Space Curvature in Real Life

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- No, it’s not.

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  What is "The Identity?"

- Bianchi Identity

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0$$

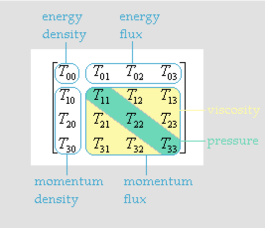
  Derived from symmetry of indices and the fact that $g^{\mu\nu}$ commutes with covariant derivatives

  For Ricci: $\nabla_\mu \left(R^\mu_\lambda - \frac{1}{2}\delta^\mu_\lambda R\right) = 0$

The importance of $\nabla_\nu G^{\mu\nu} = 0$

Generic description of a fluid in space

$$T^{\mu\nu}(t,\vec{x}) = \begin{pmatrix} \rho & p_x & p_y & p_z \\ p_x & P_x & \tau_{xy} & \tau_{xz} \\ p_y & \tau_{xy} & P_y & \tau_{yz} \\ p_z & \tau_{xz} & \tau_{yz} & P_z \end{pmatrix} (t,\vec{x})$$



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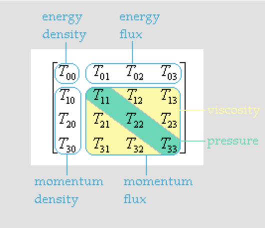

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- $\nabla_i T^{\mu i} = 0$ is $\rho\left(\dfrac{\partial \vec{p}}{\partial t} + (\vec{p}\cdot\nabla)\,\vec{p}\right) = -\nabla P$, Navier-Stokes

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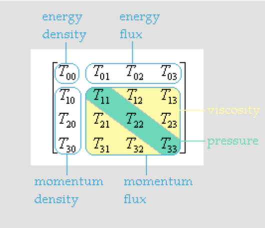

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The Einstein equation is

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \boxed{G^{\mu\nu} = k T^{\mu\nu}}$$

The Einstein Equation

- Motivated by aesthetics $$\underbrace{G^{\mu\nu}}_{\text{geometry}} = \underbrace{kT^{\mu\nu}}_{\text{matter}}$$

- While we're equating things like this, note that $\nabla_\nu g^{\mu\nu} = 0$

  $$G^{\mu\nu} = kT^{\mu\nu} + \Lambda g^{\mu\nu}$$

  Λ, the cosmological constant, is approximately 10^{-29} g/cm^3

- Can also be derived from an action principle, yielding k and Λ as constants of integration

- Calculations go from matter distribution $\leftrightarrow$ $G^{\mu\nu}$ $\leftrightarrow$ metric $\leftrightarrow$ geodesics $\leftrightarrow$ matter distribution

History (not quite fully-formed from Einstein's head)

- William Clifford wrote about a possible connection between matter and curved space in 1873, but he did not include time in the manifold

- In 1905, Hermann Minkowski and Henri Poincaré reformulated Einstein's special relativity into ordinary mechanics in a spacetime with a pseudometric
$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Albert Einstein and Marcel Grossmann worked closely on the development of the Einstein equation, the most difficult part being the choice of curvature tensor $G_{\mu\nu}$

- In 1913, Grossmann distances himself from the physical interpretation; in 1915, Einstein presents the theory in its final form (except for Λ)

Newtonian limit (1): apparent forces from curvature

- Take $g_{\mu\nu} = \begin{pmatrix} -1 - 2\phi(\vec{x})/c^2 & 0 \\ 0 & 1 - 2\phi(\vec{x})/c^2 \end{pmatrix}$ in ct and $\vec{x}$

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$$p^\nu \partial_\nu p^\mu + \Gamma^\mu_{\nu\lambda} p^\nu p^\lambda = 0$$

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$$mc\frac{\partial \vec{p}}{\partial ct} + \vec{p}\frac{\partial \vec{p}}{\partial ct} + m^2c^2\nabla\phi/c^2 - |\vec{p}|^2\nabla\phi/c^2 = 0$$

which is $\dfrac{\partial \vec{p}}{\partial ct} + mc\nabla\phi/c^2 = 0$ or $\boxed{-m\nabla\phi = \dfrac{\partial \vec{p}}{\partial t} = \vec{F}_{\text{gravity}}}$

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R_{tt} &= \nabla^2\phi/c^2 - 2(\nabla\phi)^2/c^4 &&\approx \nabla^2\phi/c^2 \\
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$$R = R_{tt} + R_{xx} + R_{yy} + R_{zz} = -2\nabla^2\phi/c^2$$

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The Schwarzschild solution: the metric we live in

$$g_{\mu\nu} = \begin{pmatrix} -c^2\left(1 - \frac{2G_N M}{c^2 r}\right) & 0 & \\ 0 & \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} & \\ & & \ddots \end{pmatrix}$$

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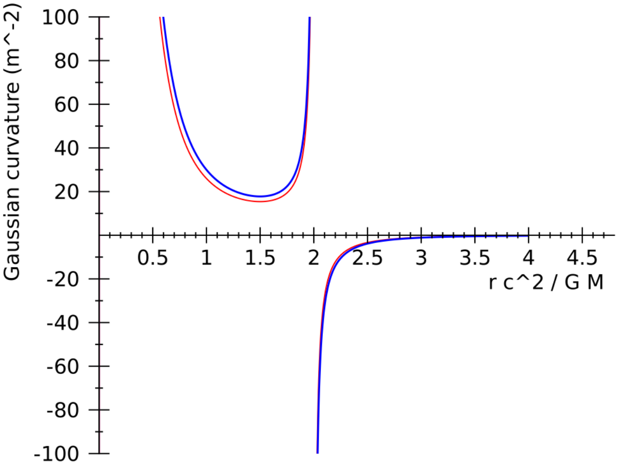
- $\dfrac{\partial}{\partial r}\left(1 - \dfrac{2G_N M}{c^2 r}\right) = \dfrac{2G_N M}{c^2 r^2} = 2.17 \times 10^{-16}/\text{m}$

  Satellites lose a microsecond every 10 minutes *(Phys. Today)*

That wierd singularity



$$R_{\text{space}} = \frac{-26 {G_N}^2 M^2}{c^4 r^4 - 2c^2 G_N M r^3} \quad \text{and} \quad R_{\text{complete}} = \frac{-30 {G_N}^2 M^2}{c^4 r^4 - 2c^2 G_N M r^3}$$

Falling into black holes

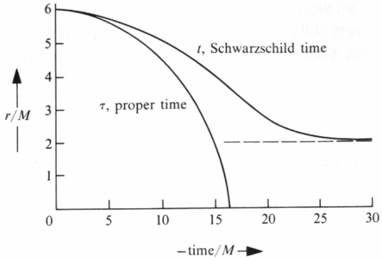
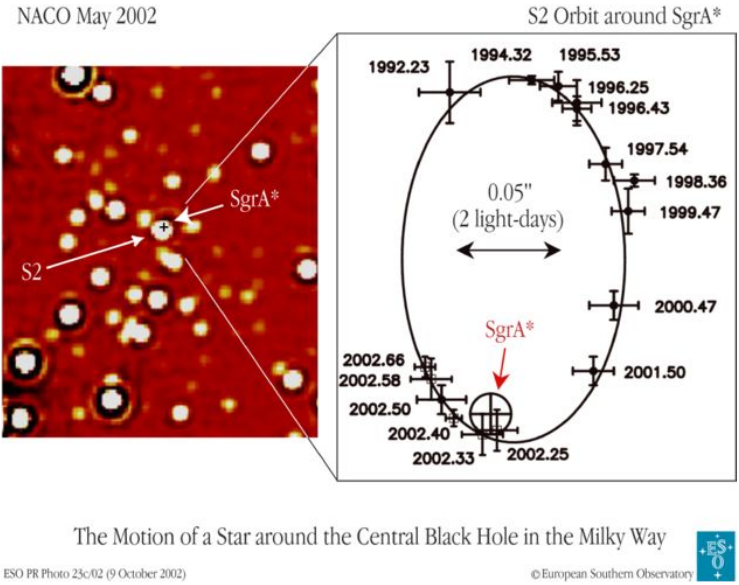


Figure 25.5.
Fall toward a Schwarzschild black hole as described (a) by a comoving observer (proper time τ) and (b) by a faraway observer (Schwarzschild-coordinate time t). In the one description, the point $r = 0$ is attained, and quickly [see equation (25.28)]. In the other description, $r = 0$ is never reached and even $r = 2M$ is attained only asymptotically [equations (25.35) and (25.37)]. The qualitative features of the motion in both cases are most easily deduced by inspection of the "effective potential-per-unit-mass" $\widetilde{V}$ in its dependence on r (Figure 25.2) when one is interested in proper time; or the same effective potential $\widetilde{V}$ in its dependence on the "tortoise coordinate" r^* [Figure 25.4 and equation (25.31)] when one is interested in Schwarzschild-coordinate time t.

- Also related to perihelion of Mercury…

A real-life black hole



Gravity waves (general idea)

- Small perterbation from flat pseudometric η_{ab}

$$g_{ab}(t,\vec{x}) = \eta_{ab} + \varepsilon h_{ab}(t,\vec{x})$$

- Keep only first-order terms in expansion (linearized gravity)

$$G_{ab} = \frac{\varepsilon}{2}(h^c_{a,bc} + h^c_{b,ac} - \eta^{cd} h_{ab,cd} - h_{,ab} - \eta_{ab} h^{cd}_{,cd} + \eta_{ab}\eta^{cd} h_{,cd}) + \mathcal{O}(\varepsilon^2)$$

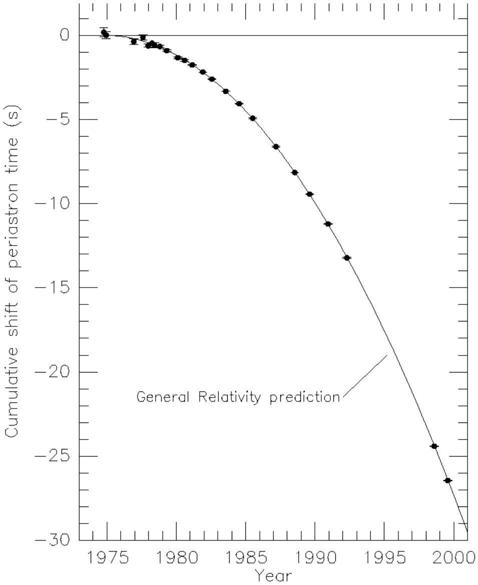
- Solve Einstein equation in vacuum: $G_{ab} = 0$ and choose coordinates (choose a gauge) such that

$$h^a_{b,a} - \frac{1}{2}\eta^a_b h_{,a} = 0$$

- It simplifies to $\eta^{cd} h_{ab,cd} = \boxed{\frac{\partial^2 h_{ab}}{\partial t^2} - \nabla^2 h_{ab} = 0}$

Gravity waves (indirect evidence)

- Binary pulsar PSR1913+16 loses rotational energy due to gravitational radiation
- Not a fit!
- Perihelion advances 4.2°/yr, Mercury 43″/yr
- No news from LIGO...



Equivalence principles

- There's no preferred origin in space (translation symmetry)
- In special relativity, there's no preferred origin in velocity ("no preferred reference frame," Lorentz symmetry)
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Two measures of angular momentum for a spinning bowl of water:

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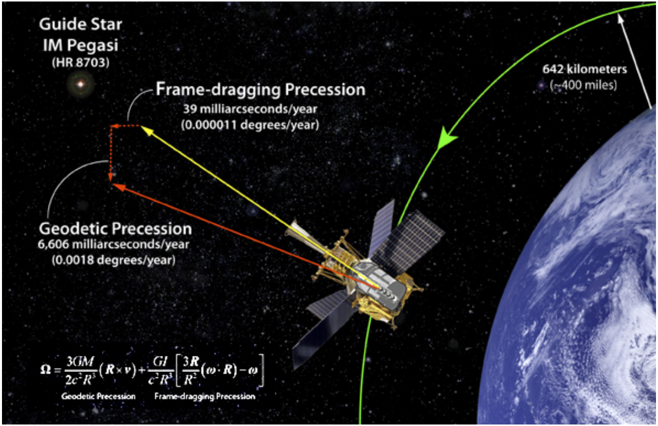
- In general relativity, these descriptions of angular momentum don't always agree. Angular momentum is relative!

Frame dragging

- Massive rotating body (Earth) causes locally inertial frame to rotate with respect to distant stars



- Gravity Probe B will release final results in December 2007

Pressure is a source for gravity

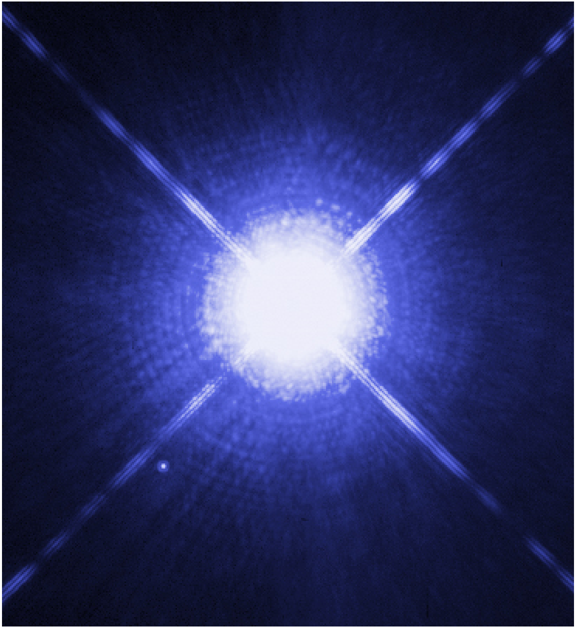
- Normal star: $G_{\mu\nu} = 8\pi G_N \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- White dwarf: $G_{\mu\nu} = 8\pi G_N \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$

- As pressure increases, gravitational force increases, increasing pressure. . .

- Tipping point is the Chandrasekhar limit, a balance of gravity and electron degeneracy pressure: all white dwarfs have masses below 1.44 solar masses

White dwarf orbiting Sirius, the brightest star



Type I-A Supernovae

- A white dwarf that accretes matter from a nearby source can slowly approach the limit
- When reached, it violently implodes, suddenly fusing most of its mass and blowing it into space at 3% of c
- Chandrasekhar limit is a universal constant, so the supernova's intrinsic brightness is precisely known. . . keep this in mind. . .

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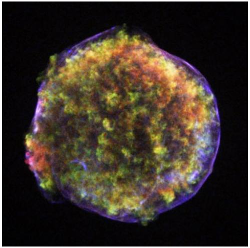
Remnant from a type I-A supernova observed by Tycho Brahe in 1572 (now a neutron star)

(Cloud of plasma surrounded by high-energy electron shell)



Curvature on a grand scale: cosmology

- "The Universe is expanding!" What does that mean?
- It means the large-scale metric is non-trivial:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}$$

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- We need to enlarge our sense of scale (map of universe)
- Derivatives such as $\Gamma^{\lambda}_{\mu\nu}$ and $R_{\mu\nu\alpha\beta}$ derive from a $\lim\limits_{\varepsilon\to 0}$
- ε must be small, like a billion light years

  (any smaller and you get into the bumpy stuff. . .)

Matter distribution tensor for the whole universe

- Impose homogeneity and isotropy:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

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- Several cases:
  - Normal, non-relativistic matter: $\rho \gg P$, so let $P = 0$
  - Nothing but photons: $\rho = P/3$

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Matter distribution tensor for the whole universe

- Impose homogeneity and isotropy:

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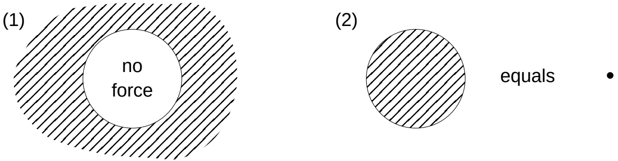
- Most general metric: $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a(t)^2 \end{pmatrix}$

  with constant spacial curvature

The universe as a simple calculus problem

- Two theorems carry over from Newtonian gravity:



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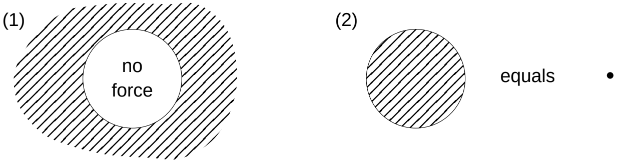

- Force law: $ma = F = -G_N mM/x^2$ or $\boxed{d^2x/dt^2 = -M_{\text{enc}}/x^2}$
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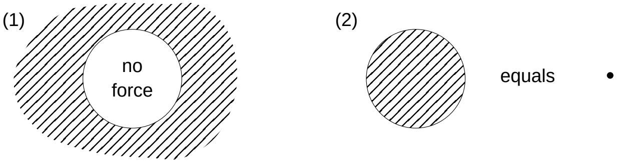

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- Imagine a ball with radius $a(t)r_0$. $M_{\text{enc}} = \frac{4}{3}\pi a^3 {r_0}^3 \rho(1 + 3w)$

$$\frac{d^2 a}{dt^2} \propto a\rho$$

Relating a and ρ through thermodynamics

- Definitions of energy density and pressure: $d(\rho V) = -P\, dV$

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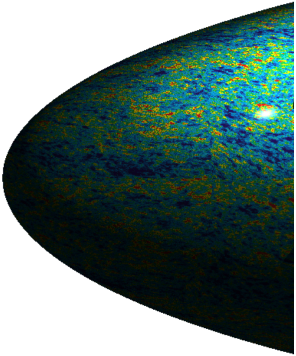
- Solve a one-variable differential equation For The Universe!

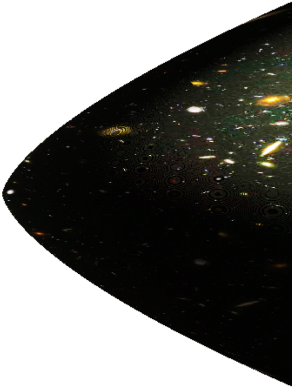
$$a \propto t^{\frac{2}{3(1+w)}}$$

The universe embedded in $\mathbb{R}^3$

- Space-time curvature we can visualize!
- Represent time with one axis and space with a periodic loop



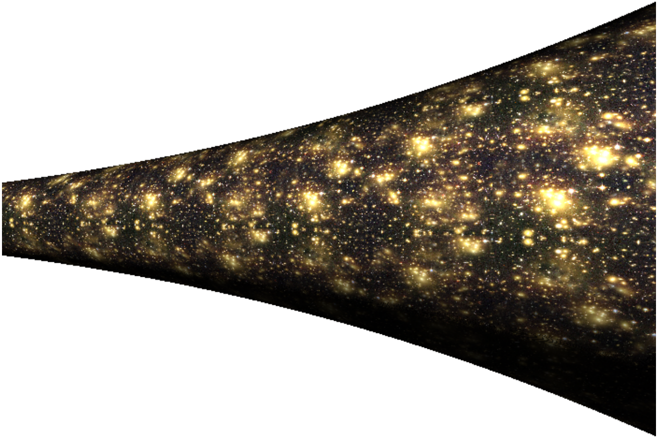
Radiation dominated early universe $a \propto t^{1/2}$



Matter dominated later universe $a \propto t^{2/3}$

Cosmological constant-dominated universe

- $w = -1$ solved by $a \propto e^t$



- This solution actually accelerates

Measuring d^2a/dt^2

- Cosmic expansion makes galaxies drift apart
- $D(t) = a(t)D(\text{now})$ and $v(t) = (da/dt)D(\text{now})$
- Measure acceleration through the departure of $v(t)$ vs. $D(\text{now})$ from proportionality
- Need to know $v(t)$ (easy) and $D(\text{now})$ (hard)

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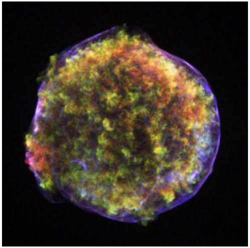
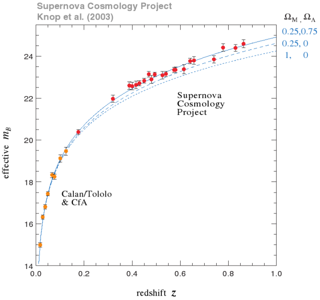
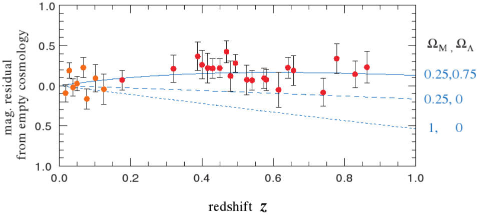
- We measure $v(t)$ from redshifted atomic spectra, knowing what the spectra are supposed to look like, what they would look like up close
- We measure $D(\text{now})$ through a galaxy's brightness. This is a problem because galaxies come in all different sizes

Type I-A supernovae save the day!

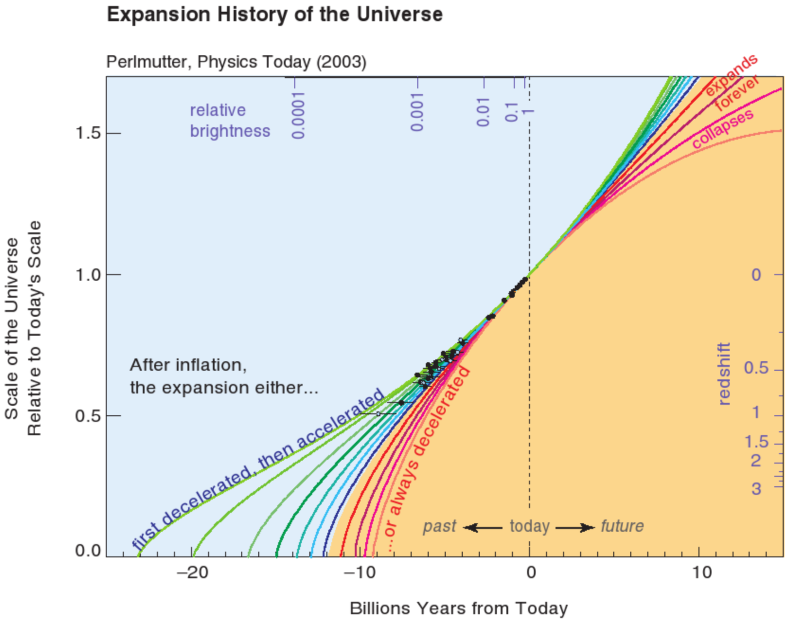


- By creeping up to the Chandrasekhar limit slowly, they guarantee that they explode with a specific energy
- Apparent brightness tells us the distance



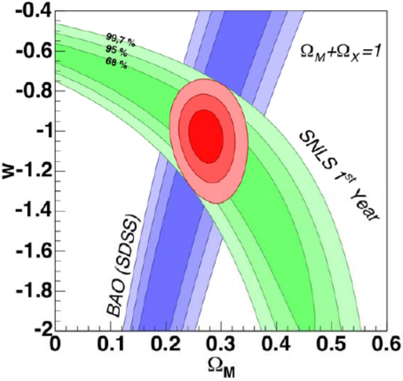


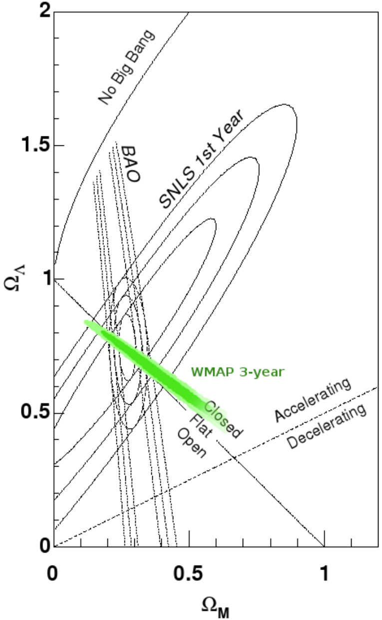
The universe is, in fact, accelerating



Constraints on cosmology!

- Spacial curvature is consistent with zero
- Accelerating source is consistent with $w = -1$





I must sleep

A mari, i a mari, i a mar,
Crep tin tas pettipace, a diea, a diea,
An aoul a esterdis as bu litten fola a veh a dusti det.
Aoght! Aoght, brefe gandele!
Fur lif ei buut a schmaga yaga,
a pur-a plaiya qi strutti-fretti heur upona stache,
an es hare na mare.
Esa talea tola bi an iti-ito,
Folla sond i fern,
Signifya noot.