Automata and Logic

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Overview

- Background and motivation
- Definitions of automata: Closure properties and class inclusions
- Definitions of automatic structures
- Model theoretic questions: Isomorphism and Scott rank
- Unary automatic structures
- Automatic decision procedures
- Directions for future work

Background and Motivation

Historical sumary

- Kleene, Myhill Nerode theorem, Pumping Lemma characterizing regular languages
- (1960s) J.R Büchi, C.C. Elgot - $S1S$ and Büchi automata
- (1969) M.O Rabin - $S2S$ and Rabin automata on infinite binary trees
  - Decidability of monadic second order theory of countable linear orders
  - Decidability of first order theory of lattice of all closed subsets of the real line
- (1984) M Vardi and P Wolper - ω-automata for program verification
- (1982, 1989) B Hodgson, B Khoussainov, A Nerode - Automatic structures

Definitions

Finite Automata

- $\mathcal{A} = (S, \Sigma, I, \delta, F)$, input in Σ^*.
- Regular language: $L \subset \Sigma^*$ such that $L = L(\mathcal{A})$ for some FA.
- Non-deterministic/ deterministic FA equally expressive. But, $2^{O(|S|)}$ cost for determinization.
- The set of regular languages is closed under union, intersection, complementation (exponential blow up), projection.
- Decidable questions:
  - Emptiness
  - Equality
  - Universality
  - Containment

ω-automata

- $\mathcal{A} = (S, \Sigma, I, \delta, Acc)$, input in Σ^ω.
  - Büchi - $F \subseteq S$ $\quad$ Acc : Inf$(r) \cap F \neq \emptyset$
  - Müller - $\mathcal{F} \subseteq \mathscr{P}(S)$ $\quad$ Acc : Inf$(r) \in \mathcal{F}$
- Non-deterministic Büchi, non-deterministic Müller, deterministic Müller equally expressive. Deterministic Büchi strictly less expressive.
- ω-regular languages: $L \subset \Sigma^\omega$ such that $L = L(\mathcal{A})$ for some non-deterministic Büchi automaton.
- The set of ω-regular languages is closed under union, intersection, complementation (hard - $2^{O(n \log n)}$ blow up).
- Decidable questions:
  - Emptiness (lasso) , etc.

Automatic structures

- As we saw, automata describe subsets of the words over a given alphabet. What about relations? Define convolution of relation.
- Automatic structure: $(A, R, \ldots, F, \ldots)$ where the domain, A, and each relation and function (replace F by $graph(F)$) is automatic – variants FA, BA, TA.
- Examples:
  - $(1^*, S)$, $(1^*, \leq)$, $(\{\lambda\} \cup \{0, \ldots, k-1\}^*\{1, \ldots, k-1\}, \mathsf{Add}_k) \cong (\mathbb{N}, +)$, $(Conf(T), E_T)$ all FA presentable structures.
  - $(\mathbb{Q}, +)$ – open.
  - $(\{0,1\}^*, \cdot)$, $(\mathbb{N}, \cdot)$ not FA presentable.
- Any automatic structure is presentable over binary alphabet
- For each structure $\mathcal{A}$, there is graph $G(\mathcal{A})$ such that $\mathcal{A}$ is automatically presentable iff $G(\mathcal{A})$ is automatically presentable.

Logical Properties

FO Decidability

- Theorem: (1) There is an algorithm which, given FO formula $\varphi(\bar{x})$ produces automaton recognizing R_φ.
- Theorem: (2) The FO theory of any automaton is decidable.

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- Theorem: (2) The FO theory of any automaton is decidable.
- Proof: (1) Use automata given for atomic sentences, then closure under Boolean operations of regular languages yields automata for arbitrary FO formulas.
- Proof: (2) To check if $\mathcal{A} \models \exists x \varphi(x)$, construct automaton for $\varphi(x)$ and check for emptiness.
- Note: the above is the basis for automatic decision procedures...more on this later.

Isomorphism problem

The complexity of the isomorphism problem for automatic...

...structures (in general) is Σ^1_1-complete

(Khoussainov, Nies, Rubin, Stephan 2004)

...ordinals is decidable.

...Boolean algebras is decidable.

...equivalence structures is at most Π^0_1.

Open whether decidable or Π^0_1-complete.

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Idea of proofs:

- To show Σ_1^1-complete, reduce isomorphism problem of c.e. downward closed trees to it.
- To show decidable, find characterization of automatic structures of the class in terms of something finite (e.g. FC-rank for ordinals, finite powers of BA of finite and cofinite subsets of ω).

Scott Rank

- Equivalence relations:
  - $\bar{a} \equiv^0 \bar{b}$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier free formulas
  - $\bar{a} \equiv^{\alpha+1} \bar{b}$ if for all c there is d, and for all d there is c such that $\bar{a}, c \equiv^{\alpha} \bar{b}, d$
  - $\bar{a} \equiv^{\beta} \bar{b}$ if for all $\gamma < \beta$, $\bar{a} \equiv^{\gamma} \bar{b}$.
- Scott rank of $\bar{a}$ in $\mathcal{A}$, $SR(\bar{a})$, is least β such that for all $\bar{b}$, $\bar{a} \equiv^{\beta} \bar{b}$ implies $(\mathcal{A}, \bar{a}) \cong (\mathcal{A}, \bar{b})$.
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Fact (Nadel, 1974): If $\mathcal{A}$ is a computable structure, then $SR(\mathcal{A}) \leq \omega_1^{CK} + 1$.

Fact (Makkai, 1981; Knight, Millar Young 2004): There are computable structures of all computable ranks, of Scott Rank ω_1^{CK}, and of Scott Rank $\omega_1^{CK} + 1$.

Scott Rank, cont'd

Given computable structure $\mathcal{A}$, we construct automatic structure $\mathcal{A}^*$ such that $\mathcal{A} \cong \mathcal{B}$ iff $\mathcal{A}^* \cong \mathcal{B}^*$ and there is a copy of $\mathcal{A}$ definable in $\mathcal{A}^*$ by an $\mathcal{L}_{\omega_1,\omega}$ formula.
(Joint with B Khoussainov)

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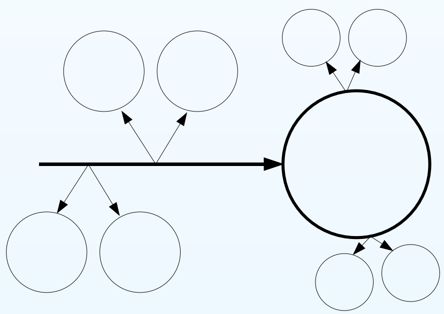
- Assume wlog that $\mathcal{A}$ is graph.
- Let $\mathcal{M}_{\mathcal{A}}$ be the reversible TM computing domain of $\mathcal{A}$, modified so it only halts in an accept state.
- Let $Conf(\mathcal{M}_{\mathcal{A}})$ be the configuration space of $\mathcal{M}_{\mathcal{A}}$. Recall, this is FA presentable. Moreover, by reversibility, consists of chains either finite or isomorphic to $(\mathbb{N}, S)$.
- The set of chains beginning with initial configurations is FA recognizable.
- Smooth out so that it preserves isomorphisms
- For R the edge relation in $\mathcal{A}$, let $\mathcal{M}_{\mathcal{R}}$ be the reversible TM computing R, modified as above.
- Have similar automatic presentation of R, but connect representations of domain and edge relation.

Unary Automata

Unary Automata

Def: A unary automatic structure is one which has a presentation over $\Sigma = \{1\}$.
The general shape of a unary automaton of a binary relation is:



Current work with B Khoussainov and J Liu: For finite degree graphs,

- Given automaton recognizing edge relation, can construct unary automaton recognizing reachability relation.
- Conjecture: isomorphism problem is decidable.
- Conjecture: if Aut$(\mathcal{A})$ is at most countable, Aut$(\mathcal{A}) =$ Aut$_a(\mathcal{A})$.

Decision Procedures

Büchi, Elgot, Rabin decision procedures

For $S1S$:

- Encode relations as tuples of characteristic functions.
- Bijection between formulae in $S1S$ and Büchi automata such that an ω-language is Büchi recognizable iff it is definable in $S1S$.
- $\varphi \in Th(S1S) \iff \varphi$ is valid $\iff$ $\neg\varphi$ is not satisfiable $\iff L(A_{\neg\varphi})$ is empty.

For $S2S$: Same idea except with sets of infinite binary trees instead of ω-languages.

Presburger arithmetic

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- Boudet, Comon (1996), Boigelot, Wolper (1995, 2000) gave automatic decision procedure
  - Given formula φ, construct FA A_φ which accepts an encoding of the set of tuples satisfying φ.
  - Atomic formula $\bar{a} \cdot \bar{x} = c$: states of FA represent integer value of $\bar{a} \cdot \bar{x}$ so far. Accept if final state is c.
  - Atomic formula $\bar{a} \cdot \bar{x} \leq c$: similar, but need to add more transitions to any state representing number greater than current value.
  - Use closure under Boolean operations of regular languages to obtain automata for non-atomic formulae.
  - φ is satisfiable iff A_φ is non-empty.

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  - Use closure under Boolean operations of regular languages to obtain automata for non-atomic formulae.
  - φ is satisfiable iff A_φ is non-empty.
- Klaedtke (2003) showed that automatic decision method has tight triple exponential bound to size of automaton.

p-adics under $+$

p-adic numbers are completion of $\mathbb{Q}$ wrt $N = |\ |_p$

$$|x|_p = \begin{cases} p^{-ord_p(x)} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

- Every p-adic has unique digit expansion $\alpha = \alpha_{-r}p^{-r} + \alpha_{1-r}p^{1-r} + \cdots + \alpha_{-1}p^{-1} + \alpha_0 + \alpha_1 p + \cdots$.
- For automatic decision method, need to recognize $x_1 + \cdots + x_n = 0$.
- Use Müller automata where states keep track of carry, except for a distinguished fail state which is excluded from any successful run.

Khoussainov-Vardi conjecture

- The decision methods we've discussed have used FA, BA, MA, Rabin Tree Automata.
- Conjecture: If $\mathcal{A}$ has a decidable FO theory then there is an automata theoretic approach to proving decidability.
  - Need to formalize ”automata theoretic approach”.
  - Do more general notions of automata need to be employed?

Future Directions

Next steps

- Transducer representable structures: functional languages.

  More in line with computational model as opposed to verification.

- Büchi automatic structures.
- Automatic model theory.
- Current open questions
  - Is $(\mathbb{Q}, +)$ FA-presentable?
  - Is $(\mathbb{N}, \cdot)$ Büchi-presentable? (known not to be FA-presentable)
  - Is the isomorphism question for unary automatic graphs decidable?