Outline

Probability models: an introduction

Random sampling

Infinitely many outcomes

Useful consequences of Kolmogorov's axioms

Random variables: an introduction

Useful consequences of the axioms

The following useful strategies are consequences of the rules of probability:

1. Partitioning an event
   - Decomposing an event into simpler pieces
2. Relating an event to its complement
   - Sometimes the probability of the complement is easier to compute
3. Leveraging the monotonicity of probability
   - If event B contains event A, then B must be as or more likely than A
4. Applying inclusion-exclusion rules
   - Relating the probability of a union to the probability of an intersection
5. Using de Morgan's law
   - Relating the complement of a union/intersection to the intersection/union of the complements

1. Partitioning an event

Definition

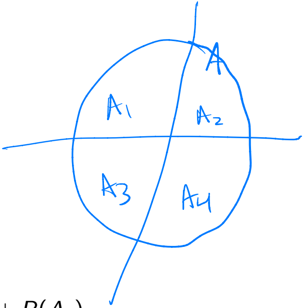
A **partition** of a set A is any collection of sets $\{A_1, ..., A_n\}$ such that

1. the sets are pairwise disjoint:

$$A_i \cap A_j = \emptyset \; \forall \; i \neq j$$

2. and their union is A:

$$\bigcup_{i=1}^{n} A_i = A.$$



Using the additivity of probabilities,

$$P(A) = P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1) + ... + P(A_n).$$

1. Partitioning an event

Let $A_W = \{1, 2\}$ Arlo wins
$E_W = \{4, 5, 6\}$ Emery wins
$|A_W| = 2$, $|A_W^c| = 4$, $|E_W| = |E_W^c| = 3$.

Example

Arlo and Emery take turns rolling a fair six-sided die.

1. Arlo wins if he rolls a 1 or 2
2. Emery wins if he rolls a 4, 5, or 6
3. Arlo rolls first

What is the probability that Arlo wins and rolls no more than 4 times?

Let $A = \{$Arlo wins & rolls ≤ 4 times$\}$
$= \{$Arlo wins on 1^{st} roll$\} \cup \{$Arlo wins on his 2^{nd} roll$\} \cup \ldots$
$= \bigcup_{i=1}^{4} \underbrace{\{\text{Arlo wins on his } i^{th} \text{ roll}\}}_{=A_i}$

Then A_1, A_2, A_3, A_4 partition A.

$\therefore P(A) = P(A_1) + P(A_2) + \ldots + P(A_4)$.

$P(A_k) = \frac{2}{6} \cdot \frac{4^{k-1} 3^{k-1}}{6^{k-1} 6^{k-1}} = \frac{1}{3^k}$

$P(A) = \sum_{k=1}^{4} \frac{1}{3^k} \approx 0.49$
$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$

Arlo	E	A	E	A	E	A

A_1: A_W $\quad |\Omega_1| = 6 \quad |A_1| = 2$

A_2: $A_W^c \; E_W^c \; A_W$ $\quad |\Omega_2| = 6 \cdot 6 \cdot 6 \quad |A_2| = |A_W^c||E_W^c| \cdot 2$

A_3: $A_W^c \; E_W^c \; A_W^c \; E_W^c \; A_W$

2. Relating an event to its complement

Either an event A happens or it doesn't happen (A^c), and this partitions the sample space:

$$A \cup A^c = \Omega$$

Therefore,

$$P(A) + P(A^c) = P(\Omega) = 1.$$

If the quantity of interest is $P(A)$, but computing $P(A^c)$ is easier, we can compute $P(A) = 1 - P(A^c)$.

Example

In five rolls of a fair six-sided die, what is the probability that any number appears more than once?

Let $A = \{$at least one number appears more than once$\}$

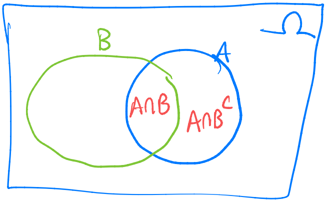
$A^c = \{$every roll is different$\}$

$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2}$

$$P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{6!}{6^5}$$

$$P(A) = 1 - P(A^c) = 1 - \frac{6!}{6^5} \approx 0.09.$$

2. Relating an event to its complement



Another use of set complements

Suppose you are interested in the probability of an event A. Sometimes it is helpful to partition A by intersecting it with another event B and its complement:

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

3. Leveraging the monotonicity of probability

If event B contains event A, then B must be as or more likely than A:

$$A \subseteq B \implies P(A) \leq P(B)$$

Example

Prove that repeated flips of a fair coin will eventually come up tails with probability 1.

$P\left(\begin{matrix}\text{eventually}\\ \text{tails}\end{matrix}\right) = 1.$

Proof.

Define

$P(A) = 0 \implies$

$$A = \text{never see tails},$$
$$A_n = \text{first } n \text{ flips come up heads}.$$

1 way for n flips to come up all heads; 2^n diff. sequences of n flips

A is the complement of the event we want to show has probability 1, so our goal is to show that $P(A) = 0$. *Which useful consequence did we just use?*

$A \implies A_n$ for any $n \in \mathbb{N}$, so which event is a subset of the other? (Look back at subset definition.) Does it go both ways, and why/why not?

$A \subseteq A_n$

we can find $\omega \in A_n$ s.t. $\omega \notin A \implies A_n \not\subseteq A$

Therefore, $P(\underline{A}) \leq P(\underline{A_n})$.

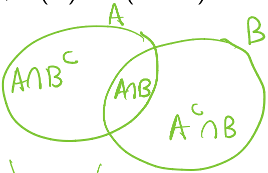
$P(A_n) = \underline{\ 2^{-n}\ } \implies \underline{P(A) \leq 2^{-n} \ \forall n \in \mathbb{N}}$

Therefore, $P(A) = 0$. □

Relating the probability of a union to the probability of an intersection:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof.
We'll use a Venn diagram.



$$P(A\cup B) = \underbrace{P(A\cap B^c) + P(A\cap B)}_{P(A)} + \underbrace{P(A^c\cap B) + P(A\cap B)}_{P(B)} \underbrace{- P(A\cap B)}_{-P(A\cap B)}$$

□

4. Applying inclusion-exclusion rules

$|\Omega| = \binom{15}{2}$ ways to choose 2 balls $\to \frac{15!}{13!\,2!} = \frac{15\cdot 14}{2} = 105.$

Example

A bag contains ~~10~~ 7 red, ~~4~~ 3 green, and ~~6~~ 5 yellow balls. Draw two without replacement. What is the probability that your sample contains exactly one red ball or exactly one yellow ball?

- Recall: "or" means "and/or", not exclusive "or"

$P(\{\text{ex. one red or exactly 1 yellow}\})$

$= P(\{\text{ex. one red}\}) + P(\{\text{ex. 1 yellow}\}) - P\left(\left\{\begin{array}{l}\text{ex. 1 red and} \\ \text{ex. 1 yellow}\end{array}\right\}\right)$

$$= \frac{\binom{7}{1}\binom{8}{1}}{\binom{15}{2}} + \frac{\binom{5}{1}\binom{10}{1}}{\binom{15}{2}} - \frac{\binom{7}{1}\binom{5}{1}}{\binom{15}{2}}$$

$$= \frac{56+50-35}{105} = \frac{71}{105} \simeq 0.68.$$

5. Using de Morgan's law

Relating the complement of a union/intersection to the intersection/union of the complements

Lemma

2D version:

For two subsets A, B of a set Ω,

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$$

nD version:

Given subsets $A_1, A_2, \ldots$ of a set Ω,

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c, \quad \left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c.$$

Proof.

$$\omega \in \left(\bigcup_i A_i\right)^c \overset{\text{iff}}{\Longleftrightarrow} \omega \notin \bigcup_i A_i$$

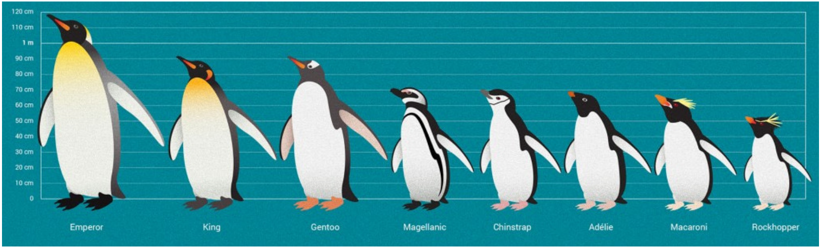
$$\Longleftrightarrow \omega \notin A_i \quad \forall i$$

$$\Longleftrightarrow \omega \in A_i^c \quad \forall i$$

$$\Longleftrightarrow \omega \in \bigcap_i A_i^c.$$

□

Putting these strategies together



Example

In a particular community of penguins, 20% of the penguins are rockhopper penguins and 15% of the penguins have interacted with humans. Rockhopper penguins who have interacted with humans make up 3% of the community. If you select one penguin uniformly at random, what is the probability of choosing a penguin that has never interacted with humans and is not a rockhopper?

- Tip: Define events and draw a Venn diagram.
- Try to solve it with and without de Morgan's law, and state which strategies you are using.