so the matrix equation is now

$$\begin{pmatrix} \alpha_{12} - \alpha_{51} \\ \alpha_{23} - \alpha_{12} \\ \alpha_{34} - \alpha_{23} \\ \alpha_{45} - \alpha_{34} \\ \alpha_{51} - \alpha_{45} \end{pmatrix} = \left[\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix} + \frac{1}{N^2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \right] \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} . \tag{19}$$

It has a unique solution in which the average correction (Eqn 17) minimized to exactly zero. Actually, adding any non-zero constant to every element would yield the same solution as the physically-motivated $\frac{1}{N^2}$.

The circular ring of chambers also provides an internal cross-check: the sum of the means of pairwise residuals must be zero. If not, no combination of alignment corrections can center all of the residuals, because

$$\text{closure} = \sum_{i=1}^{N} \alpha_{i,\,i+1} - (A_i - A_{i+1}) = \sum_{i=1}^{N} \alpha_{i,\,i+1}$$

is independent of $\{A_i\}$. (Note that $\sum_{i=1}^{N} A_{i+1}$ is just a reindexing of $\sum$ arithmetic is understood to be mod N.) With non-zero closure, th uniformly distributes residuals so that they all have non-zero means, chamber disagrees with its neighbor about where the tracks are. Unclos imply

- the average distance of the chambers from the beamline is incorr
- the presumed width of the chambers is incorrect.

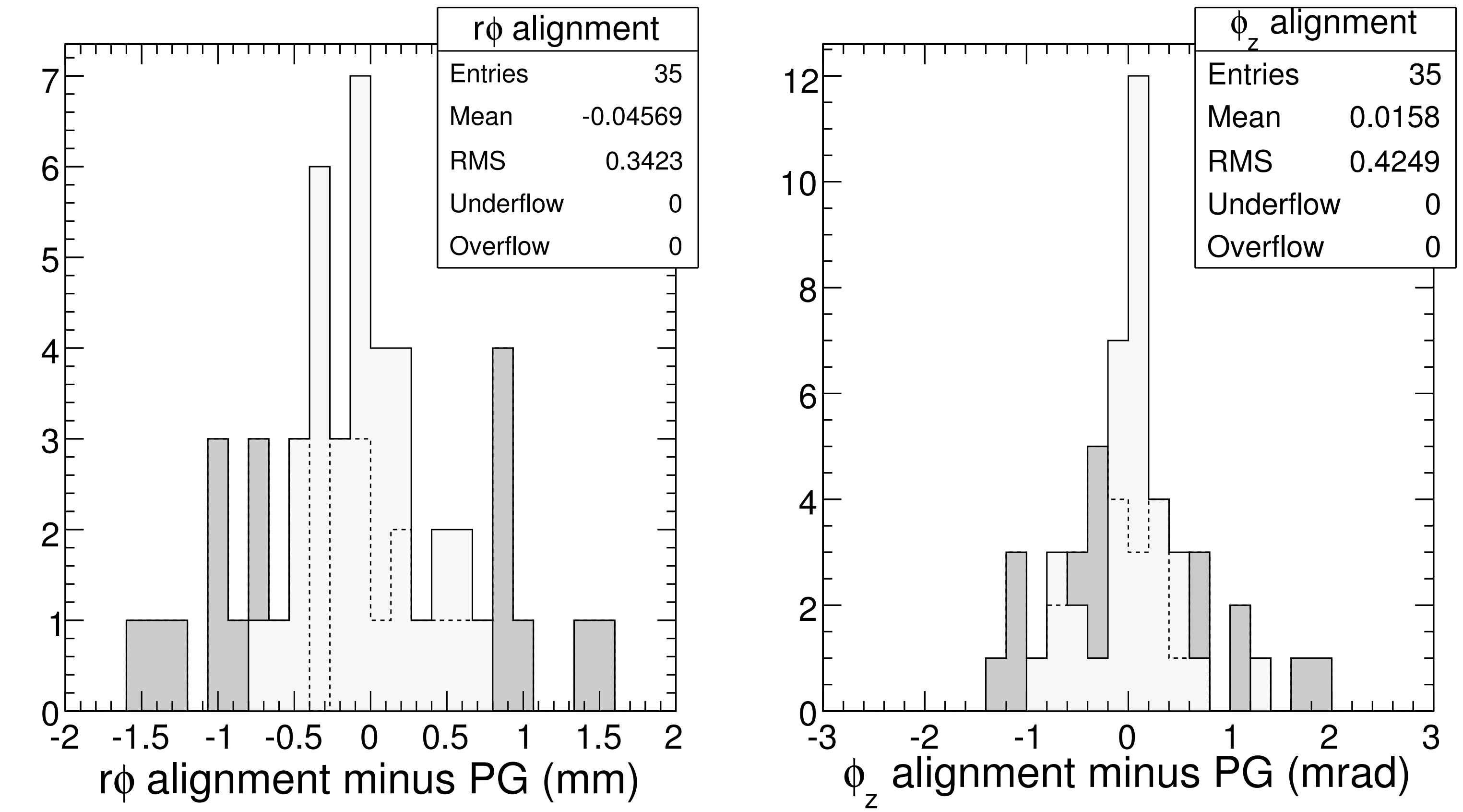

Figure 25: Chamber-by-chamber verification of the beam-halo alignment with photogrammetry. The dark histogram is before alignment; the light histogram and statistics box are after alignment.