Chapter 2 Part 2: Conditional probability and independence

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MATH/STAT 394: Probability I (Summer 2022 A-term)

Outline

The full house problem

Recap of Monday's lecture

Conditional independence

Independent random variables, independent trials

Some common discrete distributions

Odds and ends

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The full house problem

Suppose you are dealt five cards from a standard 52-card deck. How many ways can you get a full house?

- Let's see two ways to solve this problem
- Start with an easier, related problem: choosing captain and co-captain from 12-member basketball team

3 cards
1 face value
3 diff suits

2 cards
1 face value
2 diff suits

12 players :

approach #1
unordered

- pick 2 leaders: $\binom{12}{2}$
- then pick which one is captain : 2
- for every choice of 2 leaders, there are 2 ways to pick which is captain

$\Rightarrow \binom{12}{2} \cdot 2 = \frac{12 \cdot 11}{2} \cdot 2 = 12 \cdot 11$

approach #2
ordered

$\underset{\text{captain}}{12} \cdot \underset{\text{co-captain}}{11}$

1st pick captain : 12
2nd pick co-capt : 11

___ ___ ___   ___ ___

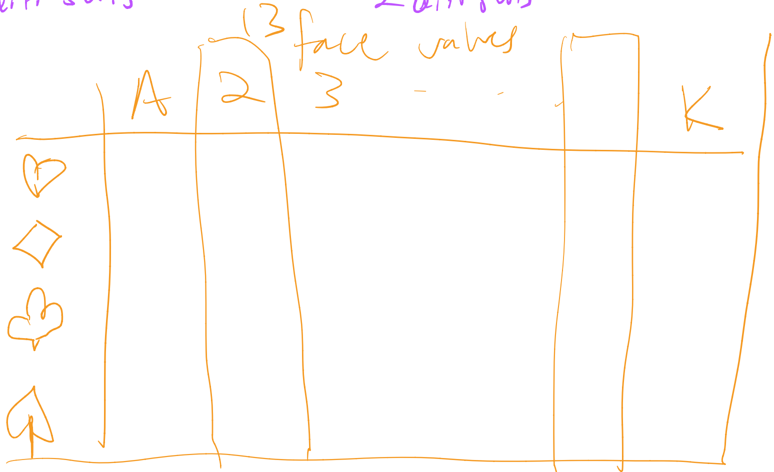
3 cards   2 cards

1 face value   1 face value

3 diff suits   2 diff suits

unordered!

4 suits



procedure: 1) pick 2 face values (columns)

$\Rightarrow \binom{13}{2}$ ways

2) choose which face value is for 3 cards

$\Rightarrow 2$ ways

3) choose 3 of 4 suits for the 3 cards $\Rightarrow \binom{4}{3}$ ways

4) choose 2 suits for the 2 cards

$\Rightarrow \binom{4}{2}$ ways

In total

$$\boxed{\binom{13}{2} \cdot 2 \cdot \binom{4}{3} \cdot \binom{4}{2}}$$

$= 3744$

ordered:

1. pick 2 cards & assign 1 to 3 card group & other to 2 card group
2. pick the other cards
3. divide by the # of ways to order the 3 cards (3!) & the 2 cards (2!)

5♡ 5♣ 5♠

5♠ 5♡ 5♣

$\underline{52} \cdot \underline{3} \cdot \underline{2} \times \underline{48} \cdot \underline{3} \div 3! \div 2!$

$= 3744$

3 cards	2 cards
1 face value	1 face value
3 diff suits	2 diff suits

like (captain) (cocaptain)

Outline

Recap of Monday's lecture

- **Conditional probability:** $P(A \mid B) = \frac{P(AB)}{P(B)}$
- **Multiplication rule:**

$$P(A_1, ..., A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2) \cdots P(A_n|A_1 \cdots A_n).$$

if $B_1, \ldots, B_n$ partition B instead of Ω, then $P(AB_1) + \ldots + P(AB_n) = P(AB)$

- **Law of total probability:** If $B_1, ..., B_n$ is a partition of Ω with $P(B_i) > 0 \ \forall\, i = 1, ..., n$, then for any event A we have

$$P(A) = \sum_{i=1}^{n} P(AB_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

- **Bayes' formula:** If $B_1, ..., B_n$ partition the sample space Ω and $P(B_i) > 0$ for all i, then for any event A with $P(A) > 0$, and any $k = 1, ..., n$,

$$P(B_k \mid A) = \frac{P(AB_k)}{P(A)} = \frac{P(A \mid B_k)P(B_k)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}.$$

- Two events A and B are **independent** if

$$P(AB) = P(A)P(B).$$

Independence

If two events A and B are independent, is their intersection empty (no overlap between A and B)?

i It has to be. (If so, show why)

ii It cannot be. (If so, show why)

iii It can be. (If so, demonstrate two events A and B for which it is and another choice of A and B for which it is not.)

Assume A & B are indep.

If $AB = \emptyset$ (no overlap), then $P(AB) = 0$

$\Rightarrow$ $P(A)$ or $P(B)$ or both equals zero,

because A & B are indep $(P(AB) = P(A)P(B))$.

when is $\underbrace{P(A|B)}$ $= P(A)$ another way to prove indep.?

undefined if $P(B) = 0$.

if $P(B) \neq 0$, then $P(B)P(A) = P(B)P(A|B) = \frac{\cancel{P(B)}P(AB)}{\cancel{P(B)}} = P(AB)$.

Recap of Monday's lecture, continued

- **Mutual** vs. **pairwise independence**
- Independence of complements
- **Random variable** $X : \Omega \to \mathbb{R}$
- The **probability distribution** of X is the collection of probabilities $P\{X \in B\}$ for sets $B \subseteq \mathbb{R}$.
- X is a **discrete random variable** if $\exists$ a finite or countably infinite set $\{k_1, k_2, ...\}$ of real numbers such that

$$\sum_i P(X = k_i) = 1.$$

- The **probability mass function** or pmf of a discrete random variable X is the function p (or p_X) defined by

$$p(k) = P(X = k)$$

  for all possible values k of X.

Outline

Conditional independence

We said before that conditioning on an event gives us a new probability measure: Given B s.t. $P(B) > 0$, then

$$P(\cdot \mid B) : A \to P(A \mid B) \text{ is a probability measure.}$$

We can now define events as conditionally independent if they are independent under this new measure.

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We can now define events as conditionally independent if they are independent under this new measure.

unconditional indep: $P(AB) = P(A)P(B)$

Definition

Let $B \subseteq \Omega$ s.t. $P(B) > 0$. The events A_1, A_2 are **conditionally independent given *B*** if

$$P(A_1 \cap A_2 \mid B) = P(A_1 \mid B)P(A_2 \mid B).$$

More generally, events $A_1, ..., A_n$ are conditionally independent given B if for any $k \in \{2, ..., n\}$ and $1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n$,

$$P(A_{i_1} \cdots A_{i_k} \mid B) = P(A_{i_1} \mid B)P(A_{i_2} \mid B) \cdots P(A_{i_k} \mid B)$$

Example

Suppose 90% of coins in the circulation are fair and 10% are biased with $P(\mathrm{T}) = \frac{3}{5}$. I have a random coin and flip it twice. Denote $A_1 = \{\text{1st flip is T}\}$ and $A_2 = \{\text{2nd flip is T}\}$. Are A_1, A_2 independent?

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- Denote $F = \{\text{the coin is fair}\}$, $B = \{\text{the coin is biased}\}$ F & B partition Ω
- By the law of total probability, for $i = 1$ or 2,

$P(A_i) = P(A_i \cap F) + P(A_i \cap B)$

$$P(A_i) = P(A_i \mid F)P(F) + P(A_i \mid B)P(B)$$

- For **a given coin** the events A_i have the same probability:

$$P(A_1 \mid F) = P(A_2 \mid F) = \frac{1}{2}, \quad P(A_1 \mid B) = P(A_2 \mid B) = \frac{3}{5}$$

- Therefore

$$P(A_i) = P(A_i \mid F)P(F) + P(A_i \mid B)P(B) = \frac{1}{2} \cdot \frac{9}{10} + \frac{3}{5} \cdot \frac{1}{10} = \frac{51}{100}.$$

$P(A_1)P(A_2) = \left(\frac{51}{100}\right)^2$. is $P(A_1 \cap A_2) = \left(\frac{51}{100}\right)^2$?

Conditional Independence

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- Now assume that for **a given coin**, the two events are conditionally independent (natural assumption), i.e.,

$$P(A_1 \cap A_2 \mid F) = P(A_1 \mid F)P(A_2 \mid F), \quad P(A_1 \cap A_2 \mid B) = P(A_1 \mid B)P(A_2 \mid B)$$

- Then by the law of total probability,

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1 \cap A_2 \mid F)P(F) + P(A_1 \cap A_2 \mid B)P(P) \\ &= P(A_1 \mid F)P(A_2 \mid F)P(F) + P(A_1 \mid B)P(A_2 \mid B)P(B) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{9}{10} + \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{1}{10} = \frac{261}{1000} \end{aligned}$$

- Then $P(A_1 \cap A_2) = \frac{261}{1000} \neq \left(\frac{51}{100}\right)^2 = P(A_1)P(A_2)$, the two events **are not independent**.

Example

Suppose 90% of coins in the circulation are fair and 10% are biased with $P(\mathrm{T}) = \frac{3}{5}$. I have a random coin and flip it twice. Denote $A_1 = \{\text{1st flip is T}\}$ and $A_2 = \{\text{2nd flip is T}\}$. Are A_1, A_2 independent?

Discussion

- Intuitively, why are the two events not independent?
- Seeing the result of one flip gives us some information about the coin, which influences the probability of getting a tail a second time
- e.g. if $A_1 = \{\text{first 100 flips are T}\}$ and $A_2 = \{\text{101th flip is T}\}$, then clearly if A_1 is true, the coin has more chances to be biased and so $P(A_2)$ is influenced by this information

Another interpretation

Conditional independence tells us that, given some information B, another event A_2 is no longer relevant

Lemma

If A_1 and A_2 are conditionally independent given B, then

$$P(A_2 \mid A_1, B) = P(A_2 \mid B).$$

Proof.

$$
\begin{aligned}
P(A_2 \mid A_1, B) &:= P(A_2 \mid A_1 \cap B) \\
&= \frac{P(A_2 \cap A_1 \cap B)}{P(A_1 \cap B)} && \text{cond. prob.} \\
&= \frac{P(A_2 \cap A_1 \mid B)\, P(B)}{P(A_1 \mid B)\, P(B)} && \text{mult. rule} \\
&= \frac{P(A_2 \cap A_1 \mid B)}{P(A_1 \mid B)} \\
&= \frac{P(A_2 \mid B)\, P(A_1 \mid B)}{P(A_1 \mid B)} && A_1, B \text{ indep.} \\
&= P(A_2 \mid B).
\end{aligned}
$$

denoted $A \perp\!\!\!\perp B$

□

Conditional Independence

Example

Every day I walk a random number of kilometers. The distance I walk one day is independent of the distance I walked another day. Let X_n the distance that I walked after n days. Are the events $\{X_1 = 10\}$ and $\{X_3 = 20\}$ conditionally independent given $\{X_2 = 15\}$?

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- This is just an intuitive example; we won't dive into this kind of problem during the course
- Yes, if we know X_2, then X_1 is not relevant:

$$P(X_3 = 20 \mid X_2 = 15, X_1 = 10) = P(X_3 = 15 \mid X_2 = 15).$$

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- Yes, if we know X_2, then X_1 is not relevant:

$$P(X_3 = 20 \mid X_2 = 15, X_1 = 10) = P(X_3 = 15 \mid X_2 = 15).$$

- This is an example of a Markov chain, a sequence of events such that the future is independent of the past given the present
- This is a very common model that can be used e.g. to predict the weather

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Recap of Monday's lecture

Conditional independence

Independent random variables, independent trials

Some common discrete distributions

Odds and ends

Independence of random variables

Definition

Random variables $X_1, ..., X_n$ defined on the same probability space $(\Omega, \mathcal{F}, P)$ —i.e. with the same sample space, event space, and probability measure — are **independent** if

$$P(X_1 \in B_1, ..., X_n \in B_n) = P(X_1 \in B_1)...P(X_n \in B_n)$$

for any (Borel) subsets $B_1, ..., B_n \subseteq \mathbb{R}$.

- This means that the distribution of the r.v. can be factorized in the distributions of each r.v.
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Specifically, discrete random variables $X_1, ..., X_n$ on the same probability space are independent if and only if

$$P(X_1 = x_1, ..., X_n = x_n) = P(X_1 = x_1)...P(X_n = x_n)$$

for any possible choices $x_1, ..., x_n$ of the values of the random variables.

Independent trials

Several of the examples we have seen can be viewed as independent repeated trials of an experiment:

- Flipping the same coin n times
- Rolling the same die n times
- Drawing a ball from the same urn n times with replacement

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Recall that we can represent the sample space in these examples as a Cartesian product of a set that represents the simpler experiment:

- Coin toss: $\Omega = \{\text{H,T}\}^n$ $\longrightarrow \{H,T\} \times \{H,T\} \times \{H,T\} \cdots$
- Die roll: $\Omega = ?$ $\{1,2,3,4,5,6\}^n$
- Drawing a ball: $\Omega = ?$ $\{\text{ball } 1, \ldots, \text{ball } k\}^n$

Identically distributed variables

This experimental setup is related to the concept of **independent and identically distributed (i.i.d.)** random variables.

Definition

Random variables $X_1, ..., X_n$ are **identically distributed** if each X_i has the same probability distribution.

- iid random variables have very nice properties and in some applications they either come up naturally or can be a reasonable assumption
  - Independent repeated trials
- In some other applications, i.i.d. is not a reasonable assumption

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Example

Consider sampling k balls one at a time from an urn with n balls labeled $1, \ldots, n$. Denote

$$X_i = \text{label of the } i\text{th ball drawn.}$$

- If sampling with replacement, then $X_1, \ldots, X_n$ are iid
- If sampling without replacement, are $X_1, \ldots, X_n$ independent? Identically distributed? Both? Neither?

iid: continuing the example

$5, \ 2, \ 3, \ 1, \ 4, \ \ldots$

$X_1 = 5$
$X_2 = 2$
$\vdots$
$X_4 = 1$

Suppose you draw k balls one at a time without replacement from an urn with n balls labeled $1, \ldots, n$. Denote

$$X_i = \text{label of the } i\text{th ball drawn.}$$

- The distribution of balls you can get changes with each draw; that means the *conditional probability* $P(X_i|X_1, X_2, ..., X_{i-1})$ is different for different i. Show this with an example.

$P(X_1)$

$P(X_2 | X_1)$ → $P(X_2 = 3 | X_1 = 3)$, $P(X_2 = 2 | X_1 = 3)$, $P(X_2 = 5 | X_1 = 1)$

e.g. $\underline{P(X_2 | X_1 = 3)}$

clearly pmf for $X_2 | X_1 = 3$ $\neq$ pmf for X_1

specifically,

$\underset{\substack{\| \\ 0}}{P(X_2 = 3 | X_1 = 3)} \neq P(X_1 = 3)$

$\vdots$

$P(X_n | X_1, \ldots, X_{n-1})$

iid: continuing the example

Suppose you draw k balls one at a time without replacement from an urn with n balls labeled $1, \ldots, n$. Denote

$n=3$

$k=2$

$P(X_1=5) = P(X_2=5) = P(X_3=5)$

$$X_i = \text{label of the } i\text{th ball drawn.}$$

- The distribution of balls you can get changes with each draw; that means the *conditional probability* $P(X_i|X_1, X_2, ..., X_{i-1})$ is different for different i. Show this with an example.
- What about $P(X_i)$? $P(X_i)$ is actually the same for all the variables

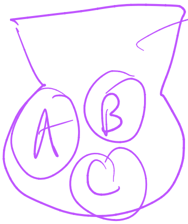
$$P(X_i=c) = \frac{\#\text{ sequences of } k \text{ draws in which } X_i=c}{\#\text{ sequences of } k \text{ draws}}$$

$\boxed{\begin{matrix} 1,3,2 \\ 1,2,3 \\ 2,1,3 \end{matrix}}$ Ω

same for all i.

X_2?

$$P(X_1=c) = \frac{1 \cdot (n-1)(n-2)\cdots(n-k+1)}{n!/k! \quad (\text{arrangement})} = \frac{(n-1)!/k!}{n!/k!} = \frac{1}{n}$$



Ω

$A^1 B^2 C^3$

$A^1 C^3 B^2$

$B^2 C^3 A^1$

$B^2 A^1 C^3$

$C^3 A^1 B^2$

$C^3 B^2 A^1$

$X_1 = 2$

$$X_i = \begin{cases} 1 & A \\ 2 & B \\ 3 & C \end{cases}$$

$$P(X_1 = 1) = \frac{2}{6} = \frac{1}{3}$$

$$P(X_2 = 1) = \frac{2}{6} = \frac{1}{3}$$

$$P(X_2 = 1 \mid X_1 = 1) = 0$$

$$P(X_2 = 1 \mid X_1 = 2) = \frac{P(X_2 = 1 \,\&\, X_1 = 2)}{P(X_1 = 2)} = \frac{1/6}{1/3} = 1/2$$

$$P(X_2 = 1 \mid X_1 = 3) = 1/2$$

iid: continuing the example

Suppose you draw k balls one at a time without replacement from an urn with n balls labeled $1, \ldots, n$. Denote

$$X_i = \text{label of the } i\text{th ball drawn.}$$

- The distribution of balls you can get changes with each draw; that means the *conditional probability* $P(X_i | X_1, X_2, ..., X_{i-1})$ is different for different i. Show this with an example.
- What about $P(X_i)$? $P(X_i)$ is actually the same for all the variables
  - This is called **marginal probability**
- The X_i are not independent

$X_i = 1 \quad \forall i$

$\therefore$ not indep.

$P(X_1 = \cdots X_k = 1) = 0$

$P(X_1 = 1) \cdots P(X_k = 1) = P(X_i = 1)^k = \frac{1}{n^k}$.

iid: continuing the example

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- The distribution of balls you can get changes with each draw; that means the *conditional probability* $P(X_i|X_1, X_2, ..., X_{i-1})$ is different for different i. Show this with an example.
- What about $P(X_i)$? $P(X_i)$ is actually the same for all the variables
  - This is called **marginal probability**
- The X_i are not independent
- Therefore $X_1, \ldots, X_n$ are identically distributed but not independent.

Takeaway: be careful about whether your intuition is based on conditional or marginal probability

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Bernoulli distribution

Definition

A random variable X has a Bernoulli distribution with success probability p if $X \in \{0, 1\}$ (binary) and satisfies $P(X = 1) = p$. We denote this by $X \sim \text{Ber}(p)$, read "X is distributed Bernoulli with success probability p."

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- What must $P(X = 0)$ be equal to? Why? $P(X=0) = 1-p$
- We call p a **parameter** of the distribution, a fixed number used as part of computing the probability of any given outcome.

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- We call p a **parameter** of the distribution, a fixed number used as part of computing the probability of any given outcome.

A sequence of independent Ber(p) trials: Let X_i denote the outcome of trial i. Then, for example,

$X_i = \begin{cases} 1 & T \\ 0 & H \end{cases}$

$$P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0, X_5 = 1) = p^3(1 - p)^2.$$

H, T, T, H, T $(1-p) \cdot p \cdot p \cdot (1-p) \cdot p$

- What is $P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 0, X_5 = 1)$?
- Suppose heads are "successes" with probability p. What is the probability of 3 heads in a row followed by 2 tails in a row?
- What is the probability of the sequence (H, T, H, T, H)?

  $p^3(1-p)^2$
- What is the probability that you get either $\underbrace{(H, H, H, T, T)}_{=A}$ or $\underbrace{(H, T, H, T, H)}_{=B}$?

$$P(X_1=1, X_2=1, X_3=1, X_4=0, X_5=1)$$

$$= \underline{P(X_1=1)} \cdot \underline{P(X_2=1)} \cdot \underline{P(X_3=1)} \cdot$$

$$\underline{P(X_4=0) \cdot P(X_5=1)}$$

$X_1, \ldots, X_5$ are indep.

$$= P(X_i=1)^4 \, P(X_i=0)$$

$X_1, \ldots, X_5$ are ident. dist.

$$= p^4 (1-p).$$

Now $X_i = \begin{cases} 1 & H \\ 0 & T \end{cases}$

P(3 heads followed by 2 tails)

$$= P(X_1 = X_2 = X_3 = 1, X_4 = X_5 = 0)$$

$$= p^3 (1-p)^2$$

$$P(A \cup B) = P(A) + P(B) = p^3(1-p)^2 + p^3(1-p)^2$$

(disjoint) $\quad = 2p^3(1-p)^2$

Binomial distribution

The binomial distribution counts the number of successes in a sequence of n independent $\mathrm{Ber}(n)$ trials

Definition

Let n be a positive integer and $0 \leq p \leq 1$. A random variable Y **is binomially distributed** or **follows a binomial distribution** with parameters n and p if the pmf of Y is given by

p = probability of success on a given trial

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

n = number of trials.

We write that $Y \sim \mathrm{Bin}(n, p)$.

Example

What is the probability that five rolls of a fair die yield two or three sixes?

- Repeated trial = die roll; success = roll a six
- Let S be the number of sixes that appear in 5 rolls
- Then S is binomial. What are n and p?
- Finish solving this problem.

Binomial distribution: Solution to previous example

$n = \#\text{ trials} = 5 \qquad p = 1/6$ for each trial

$$P(S=2 \text{ or } S=3) = P(S=2) + P(S=3)$$

disjoint (either $S=2$ or $S=3$, but not both)

$$= \binom{5}{2} p^2 (1-p)^3 + \binom{5}{3} p^3 (1-p)^2$$

$$\approx 0.193$$

Geometric distribution

In a series of independent repeated trials, the geometric distribution gives the probability that the kth trial is the first success. Let's compute its pmf:

Previously:

let's say success = H

$$\underset{1}{\underline{T}}\ \underset{2}{\underline{T}}\ \underset{3}{\underline{T}}\ \cdots\ \underset{k-1}{\underline{T}}\ \underset{k}{\underline{H}}$$

$$P(\underbrace{TT\cdots T}_{k-1}\underset{k}{H}) = \frac{1}{2^k}$$

this assumes a fair coin

Now in general,

$$P(X_1 = \cdots = X_{k-1} = 0,\ X_k = 1)$$

$$= P(X_i = 0)^{k-1} P(X_i = 1)$$

$$= (1-p)^{k-1} p$$

Geometric distribution

In a series of independent repeated trials, the geometric distribution gives the probability that the kth trial is the first success. Let's compute its pmf:

Definition

A random variable G follows the **geometric distribution** with success probability $0 \leq p \leq 1$ if the pmf of G is given by

$$P(G = g) = (1-p)^{g-1}p \quad \forall\, g \in \mathbb{N}.$$

(Recall that $\mathbb{N}$ is the set of positive integers.) We write $G \sim \text{Geom}(p)$.

Geometric distribution

$p = P(T)$

1. What is the probability that the first tails happens on the 5th flip of a fair coin?

$P(5^{th} \text{ flip is } 1^{st}\ T) = P(G=5) = (1-p)^4 p = \left(1-\frac{1}{2}\right)^4 \frac{1}{2} = \frac{1}{2^5}$

≈ 0.031

2. What if the coin is biased and $P(\mathrm{H}) = 0.3$? (Tip: what is a "success" here?)

$P(T) = 0.7$

   - Should this value be larger or smaller than the answer to #1?

$\ldots = (1-p)^4 p = (1-0.7)^4 (0.7) \approx 0.0057$

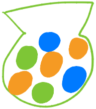
smaller

3. What is the probability that it takes more than 7 rolls of a fair die to roll a six?

Hypergeometric distribution

- Draw n things without replacement from a finite population of N things that contains K things with some feature of interest
- What is the probability that k of the n things you drew have that feature?

e.g. orange



$n=3 \quad k=2$

$N=7_{\text{balls}} \quad K=3$

Hypergeometric distribution

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- Example: urn contains 5 balls of each of 4 colors (green, yellow, blue, red), and you draw 3 balls. $P(2 \text{ of the } 3 \text{ balls are green}) =?$

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Definition

A random variable H follows a **hypergeometric distribution** with parameters N, K, and n if the pmf of H is

$$P(H = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \quad \text{for } k = 0, 1, ..., n.$$

Outline

The birthday problem

You meet someone at a party and find out they have the same birthday as you (month and day)! What is the probability of that happening?
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These two questions have very different answers; see the book for details if you are curious.

Degenerate random variables

(not on midterm)

Definition

A random variable is **degenerate** if $\exists\ b \in \mathbb{R}$ such that $P(X = b) = 1$.

Example

Consider drawing a number uniformly at random from the interval $[0, 10]$.

- Define a random variable X such that X is degenerate.