CSSS Math Camp Lecture 4

Integral Calculus

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September 14, 2023

Lecture 4: Integral Calculus

- Motivation for Integrals
- Definition of integration
- Rules of Integration

Differentiation Example

distance, velocity, acceleration

Let's take d=distance, v=velocity, a=acceleration. You may remember from physics, the distance traveled after time t

$$d(t) = \frac{a}{2}t^2$$

The velocity at any time t is the instantaneous rate of change of the distance, $v(t) = d'(t)$:

$$v(t) = 2 \cdot \frac{a}{2}t = at$$

The acceleration at any time t is the instantaneous rate of change of the velocity, $a(t) = v'(t) = d''(t)$:

$$a(t) = a$$

Distance

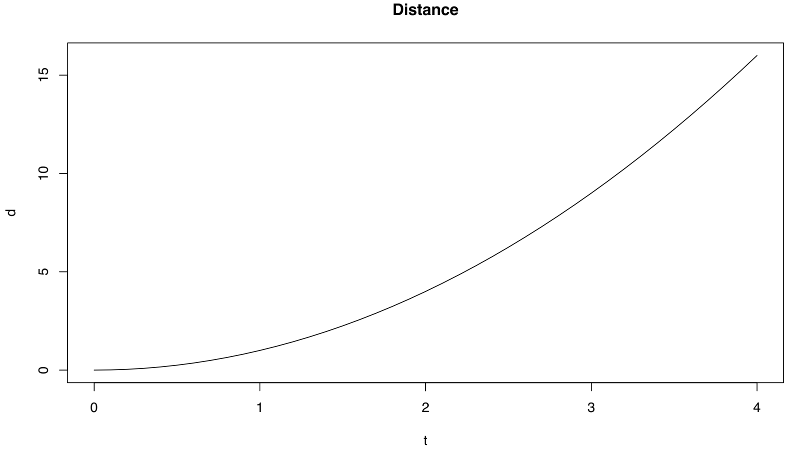


Figure: Distance over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Velocity

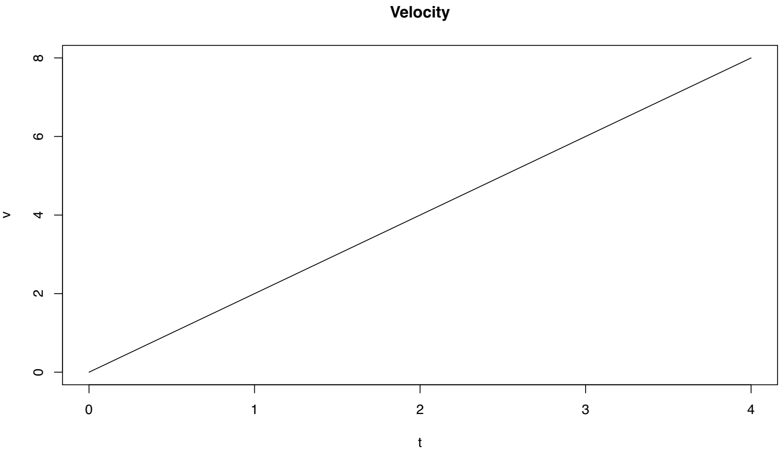


Figure: Velocity over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Acceleration

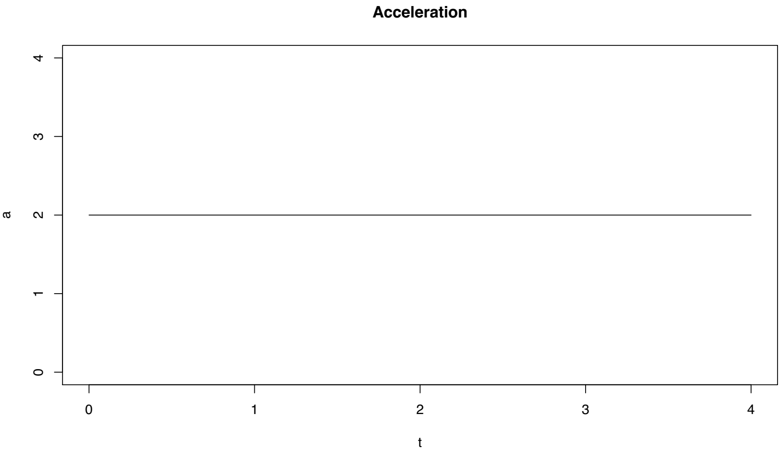


Figure: Acceleration over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

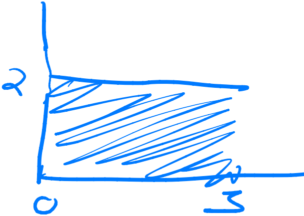
What is the velocity at t=3 when a=2?

We know that $v(t) = 2t$, so clearly

$$v(3) = 2 \cdot 3 = 6.$$

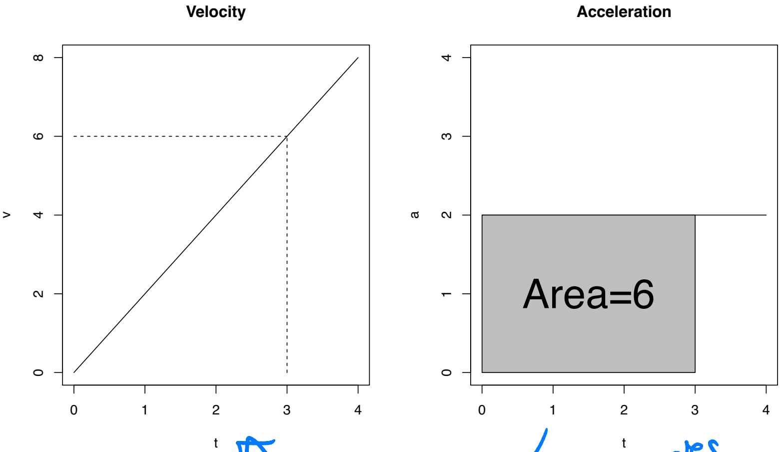
However we can also find the velocity, by looking at the area under the acceleration curve from $t = 0$ to $t = 3$. This would just be the area of a rectangle (`base X height`),

$$(3 - 0) \cdot 2 = 3 \cdot 2 = 6.$$



What is the velocity at t=3 when a=2?

derivative



gives change in the function

area undoes derivative

What is the distance at t=3 when a=2?

$3 \cdot 6/2$

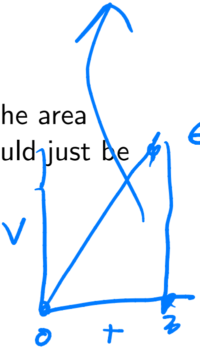
area $= 9$

We know that $d(t) = 2/2t^2 = t^2$, so clearly

$$d(3) = 3^2 = 9.$$

However we can also find the distance, by looking at the area under the velocity curve from $t = 0$ to $t = 3$. This would just be the area of a triangle (1/2 `X base X height`),

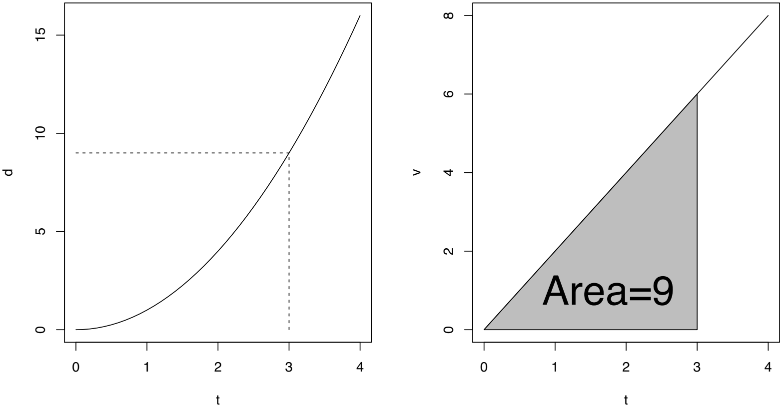
$$1/2 \cdot (3 - 0) \cdot 6 = 3/2 \cdot 6 = 18/2 = 9.$$



$d(3) = d(3) - d(0)$

$=$ area under velocity from 0 to 3

What is the distance at t=3 when a=2?



Motivation for Integrals in Statistics

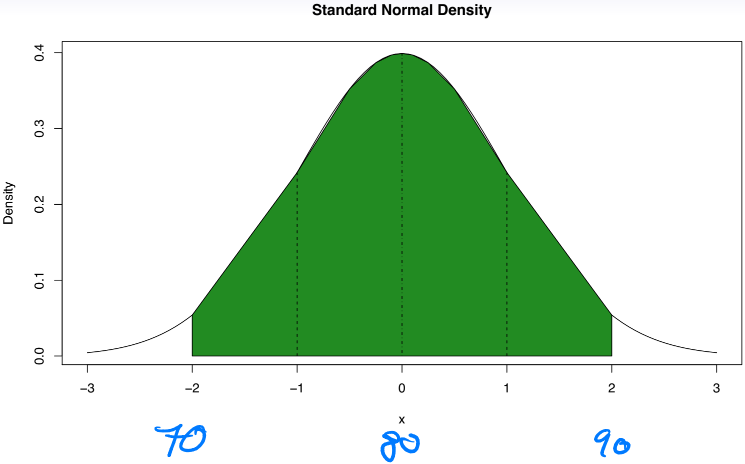


Figure: Standard Normal Density (N(0,1)). Approximately 68% of the probability lies within 1 standard deviation and 95% within 2 standard deviations. The area under the whole curve (from $-\infty$ to ∞) is 1.

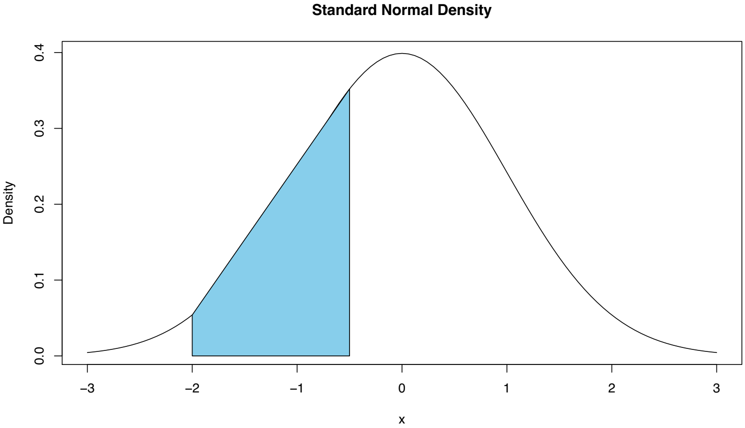
Motivation for Integrals in Statistics

Integral caclulus...

- is a tool for computing areas under curves.
- is used to compute probabilities of events.
- will be needed to compute the expected values and variance of probability distributions.
- is heavily used in statistical theory!

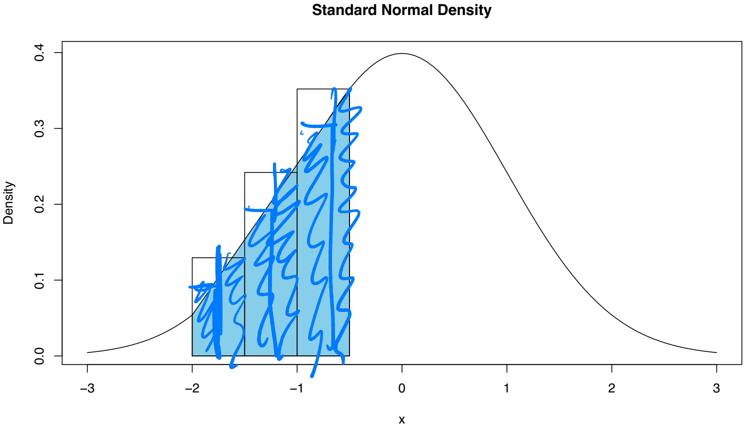
Motivation for Integrals in Statistics

What if we wanted to find the area under the curve from -2 to -0.5?

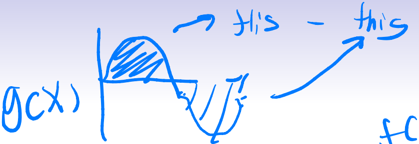


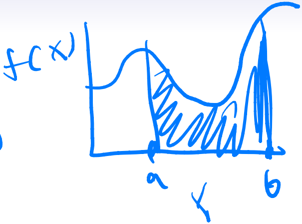
Motivation for Integrals in Statistics

We could approximate with rectangles or trapezoids. Narrower rectangles would give better approximations.



Integration





The area under a curve is written:

$$\int_a^b f(x)dx$$

This formula is called the *definite integral* of $f(x)$ from a to b.

Here a and b are our endpoints of interest. You can think of the integral as the 'opposite' of the derivative.

Integration

area under curve between a and b

how much a function changes between a and b

More specifically,

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

$F(x)$ is called the *indefinite integral* of $f(x)$. The important relationships between derivatives and integrals are:

$$F'(x) = f(x) \quad \& \quad \int f(x)dx = F(x)$$

↳ derivative of the indefinite integral is the function we are taking integral of

What is an integral?

You can think of integrating as looking at a derivative and trying to find the original function.

- $\int 3dx$. What function has a derivate equal to 3? $3x$
- $\int 2xdx$. What function has a derivate equal to $2x$? x^2
- $\int e^x dx$. What function has a derivate equal to e^x? e^x

In practice, you don't have to search for the right function. We have handy shortcuts (rules).

$\int x\,dx$

guess 1 : $f(x) = x^2$

$f'(x) = 2x$

guess 2 : $f(x) = \frac{1}{2}x^2$

$f'(x) = \frac{1}{2}(2x) = x$

$$\int c dx = cx$$

Examples:

- $\int 1 dx = x$
- $\int 6 dx = 6x$
- $\int y dx = yx$

constant

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

Examples:

- $\int x dx = \int x^1 dx = \frac{1}{1+1} x^2 = \frac{1}{2} x^2$
- $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-2+1} x^{-2+1}$

$$= \frac{1}{-1} x^{-1} = -x^{-1} = -\frac{1}{x}$$

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}} dx = \frac{1}{(1/2)+1} x^{(1/2)+1} = \frac{1}{3/2} x^{3/2} = \frac{2}{3} x^{3/2}$$

Exponential:

$$\int e^x dx = e^x$$

(Natural) Logarithm:

$$\int \frac{1}{x} dx = log(x)$$

Integration Rules

Multiple of a Function

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

$$\int af(x)dx = a \cdot \int f(x)dx = aF(x)$$

Examples:

- $\int 4x^2 dx =$ $4\int x^2 dx = 4\frac{1}{1+2}x^{1+2} = 4\frac{1}{3}x^3 = \frac{4}{3}x^3$
- $\int \frac{3}{x^2} dx =$ $3\int x^{-2}dx = 3\left(\frac{1}{-2+1}x^{-2+1}\right)$ $= 3(-x^{-1}) = \frac{-3}{x}$
- $\int \mu y dy =$

$$= \mu \int y\,dy = \mu\left(\frac{1}{2}y^2\right) = \frac{\mu}{2}y^2$$

$$\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx = F(x) + G(x)$$

Examples:

- $\int \left(4x + 3x^2\right) dx$

$= \int 4x\, dx + \int 3x^2 dx = 4\int x\, dx + 3\int x^2 dx$

$= 4\left(\frac{1}{2}x^2\right) + 3\left(\frac{1}{3}x^3\right)$

$= 2x^2 + x^3$

- $\int \left(e^x - \frac{2}{x}\right) dx =$

incorrect

$\boxed{\frac{1}{-1+1} x^{-1+1} \quad \frac{1}{0}x^0}$

$\int \frac{1}{x} dx = \int x^{-1} dx$

$= \log(x)$

$u = x^2$

$du = 2x\,dx$

$u = \frac{x}{3}$

$du = \frac{1}{3}dx$

Sometimes the function we are integrating is similar to a simpler function with an easy derivative.

For example, $\int \frac{1}{1-x}dx$ is similar to $\int \frac{1}{x}dx$ which we know is $log(x)$. Similar to the chain rule, we can think about functions within functions.

$u = 1 - x$

$du = -1 \cdot dx$

$du = -dx \rightarrow dx = -du$

$$\int \frac{1}{1-x}dx = \int \frac{1}{u}(-du) = -\int \frac{1}{u}du$$

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For example, $\int \frac{1}{1-x}dx$ is similar to $\int \frac{1}{x}dx$ which we know is $log(x)$. Similar to the chain rule, we can think about functions within functions.

Let's set $u = 1 - x$. If we differentiate the left with respect to u and the right with respect to x we have $du = -1dx$. Solving for dx we have $dx = -1du$. Now we can substitute these values into our original integral.

Integration Rules

u-substitution continued

Example: Find $\int \frac{1}{1-x} dx$ using the substitution $u = 1 - x$ (and so $du = -dx$).

First substitute $u = 1 - x$ and $du = -dx$ into the integral:

$$\int \frac{1}{1-x} dx = \int \frac{1}{u} \cdot (-1) du = -1 \int \frac{1}{u} du$$

Example: Find $\int \frac{1}{1-x} dx$ using the substitution $u = 1 - x$ (and so $du = -dx$).

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Now let's take the integral with respect to u:

$$-1 \int \frac{1}{u} du = -log(u)$$

Integration Rules

u-substitution continued

Example: Find $\int \frac{1}{1-x} dx$ using the substitution $u = 1 - x$ (and so $du = -dx$).

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$$\int \frac{1}{1-x} dx = \int \frac{1}{u} \cdot (-1) du = -1 \int \frac{1}{u} du$$

1 make substitution

Now let's take the integral with respect to u:

$$-1 \int \frac{1}{u} du = -log(u)$$

2 take integral

Then we can plug in the value for $u = 1 - x$:

$$-log(u) = -log(1-x)$$

3 substitute back

guess and check

guess 1: $F(x) = \frac{1}{4}(2x+4)^4$

$f'(x) = 4(\frac{1}{4})(2x+4)^3 \cdot 2$

$2(2x+4)^3$

Example:

$$\int (2x+4)^3 dx$$

1 $u = 2x+4$ $du = 2dx \longrightarrow \frac{1}{2}du = dx$

$\int (2x+4)^3 dx = \int u^3 (\frac{1}{2}du) = \frac{1}{2}\int u^3 du$

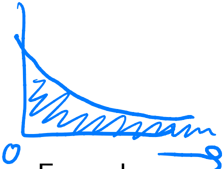
2 $\frac{1}{2}\int u^3 du = \frac{1}{2}\left(\frac{1}{4}u^4\right) = \frac{1}{8}u^4$

3 $\frac{1}{8}u^4 = \frac{1}{8}(2x+4)^4$

Finding Definite Integrals

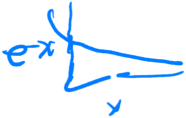
$$\int_0^{\infty} f(x)dx = \left(\lim_{x \to \infty} F(x)\right) - F(0)$$

Often we will be interested in knowing the exact area under the curve $f(x)$, not just the function $F(x)$.



$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

$\rightarrow F(x)\Big|_a^b$



$$\int x^2 dx = \frac{1}{3}x^3$$

Examples:

- $\int_0^1 x^2 dx = \left(\frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3} - 0 = \frac{1}{3}$
- $\int_0^{\infty} e^{-x} dx = \left(\int e^{-x}dx\right)\Big|_0^{\infty} = \left(-e^{-x}\right)\Big|_0^{\infty}$

$$= \lim_{x \to \infty}\left(-e^{-x}\right) - \left(-e^{-0}\right)$$

$$\frac{1}{e^x} \approx \frac{1}{\infty}$$

$$= 0 - (-1) = 1$$

- $\int_2^8 \frac{1}{x} dx =$

Back to our original example, with $a = 2$. The velocity at time $t = 3$ is the definite integral of of the acceleration, $v(3) = \int_0^3 a(t)dt$:

$$v(3) = \int_0^3 2dt = 2t|_0^3 = 2 \cdot 3 - 2 \cdot 0 = (3 - 0) \cdot 2 = 6$$

Integration Example

distance, velocity, acceleration

Back to our original example, with $a = 2$. The velocity at time $t = 3$ is the definite integral of of the acceleration, $v(3) = \int_0^3 a(t)dt$:

$$v(3) = \int_0^3 2dt = 2t\big|_0^3 = 2 \cdot 3 - 2 \cdot 0 = (3 - 0) \cdot 2 = 6$$

Similarly, the distance at time $t = 3$ is the definite integral of of the velocity, $d(3) = \int_0^3 v(t)dt$:

$$d(3) = \int_0^3 v(t)dt = \int_0^3 2tdt = t^2\big|_0^3 = 3^2 - 0^2 = 9$$

Example

$$\int_0^3 e^{x/3} dx$$

$u = x/3 \quad du = \frac{1}{3} dx \rightarrow 3du = dx$

Need to translate limits as well

original bounds: $x = 0$, $x = 3$

new bounds: $u = 0/3 = 0$, $u = 3/3 = 1$

$$\int_0^3 e^{x/3} dx = \int_0^1 e^u (3du) = 3\int_0^1 e^u du$$

$$= 3(e^u)\Big|_0^1 = 3(e^1 - e^0)$$

$$= \boxed{3(e-1)}$$

The End

Questions?