_3, y_4

Fractions

Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

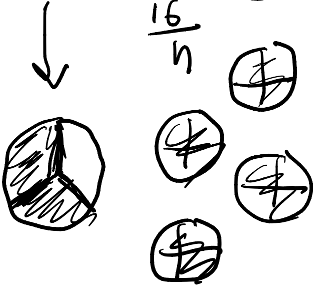
numerator → parts

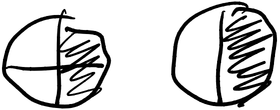
denominator → of a whole

All numbers can be written as fractions. Examples:
$\frac{2}{3}$, $\frac{16}{4}(=4)$, $\frac{2}{4}=\frac{1}{2}$, $\frac{8}{1}(=8)$.

$\frac{2}{4} = \frac{1}{2}$

$\frac{16}{4}$





$\frac{2}{4} = \frac{2/2}{4/2} = \frac{1}{2}$

combine 2 pieces into 1

Fractions

Multiplying & Dividing

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$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

All numbers can be written as fractions. Examples: $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4} = \frac{1}{2}, \frac{8}{1}(=8)$.

Multiplication: Multiply the numerators; multiply the denominators. Examples: $\frac{1}{2} \times \frac{3}{4} = \frac{1 \times 3}{2 \times 4} = \frac{3}{8}$

Fractions

Multiplying & Dividing

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$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

All numbers can be written as fractions. Examples: $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4} = \frac{1}{2}, \frac{8}{1}(=8)$.

Multiplication: Multiply the numerators; multiply the denominators. Examples: $\frac{1}{2} \times \frac{3}{4} =$

Division: Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction. Examples: $\frac{1/2}{7/8} =$ $\frac{1}{2} \times \frac{8}{7} = \frac{1 \times 8}{2 \times 7} = \frac{8}{14}$

= reciprical

flip numerator and denominator

Fractions

Multiplying & Dividing

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$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

All numbers can be written as fractions. Examples: $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4}=\frac{1}{2}, \frac{8}{1}(=8)$.

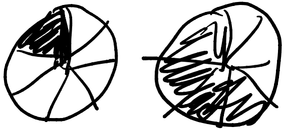
Multiplication: Multiply the numerators; multiply the denominators. Examples: $\frac{1}{2}\times\frac{3}{4}=$

Division: Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction. Examples: $\frac{1/2}{7/8}=$

Simplify: $\boxed{\frac{8}{14}} = \frac{8/2}{14/2} = \frac{4}{7}$

Fractions

Adding & Subtracting



Adding and subtracting requires that fractions must have the same denominator. If not, we need to find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add (or subtract) the two numerators.

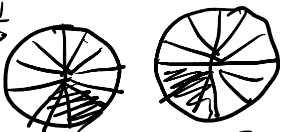
Examples:

$\frac{1}{7} + \frac{4}{7} =$ $\frac{1+4}{7} = \frac{5}{7}$

$\frac{1}{3} + \frac{1}{4} =$ $\frac{1\times 4}{3\times 4} + \frac{1\times 3}{4\times 3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

$\frac{17}{20} - \frac{3}{4} =$

$\frac{2/2}{20/2} = \frac{1}{10}$



$\frac{17}{20} - \frac{3\times 5}{4\times 5} = \frac{17}{20} - \frac{15}{20} = \frac{2}{20}$

$\frac{1\times 4}{3\times 4} = \frac{4}{12}$

$\frac{1}{4} \times \frac{3}{3} = \frac{3}{12}$

Exponents

$2+2+2 = 2 \times 3$ [mult is repeated addition]

$2 \times 2 \times 2 = 2^3$

a^n is 'a to the power of n'. a is multiplied by itself n times. Often a is called the base, n the exponent. Examples:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

$$6^4 = 6 \cdot 6 \cdot 6 \cdot 6 = 1296$$

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Exponents do not have to be whole numbers. They can be fractions or negative.
Examples:

$$4^{1/2} = \sqrt{4} = 2$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Common Rules

- $a^1 = a$

Common Rules

- $a^1 = a$
- $a^k \cdot a^l = a^{k+l}$

$$\underbrace{\underbrace{(a \times \cdots \times a)}_{k} \times \underbrace{(a \times \cdots \times a)}_{l}}_{k+l}$$

Common Rules

- $a^1 = a$
- $a^k \cdot a^l = a^{k+l}$
- $(a^k)^l = a^{kl}$

$$\underbrace{\underbrace{(\underbrace{a \times \cdots \times a}_{k}) \times \cdots \cdots \times (\underbrace{a \times \cdots \times a}_{k})}_{l}}_{k \times l}$$

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- $\left(\frac{a}{b}\right)^k = \left(\frac{a^k}{b^k}\right)$

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- $a^{-k} = \frac{1}{a^k}$ → definition

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- $\frac{a^k}{a^l} = a^{k-l}$

$$\frac{\overbrace{a \times \cdots \times a}^{k}}{\underbrace{a \times \cdots \times a}_{l}} = \underbrace{a \times \cdots \times a}_{k-l}$$

ex

$k=3$

$l=2$

$$\frac{a \times a \times a}{a \times a} = a$$

Common Rules

- $a^1 = a$
- $a^k \cdot a^l = a^{k+l}$
- $(a^k)^l = a^{kl}$
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- $\left(\frac{a}{b}\right)^k = \left(\frac{a^k}{b^k}\right)$
- $a^{-k} = \frac{1}{a^k}$
- $\frac{a^k}{a^l} = a^{k-l}$
- $a^{1/2} = \sqrt{a}$ $\longrightarrow$ definition

ex $4^{1/2} = \sqrt{4} = 2$

Common Rules

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- $a^k \cdot a^l = a^{k+l}$
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- $a^{-k} = \frac{1}{a^k}$
- $\frac{a^k}{a^l} = a^{k-l}$
- $a^{1/2} = \sqrt{a}$
- $a^{1/k} = \sqrt[k]{a}$

general version

$2 \times 2 \times 2 = 8$

ex $8^{1/3} = \sqrt[3]{8} = 2$

Common Rules

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- $a^k \cdot a^l = a^{k+l}$
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- $\frac{a^k}{a^l} = a^{k-l}$
- $a^{1/2} = \sqrt{a}$
- $a^{1/k} = \sqrt[k]{a}$
- $a^0 = 1$ → definition

$2^0 = 1 \qquad 37^0 = 1$

Logarithms

A logarithm is the power (x) required to raise a base (c) to a given number (a).

answer   exp

$$\log_c(a) = x \Rightarrow c^x = a \rightarrow \text{answer}$$

base   pows   base

Examples:

- $2^3 = 8 \Rightarrow \log_2(8) = 3$

↳ what power do we raise 2 to to get 8

Logarithms

A logarithm is the power (x) required to raise a base (c) to a given number (a).

$$\log_c(a) = x \Rightarrow c^x = a$$

Examples:

- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $4^6 = 4096 \Rightarrow \log_4(4096) = 6$

Logarithms

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$$\log_c(a) = x \Rightarrow c^x = a$$

Examples:

- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $4^6 = 4096 \Rightarrow \log_4(4096) = 6$
- $\underbrace{9^{1/2}}_{\sqrt{9}} = 3 \Rightarrow \log_9(3) = \frac{1}{2}$

Logarithms

The three most common bases are 2, 10, and $e = 2.718$, the natural logarithm. It is often called Euler's number after Leonhard Euler.

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Examples:

- $10^2 = 100 \Rightarrow \log_{10}(100) = 2$

Logarithms

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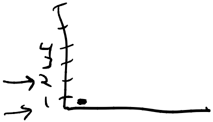
Examples:

- $10^2 = 100 \Rightarrow \log_{10}(100) = 2$
- $2^3 = 8 \Rightarrow \log_2(8) = 3$

Logarithms

10^1 10^2



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$10^1\ 10^2\ 10^3$ 10^{72}

The three most common bases are 2, 10, and $e = 2.718$, the natural logarithm. It is often called Euler's number after Leonhard Euler.

Examples:

- $10^2 = 100 \Rightarrow \log_{10}(100) = 2$
- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $e^2 = 7.3891... \Rightarrow \log(7.3891) = 2$

$\log = \log_e$

natural log

Logarithms

The three most common bases are 2, 10, and $e = 2.718$, the natural logarithm. It is often called Euler's number after Leonhard Euler.

Examples:

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- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $e^2 = 7.3891... \Rightarrow \log(7.3891) = 2$

The natural logarithm ($\log_e$) is the most common; used to model exponential growth (populations, etc). If no base is specified, i.e. $\log(a)$, most often the base is e. Sometimes written as $\ln(a)$.

$\underline{ex}$ $\log(e)$ $\rightarrow \log(e) = \log_e(e) = 1$ $e^1 = e$

Logarithms

What is e?

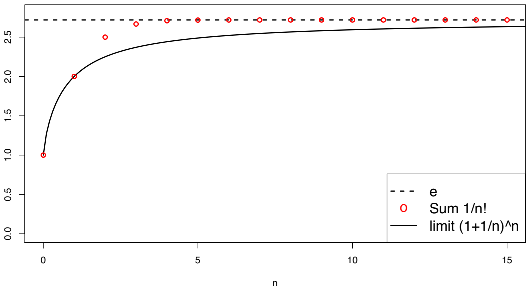
factorials

$5! = 5 \times 4 \times 3 \times 2 \times 1$ $3! = 3 \times 2 \times 1 = 6$ $0! = 1$

The number e is a famous irrational number. The first few digits are $e = 2.718282...$

Two ways to express e:

- $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
- $\sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e$



Logarithms

Rules

$$a^{k+l} = a^k a^l$$

$$\log(a \cdot b) = x$$
$$= x_1 + x_2$$
$$= \log_c(a) + \log_c(b)$$

$$\log_c(a \cdot b) = \log_c(a) + \log_c(b)$$

Proof:

$$\log_c(a \cdot b) = x \longrightarrow c^x = a \cdot b$$

$$x = x_1 + x_2$$

$$\longrightarrow c^{x_1 + x_2} = a \cdot b$$

$$\longrightarrow \underbrace{c^{x_1}}_{a} \underbrace{c^{x_2}}_{b} = a \cdot b$$

define

$$\log_c(a) = x_1 \qquad \log_c(b) = x_2 \longleftarrow\!\longrightarrow c^{x_1} = a \qquad c^{x_2} = b$$

Logarithms

Rules

$$\boxed{\log_c(a^n) = n \cdot \log_c(a)}$$

Proof for $n = 2$:

$$\begin{aligned}
x &= \log_c(a^2) \iff c^x = a^2 \\
&\Rightarrow c^{x_1+x_2} = a \cdot a \text{ where } x_1 + x_2 = x \\
&\Rightarrow c^{x_1} \cdot c^{x_2} = a \cdot a \Rightarrow c^{x_1} = a; c^{x_2} = a \\
&\Rightarrow x_1 = \log_c(a); x_2 = \log_c(a) \\
&\Rightarrow x = x_1 + x_2 \Rightarrow \log_c(a^2) = \log_c(a) + \log_c(a) = 2 \cdot \log_c(a)
\end{aligned}$$

Logarithms

Rules

$$\boxed{\log_c\left(\frac{a}{b}\right)} = \log_c(a) - \log_c(b)$$

Proof:

$$\begin{aligned}
x &= \log_c\left(\frac{a}{b}\right) \iff c^x = \frac{a}{b} \\
&\Rightarrow c^{x_1+x_2} = \frac{a}{b} \text{ where } x_1 + x_2 = x \\
&\Rightarrow c^{x_1} \cdot c^{x_2} = \frac{a}{b} \Rightarrow c^{x_1} = a; c^{x_2} = \frac{1}{b} = b^{-1} \\
&\Rightarrow x_1 = \log_c(a); x_2 = (-1) \cdot \log_c(b) \\
&\Rightarrow x = x_1 + x_2 \Rightarrow \log_c\left(\frac{a}{b}\right) = \log_c(a) - \log_c(b)
\end{aligned}$$

Logarithms

Examples

- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$

Logarithms

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- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$
- $\log_{10}\left(\frac{1000}{10}\right) = \log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$

Logarithms

Examples

- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$
- $\log_{10}\left(\frac{1000}{10}\right) = \log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$
- $\log_4(6^4) = 4 \cdot \log_4(6)$

Logarithms

Examples

- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$
- $\log_{10}\left(\frac{1000}{10}\right) = \log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$
- $\log_4(6^4) = 4 \cdot \log_4(6)$
- $\log(x^3) = 3 \cdot \log(x)$

Order of Operations

$1 + 1/2$

ex

$(1+1)/2$

$= 2/2$ P

$= 1$ D

Please **E**xcuse **M**y **D**ear **A**unt **S**ally

- **P**arentheses
- **E**xponents
- **M**ultiplication] together
- **D**ivision
- **A**ddition] together
- **S**ubtraction

$1 + 1/2$ D

$= 1 + 0.5$ A

$= 1.5$

Order of Operations

Examples

When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

- $\left((\underbrace{1+2})^3\right)^2 = (3^3)^2 = 27^2 = 729$

P (A) E E

Order of Operations

Examples

When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

- $\left((1+2)^3\right)^2 = (3^3)^2 = 27^2 = 729$
- $4^3 \cdot 3^2 - 10 + 27/3 = 64 \cdot 9 - 10 + 9 = 576 - 10 + 9 = 575$

Order of Operations

Examples

When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

- $\left((1+2)^3\right)^2 = (3^3)^2 = 27^2 = 729$
- $4^3 \cdot 3^2 - 10 + 27/3 = 64 \cdot 9 - 10 + 9 = 576 - 10 + 9 = 575$
- $(x+x)^2 - 2x + 3 = (2x)^2 - 2x + 3 = 4x^2 - 2x + 3$

Order of Operations

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When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

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