

A MODERN INTRODUCTION TO QUARKONIUM THEORY

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Recent advances in lattice and continuum QCD have given us new insights into quarkonium physics. These set of lectures are intend for the uninitiated. We first give a physical picture of quarkonium and describe the hybrids states established in lattice QCD. Then we give an unorthodox presentation of Non-Relativistic QCD (NRQCD) including a novel method for the application of spin-symmetries. Finally we describe the prototypical application of NRQCD: cancellation of infrared divergences in decays of P-wave quarkonia.

1 Introduction

1.1 About these lectures

I have put together a little material on the theory of quarkonia that I hope will be a good introduction to the subject for novices, be them graduate students or researchers outside this field. I must warn the readers that I give a personal view of the subject. There are many excellent, more orthodox reviews available. However, I have combined material in a way that I think brings out the physics more clearly. For example, although my aim is to present the advances in NRQCD of the last few years, at the expense of the whole subject of quarkonium production I have used much space and effort in giving a physical picture of quarkonium and hybrids based on the rather good recent calculations of inter-quark potentials in lattice QCD. I also have injected here and there little tricks that help me understand or calculate. For example, I have given explicitly a foolproof method of calculating spin-symmetry relations using the Wigner-Eckart theorem by means of a trace formula, akin to that used in HQET.

There are three parts to these lectures. First we discuss the physical picture. To this end we briefly use the bag-model picture to motivate a non-trivial inter-quark potential, linear at long distances. Then the role of octets in the description creeps in through the discussion of hybrids. Hybrids are the particle physics analogs of molecules, so it should be no surprise that a Born-Oppenheimer approximation can be used to describe them (the BO is reviewed). This immediately suggests we study QCD in the Non-Relativistic limit, or ‘NRQCD’, which is the subject of the middle part of these lectures. The last part describes the application of NRQCD to the decays of S and P-wave quarkonia.

Since these lectures are intended for learning students, I have toiled to include many exercises. They are included as separate paragraphs, and most often the answer is given.

I have left many loose ends untied. This is not only because I have been careless. There are many questions that are unresolved, and I hope the alert reader will spot

them and hopefully solve them. In the conclusions I attempt to list some of these questions, but offer little in the way of solutions.

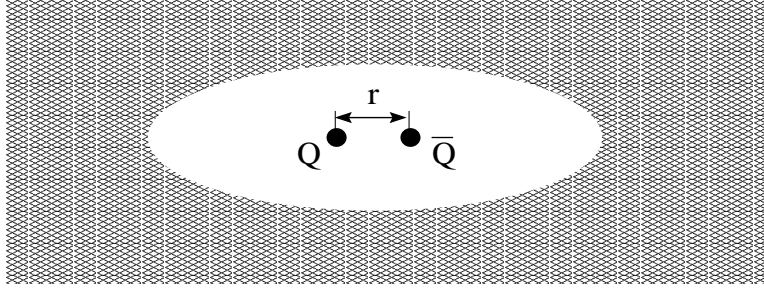


Fig. 1. Static quarks at small separation. The shaded region represents the non-perturbative QCD vacuum. The ‘bag’ of perturbative vacuum has radius R_b given in Eq. (2)

1.2 Quarkonium: A Physical Picture

Consider a quark Q and an antiquark \bar{Q} pinned down a distance r from each other. Let us begin our considerations by assuming that this distance r is much smaller than the typical size of a hadron $\Lambda_{\text{QCD}}^{-1} \sim 1$ Fermi. See Fig. 1. Far away from the pair, at a distance $R \gg r$, the chromo-electric field is that of a dipole,

$$E_\theta \sim g(r) \frac{r \cos \theta}{R^3}, \quad (1)$$

where $g(r)$ is the coupling constant of QCD at distance scales of order r . This description in terms of a perturbative vacuum cannot be correct at very large distances from the dipole. When the field strength drops to a critical value $E_{\theta, \text{crit}} \sim \Lambda_{\text{QCD}}^2$ the vacuum quickly turns non-perturbative. Thus the dipole can be described as a perturbative “bag” inside the non-perturbative vacuum.^{3,4,2} The radius R_b of this bag is thus determined

$$\frac{g(r)}{R_b^3} \sim \Lambda_{\text{QCD}}^2. \quad (2)$$

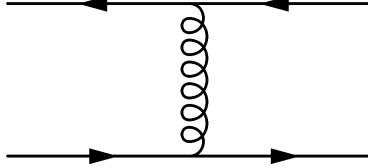


Fig. 2. At short distances the inter-quark force is Coulomb-like, and arises from this one gluon exchange diagram.

We will be interested in dynamical bound states, with quarks free to move. For this we need an understanding of the force between static quarks as a function of distance r . For $r \ll R_b$ the interaction is well approximated as a Coulomb force. It arises from perturbative single gluon exchange; see Fig. 2. The energy of the state is the energy of the Coulomb dipole, $\sim 1/r$. At the opposite extreme, shown in Fig. 3, when $r \gg R_b$ non-perturbative effects become important. The perturbative vacuum is squashed out into a cigar shape; this is the “string limit”. The bag carries volume energy density \mathcal{E} . If the cross sectional area of the cigar in the string limit is A , the energy of the state is $\mathcal{E} \cdot (Ar)$, *ie*, it increases linearly with r . The linear slope of this potential energy is the “string tension” $\mathcal{E}A$.^a

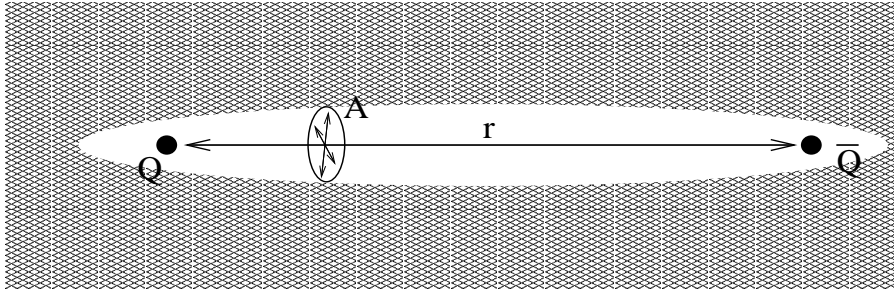


Fig. 3. For large inter-quark separation r the perturbative vacuum ‘bag’ takes on a cigar shape, of fixed cross sectional area A . The energy of the configuration is proportional to the volume of the bag, Ar , giving rise to a linear inter-quark potential.

^aIn fact the cross sectional area of the string solution is not an independent parameter, $A \sim \sqrt{g^2/\mathcal{E}}$, so the string tension $\sim \sqrt{g^2\mathcal{E}}$.

1.3 Digression: The strength of the Coulomb Gluon

In the perturbative vacuum the inter-quark force is dominated by single gluon exchange, as in the following figure:

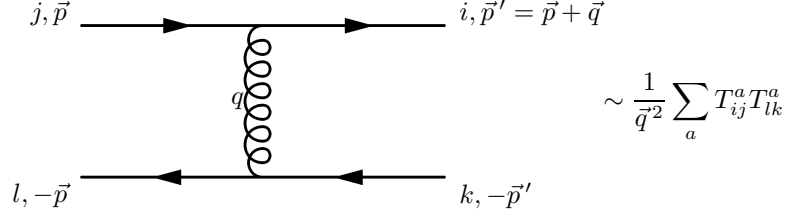


Fig. 4. The gluon exchange gives rise to a Coulomb interaction.

The momentum dependence gives rise to a Coulomb potential (the Fourier transform of $1/\vec{q}^2$ is proportional to $1/r$). Here we will focus on the color factor, $\sum_a T_{ij}^a T_{lk}^a$. The matrices T^a are the generators of $SU(3)$ in the fundamental representation, that is, Gell-Mann matrices. The indices i, j, k, l refer to the external quarks, as in Fig. 4.

It is now easy to see that the strength of the interaction depends on the relative color of the $Q\bar{Q}$ state. Since Q is in the $\mathbf{3}$ (fundamental representation) of $SU(3)$, and

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8},$$

the $Q\bar{Q}$ force is in either of these two channels. It will be instructive to carry the calculation for the more general case of $SU(N)$, so the representation $\mathbf{8}$ is understood as the adjoint representation of $SU(N)$. To find the $\mathbf{1}$ and $\mathbf{8}$ components of the force we prepare the initial $Q\bar{Q}$ state to be purely $\mathbf{1}$ or $\mathbf{8}$:

$$(Q_j \bar{Q}_l)_1 \equiv \delta_{jl} (Q\bar{Q}) \quad (3)$$

$$(Q_j \bar{Q}_l)_8 \equiv Q_j \bar{Q}_l - \frac{1}{N} \delta_{jl} (Q\bar{Q}). \quad (4)$$

Contracting this with

$$\sum_a T_{ij}^a T_{lk}^a = \frac{1}{2} (\delta_{ik} \delta_{jl} - \frac{1}{N} \delta_{ij} \delta_{kl})$$

we get

$$(Q_j \bar{Q}_l)_1 \sum_a T_{ij}^a T_{lk}^a = \frac{N^2 - 1}{2N} (Q_i \bar{Q}_k)_1 \quad (5)$$

$$(Q_j \bar{Q}_l)_8 \sum_a T_{ij}^a T_{lk}^a = -\frac{1}{2N} (Q_i \bar{Q}_k)_8. \quad (6)$$

We see that the force in the color octet channel is of opposite sign (in fact, repulsive) and weaker than in the singlet channel by a factor of $1/(N^2 - 1)$.

In atomic physics the $\vec{L} \cdot \vec{S}$ interaction is computed most easily via the well known formula

$$\langle 2\vec{L} \cdot \vec{S} \rangle = j(j+1) - l(l+1) - s(s+1).$$

The reader may verify the analogous formula

$$\langle \sum_a T^a T^a \rangle_R = C(R) - 2C(\text{fund})$$

where $C(R)$ is the Casimir invariant of the representation R ,

$$\sum_a T^a T^a = C(R)\mathbf{1}.$$

For $SU(3)$ if we normalize $C(R)$ so that $C(\mathbf{8}) = 3$, then $C(R) = 4/3, 10/3$ and 6 , for $R = \mathbf{3}, \mathbf{6}$ and $\mathbf{10}$, respectively.

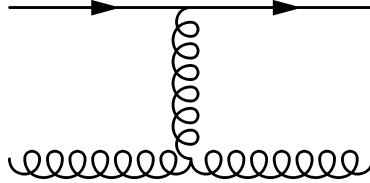


Fig. 5. One gluon exchange gives rise to gluon-quark force. The force is attractive in the $\mathbf{3}$ and $\mathbf{6}$ channels, but repulsive in the $\mathbf{15}$ channel.

One can compute the force between a gluon and a quark similarly; see Fig. 5. The available channels are given by

$$\mathbf{3} \times \mathbf{8} = \mathbf{15} \oplus \bar{\mathbf{6}} \oplus \mathbf{3}.$$

The force is attractive in the $\mathbf{3}$ channel,

$$C(\mathbf{3}) - C(\mathbf{8}) - C(\mathbf{3}) = -3,$$

and in the $\bar{\mathbf{6}}$ channel,

$$C(\bar{\mathbf{6}}) - C(\mathbf{8}) - C(\mathbf{3}) = -1,$$

but repulsive in the $\mathbf{15}$ channel.

1.4 Hybrids: Octet Quarkonia

As we have seen a quark-antiquark pair in a color octet configuration repel each other, and a quark-gluon pair attract each other in either the $\mathbf{3}$ or $\bar{\mathbf{6}}$ configurations. A state of quarks in an octet configuration can bind to a gluon, provided the gluon

attraction to both quarks overcomes the repulsion between quarks. States like this in which the glue state plays a particle-like role in binding are called “hybrids”, because they are a hybrid of a glueball and a pure quarkonium state. Lattice calculations ¹ convincingly indicate that QCD predicts such states, but experimentally no such state has been demonstrated. The state is very non-perturbative in nature, but can be empirically understood in terms of a bag picture in the string limit with excited chromo-electric and magnetic cavity modes in the bag.

Hybrids have a useful analog in atomic physics: the H_2 molecule. There two heavy protons repel each other, but the intervening electronic cloud attracts them both and overcomes the repulsion. The physics of the molecule is well described by the Born-Oppenheimer (BO) approximation, which we now review.

1.5 Quick Review: Born-Oppenheimer Approximation

Consider ⁵ the molecular Hamiltonians

$$H = T_e + T_N + V$$

where the kinetic energies for electrons and nuclei are, respectively,

$$T_e = -\frac{1}{2m} \sum_i \frac{\partial^2}{\partial x_i^2} \quad , \quad T_N = -\sum_j \frac{1}{2M_j} \frac{\partial^2}{\partial X_j^2}$$

and the potential energy $V = V(X_j, x_i)$ is a function of all particle locations, in general.

In the BO approximation one considers first the simpler Hamiltonian

$$H^{(0)} = T_e + V$$

obtained formally by taking the infinite nuclear mass limit. Since $[X_j, H^{(0)}] = 0$ we can simultaneously diagonalize X_j and $H^{(0)}$. Set the values of X_j to X'_j and notice that X' appear as parameters of the Hamiltonian for the electrons. We include a label X' in eigenstates and eigenvalues,

$$H^{(0)}|nX'\rangle = W_n(X')|nX'\rangle,$$

to remind us that they carry implicit dependence on X' . In the $|xX\rangle$ representation,

$$\langle xX|nX'\rangle = \phi_n(x, X')\delta(X - X')$$

this is just the Shrodinger equation for the electrons in the field of fixed point charges

$$[T_e + V(x, X')]\phi_n(x, X') = W_n(X')\phi_n(x, X').$$

Next we let the nuclei move, slowly. We replace above $X' \rightarrow X'(t)$. If the motion is slow enough we can solve the new Schrodinger equation

$$[T_e + V(x, X'(t))]\tilde{\phi}_n = \tilde{W}_n\tilde{\phi}_n$$

in the adiabatic approximation:^b

$$\tilde{\phi}_n = \phi_n(x, X'(t)) \quad (\text{up to a phase})$$

$$\tilde{W}_n = W_n(X'(t)) .$$

The condition for the validity of the adiabatic approximation is that the probability P_n of electrons jumping out of state $|nX'\rangle$ be small:

$$P_n = \left| \frac{\langle nX' | (\frac{d}{dt} | nX' \rangle)}{\min \Delta W} \right|^2 \ll 1, \quad (7)$$

where ΔW stands for the electronic energy spacings.

To estimate P_n , let a be the molecular size, that is, the typical separation between nuclei. By the uncertainty principle the electronic momentum is $p_e \sim 1/a$ and thus $\Delta W \sim E_e \sim p_e^2/m \sim 1/ma^2$. For the numerator we argue as follows. To remove one atom from the molecule we need to move that nucleus by $\delta X' \sim a$. At this point the new state is essentially orthogonal to the original one,

$$\int dx \phi_n(x, X' + \Delta X') \phi_n(x, X') \approx 0 \quad \text{for } \Delta X' \sim a$$

or, since ϕ_n is normalized,

$$\int dx a \frac{\partial \phi_n}{\partial X'} \phi_n \sim 1$$

Now, denote the slow nuclear velocity by $V_j = dX'_j/dt$. We have, roughly,

$$\langle \phi | \frac{d\phi}{dt} \rangle \sim \frac{1}{a} V_{\text{rms}} \sim \frac{1}{a} \sqrt{\frac{t_N}{M}}$$

where the rms velocity is $V_{\text{rms}} = \sqrt{\sum_j \langle V_j^2 \rangle}$ and the kinetic energy $t_N = \sum_j \frac{1}{2} M_j \langle V_j^2 \rangle$. Thus

$$P_n \sim \frac{t_N / Ma^2}{E_e (1/ma^2)} = \left(\frac{m}{M} \right) \frac{t_N}{E_e}.$$

Now, t_N has contributions from rotational and vibrational modes. In most cases the latter is dominant, and we write $E_{\text{vib}} = \omega$. Since the molecule is bound by the electronic cloud, the vibrational potential energy $V_N = \frac{1}{2} M \omega^2 X^2$ at $X \sim a$ should be comparable to the electronic energy, $\frac{1}{2} M \omega^2 a^2 \sim \frac{1}{2} \frac{1}{ma^2}$, or

$$t_N \sim E_{\text{vib}} \sim \omega \sim \frac{1}{\sqrt{Mm}} \frac{1}{a^2}$$

Using this in P_n ,

$$P_n \sim \frac{m}{M} \sqrt{\frac{m}{M}}. \quad (8)$$

^bThe phase, here ignored, can play an important role.^{6,7}

Exercise: Show that rotational modes give $P_n \sim (m/M)^2$.

In the adiabatic approximation n is a good quantum number (the coupling between electronic levels is neglected). Eigenvectors of H are linear combinations of $|nX'\rangle$ with n fixed:

$$\int |nX'\rangle \psi(X') dX',$$

or, in the $|xX\rangle$ representation,

$$\Phi_n(x, X) = \phi_n(x, X) \psi(X)$$

The eigenfunctions of H are found by expanding in terms of these. Let's use these as trial wavefunction in the variational method. The energy functional is

$$E(\Phi_n) = \frac{\langle \Phi_n | H | \Phi_n \rangle}{\langle \Phi_n | \Phi_n \rangle} = \frac{\langle \psi | H_n | \psi \rangle}{\langle \psi | \psi \rangle}$$

where

$$H_n \psi(X) = \int \phi_n(x, X) [H \phi_n(x, X) \psi(X)] dx,$$

and the condition $\delta E(\Phi_n) = 0$ is equivalent to

$$H_n \psi = E \psi$$

This is nothing but the Schroedinger equation for the nuclei in the background Coulomb field of the electrons in the n -state.

Exercise: Show that

$$H_n = T_N + W'_n + W_n$$

where

$$W'_n = \sum_j \frac{1}{2M_j} \int \left(\frac{\partial \phi_n}{\partial X_j} \right)^2 dx$$

so the nucleus moves in a potential $W_n(X) + W'_n(X)$. Show $W'_n \sim 1/Ma^2$ and is therefore a small correction.

1.6 Born-Oppenheimer in Quarkonium and the Relevant Distance Scales

It seems obvious that the Born-Oppenheimer approximation should be adequate in the description of hybrids. There we would choose the quark and antiquark to be the slow degrees of freedom while the glue would be the fast degrees of freedom. One would guess that the probability P_g of Eq. (7), that the glue stays in a fixed state, is approximated by Eq. (8), $P_g \sim (\Lambda_{\text{QCD}}/M_Q)^{3/2}$, so for large M_Q the adiabatic approximation will be very good.

It is not so obvious that the color singlet states can also be treated in this approximation. This is a subtle issue which I won't discuss, save for the following guess based on atomic physics analogs. Since the $(Q\bar{Q})_1$ acts like a bound state in a mutually attractive Coulomb field, one could guess that the average kinetic energy $t_{Q\bar{Q}} \sim 1/Ma^2 \sim \text{"Rydberg"}$, where $a \sim (\alpha_s M)^{-1}$ is the "Bohr Radius",

and if the gluonic energy (or “mass”) is fixed $E_g \sim m_g \sim \Lambda_{\text{QCD}}$, then $P_g \sim (m_g/M)(t_{Q\bar{Q}}/E_g) \sim (\Lambda_{\text{QCD}}/M)(\alpha_s^2 M/\Lambda_{\text{QCD}}) \sim \alpha_s^2$. Therefore the approximation will work to the extent that α_s^2 is small.

The scales that appear in this discussion play an important role throughout, so we pause to revisit them. The overall mass of the system is dominated by M ; the size of the system is its Bohr radius, a , with $a^{-1} \sim M\alpha_s \sim Mv$; the “Rydberg” is the gross measure of energy spacings and is the (inverse of the) primary time scale in the system, $\sim Mv^2$; and the strong interactions become relevant at distances greater than $1/\Lambda_{\text{QCD}}$. Just as in the Hydrogen atom, we have used $v \sim \alpha_s$ as the velocity of the quarks. So we have the hierarchy of scales

$$M \gg Mv \gg Mv^2 \gg \Lambda_{\text{QCD}}.$$

Whether this is realized in nature for charm or beauty is a subject much debate, so we will develop the theory hoping that it will be appropriate for at least one of these systems. Notice also that theoretically one could have such large masses that even $Mv^n \gg \Lambda_{\text{QCD}}$ for some $n \geq 3$. Such scales dictate the physics of, eg, hyperfine splittings. In such a world one would be able to compute reliably even such small effects.

It is now clear how our discussion of the static quarks in a perturbative bag plays a role: it just corresponds to the first step in the BO approximation. As argued above, the inter-quark potential is Coulombic at short distances and increases linearly with separation at long distances. In the BO approximation this potential is used in a Schroedinger equation for the quarks to describe the bound state.

1.7 Towards NRQCD

Now that we have decided to apply the Born-Oppenheimer approximation to heavy quarkonia we find difficulties implementing this program. Consider the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot D - M)\psi$$

where the covariant derivative is $D_\mu = \partial_\mu - i\frac{g}{c}A_\mu$. Dirac’s Hamiltonian density follows

$$\mathcal{H} = -\bar{\psi}(-i\vec{\gamma} \cdot \vec{\partial} + \frac{g}{c}A^0\gamma^0 - \frac{g}{c}\vec{\gamma} \cdot \vec{A} - M)\psi.$$

We immediately face two difficulties. In the standard application of Born-Oppenheimer we need to write a Hamiltonian that includes a kinetic energy term for the slow degrees of freedom. Our Hamiltonian has no obvious kinetic energy term, and it does not account separately for the Q and \bar{Q} variables.

The solution to this problem is to work with a non-relativistic version of \mathcal{H} , by doing a $1/c$ expansion. This immediately leads to what is known as “Non-Relativistic QCD” or NRQCD.

Now, the $1/c$ expansion in QED is old hat. You can find detailed discussions in famous texts.⁸ In fact, the Born-Oppenheimer approximation and the non-relativistic expansion applied to $Q\bar{Q}$ states in QCD is also old hat, first considered two decades ago.^{9,10,11} It will become clear what new developments have re-ignited interest in this field.

2 NRQCD

I will present NRQCD as an expansion in a parameter, namely $1/c$. The unattractive alternative is to expand in a kinematic variable, v_{rel} —the relative velocity of the $Q\bar{Q}$ pair in a slowly moving CM frame.

2.1 Notation

Our metric is

$$\eta^{\mu\nu} = \text{diag}(+ - - -).$$

The Dirac 4-spinor is broken up into two 2-spinors

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}. \quad (9)$$

The Dirac gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

and are given, in the Dirac representation, by

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where the 2×2 Pauli matrices σ^i satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ and $\sigma^3 = \text{diag}(1, -1)$. We will also use

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

and the invariant four index tensor has

$$\epsilon^{0123} = -\epsilon_{0123} = 1.$$

Finally,

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

is, in the Dirac representation,

$$\sigma^{0i} = i\gamma^0\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \quad \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

2.2 Classical NRQCD

In the Dirac representation the two 2-spinors in Eq. (9) roughly correspond to the particle and anti-particle components of the 4-spinor. We'd like to make this explicit by rewriting the Dirac lagrangian in terms of variables appropriate for a non-relativistic approximation about a frame of reference. The appropriate frame is, of course, that for which the particles are at, or almost at, rest. To this end shift the field so that the trivial (but dominant) time evolution factor is taken out,

$$\Psi = e^{-iMc^2t} \tilde{\Psi},$$

in the Dirac Equation

$$[i\gamma \cdot D + Mc(\gamma^0 - 1)]\tilde{\Psi} = 0$$

and write out the components $\tilde{\Psi} = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{c}iD_t & -i\vec{\sigma} \cdot \vec{D} \\ i\vec{\sigma} \cdot \vec{D} & -\frac{1}{c}(2Mc^2 + iD_t) \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0.$$

The field χ is suppressed by a power of $1/c$ so one may solve this equation recursively. Formally, we write

$$\chi = \frac{1}{2Mc + \frac{1}{c}iD_t} i\vec{\sigma} \cdot \vec{D} \psi$$

and use the field

$$\tilde{\Psi} = \begin{pmatrix} \psi \\ \frac{1}{2Mc + \frac{1}{c}iD_t} i\vec{\sigma} \cdot \vec{D} \psi \end{pmatrix}$$

to describe the quark Q being nearly on-shell, in the rest frame of the $Q\bar{Q}$ system, in the $1/c$ expansion.

Exercise: Carry out a similar expansion, starting from

$$\Psi = e^{+iMc^2 t} \hat{\Psi},$$

and show how to describe \bar{Q} in terms of $\hat{\Psi}$ in the $1/c$ expansion.

We can now easily compute the lagrangian that describes the quark state,

$$\begin{aligned} c\mathcal{L} &= c\bar{\Psi}[i\gamma \cdot D + Mc(\gamma^0 - 1)]\tilde{\Psi} \\ &= \psi^\dagger iD_t \psi - \psi^\dagger i\vec{\sigma} \cdot \vec{D} \frac{c}{2Mc + \frac{1}{c}iD_t} i\vec{\sigma} \cdot \vec{D} \psi \end{aligned}$$

This is not yet written as an expansion in $1/c$. To this end we expand

$$(2Mc + \frac{1}{c}iD_t)^{-1} = (2Mc)^{-1}(1 - iD_t/2Mc^2 + \dots)$$

and use

$$\begin{aligned} \sigma^i \sigma^j D_i D_j &= (\delta^{ij} + i\epsilon^{ijk} \sigma^k) D_i D_j \\ &= \vec{D}^2 + \frac{g}{2c} \vec{\sigma} \cdot \vec{B} \end{aligned}$$

to write a $1/c$ -expanded lagrangian

$$c\mathcal{L} = \psi^\dagger (iD_t - \frac{1}{2M}(i\vec{D})^2) \psi + \frac{g}{2Mc} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi + \dots$$

Exercise: The first term in the ellipsis is $\vec{\sigma} \cdot \vec{D} D_t \vec{\sigma} \cdot \vec{D}$. Consider the piece $\vec{\sigma} \cdot \vec{D} \sigma^i [D_t, D_i]$. Show it gives $\vec{D} \cdot \vec{E} + i\sigma \cdot \vec{D} \times \vec{E}$.

Exercise: Let $A = \vec{\sigma} \cdot \vec{D}$. Write $AD_t A = \frac{1}{2}A[D_t, A] - \frac{1}{2}[D_t, A] + \frac{1}{2}D_t A^2 + \frac{1}{2}A^2 D_t$. As in the previous exercise, the first two terms give no time derivatives D_t acting

on ψ . We'd like to have time derivatives appear in the lagrangian only as $\psi^\dagger D_t \psi$. Show that this can be accomplished, to this order, by redefining $\psi \rightarrow (1 - \frac{1}{2}A^2)\psi$.

Exercise: Determine the $1/c$ -expanded lagrangian for the antiquark \bar{Q} .

Putting the result of these exercises together, and incorporating the lagrangian for the fast degrees of freedom we have the following lagrangian for NRQCD: ¹²

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}(\partial_i A_0^a - \frac{1}{c}\partial_t A_i^a - \frac{g}{c}f_{abc}A_i^b A_0^c)^2 \\
& - \frac{1}{4}(\partial_i A_j^a - \partial_j A_i^a - \frac{g}{c}f_{abc}A_i^b A_j^c)^2 \\
& + \psi^\dagger (iD_t - \frac{1}{2M})(i\vec{D})^2 \psi \\
& + \frac{c_F g}{2Mc} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi \\
& + \frac{1}{8M^3 c^2} \psi^\dagger (\vec{D}^2)^2 \psi \\
& + \frac{c_D g}{8M^2 c^2} \psi^\dagger (\vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D}) \psi \\
& + \frac{c_S g}{8M^2 c^2} \psi^\dagger \vec{\sigma} \cdot (\vec{D} \times \vec{E} - \vec{E} \times \vec{D}) \psi \\
& + \mathcal{O}(1/c^3)
\end{aligned} \tag{10}$$

Here we have rescaled ψ by \sqrt{c} and we have generalized the coefficient of the fourth, sixth and seventh lines with hindsight (although, for now, $c_F = c_D = c_S = 1$). We have omitted the antiquark terms, which are left to the student as an exercise. The terms in this lagrangian have well known physical interpretation. The fourth line is the chromo-magnetic moment interaction, the fifth line is the first relativistic correction, the sixth line is the Darwin term and the seventh is the spin orbit coupling.

It would be desirable to have a lowest order NRQCD lagrangian that would be completely independent of $1/c$. One would then use the $1/c = 0$ spectrum as a solid basis for perturbation theory in the small parameter $1/c$, much as it is done in the $1/M$ expansion for B -mesons. If this cannot be done, that is, if our lowest order lagrangian contains explicitly the parameter $1/c$, then the lowest order states will depend implicitly and non-trivially on $1/c$, and an explicit expansion in $1/c$ will be impossible.

What are the leading terms in the $1/c$ expansion? To this end one is tempted to omit from the lagrangian above all terms with powers of $1/c$. This is almost correct except for one subtlety. The second term on the first line of (10) must be kept. Recall, for a Hamiltonian we need the generalized momentum

$$\pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} = \frac{1}{c^2}(\partial_t A_i - c\partial_i A_0)$$

so that the Hamiltonian starts at order c^2 :

$$\mathcal{H} = \pi_i \partial_t A_i - \mathcal{L} = \frac{1}{2}c^2 \pi^2 + \dots$$

Also, in the third line, the covariant derivative contains hidden dependence on $1/c$ which should be dropped. Hence, we have the leading order lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_i A_0^a - \frac{1}{c}\partial_t A_i^a)^2 - \frac{1}{4}(\partial_i A_j^a - \partial_j A_i^a)^2 + \psi^\dagger(iD_t + \frac{\vec{\nabla}^2}{2M})\psi$$

2.3 Interlude: Momentum Space Derivation

It is instructive to derive the effective lagrangian of NRQCD by analyzing Green functions (Feynman diagrams) in momentum space. This may be less elegant than the derivation above, but it has several advantages. First, it is more intuitive, since most of us have spent a great deal of time computing Feynman diagrams. Second, it is a better starting point towards including quantum effects. Third, it's easier.

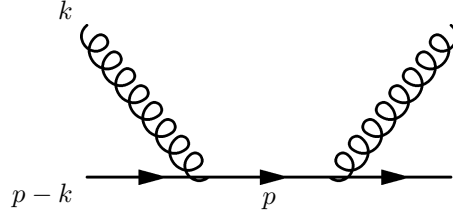


Fig. 6. Feynman diagram giving the leading contribution to Compton scattering of a gluon off a quark.

Consider Compton scattering of a gluon off a quark; see Fig. 6. The momentum p of the internal propagator will be conveniently taken to have components

$$p = (Mc + \frac{1}{c}E, \vec{p}).$$

This choice is the right starting point for a $1/c$ expansion; we have written the time-like component as an energy, and shifted it by the dominant term Mc . Now, we expand the intermediate propagator about $1/c = 0$; we assume the momentum p “scales” with c as given, *ie*, we take the large c limit keeping M , E and \vec{p} fixed:

$$\begin{aligned} \frac{1}{c} \frac{\gamma \cdot p + Mc}{p^2 - (Mc)^2} &= \frac{Mc(1 + \gamma_0) + \frac{1}{c}E\gamma_0 - \vec{\gamma} \cdot \vec{p}}{(Mc + \frac{1}{c}E)^2 - \vec{p}^2 - M^2c^2} \\ &= \frac{Mc(1 + \gamma_0) + \dots}{2ME - \vec{p}^2 + \dots} \\ &= \left(\frac{1 + \gamma_0}{2}\right) \frac{1}{E - \vec{p}^2/2M} + \mathcal{O}(\frac{1}{c}) \end{aligned}$$

The factor $(1 + \gamma_0)/2$ in last line projects out the ψ component of the 4-spinor, and the second factor, $1/(E - \vec{p}^2/2M)$, is the non-relativistic propagator. This is precisely the propagator for ψ that follows from \mathcal{L}_0 .

For this expansion to work we need to insist that the momenta of any quark scales with c just like p does, so both p and $p-k$ scale at most as $p = (Mc + \frac{1}{c}E, \vec{p})$, and therefore k must scale at most as $k = (\frac{1}{c}k_0, \vec{k})$.

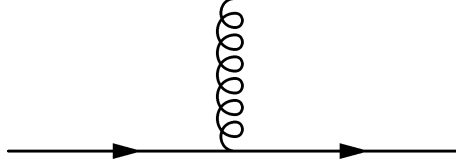


Fig. 7. Quark-gluon vertex. In NRQCD the vertex is spin independent.

Consider next interactions. Since the propagators on either side of the vertex in Fig. 7 have a projector operator P_+ , where $P_{\pm} = (1 \pm \gamma_0)/2$, $P_{\pm}^2 = P_{\pm}$, $P_+ + P_- = 1$, the vertex can be simplified,

$$P_+(ig_s T^a \gamma_{\mu})P_+ = ig_s T^a \delta_{\mu 0} \gamma_0 P_+ = ig_s T^a P_+ \delta_{\mu 0}.$$

Therefore only A_0 couples to the quark, and the coupling is a term in the lagrangian

$$\delta\mathcal{L} = \psi^\dagger g_s T^a \psi A_0^a.$$

The propagator and interaction of the quark are given therefore by the Lagrangian

$$\mathcal{L} = \psi^\dagger (iD_t + \frac{\vec{\nabla}^2}{2M})\psi,$$

which is precisely the leading term in the quark lagrangian in the previous section.

We can now try to construct an effective field theory in this limit. I like to think of an effective field theory as a factorization theorem for Green functions: ¹³

$$G\left(p_Q = (Mc + \frac{1}{c}E, \vec{p}), k_g = (\frac{1}{c}k^0, \vec{k})\right) = A(c)\tilde{G}\left((E, \vec{p}), (k^0, \vec{k}); M\right) + \mathcal{O}(\frac{1}{c}).$$

There is much explaining to do. The left hand side is the Green function for the full theory (QCD), with the kinematic variables as chosen, *ie*, in the scaling regime discussed earlier. The variables p_Q and k_g are generic for the 4-momenta of quarks and gluons, of which there may be several. The expression at right shows how the full Green function *factorizes* into a term, $A(c)$, that contains all the dependence on the small parameter, and a term that contains the dynamical information, \tilde{G} . What

makes the effective theory useful is that \tilde{G} can be computed as the corresponding Green function of a theory based on an “effective” lagrangian.

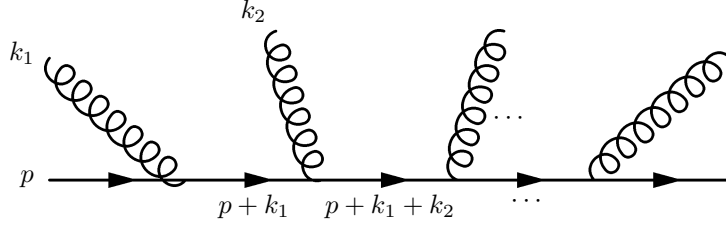


Fig. 8. That a diagram like this, with an arbitrary number of gluons, is reproduced correctly by NRQCD follows from repeated application of the analysis for the propagator and vertex diagrams.

In fact, we have just proved this identity at tree level for the class of Green functions which contain one quark line and any number of external gluons, as displayed in Fig. 8.

Let’s explore what interesting things may happen at higher order in the loop expansion and in higher order in $1/c$, as well as for other classes of Green functions.

2.4 Loops

Let us take a convergent example, like the Feynman diagram of Fig. 9. This avoids unnecessary confusion from ultraviolet divergences. We have seen that the propagators of QCD go over into those of NRQCD if the loop momentum scales as $k = (\frac{1}{c}k^0, \vec{k})$. However this means that the gluon propagator becomes, in leading order in $1/c$,

$$\frac{1}{\frac{1}{c^2}k_0^2 - \vec{k}^2} \rightarrow -\frac{1}{\vec{k}^2} \quad (11)$$

This is problematic, since it truncates the theory in a drastic way: it suppresses all retardation effects. However, it seems that if we keep the $\frac{1}{c^2}k_0^2$ in the propagator we fail to organize the expansion in powers of $1/c$.

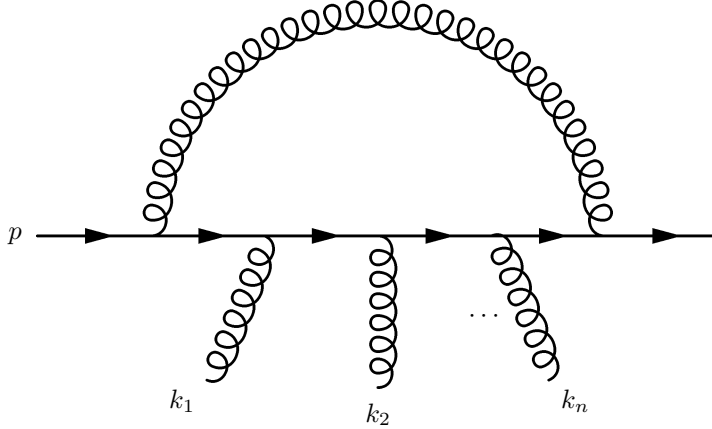


Fig. 9. This one loop diagram is convergent. One must show that NRQCD reproduces QCD's non-analytic behavior at small momenta.

The solution to this puzzle is to measure internal gluon momentum in *energy* units. That is, we take the gluon momentum to scale as $k = (\frac{1}{c}k_0, \frac{1}{c}\vec{k})$ where k_0 and \vec{k} have units of energy. This is not inconsistent with our previous scaling rules, it is just more restrictive. The propagator is then

$$\frac{c^2}{k_0^2 - \vec{k}^2}$$

and the loop integral is over

$$\int dk_0 \frac{d^3k}{c^3}.$$

This modification to our scaling rules introduces a surprise. The internal quark propagators are now

$$\frac{1}{E + k_0 - \frac{(\vec{p} + \frac{1}{c}\vec{k})^2}{2M}} = \frac{1}{E + k_0 - \frac{\vec{p}^2}{2M}} + \mathcal{O}\left(\frac{1}{c}\right).$$

In NRQCD this further expansion of the quark propagator is called a “multipole expansion”, because it is just what we would do if the gluon were real and we expanded in inverse powers of the large wavelength $\lambda \sim 1/|\vec{k}|$. We emphasize that the previous, tree level, success remains valid.

It is worth dwelling a bit on why we insist in retaining the k_0 in the gluon propagator, rather than following our nose through the $1/c$ expansion and using the propagator in (11). The problem is that this amounts to making static our fast moving modes in the Born-Oppenheimer approximation. There are, for sure, static components to the interaction — in Coulomb gauge the potential A_0 is instantaneous. But using (11) removes all retardation effects.

2.5 Other Green Functions

Since we are primarily interested in bound states of heavy quark and antiquarks, we should study almost forward $Q - \bar{Q}$ scattering amplitudes; see Fig. 10.

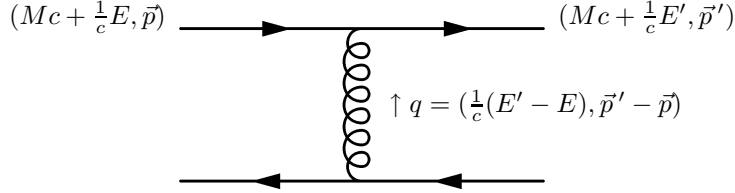


Fig. 10. One gluon exchange. The gluon is harder than the internal (loop) gluons in NRQCD.

The external initial and final quark momenta are chosen according to our scaling rules, $p = (Mc + \frac{1}{c}E, \vec{p})$ and $p' = (Mc + \frac{1}{c}E', \vec{p}')$. This implies the momentum transfer $q = p' - p = (\frac{1}{c}(E' - E), \vec{p}' - \vec{p})$, carried by the gluon has a spatial component that scales as c^0 rather than $1/c$. Is this a problem?

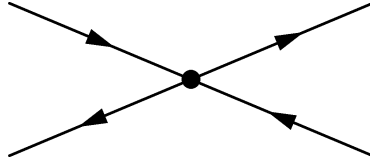
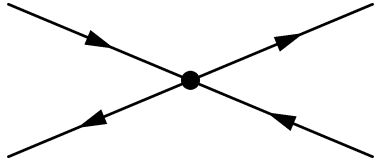


Fig. 11. Local interaction reproducing the hard gluon exchange of Fig. 10.

Our rules state that the internal integration variables for gluon momenta should have units of energy, and therefore come accompanied by appropriate factors of $1/c$. But in the diagram of Fig. 10 the momenta in the gluon propagator is fixed externally and scales as c^0 , while the energy scales as $1/c$. So the amplitude is



$$\sim -\frac{1}{q^2} \cdot g_s^2 T^a \otimes T^a .$$

This can be reproduced by our effective theory as a spatially non-local 4-quark vertex. Since the Fourier transform of $1/|\vec{q}|^2$ is $1/|\vec{x}|$, the effective lagrangian should be augmented by terms that are schematically of the form

$$\psi^\dagger \psi(t, \vec{x}) \int d^3 y \frac{1}{|\vec{x} - \vec{y}|} \chi^\dagger \chi(t, \vec{y}).$$

This interaction is responsible for, among others, the Coulomb-like potential between Q and \bar{Q} .

Note that the interaction is local in time, as it must for an appropriate Hamiltonian formulation of the problem.

Exercise: Compute explicitly the Coulomb term that needs to be added to the effective theory.

2.6 The NRQCD Lagrangian Revisited

After our exploration in momentum space we return to a derivation of the effective lagrangian directly in configuration space. Our aim is to understand in terms of a Lagrangian formulation the origin of the new rules that we derived by considering Feynman diagrams. From here on we will adopt Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$. As we saw earlier the lowest order lagrangian density

$$\mathcal{L}_0 = \frac{1}{2}(\partial_i A_0^a - \frac{1}{c}\partial_t A_i^a)^2 - \frac{1}{2}(\partial_i A_j^a)^2 + \psi^\dagger(iD_t + \frac{\vec{\nabla}^2}{2M})\psi$$

still contains c dependence. Defining ¹⁴

$$\tilde{A}_i(\vec{y} = \frac{\vec{x}}{c}, t) = \sqrt{c}A_i(\vec{x}, t)$$

and writing the lagrangian in terms of this new variable, we obtain

$$L_0 = \int d^3y \left[\frac{1}{2}(\partial_0 \tilde{A}_i)^2 - \frac{1}{2}(\partial_i \tilde{A}_j)^2 \right] + \int d^3x \left[\psi^\dagger(iD_t + \frac{\vec{\nabla}^2}{2M})\psi + \frac{1}{2}(\partial_i A_0)^2 \right]$$

This is independent of c and can be used as a starting point for an expansion in $1/c$.

In particular:

- The states of L_0 are c -independent
- These states can be used to formulate perturbation theory in $1/c$
- All $1/c$ corrections can be expressed as matrix elements of operators between these states. That is, we have *explicit power counting*. The engineering dimensions of an operator can be determined, and compensated for with the appropriate inverse powers of mass, M , and speed of light, c .

Consider the $1/c$ corrections. The terms of order $1/c$ in the Lagrangian density of Eq. (10) are

$$L_{c^{-1}} = \int d^3x \psi^\dagger(\vec{x}, t) \left[\frac{g}{Mc^{3/2}} \tilde{A}_i(\frac{1}{c}\vec{x}, t) \frac{\partial}{\partial x^i} + \frac{c_F g}{2Mc^{5/2}} \vec{\sigma} \cdot \tilde{\mathbf{B}}(\frac{1}{c}\vec{x}, t) \right] \psi(\vec{x}, t),$$

where $\tilde{B}_i = \epsilon_{ijk} \frac{\partial}{\partial y^j} \tilde{A}_k(\vec{y}, t)$. As we see, in these interaction terms the fields have dependence both on \vec{x} and on $\vec{y} = \frac{1}{c}\vec{x}$. Changing variables to \vec{y} does not eliminate the dependence on c . Instead we write all field dependence in terms of \vec{x} and expand in a Taylor series the terms that depend on $\frac{1}{c}\vec{x}$. This is just the multipole expansion: ^{14,15}

$$L_{c^{-1}\text{MP}} = \int d^3x \psi^\dagger(\vec{x}, t) \left\{ \frac{g}{Mc^{3/2}} \left[\tilde{A}_i(\vec{0}, t) + \frac{x_j}{c} (\partial_j \tilde{A}_i(\vec{0}, t) + \dots) \right] \frac{\partial}{\partial x^i} + \frac{c_F g}{2Mc^{5/2}} \left[\vec{\sigma} \cdot \tilde{\mathbf{B}}(\vec{0}, t) + \dots \right] \right\} \psi(\vec{x}, t). \quad (12)$$

Although this expression seems unfamiliar, it is the configuration space version of the multipole expansion performed earlier in momentum space where it appeared in a natural and intuitive form.

2.7 Technical Note On Explicit Power Counting

The multipole expansion makes power counting straightforward, and this is why we have introduced it here. Let us see how it works in more detail in a somewhat non-trivial example.

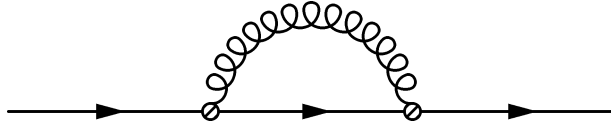


Fig. 12. Contribution to the quark self-energy from two insertions of the operator $\psi^\dagger \vec{\sigma} \cdot \vec{B} \psi$, represented by a shaded circle. Without a multipole expansion this graph gives contributions of lower order in $1/c$ than the order of the operators inserted.

Consider a contribution to the quark self-energy from two insertions of the operator $\psi^\dagger \vec{\sigma} \cdot \vec{B} \psi$. This is one of the terms of order $1/c$ in the lagrangian density in Eq. (10).

If we use this operator in the lagrangian as it stands in Eq. (10), that is, before multipole expanding, the one loop contribution to the dimensionally regularized two-point function is

$$i\Gamma^{(2)} = \frac{c_F^2 C(R) g^2}{4M^2 c^2} \int \frac{d^D k}{(2\pi)^D} \frac{(\vec{k} \times \vec{\sigma}) \cdot (\vec{k} \times \vec{\sigma})}{\left(\frac{1}{c^2} k_0^2 - \vec{k}^2 + i\epsilon\right) \left(E + k_0 - \frac{(\vec{p} + \vec{k})^2}{2M} + i\epsilon\right)}.$$

With $D = 4 - 2\varepsilon$, the divergent pieces as $\varepsilon \rightarrow 0$ are,

$$i\Gamma_{\text{div}}^{(2)} = \frac{c_F^2 C(R) \alpha_s}{\pi} \Gamma(2\varepsilon) \left[2 \left(E - \frac{\vec{p}^2}{2M} \right) + 4Mc^2 + \frac{4}{3} \frac{\vec{p}^2}{2M} \right].$$

Other schemes yield analogous results.

As an exercise the reader should compute the loop integral with a cut-off $|\vec{k}| < \Lambda$. Clearly he or she will find a term of order $\frac{c_F^2 C(R) \alpha_s}{\pi} \Lambda^2$. This result, and

others like it, has led many (including me) to make the ridiculous statement that the effective theory must be defined with a *low* cut-off, say, $\Lambda \sim \text{Rydberg} \sim 1/Ma^2$.

This is, of course, non-sense. One should take the cutoff Λ to be arbitrarily large, and make the necessary subtractions of divergent and finite terms. However making subtractions is a bit of a pain. Firstly, since the lower order terms in the $1/c$ expansion are modified by the inclusion of any higher order term, one must recalculate counterterms every time a new higher order term is added. Secondly, this means that to restore power counting one must tune coefficients carefully.

If instead one computes with the multipole expanded lagrangian, the result in dimensional regularization is

$$i\Gamma^{(2)} = \frac{c_F^2 C(R) g^2}{2M^2 c^5} \int \frac{d^D k}{(2\pi)^D} \frac{\vec{k}^2}{\left(k_0^2 - \vec{k}^2 + i\epsilon\right) \left(E - \frac{\vec{p}^2}{2M} + k_0 + i\epsilon\right)}.$$

Note that, by design, the whole dependence on c is explicit in the coefficient, $1/c^5$. The divergent part is now

$$i\Gamma_{\text{div}}^{(2)} = \frac{c_F^2 C(R) \alpha_s}{2\pi} \Gamma(2\epsilon) \frac{(E - \vec{p}^2/2M)^3}{M^2 c^4}.$$

2.8 NRQCD – An Effective Field Theory: Is All This Really Necessary?

Although nobody has bothered to proof the validity of NRQCD as an effective theory (in the sense of a factorization theorem, as explained above), we believe this to be true. Why?

The answer is that NRQCD has the right ingredients. That is, one could simply ask what terms should be contained in a local lagrangian field theory describing gluons and non-relativistic quarks. The answer ¹⁶ is the lagrangian we have been considering,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \psi^\dagger (iD_t + \frac{1}{2M} \vec{D}^2) \psi + \frac{c_F g}{2Mc} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi + \dots \quad (13)$$

The coefficient of the F^2 and $\psi^\dagger D_t \psi$ terms have been scaled to one by wave function renormalization, as usual. The coefficient of the $\psi^\dagger \vec{D}^2 \psi$ term *defines* the mass M . Coefficients of higher dimension operators (c_F, c_D, \dots) remain undetermined. In a non-perturbative treatment of the theory, as for example on the lattice, one can treat c_F, c_D, c_S as additional free couplings to be determined by comparing with experiment (or with full, continuum QCD, in which case we call the procedure “matching”).

It would appear then that all the careful analysis of the previous sections, and in particular the multipole expansion, is really unnecessary. The lagrangian of Eq. (13) describes the same physics as our multipole expanded lagrangian of Eq. (12). The only difference is that in the first case one may not systematically study each order of the $1/c$ expansion. This may not be important for many applications, and in such cases the simpler formalism should be employed. But, as we will see, this is not always the case. An important example is given in the next chapter. The calculation of the decay of P -wave quarkonium exhibits infrared divergences in

the matching to the effective theory that can be eliminated properly only if one includes the correct set of operators. Counting of powers of $1/c$ singles out the correct operators. Moreover, if some of the $1/c$ effects were left implicit in the lowest order states (or elsewhere) one could in principle have some of the infrared cancellation take place against part of the state definition: this would make such a cancellation practically intractable.

2.9 Spin Symmetry

The Lagrangian

$$\mathcal{L} = \psi^\dagger (iD_t + \frac{1}{2M} \vec{D}^2) \psi - \chi^\dagger (iD_t + \frac{1}{2M} \vec{D}^2) \chi$$

has a $U(2) \times U(2)$ symmetry

$$\psi_\alpha \rightarrow R_{\alpha\beta} \psi_\beta \quad \chi_{\dot{\alpha}} \rightarrow T_{\dot{\alpha}\dot{\beta}} \chi_{\dot{\beta}}$$

with $R^\dagger R = T^\dagger T = \mathbf{1}$. Please note that while the field ψ annihilates a heavy quark, the field χ creates a heavy anti-quark. The abelian factors $U(1) \times U(1)$ correspond to separate conservation of heavy quark and anti-quark numbers. The $SU(2)$'s are *spin symmetries*. They are broken by the spin-flip magnetic moment interaction that appears in $L_{c^{-1}}$, and which is of order $c^{-5/2}$ in the multipole expansion:

$$L_{c^{-1}\text{MP}} = \int d^3x \frac{c_F g}{2Mc^{5/2}} \psi^\dagger(\vec{x}, t) \vec{\sigma} \cdot \tilde{\mathbf{B}}(\vec{0}, t) \psi(\vec{x}, t) + \dots$$

None of these observations come as a surprise to the reader who has kept in mind the molecular physics analogue of quarkonium.

In non-relativistic quantum mechanics orbital angular momentum \vec{L} and spin \vec{S} are separately conserved, and therefore so is the total angular momentum $\vec{J} = \vec{L} + \vec{S}$. States are specified by $^S L_J$, where in the standard spectroscopic notation S takes values denoting the dimension of the $SU(2)$ representation ($S = 1$ for the singlet, $S = 3$ for triplet (adjoint), etc), $L = S, P, D, \dots$ and $J = 0, 1, 2, \dots$. In the following table we give the possible quark–anti-quark states for the lowest few values of L , and the name of the state for the case of charm quarks:

$L = 0$	
1S_0	η_c
3S_1	J/ψ
$L = 1$	
1P_1	h_c
3P_0	χ_{c0}
3P_1	χ_{c1}
3P_2	χ_{c2}

In leading order in the $1/c$ expansion states are classified this way, and their properties are related by spin symmetry. The names of b-quark states are obtained by replacing $c \rightarrow b$, except for the 3S_1 which is called “ Υ .”

The spin symmetry is most easily applied by use of the Wigner-Eckart theorem. This is best explained through examples.

Example 1: S-states

Construct 2×2 matrix $H_{\alpha\dot{\beta}}$ representing the four $^1S_0, ^3S_1$ states. Under spin symmetry

$$H \rightarrow RHT^\dagger.$$

Then, for any matrix Γ and any state X free of heavy quarks (and therefore a singlet under spin symmetry), the Wigner-Eckart theorem gives

$$\langle X|\chi^\dagger\Gamma\psi|H\rangle = \xi_X \text{Tr}\Gamma H.$$

Here ξ_X is the reduced matrix element; generally it depends on kinematic variables and the parameters of the theory. We have used the same letter H to denote the specific quarkonium state on the left hand side and the corresponding matrix on the right. Explicitly (for b -quarkonium)

$$H = \sigma^0 \eta_b + \sum_{a=1}^3 \sigma^a \Upsilon^a.$$

Here and below $\sigma^0 = \text{diag}(1, 1)$.

This implies, for example

$$\begin{aligned} \langle X|\chi^\dagger\psi|\eta_b\rangle &= \xi_X \\ \langle X|\chi^\dagger\sigma^a\psi|\Upsilon(\epsilon)\rangle &= \xi_X \epsilon^a \end{aligned}$$

where $\vec{\epsilon}$ is the polarization vector of the spin-1 Υ .

Example 2: P-states

More interesting is the case of the 12 P-wave states. The $12 = 3 \times 2 \times 2$ states can be represented by

$$\tilde{H}_{m\alpha\dot{\beta}}, \quad m = 1, 2, 3 \quad \alpha, \dot{\beta} = 1, 2$$

and the components are found by insisting they transform correctly under rotations (index m) and spin symmetry (α and $\dot{\beta}$), combined into appropriate J . Without proper normalization:

$$\tilde{H}_{m\alpha\dot{\beta}} = \sigma_{\alpha\dot{\beta}}^m \chi_0 + \epsilon_{mjk} \sigma_{\alpha\dot{\beta}}^j \chi_1^k + \sigma_{\alpha\dot{\beta}}^j \chi_2^{mj} + \sigma_{\alpha\dot{\beta}}^0 h_m$$

Here χ_2^{mj} is symmetric and traceless.

Exercise: (a) Normalize these properly. To this end compute $\text{Tr}H^\dagger H$ and verify that the coefficients of all states are unity.

(b) In the rest frame of the quarkonium state ($\vec{p} = 0$) what is $\langle 0|\chi^\dagger\Gamma\psi|\tilde{H}\rangle$? Calculate $\langle 0|\chi^\dagger\vec{D}_m\Gamma\psi|\tilde{H}\rangle$. Find the relations between these matrix elements of χ_J and h .

3 Annihilation Decays of Quarkonium

3.1 Color Singlet Model

Consider the decay of η_c into light hadrons. To this effect model the state by a pure singlet, $\eta_c = (Q\bar{Q})_1$, that is, with no $(Q\bar{Q})_8$ component. Let us represent the $Q\bar{Q}$ in the 1S_0 state as in the quark model

$$|\eta_c\rangle = \frac{1}{\sqrt{2M_\eta}} \int \frac{d^3q}{(2\pi)^3} \psi(\vec{q}) \frac{\delta^{ij}}{\sqrt{3}} \frac{\epsilon^{mn}}{\sqrt{2}} |c^{im}(\vec{q}) \bar{c}^{jn}(-\vec{q})\rangle,$$

where i, j are color indices, m, n are spin indices and $\psi(\vec{q})$ is the Fourier transform of the coordinate space wavefunction

$$\psi(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \psi(\vec{q}).$$

By spherical symmetry

$$\psi(\vec{x}) = \frac{1}{\sqrt{4\pi}} R(r)$$

where $r = |\vec{x}|$.

We stop here for a small digression. We note that something is missing. Even in the molecular physics analogy of Sect. 1 the state would be represented by a wave-function $\phi_n(x, X)\psi(X)$, with $\phi_n(x, X)$ the wavefunction for the fast degrees of freedom. We are missing the wavefunction for the glue! We will look the other way and carry on, as if the glue component were trivial.

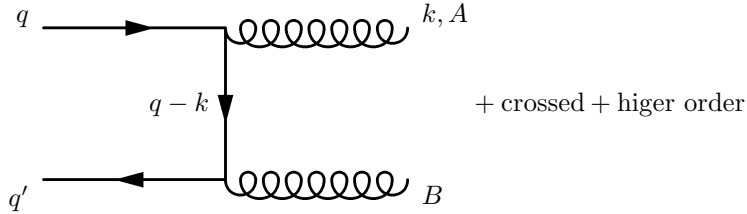


Fig. 13. Feynman diagrams for the $cc \rightarrow gg$ amplitude, used in the calculation of the η_c width.

In order to calculate the total inclusive width $\Gamma(\eta_c \rightarrow \text{hads})$ we use *duality*. This is the physically plausible assumption that the sum over all final hadronic states should be well approximated by the rate into partons,

$$\Gamma(\eta_c \rightarrow \text{hads}) = \sum_X \Gamma(\eta_c \rightarrow X) \approx \Gamma(\eta_c \rightarrow gg).$$

With our representation of the η_c state we then have

$$\Gamma(\eta_c \rightarrow \text{hads}) = \frac{1}{2M_\eta} \int \frac{d^3k}{(2\pi)^3 2k_0} \frac{2\pi\delta(M_\eta - 2|\vec{k}|)}{M_\eta} |\mathcal{M}(\eta_c \rightarrow g(\vec{k})g(-\vec{k}))|^2.$$

Here \mathcal{M} is the invariant (T-matrix) amplitude. Suppressing color and spin indices

$$\mathcal{M}(\eta_c \rightarrow g(\vec{k})g(-\vec{k})) = \frac{1}{\sqrt{2M_\eta}} \int \frac{d^3q}{(2\pi)^3} \psi(\vec{q}) \mathcal{M}(c(\vec{q})\bar{c}(-\vec{q}) \rightarrow g(\vec{k})g(-\vec{k})).$$

In perturbation theory the amplitude $\mathcal{M}(c(\vec{q})\bar{c}(-\vec{q}) \rightarrow g(\vec{k})g(-\vec{k}))$ is computed from the diagram in Fig. 13, and is given by

$$\mathcal{M}(c(\vec{q})\bar{c}(-\vec{q}) \rightarrow g(\vec{k})g(-\vec{k})) = -ig_s^2 (T^B T^A) \bar{v}\gamma \cdot \epsilon^*(k') \frac{1}{\gamma \cdot (q - k) - m_c} \gamma \cdot \epsilon^*(k) u$$

where $q = (m_c + E, \vec{q})$, $q' = (m_c + E, -\vec{q})$ and $k = \frac{1}{2}M_\eta(1, \hat{n})$ ($|\hat{n}| = 1$). Note that, for now, I have suppressed powers of c which can be restored by dimensional analysis. The denominator

$$(q - k)^2 - m_c^2 \approx -2q \cdot k = -M_\eta[(m_c + E) - \vec{q} \cdot \hat{n}] \quad (14)$$

is dominated by the mass term for $|\vec{q}| < m_c$ and there it is approximately constant (or “flat”) and $\approx -2m_c^2$. The wavefunction $\psi(\vec{q})$ has support within one inverse Bohr radius, $1/a_{\text{Bohr}} \sim \alpha_s m_c$, which is indeed smaller than m_c if a perturbative value for α_s can be used.

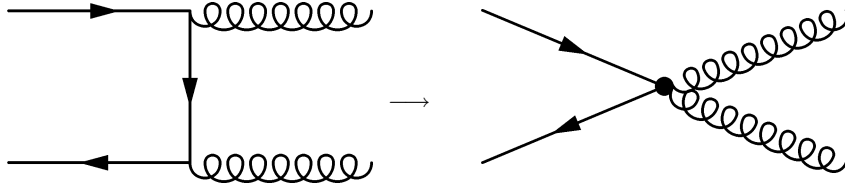


Fig. 14. The amplitude for $cc \rightarrow gg$ is effectively local: when the external charm quarks are non-relativistic the internal quark must be off shell by an amount of order of the charm mass.

We therefore approximate the integral by replacing the propagator by its flat value,

$$\Gamma(\eta_c \rightarrow \text{hads}) \approx \frac{1}{(2M_\eta)^2} \left| \int \frac{d^3q}{(2\pi)^3} \psi(\vec{q}) \right|^2$$

$$\times \int \frac{d^3 k}{(2\pi)^3 2k_0} \frac{2\pi\delta(M_\eta - 2|\vec{k}|)}{M_\eta} |\mathcal{M}(c(0)\bar{c}(0) \rightarrow g(\vec{k})g(-\vec{k}))|^2$$

Exercise: Complete the calculation. Show this gives

$$\Gamma(\eta_c \rightarrow \text{hads}) \approx \frac{8\alpha_s^2}{3M_\eta^2} |R(0)|^2.$$

3.2 η_c Decay in NRQCD

We would like to carry out the calculation of the previous section in a more rigorous manner. To this end we would like to use the effective theory, NRQCD. However, several problems immediately come to mind:

- The effective theory cannot describe properly gluons or quarks with $E, |\vec{p}| \sim m_c$ (because the $1/c$ expansion will result in an expansion in velocity v with $v/c \sim |\vec{p}|/m_c c$ or $v^2 \sim E/m_c c^2$).
- The effective theory has separate conservation of Q and \bar{Q} numbers. How can it possibly describe $Q\bar{Q}$ annihilation in η_c decay?

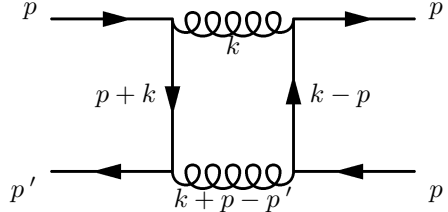


Fig. 15. Feynman diagram for the forward $c\bar{c}$ scattering amplitude.

Fortunately there is a common solution to these problems.^{17,18} Consider the Feynman diagram for $c\bar{c}$ annihilation in Fig. 13. The external quark lines, representing the quark-antiquark pair in charmonium, are almost on-shell. But the internal quark line is off the mass shell by the mass of the quark itself! Effectively the interaction is local; see Fig. 14. Formally the denominator in Eq. (14) can be expanded

$$\frac{1}{m_c + E - \vec{q} \cdot \hat{n}} = \frac{1}{m_c} \left(1 - \frac{E - \vec{q} \cdot \hat{n}}{m_c} + \dots \right)$$

and the leading term gives a local non-derivative quark-gluon interaction for $c\bar{c} \rightarrow gg$.

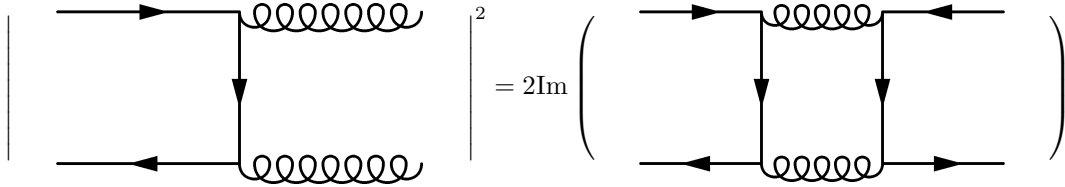


Fig. 16. The optical theorem relates the total $c\bar{c}$ annihilation cross section, which enters the calculation of the width of the η_c , to the $c\bar{c}$ forward scattering amplitude, which can be modeled by a local interaction in NRQCD.

Notice however that the external gluons have energy and momenta of order m_c . This, of course, is not appropriate for treatment in the effective theory. So we are not out of the woods yet. The proposed solution is to use the optical theorem which gives the rate for $c\bar{c} \rightarrow gg$ from the imaginary part of the forward $c\bar{c}$ scattering amplitude. The lowest order Feynman diagram for $c\bar{c} \rightarrow c\bar{c}$ is shown in Fig. 15. The imaginary part has the internal lines off the mass shell by about m_c , so the whole loop is local on the scale of $|\vec{q}| \sim a_{\text{Bohr}}^{-1} \sim m_c \alpha_s \sim m_c v$ and we may replace it by a local interaction. This is shown in Fig. 17 where κ is a constant and the ellipsis indicate terms of order $|\vec{q}|/m_c$. The constant κ can be easily computed in the limit $\vec{q} = 0$.

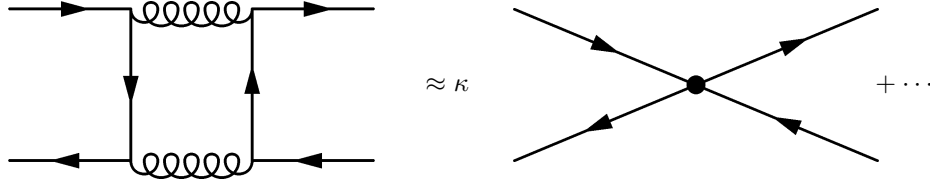


Fig. 17. The $c\bar{c}$ forward scattering amplitude can be modeled by a local interaction in NRQCD. The constant κ is determined by “matching” to QCD.

To get some better understanding of how this works, let’s look at the propagators in the box diagram of Fig. 15. Let k stand for the loop momentum and p and p' the external momenta. The quark propagators give

$$\frac{1}{k^2 + 2p \cdot k} \frac{1}{k^2 - 2p' \cdot k}$$

and the gluon propagators

$$\frac{1}{k^2} \frac{1}{(k + p - p')^2}.$$

Now, the imaginary part of the diagram gives a cut on the gluon lines,

$$\text{Im} \frac{1}{k^2 + i\epsilon} = -\pi \delta(k^2)$$

so the product of internal propagators reduces to

$$-\frac{\pi^2}{4(p \cdot k)^2} \delta(k^2) \delta((k + p - p')^2).$$

This is easy to interpret: the two gluons are on shell, the two quark propagators are just as before, Eq. (14).

In NRQCD the annihilation process is modeled by adding to the lagrangian a non-hermitian term that gives the imaginary part of the $Q\bar{Q}$ forward scattering amplitude. That is, the lagrangian is augmented by a local four-fermion operator

$$\Delta\mathcal{L} = \frac{f}{M^2 c} \psi^\dagger \chi \chi^\dagger \psi$$

with f a dimensionless constant chosen judiciously to give the right answer to this order in the $1/c$ expansion. This operator is the product of two $(^1S_0)_1$ bilinears (the subindex refers to the color channel, the spin quantum numbers are inside the parenthesis). The first bilinear annihilates the $(^1S_0)_1$ $c\bar{c}$, while the second creates it from the vacuum.

The constant f is determined by “perturbative matching”. This means that the rate should correspond to that of full QCD to the accuracy of the order of relevance of the NRQCD approximation. But the same constant f should reproduce the perturbative QCD calculation of a partonic cross section. Let’s consider this procedure in some detail. In QCD the forward scattering amplitude is given by the box diagram of Fig. 15. The spinors come in as

$$\bar{v}\gamma^\mu[\gamma \cdot (p + k) + m]\gamma^\nu u \bar{u}\gamma^\nu[\gamma \cdot (p - k) + m]\gamma^\mu v.$$

In the NR-limit we can replace

$$u = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{v} = \begin{pmatrix} 0 \\ -\chi^\dagger \end{pmatrix}$$

The product of Dirac matrices is reduced by $\gamma^\mu \gamma^\lambda \gamma^\nu = \gamma^\mu \eta^{\lambda\nu} + \gamma^\nu \eta^{\lambda\mu} - \gamma^\lambda \eta^{\mu\nu} - i\epsilon^{\mu\nu\lambda\rho} \gamma_\rho \gamma_5$ and $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu}$. The needed binomials are

$$\begin{aligned} \bar{v}\gamma^\mu u &= \begin{cases} 0 & \mu = 0 \\ \chi^\dagger \sigma^i \psi & \mu = i \end{cases} \\ \bar{v}\gamma^\mu \gamma_5 u &= \begin{cases} \chi^\dagger \psi & \mu = 0 \\ 0 & \mu = i \end{cases} \\ \bar{v}\sigma^{0i} u &= -i\chi^\dagger \sigma^i \psi \\ \bar{v}\sigma^{ij} u &= 0 \end{aligned}$$

To count dimensions, recall $S = \int dt d^3x \mathcal{L}$ is dimensionless, $\mathcal{L} = \psi^\dagger \partial/\partial t \psi$, so $[\psi^\dagger \psi] \sim L^{-3}$. Since $\hbar = 1$, $[Mc] \sim L^{-1}$ so we have $[\int dt d^3x \frac{c}{(Mc)^2} \psi^\dagger \chi \chi^\dagger \psi] \sim L^4 L^2 L^{-6} \sim 1$.

Notice that when we construct singlet combinations we have, in addition,

$$\sum_{\text{singlet}} \chi^\dagger \sigma^i \psi = 0.$$

Therefore the only non-vanishing singlet-singlet contribution is from

$$i\epsilon^{\mu\lambda\nu 0} i\epsilon_{\mu\sigma\nu 0} (p+k)_\lambda (p-k)^\sigma \chi^\dagger \psi \psi^\dagger \chi$$

The coefficient is just $[(p+k) \cdot (p-k) - (p+k)^0(p-k)^0]$ and this, on-shell (when we take the imaginary part of the amplitude), is m_c^2 .

Exercise: Complete the calculation. Show that the spin singlet, color singlet part of the cut diagram is

$$\frac{2\pi\alpha_s^2}{9m_c^2} \chi^\dagger \psi \psi^\dagger \chi.$$

In NRQCD this is to be reproduced by a term

$$\frac{f}{m_c^2} \chi^\dagger \psi \psi^\dagger \chi.$$

in the Lagrangian. The symbols χ and ψ are fields, but the tree level amplitude corresponds to replacing them by spinors. The matching condition is

$$\text{Im } f = \frac{2\pi\alpha_s^2}{9}$$

Finally, we compute the rate $\Gamma(\eta_c)$ in NRQCD. Treating \mathcal{L} as a perturbation, the energy shift is ^d

$$\Delta E_{\eta_c} = -\frac{f}{m_c^2 c} \frac{\langle \eta_c | \psi^\dagger \chi \chi^\dagger \psi | \eta_c \rangle}{2M_{\eta_c}}.$$

The imaginary part of ΔE is just $\frac{1}{2}\Gamma$, so we have finally

$$\Gamma(\eta_c) = \frac{1}{2M_{\eta_c}} \frac{4\pi\alpha_s^2}{9m_c^2} \langle \eta_c | \psi^\dagger \chi \chi^\dagger \psi | \eta_c \rangle.$$

We can find the relation between this result and that of the color-singlet model by using the vacuum insertion approximation

$$\langle \eta_c | \psi^\dagger \chi \chi^\dagger \psi | \eta_c \rangle \approx |\langle 0 | \chi^\dagger \psi | \eta_c \rangle|^2$$

with

$$\langle 0 | \chi^\dagger \psi | \eta_c \rangle \approx \sqrt{2M_{\eta_c}} \sqrt{\frac{3}{2\pi}} R(0)$$

We close this section with a short digression. The reader may be concerned with the appearance of complex coefficients of hermitian operators in the lagrangian.

^dIn the standard formula $\Delta E = \frac{\langle \psi | H' | \psi \rangle}{\langle \psi | \psi \rangle}$ go to relativistic normalization: $\langle \psi | \psi \rangle = 2M_{\eta_c} \delta^3(0) = 2M_{\eta_c} V$. The factor of V^{-1} is used in $H' V^{-1} = \mathcal{H}' = -\Delta \mathcal{L}$.

This renders the Hamiltonian non-hermitian and jeopardizes conservation of probability. A little thought shows that this is as expected. The situation is similar to that encountered in the effective two level Hamiltonian for $K^0 - \bar{K}^0$ mixing. There the Hamiltonian is given in terms of hermitian matrices M and Γ as $H = M + \frac{i}{2}\Gamma$. The loss of hermiticity encoded in Γ arises because the states into which the K -mesons may decay are not incorporated in the theory. Similarly, in NRQCD we have truncated the theory by excluding energetic gluons and light quarks, but these are the decay products of heavy quarkonia.

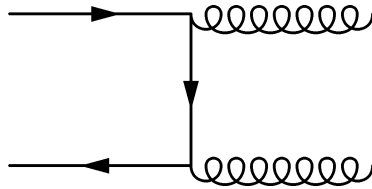


Fig. 18. Feynman diagram for $c\bar{c}$ annihilation entering the calculation of the η_c decay width.

3.3 Radiative Corrections and Decays of P-Waveonium

In the $c\bar{c}$ annihilation diagram of Fig. 18, used to describe η_c decay, the gluons have momentum $|\vec{k}| = \frac{1}{2}M_{\eta_c}$. Consider now the radiative corrections shown in the diagrams of Fig. 19. We are particularly interested in the kinematics where the momentum s is soft, that is, small. In both diagrams the sub-diagram corresponding to $c\bar{c} \rightarrow gg$ of Fig. 18 is much as before. That is, the internal quark is far off the mass shell, and the two external gluons are hard. As before we attempt to replace the interaction by a local $c\bar{c}gg$ vertex.

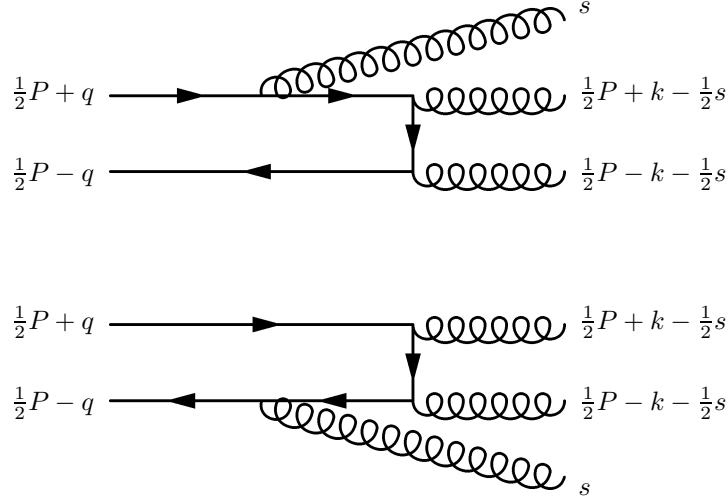


Fig. 19. Feynman diagrams representing soft gluon radiation as a correction to the amplitude for $c\bar{c}$ annihilation of Fig. 18. The soft gluon carries momentum s .

Let us analyze the emission of the soft gluon. In the first diagram of Fig. 19 the quark line emitting the gluon gives

$$\begin{aligned} & \frac{\gamma \cdot (\frac{1}{2}P + q - s) + m_c}{(\frac{1}{2}P + q - s)^2 - m_c^2} \gamma^\mu \frac{\gamma \cdot (\frac{1}{2}P + q) + m_c}{(\frac{1}{2}P + q)^2 - m_c^2} \\ & \approx \frac{(1 + \gamma^0)}{-2s^0 + s^2/m_c} \gamma^\mu \frac{(1 + \gamma^0)m_c}{(\frac{1}{2}P + q)^2 - m_c^2} \\ & \rightarrow \frac{-\gamma^0}{s^0} \left[\left(\frac{1 + \gamma^0}{2} \right) \frac{m_c}{(\frac{1}{2}P + q)^2 - m_c^2} \right]. \end{aligned}$$

We have left the propagator on the right side unexpanded since it will be amputated when we compute an amplitude. Similarly, the second diagram in Fig. 19 gives

$$\frac{\gamma \cdot (-\frac{1}{2}P + q) + m_c}{(-\frac{1}{2}P + q)^2 - m_c^2} \gamma^\mu \frac{\gamma \cdot (-\frac{1}{2}P + q + s) + m_c}{(-\frac{1}{2}P + q + s)^2 - m_c^2}$$

$$\rightarrow \left[\frac{m_c}{(\frac{1}{2}P - q)^2 - m_c^2} \left(\frac{1 - \gamma^0}{2} \right) \right] \frac{+\gamma^0}{s^0}.$$

We see immediately that although each diagram is separately infrared divergent, *ie*, they contain a singularity as $s^0 \rightarrow 0$, the divergence cancels in the sum. Physically the cancellation can be easily understood: the coupling involves γ^0 , which corresponds to the “charge” of the quark, but c and \bar{c} have opposite charges.

To see that this physical intuition is correct, let’s compute the next term, involving the current γ^i . Using $\frac{1}{2}P + q = (m_c c, \vec{0}) + (\frac{1}{c}E, \vec{q})$ and neglecting q^0 and s , the Dirac structure of the soft gluon-quark interaction is

$$\begin{aligned} & [\gamma \cdot (\frac{1}{2}P + q - s) + m_c] \gamma^\mu [\gamma \cdot (\frac{1}{2}P + q) + m_c] \\ & \rightarrow [m_c(1 + \gamma^0) + \gamma \cdot q] \gamma^\mu [m_c(1 + \gamma^0) + \gamma \cdot q] \\ & = 2m_c^2(1 + \gamma^0)\delta_0^\mu + m_c[\gamma \cdot q \gamma^\mu(1 + \gamma^0) + (1 + \gamma^0)\gamma^\mu \gamma \cdot q] + \dots \\ & = \begin{cases} 2m_c^2(1 + \gamma^0) - 4m_c \vec{\gamma} \cdot \vec{q} & \text{for } \mu = 0 \\ -2m_c(1 + \gamma^0)q^i & \text{for } \mu = i \end{cases} \end{aligned}$$

The leading term (for $\mu = 0$) is what we computed above. The new terms of interest are the ones of order m_c . The second diagram gives, similarly

$$\begin{aligned} & [\gamma \cdot (-\frac{1}{2}P + q) + m_c] \gamma^\mu [\gamma \cdot (-\frac{1}{2}P + q + s) + m_c] \\ & \rightarrow \begin{cases} -2m_c^2(1 - \gamma^0) + 4m_c \vec{\gamma} \cdot \vec{q} & \text{for } \mu = 0 \\ -2m_c(1 - \gamma^0)q^i & \text{for } \mu = i \end{cases} \end{aligned}$$

So we see that there is at this order an infrared divergence in these diagrams. It appears only in the $\mu = i$ terms. However, it does not afflict decays of S -wave charmonium since the factor q^i changes L by one.^eFormally, the operator with the divergent coefficient is $\chi^\dagger \overleftrightarrow{D}\psi$, rather than $\chi^\dagger \psi$. But

$$\langle 0 | \chi^\dagger \overleftrightarrow{D}\psi | \eta_c \rangle = 0.$$

However $L = 1$ operators do interpolate for P -wave quarkonium.

Exercise: Using the methods of Sec. 2.9 show that

$$\langle 0 | \chi^\dagger \vec{\sigma} \cdot \overleftrightarrow{D}\psi | h_c \rangle = 0$$

$$\langle 0 | \chi^\dagger \vec{\sigma} \cdot \overleftrightarrow{D}\psi | \chi_1 \rangle = 0$$

and find the relation between $\langle 0 | \chi^\dagger \vec{\sigma} \cdot \overleftrightarrow{D}\psi | \chi_0 \rangle$ and $\langle 0 | \chi^\dagger \vec{\sigma} \cdot \overleftrightarrow{D}\psi | \chi_2 \rangle$.

^eThe amplitude has $\vec{q} \cdot \vec{\epsilon}$, where $\vec{\epsilon}$ is the gluon polarization vector.

In the color singlet model there seems to be no way out of this problem with infrared divergences in the decay of P -wave quarkonium. Let us emphasize that the problem is that we have found an infrared divergence in the calculation of the matching of a coefficient, which should be given by short distance physics. Usually one finds a way to shove the infrared divergences into matrix elements of operators, but there is no candidate for an operator into which to shove this infrared divergence.

However, in NRQCD there is a neat resolution to this problem.¹⁸ In leading order, the decay is described by local interactions

$$\delta\mathcal{L} = \frac{f_1(^3P_0)}{m^4c^3}\mathcal{O}_1(^3P_0) + \frac{f_8(^3S_1)}{m^2c}\mathcal{O}_8(^3S_1) \quad (15)$$

where f 's are coefficients determined by matching and \mathcal{O} are operators. The subscript denotes the relative color of the quark bilinears, singlet or octet, and in parenthesis we have their JL_S number. The operators are

$$\begin{aligned} \mathcal{O}_1(^3P_0) &= \frac{1}{3}\psi^\dagger(\frac{i\leftrightarrow}{2}\vec{D}\cdot\sigma)\chi\chi^\dagger(\frac{i\leftrightarrow}{2}\vec{D}\cdot\sigma)\psi \\ \mathcal{O}_8(^3S_1) &= \psi^\dagger(\sigma^iT^a)\chi\chi^\dagger(\sigma^iT^a)\psi \end{aligned}$$

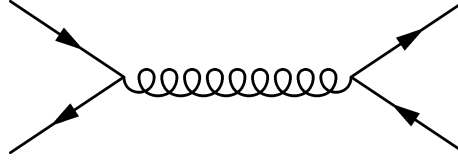


Fig. 20. Feynman diagram for $c\bar{c}$ annihilation into light quarks. It represents the leading contribution to the matching of the color octet operator $\mathcal{O}_8(^3S_1)$.

Exercise: Show that at tree level the diagram of Fig. 18 gives the matching to the P wave operator

$$\text{Im } f_1(^3P_0) = \frac{3\pi C(R)}{2N_c}\alpha_s^2 \quad (16)$$

while the diagram of Fig. 20, where there are n_f species of light quarks in the possible final state on the right, gives the S -wave coefficient

$$\text{Im } f_8(^3S_1) = \frac{\pi n_f}{6}\alpha_s^2 \quad (17)$$

3.4 Power Counting

Decays of S -wave Quarkonium

For this discussion to make any sense we must show that the two operators in Eq. (15) contribute to the same (leading) order in $1/c$. It is instructive though to

begin by revisiting the decay of the S -wave quarkonium. Let us set the normalization arbitrarily by saying that the contribution from the operator $\mathcal{O}_1(^1S_0)$ is of order one:

$$\mathcal{O}_1(^1S_0) : \quad \langle \eta_c | \psi^\dagger \chi \chi^\dagger \psi | \eta_c \rangle \sim 1$$

Consider other possibilities for $\delta\mathcal{L}$:

$$\mathcal{O}_1(^3S_1) : \quad \langle \eta_c | \psi^\dagger \sigma^i \chi \chi^\dagger \sigma^i \psi | \eta_c \rangle$$

This vanishes by spin-symmetry. However, we can include, perturbatively, interactions that break spin-symmetry and then the result need not vanish. Recall

$$\mathcal{L} = \dots + \frac{c_F g}{2mc^{5/2}} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi.$$

Now $\vec{\sigma} \cdot \vec{B}$ flips spins, but also changes relative color from singlet to octet. One can restore the quarks into a singlet color combination by inserting, in addition, an electric dipole (E1) operator

$$\mathcal{L} = \dots + \frac{g}{2mc^{3/2}} \psi^\dagger \vec{A} \cdot \vec{\nabla} \psi.$$

Thus we estimate

$$\langle \eta_c | \mathcal{O}_1(^3S_1) | \eta_c \rangle \sim \left(\frac{g}{c^{5/2}} \frac{g}{c^{3/2}} \right)^2 \sim \left(\frac{g^2}{c} \right)^2 \frac{1}{c^6}.$$

Consider next

$$\mathcal{O}_8(^1S_0) : \quad \langle \eta_c | \psi^\dagger T^a \chi \chi^\dagger T^a \psi | \eta_c \rangle.$$

Inserting the E1 operator twice (once is not enough because it has $L = 1$), we estimate its order to be

$$\left(\frac{g^2}{c^3} \right)^2 \sim \left(\frac{g^2}{c} \right) \frac{1}{c^4}$$

The operator $\mathcal{O}_8(^1P_1)$ requires only one E1 insertion, but comes built in with a $1/c^2$ suppression:

$$\mathcal{O}_8(^1P_1) : \quad \frac{1}{m^2 c^2} \langle \eta_c | \psi^\dagger T^a \overleftrightarrow{D} \chi \chi^\dagger T^a \overleftrightarrow{D} \psi | \eta_c \rangle \sim \frac{1}{c^2} \left(\frac{g}{c^{3/2}} \right)^2 \sim \left(\frac{g^2}{c} \right) \frac{1}{c^4}.$$

For these non-perturbative matrix elements the strong coupling constant should be taken large, $g^2/c \sim \alpha_s \sim 1$. We conclude that the corrections to the η_c decay width from $\mathcal{O}_8(^1S_0)$ and $\mathcal{O}_8(^1P_1)$ are suppressed by $1/c^4$, and from $\mathcal{O}_1(^3S_1)$ by $1/c^6$. The leading correction is in fact of order $1/c^2$ and arises from an operator with the same quantum numbers as the leading operator, but with two derivatives:

$$\mathcal{P}_1(^1S_0) : \quad \frac{1}{m^2 c^2} \langle \eta_c | \psi^\dagger \overleftrightarrow{D}^2 \chi \chi^\dagger \psi | \eta_c \rangle \sim \frac{1}{c^2}$$

[†]The matrix elements are considered to all orders in the zeroth order lagrangian of NRQCD. We have used perturbation theory only with respect to the dimensionful parameter $1/c$.

Decays of P -wave Quarkonium

We now perform an analogous analysis for the case of decays of P -wave states. The operators we focus on are the ones in the effective lagrangian of Eq. (15). $\mathcal{O}_1(^3P_0)$ has just the right quantum numbers to interpolate the color singlet P -wave states,

$$\frac{1}{m^4 c^3} \langle P | \mathcal{O}_1(^3P_0) | P \rangle \sim \frac{1}{c^3}$$

while $\mathcal{O}_8(^3S_1)$ requires an E1 transition,

$$\frac{1}{m^2 c} \langle P | \mathcal{O}_8(^3S_1) | P \rangle \sim \frac{1}{c} \left(\frac{g}{c^{3/2}} \right)^2 \sim \left(\frac{g^2}{c} \right) \frac{1}{c^3}.$$

We see that these two operators contribute to the P -wave decay width at the same order in the $1/c$ expansion. It is easy to see that there are no other operators that contribute at this order.

We can now use the Wigner-Eckart theorem to express the decay widths in terms of reduced matrix elements

$$\begin{aligned} \langle \mathcal{O}_1 \rangle &\equiv \frac{|\langle 0 | \chi^\dagger (\frac{i}{2} \overleftrightarrow{D} \cdot \vec{\sigma}) \psi | \chi_{c0} \rangle|^2}{2M_\chi} \\ \langle \mathcal{O}_8 \rangle &\equiv \frac{\langle \chi_{c0} | \psi^\dagger \sigma^i T^a \chi \chi^\dagger \sigma^i T^a \psi | \chi_{c0} \rangle}{2M_\chi} \end{aligned}$$

Using the results of matching at leading order in Eqs. (16) and (17) the widths for the four P wave particles are given in terms of these two reduced matrix elements:

$$\begin{aligned} \Gamma(h_c) &= \frac{5\pi\alpha_s^2}{6m_c^2} \langle \mathcal{O}_8 \rangle \\ \Gamma(\chi_{c0}) &= \frac{4\pi\alpha_s^2}{m_c^4} \langle \mathcal{O}_1 \rangle + \frac{n_f \pi \alpha_s^2}{3m_c^2} \langle \mathcal{O}_8 \rangle \\ \Gamma(\chi_{c1}) &= \frac{n_f \pi \alpha_s^2}{3m_c^2} \langle \mathcal{O}_8 \rangle \\ \Gamma(\chi_{c2}) &= \frac{16\pi\alpha_s^2}{45m_c^4} \langle \mathcal{O}_1 \rangle + \frac{n_f \pi \alpha_s^2}{3m_c^2} \langle \mathcal{O}_8 \rangle \end{aligned}$$

We are now in a position to understand the cancellation of infrared divergences. In the expression

$$\Gamma(\chi_{c0}) = \frac{\text{Im } f_1(^3P_0)}{m^4 c^3} \langle \mathcal{O}_1 \rangle + \frac{\text{Im } f_8(^3S_1)}{m^2 c} \langle \mathcal{O}_8 \rangle$$

the infrared divergence in $\text{Im } f_1(^3P_0)$ is precisely cancelled by a divergence in the matrix element $\langle \mathcal{O}_8 \rangle$. The resolution of this problem is responsible for much of the renewed interest in NRQCD.

3.5 Decays: Summary

In computing decays of quarkonium into light hadrons we include in the effective lagrangian local four-fermion operators

$$\delta\mathcal{L} = \sum_n f_n \mathcal{O}_n$$

and compute

$$\frac{1}{2}\Gamma = \sum_n \text{Im} f_n \langle \mathcal{O}_n \rangle,$$

where $\text{Im} f_n$ is computed by matching to QCD. The relative importance of these terms is determined by power counting in $1/c$, using the minimal number of required interactions from the multipole expanded NRQCD effective lagrangian, *and using* $g^2/c \sim \alpha \sim 1$.

The operators \mathcal{O}_n are of the form

$$\psi^\dagger \kappa_m \chi \chi^\dagger \kappa_n \psi$$

with κ_m a tensor product of \overleftrightarrow{D} , T^a and σ^i .

4 Concluding Remarks

There are many open questions that remain un-addressed. In the standard presentation of the subject¹⁸ the color octet bilinears in the lagrangian of NRQCD, such as $\mathcal{O}_8(^3S_1)$ of Eq. (15), are interpreted as literally producing and annihilating color octet states. It is not quite clear what these states are. It is reasonable to assume they correspond to the hybrid states of Sec. 1.4. But doing so would invalidate the standard analysis for power counting,¹⁸ which requires the size of this states to still be of the order of a_{Bohr} rather than the bag radius $\Lambda_{\text{QCD}}^{-1}$.

A closely related question is whether the non-perturbative potential, which grows linearly with distance at large distances, should be used in NRQCD. The Born-Oppenheimer approximation certainly suggests this is the right approach. However, it remains possible that the condensates of the higher dimension operators fully incorporate the non-perturbative information relevant to all the questions that NRQCD attempts to answer. In that case, including both the non-perturbative potential and the condensates of higher dimension operators would result in double-counting.

Another question that remains unsolved is the rather arbitrary assumption that g^2/c must be taken as of order unity in order for power counting to work, as presented in Sec. 3.4. It is not enough to insist that $g^2 \sim c$ in general, for this must not be the case in calculations of matching coefficients of the operators in $\delta\mathcal{L}$. The special rule $g^2/c \sim 1$ applies only when calculating matrix elements of operators.

Because of time and space limitations I have not included in these lectures a discussion of the analysis of quarkonium production in NRQCD. Yet most of the ongoing work on the subject is in this realm. There are many good reviews the reader may consult.¹⁹

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References

1. K. J. Juge, J. Kuti and C. J. Morningstar, Nucl. Phys. Proc. Suppl. **63**, 326 (1998).
2. K. J. Juge, J. Kuti and C. J. Morningstar, Nucl. Phys. Proc. Suppl. **63**, 543 (1998).
3. P. Hasenfratz, R.R. Horgan, J. Kuti, and J.M. Richard, *Phys. Lett.. B* **95**, 299 (1980).
4. S. Ono, *Z. Phys. C* **26**, 307 (1984).
5. For details see: A. Messiah, *Quantum Mechanics*, vol. 2, John Wiley, New York, 1958.
6. H. K. Lee and M. Rho, *Phys. Rev. D* **48**, 2329 (1993).
7. H. K. Lee, M. A. Nowak, M. Rho and I. Zahed, *Ann. Phys* **227**, 175 (1993).
8. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd ed., Pergamon Press, New York, 1982.
9. V. A. Novikov, L. B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, *Phys. Rept.* **41**, 1 (1978).
10. M. E. Peskin, *Nucl. Phys. B* **156**, 365 (1979).
11. G. Bhanot and M. E. Peskin, *Nucl. Phys. B* **156**, 391 (1979).
12. W. E. Caswell and G. P. Lepage, *Phys. Lett.. B* **167**, 437 (1986).
13. E. Witten, *Nucl. Phys. B* **104**, 445 (1976); *idem* **122**, 109 (1977).
14. B. Grinstein and I. Z. Rothstein, *Phys. Rev. D* **57**, 78 (1998).
15. P. Labelle, *Phys. Rev. D* **58**, 093013 (1998).
16. G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea K. Hornbostel, *Phys. Rev. D* **46**, 4052 (1992).
17. G. T. Bodwin, E. Braaten, T. C. Yuan and G. P. Lepage, *Phys. Rev. D* **46**, 3703 (1992).
18. G. T. Bodwin, E. Braaten and G. P. Lepage, *Phys. Rev. D* **51**, 1125 (1995).
19. For example, M. Beneke, *Nonrelativistic Effective Theory For Quarkonium Production In Hadron Collisions*, Lectures given at 24th Annual SLAC Summer Institute on Particle Physics: The Strong Interaction, From Hadrons to Protons (SSI 96), Stanford, CA, 1996