Statistical Mechanics of Economic Systems

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ABSTRACT

Short summary of the contents of your thesis.

To someone special

ACKNOWLEDGEMENTS

Put your acknowledgements here.

DECLARATION

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ECONOMICS AND THE GENERAL EQUILIBRIUM THEORY

In this chapter we will discuss some of the economic theories behind the work of the thesis. Mainly, we will introduce General Equilibrium Theory and some of its main results and statements. We will then discuss how it relates to Statistical Physics and some of its criticism in the Economics literature.

General Equilibrium Theory is a mature and consolidated field of Economics [1, 8, 9] which aims to characterize the existence and properties of equilibria in certain market settings. Economic systems are assumed to frequently have actors with opposing goals: the owner of a good wants to sell it for the highest possible price, while its potential buyers would like to purchase it for as low as possible. Fishermen would like to catch as many fishes as possible as long as their peers care to not also overdo it otherwise they may extinguish the oceans. In this way, one expects economies and markets to converge to a certain steady state and among other things, General Equilibrium Theory characterizes these steady states in a rigorous manner. In this sense, it's also a theory in Microeconomic, because it explains macrobehavior from the incentives of microscopic agents.

While the work in this thesis does not aim to strictly expand economies as they are describe in General Equilibrium Theory, it's purposes and questions are very similar to the ones physicists usually go for when studying an economic system: namely, the characterization of it's equilibrium state. It's important, therefore, to have a deeper understanding of how it's done in economics, what are the main concerns and assumptions.

3.1 A BRIEF EXPOSITION

The exposition in this chapter is mostly adapted and simplified from [8] and will be considerably more formal than the rest of this thesis. This is due to the way the discipline is commonly studied.

In General Equilibrium Theory, an economy is defined through the following components: we assume J consumers, N firms and M goods. Each consumer has a consumption set X_j which contains all possible consumption bundles $x_j = (x_j^1, \dots x_j^M)$ that the consumer has access to, ie, each bundle x_j is a $M-$dimensional vector with nonnegative entries (we are assuming he cannot consume a negative amount of a good). X_j is limited by "physical" contraints, such as no access to water or bread, but not monetary constraints, which will arise later.

The consumer also has an utility function $U_j(x)$ that takes every element $x_j \in X_j$ to a real number, representing how much the consumer values each bundle of his consumption set. This allows us to define define a preference relationship over the elements in X_j (ie, if the consumer prefers bundle x to x'), which is **complete**[1] and **transitive**[2], two standard requirements in Economics for rational behavior.

Finally, the consumer is also endowed with an initial bundle of goods $\omega_j = (\omega_j^1, \ldots, \omega_j^M)$, $\omega_j^\mu \geq 0$ which will define his budget given a set of prices for the goods and will constraint his choices on X_j.

Each firm i has a production set Ξ_i of technologies $\xi_i = (\xi_i^1, \ldots, \xi_i^M)$ which it is able to operate. Unlike consumption bundles, which are final allocations and therefore must be nonnegative, technologies can be any real number: the negative entries are inputs and the positive entries are outputs that the firm can operate. Ξ_i is also limited only by "physical" constraints, not by monetary constraints. A firm that has $\xi_i = (-1, 2)$ in its production set is able to transform one unit of good 1 into two units of good 2. It won't necessarily be able to transform two units of good 1 into four units of good 2, for that it must also have $\xi_i' = (-2, 4)$ in Ξ_i. It might be the case, for example, that companies get more efficient with production and therefore it might have $\xi_i'' = (-2, 6)$ in its production set.

In General Equilibrium Theory, an economy is formally defined as the tuple

$$E = \left(\{(X_j, U_j)\}_{j=1}^J, \{\Xi_i\}_{i=1}^N, \{\omega_j\}_{j=1}^J \right). \tag{1}$$

One of the theory's assumptions is that the economy described is **complete**, that is, every agent can exchange every good with no transaction costs and complete information about the firm's technologies, other consumer's consumption, etc. Also, a good μ contains all the possible information that a consumer would take into account when making his choice. That is, among the space of goods we could have "umbrella" and "chocolate", or we could also have "an umbrella on August 13th, 2016 in Sao Paulo with 50% chance of rain" and "an umbrella on December 12th, 2016 in Chicago with 90% chance of rain".

It's assumed that agents are **price-takers**, that is, they are unable to affect the market prices and therefore take them as a given. The prices of the goods are given by a $M-$dimensional vector $p = (p_1, \ldots, p_M)$, where each price is a strictly positive quantity, ie, $p_\mu > 0$ for all μ. This assumes that goods have global prices, which is consistent with the completeness assumption: there is no reason why the market prices should be different for certain consumers or firms if they have complete knowledge and no transaction costs.

With a price vector p defined, we say the consumer j has a budget $B_j = p \cdot \omega_j$, which is the monetary value of his initial endowment. Any bundle he chooses to purchase will cost him $p \cdot x_j$. His objective, therefore, is to find the best bundle x_j he is able to afford, that is:

$$\max_{x_j \in X_j} U(x_j), \quad \text{s.t. } p \cdot x_j \leq p \cdot \omega_j \tag{2}$$

1 For every $x, x' \in X_j$, either $U_j(x) \geq U_j(x')$ or $U_j(x) \leq U_j(x')$.
2 For every $x, y, z \in X_j$, if $U_j(x) \geq U_j(y)$ and $U_j(y) \geq U_j(z)$, then $U_j(x) \geq U_j(z)$.

The firms, on the other hand, have an operating profit for each technology given by $p \cdot \xi_i$, which is how much money they earn by selling their outputs ($\xi_i^\mu > 0$) minus how much they spend purchasing the inputs ($\xi_i^\mu < 0$). Their objective is to maximize their profits, that is:

$$\max_{\xi_i \in \Xi_i} p \cdot \xi_i \tag{3}$$

With these ingredients laid out, we define an **allocation** of the economy as a set of specific choices for consumption bundles and technologies, ie, an allocation a of an economy E is

$$a = (x_1, \ldots, x_J, \xi_1, \ldots, \xi_N),\ x_j \in X_j,\ \xi_i \in \Xi_i \tag{4}$$

The economy is closed, and therefore all that is produced must come from the initial endowments and be consumed by the consumers. We therefore say an allocation is **feasible** if it satisfies **market clearing** for all the goods:

$$\sum_{j=1}^{J} x_j^\mu = \sum_{j=1}^{J} \omega_j^\mu + \sum_{i=1}^{N} \xi_i^\mu, \quad \forall\, \mu \in \{1, \ldots, M\} \tag{5}$$

This is a strong condition which constraints many quantities in the economy. In particular, if we multiply both sides of the equation by p^μ and sum then in μ we get, in vector notation,

$$\sum_{j=1}^{J} p \cdot (x_j - \omega_j) = \sum_{i=1}^{N} p \cdot \xi_i \tag{6}$$

The left handside is the leftover money the consumers have after making their choice of consumption, also called the value of excess demand, whereas the right handside is the firms aggregate profit, also known as the value of excess suply. Because we assume that the consumer may not spend more than his budget, the value of each consumer's individual excess demand has to be non positive. Simultaneously, if we assume that the firms always have $\xi_i = 0$ in their production set, ie, we assume that they can always opt to not produce at all and leave the market, then the value of excess supply for each firm has to be non negative. Because they must be equal, we conclude that in an economy for which market clearing holds, the consumer spends all his available budget and the firms all have zero profit, a result known as **Walras' Law**.

Given a set of possible feasible allocations $\{a_k\}$, we may wonder if there is any allocation we desire most over the other. This of course depends on the criteria we use to judge them: we may like allocations with less inequality, with the most aggregate utility, with the smallest minimum utility, etc. Economist opt to use one particular condition which is called **Pareto optmality**.

Intuitively, a **Pareto optimal** (or **Pareto efficient** allocation is one that you can't make a consumer better without making another consumer worse off. The idea is that, a non Pareto optimal allocation has some waste in it: one could change the consumption bundles in order to increase some utilities and no other consumer would complain. Because firms have zero profit in feasible allocations, they wouldn't mind the change.

More formally, a feasible allocation $a = (x, \xi)$ is said to be **Pareto optimal** if there is no other allocation that **Pareto dominates** it, that is, no allocation $a' = (x', \xi')$ such that $U(x'_j) \geq U(x_j)$ for all j and $U(x'_j) > U(x_j)$ for at least one j.

The Pareto optimality concept therefore defines a socially desireble outcome in a "non-controversial" way, by definition no agent in the economy would have a problem with policies or actions taken to make it more Pareto efficient. However, it says nothing about equality: an allocation in which one consumer has all the goods and no other consumer has any goods is Pareto optimal.

We finally arrive to the concept of equilibrium in an economy. A **Walrasian equilibrium** (or competitive equilibrium or simply equilibrium) in an economy E is an allocation (x^*, ξ^*) and a price vector p such that

1. Every firm i maximizes it's profits in its production set Ξ_i, that is
$$p \cdot \xi_i^* \geq p \cdot \xi_i, \quad \forall \xi_i \in \Xi_i, \quad \forall i in \{1, \ldots, N\} \tag{7}$$

2. Every consumer j maximizes his utility in his consumption set X_j, that is
$$U(x_j^*) \geq U(x_j), \quad \forall x_j \in X_j, \quad \forall j \in \{1, \ldots, J\} \tag{8}$$

3. The allocation (x^*, ξ^*) is feasible, that is,
$$\sum_{i=j}^{J} x_j^* = \sum_{j=1}^{J} \omega_j + \sum_{i=1}^{N} \xi_i^* \tag{9}$$

The Walrasian equilibrium is essentially a pair allocation - prices in each all the optimization problems are solved at once. Although we have made no mention of dynamics in this economy, it's considered an equilibrium because all agents are as satisfied as possible with their allocation given the prices, which we have assumed to be global and unchangeable by any agent's action. This is not exactly a definition of equilibrium as used in Physics, but we will discuss this point later. For now, we point out that a Walrasian equilibrium is in some sense stable.

One

We have thus defined two desirable properties of an allocation: efficiency and equilibrium. The fundamental results of General Equilibrium Theory are the **welfare theorems**, which define the conditions in which an equilibrium is Pareto optimal and vice versa.

The **First Fundamental Welfare Theorem** asserts that if the consumers have a utility function continuous on X_j[3], then all Walrasian equilibria are Pareto optimal. This result is simple yet useful, because it tells us that if our economy is in equilibrium, we don't have to care about checking if it's efficient. The violation is also important: if a given economy we are studying is in an inefficient equilibrium, then it must be that one of the theorem's condition was violated. This sheds light in where to look for market failures. We remind the reader, however, that some extra strong assumptions were made for the economies described by this therorem, namely, completeness of market and global prices that no single agent is capable of influencing.

The **Second Fundamental Welfare Theorem** requires extra assumptions: it afirms that if an economy satisfies the condition of the first fundamental theorem, the utility functions U_j plus all sets X_j and Y_i are convex and if we are able to redistribute the initial endowments at will while keeping the total amount $\sum_{j=1}^{J} \omega_j$ constant, then for every Pareto efficient allocation there exists a wealth allocation ω and price vector p^* such that (x^*, ξ^*, p^*) is a Walrasian equilibrium.

The second theorem is considerably more interesting than the first one: any Pareto optimal allocation we would like in an economy can be an equilibrium given the appropriate price vector and a possible wealth transfer, albeit under a stronger set of conditions.

3.2 THE DYNAMICS OF GENERAL EQUILIBRIUM

A conspicuous element was missing from the exposition above: there are no rules for the dynamics of the economies described above. The prices are taken as a given, as are the consumer and firm choices. What happens if a firm closes? What happens if a new firm appears? The equilibrium simply "recalculates" and the economy moves to the new one?

Indeed, this is a long standing criticism of General Equilibrium Theory. Walras proposed it as a process of **tatônnement**[4]: a central figure, known as the Walrasian auctioneer, sugests a price and asks all the firms and consumers how much would they like to produce and buy at these given prices, but without any transaction taking place at the out of equilibrium prices. The auctioneer updates the prices in the direction of diminishing excess demand or supply, a "gradient descent" of sorts, until equilibrium is reached.

However, this process is quite indetermined. Chiefly, this auctioneer figure doesn't exist in most decentralized markets: goods are traded at agreed prices by both parts, which do not wait until their transaction

3 The actual theorem asserts a weaker condition, that the preferences be locally non-satiated, that is, for every $x \in X$, there is an $x' \in X$ such that $\|x - x'\| < \varepsilon$ and x' is preferred to x.

4 From "trial and error" in french

is authorized by some central authority. Even if they did, such an auctioneer would require an infinitely large computational capability to compute the excess demand and supply of every consumer and firm and for every good in a modern economy [2, 10]. Worst of all, even if there was such central figure with such an arbitrary large amount of computing power, the price updating dynamic is not guaranteed to converge [5]. Finally, even if it converges, we have no assurance that it will converge in finite time.

[13] [12] [11] [8]

[14, 6, 15]

4

THE RANDOM LINEAR ECONOMY MODEL

In this chapter we will present in detail and discuss the Random Linear Economy model [4] developed by Andrea De Martino, Matteo Marsili and Isaac Pérez Castillo which will be the basis for some of the applications discussed in the second part of this thesis.

There are some reasons why we chose to work with this model in particular: first, it presents a General Equilibrium Model which has few ingredients but already displays a rich behavior, including phase transitions which depend on the number of firms in the market. Secondly, it is analytically solvable using statistical mechanics techniques, such as using the replica trick to calculate the partition function. Therefore, it was ideal for trying new venues of exploration without the difficulty imposed in trying to prove general phenomena.

4.1 THE MODEL INGREDIENTS

An economy in the model is, like the General Equilibrium setting, composed by two distinct actors: consumers and firms. We assume N firms and one single representative consumer with utility function $U(x)$ and initial endowment x_0. This is a common approximation when doing equilibria calculation in Economics due to the simplicity: if we have J consumers with independent utility functions U_j (ie, U_j never depends on x_k, $k \neq j$) and initial endowments ω_j, then either we do not allow wealth transfers of ω_j and the optimization problem becomes very complicated, or we allow the central authority to carry out wealth transfers prior to allocation, and then the the demands generated by the consumers in this scenario is equivalent to that of a single representative consumer with utility function $U_R = \sum_{j=1}^{J} U_j$ and wealth $\omega_R = \sum_{j=1}^{J} \omega_j$.

The representative consumer assumption receives considerable criticism [7], chiefly because disregarding interaction among agents (via the utility of one depending on the decisions of the others) washes out the possibility of interactions and the wide range of important and interesting phenomena that in the statistical physics community we know to be generated precisely by these interactions [3], whereas the representative agent is a mean field approximation for consumers.

That said, the representative consumer is used in this model precisely because it generates an energy function which is convex and has a well defined, unique minimum and the resulting partition function can be calculated analytically in the zero temperature limit, while at the same time generating interesting behavior.

The consumer and the N firms will trade M goods, with a technological density parameter given by $n = N/M$. We assume as before that the consumer has an initial wealth $x_0 = (x_0^1, \ldots, x_0^M)$, $x_0^\mu \geq 0$, and wishes to improve its welfare in the market by using his endowment x_0 to purchase a consumption bundle x according to a separable utility function $U(x) = \sum_{\mu=1}^{M} u(x^\mu)$. His initial endowment, however, is assumed to be random, each x_0^μ drawn independently from a exponential distribution with unitary scale, ie,

$$P(x_0^\mu) = e^{-x_0^\mu} \tag{10}$$

As before, the aim of the consumer in this economy is to solve the maximization problem

$$x^* = \arg\max_x U(x) \text{ s. t. } p \cdot x \leq p \cdot x_0 \tag{11}$$

In most of the analysis done in this thesis we will treat the particular case of the consumer's separable utility function as $u(x_\mu) = \log x_\mu$, although any concave function would work exhibit similar qualitative behavior. The logarithm is a common choice for the consumer's utility function because it satisfies some of the usual properties desired for the consumer behavior in economics: first, the consumer is **loss averse**, which means that he will always prefer a guaranteed amount a of any good to a lottery in which he can win $a + \delta$ with probability 0.5 and $a - \delta$ with probability 0.5, for any $\delta > 0$. He is loss averse because the disutility losing δ is larger than the utility of gaining δ. In our case, we don't have lotteries, but the principle holds for two goods: if he has $\bar{x} + \delta$ of good μ and $\bar{x} + \delta$ of good ν, he will try to find a company that trades this excess of good ν so he can average both goods and in fact, may even do so at a loss (ie, he ends up with $\bar{x} - \varepsilon$ for both goods, for some $\varepsilon < \delta$). Also, with the separable utility as chosen, there are not complementary or substitute goods, ie, goods for which the consumer prefers to have more (or less) of one if he has another. Finally, because $u(0) = -\infty$, the consumer will always try to obtain a little bit of every good, even if at a great cost, because nothing is worse than having none of a particular good.

The firms on the other hand have each an M-dimensional random technology $\xi_i = (\xi_i^1, \ldots, \xi_i^M)$, where $\xi_i^\mu < 0$ represents an input and $\xi_i^\mu > 0$ represents an output. The production set of each firm is the space of all vectors which are proportional to ξ_i, that is, $\Xi_i = s\xi_i$, $s \geq 0$. This means that each firm i only has one technology and its only decision is the scale s_i at which it operates this technology. Once chosen the scale s_i, a company will consume $s_i\xi_i^-$ goods and produce $s_i\xi_i^+$ goods, where $\xi_i^\pm$ are the positive and negative entries of the ξ_i vector.

The elements ξ_i^μ are independently drawn from a normal distribution with zero mean and Δ/M variance, where $\Delta > 0$, and are normalized so that the sum over all the goods for a company is fixed at a negative value and all technologies are a little inefficient. We must have then:

$$P(x_i^\mu) = \mathcal{N}(x_i^\mu|0, \Delta M^{-1}), \quad \sum_{\mu=1}^{M} \xi_i^\mu = -\epsilon \tag{12}$$

We normalize the technologies to be inefficient so that we don't have a combination of firms producing infinite goods, ie, firm i and j can produce infinite amounts of certain goods by each feeding its output to be used as the other's input.

The objective of each company in the market is the same as before: each firm i tries independently to choose it's production scale s_i as to maximize it's profits:

$$s_i^* = \arg\max_{s_i > 0} p \cdot (s_i \xi_i) \tag{13}$$

Other underlying assumptions of General Equilibrium Theory are valid here: we assume a complete market, where each agent knows the offer and demand of all other agents, there is no transaction costs and a good is uniquely defined. Also, agents are price-takers, which means that they have no power over the prices and must accept them as given.

We also treat the economy as closed and therefore it must satisfy the market clearing condition. Because we have just one consumption bundle, then the N dimensional production scale vector s has to be such that

$$x = x_0 + \sum_{i=1}^{N} s_i \xi_i \tag{14}$$

ie, all the inputs the firms use have to come from the consumer's initial endowment.

Because market clearing hold and agents are price takers, we can also derive the strong restriction on profits discussed before. If we multiply both sides of the equation (14) by p, we get

$$p \cdot (x - x_0) = \sum_i s_i p \cdot \xi_i, \tag{15}$$

The left side of the above equation has to always be smaller or equal to zero, because of the budget condition. But the right hand side has to be always greater or equal to zero, because this term represents the sum of the individual firms' profits and if a firm is losing money they can always choose to set $s_i = 0$ and leave the market. Therefore, we must have that both sides are equal to zero, and the consequence is that the agent completely spends all his available budget (ie, $p \cdot x = p \cdot x_0$, he has not "leftover" cash after choosing x) and that the firms either have zero profit ($p \cdot \xi_i = 0$) or leave the market ($s_i = 0$).

One of the important implications of equation (15) for the Random Linear Economy model is that we may not have more than M firms active at any given realization of equilibrium. If the right hand side of equation (15) has to be zero, then for every firm either $s_i = 0$ or $p \cdot \xi_i = 0$. If ϕ is the fraction of firms active in the market, that is

$$\phi = \frac{\sum_{i=1}^{N} \mathbb{I}(s_i > 0)}{N}, \tag{16}$$

then all of them have $p \cdot \xi_i = 0$. Because the price is the same for all of them, we have ϕN equations of this type, and M unknowns. For this system to have a non-trivial solution (ie, $p_\mu > 0$ for all μ), it must be that $\phi N \leq M$, which implies that

$$\phi leq \frac{1}{n} \tag{17}$$

Having a single representative consumer (or many consumers but with wealth transfers) has two important consequences: first, the price vector is entirely determined by the consumer's demand. This is a result of the first order condition for the maximization problem. By taking the derivative of equation (11) with the proper Lagrange multiplier we get

$$\frac{\partial U(x)}{\partial x_\mu} - \lambda p_\mu = 0 \Rightarrow p_\mu = \frac{1}{\lambda x_\mu} \tag{18}$$

Furthermore, the market clearing condition binds the optimization problem of the consumer and the firms. If we substitute equation (14) in the consumer's utility, we get:

$$s^* = \arg\max_{s:s_i \geq 0} U(x_0 + \sum_{i=1}^{N} s_i \xi_i) \tag{19}$$

We can easily check that the zero profit condition is preserved with this solution. If s_i is in s^*, the solution for the consumer's maximization problem, then either $s_i = 0$ or $s_i > 0$. If $s_i = 0$, the condition is satisfied. Otherwise, if $s_i > 0$, it means that the constraint $s_i \geq 0$ was not enforced and the derivative at s_i must be zero. We then have

$$0 = \frac{\partial U}{\partial s_i} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial s_i} = p \cdot xi_i \tag{20}$$

Our problem is now considerably reduced: to find the equilibria in this model economy all we have to do is solve the maximization problem in equation (19).

4.2 THE ROLE OF STATISTICAL MECHANICS

If we were to employ standard convex optimization techniques to solve to solve (19), we would be able to find the solution for a specific realization of x_0 and ξ given a fixed N and M. But if we were to calculate quantities of interest such as consumer utility, average good consumption, average good price, price deviation among goods, number of active firms, etc, these would all be random variables which depend on the realization of endowments and technologies.

This is, of course, a well known behavior in statistical mechanics. We solve this by treating the case where the system size is very large, so that these average quantities converge to a single value. This isn't always the case, but holds for the so called *self-averaging* systems. In these systems, these average quantities for large systems converge to an average over the realizations for smallers systems.

The general approach to finding the equilibrium properties of a physical system is is to calculate the partition function for a specific realization of ξ, x_0:

$$Z(\beta|\xi, x_0) = \int dx e^{\beta U(x|x_0,\xi)}, \tag{21}$$

where β is the inverse value of the temperature. From this, we can calculate the average value of the utility function by taking the derivative of $\log Z$:

$$\langle U \rangle (\beta|\xi, x_0) = \int_0^\infty dx \frac{e^{\beta U(x|\xi,x_0)}}{Z(\beta|\xi, x_0} U(x|\xi, x_0) = \frac{\partial}{\partial\beta} \log Z(\beta|\xi, x_0) \tag{22}$$

The maximum value for the utility $U(x)$ is equivalent to the average value on the ground state[1], ie:

$$\max_x U(x|\xi, x_0) = \lim_{\beta\to\infty} \langle U \rangle (\beta|\xi, x_0) \tag{23}$$

However, we are still calculating the maximum as a function of the samples x_0 and ξ. In order to get the average behavior, which holds for a large system, we must average the utility over the disorder. Assembling all pieces together, we finally get the solution to equation (19):

$$\max_x U(x) = \int d\xi dx_0 P(\xi) P(x_0) \lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \log \int dx e^{\beta U(x|x_0,\xi)} \tag{24}$$

The explicit calculation of the expression above is considerably involved and makes use of a method commonly known as **replica trick** in the statistical physics community. We proceed with the detailed calculation on Appendix **??**. The solution of this calculation is

$$\lim_{N\to\infty} \frac{1}{N} \max_x U(x) = \max h(\Omega, \kappa, p, \sigma, \chi, \hat{\chi}), \tag{25}$$

$$\begin{aligned} h(\Omega, \kappa, p, \sigma, \chi, \hat{\chi}) = {} & \left\langle \max_s \left[-\frac{\hat{\chi}}{2} s^2 + (t\sigma - \epsilon p) s \right] \right\rangle_t + \frac{1}{2} \left(\Omega\hat{\chi} + \frac{\kappa p}{n} \right) - \frac{1}{2n\Delta} \chi\sigma^2 - \frac{1}{2n} \chi p^2 + \\ & + \frac{1}{n} \left\langle \max_x \left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega} t)^2}{2\chi} \right] \right\rangle_{t,x_o} \end{aligned} \tag{26}$$

1 This is true because $U(x)$ is convex and therefore has only one maximum.

4.3 RESULTS

The model has some very interesting properties which are described at length in [4]. In particular, it's possible to analytically calulate the distribution probabilities of x and s (and therefore of p) and see that all macroscopic quantities derived from these two quantities depend on the number of firms per good $n = N/M$. The model displays a regime change at $n = 2$, ie, two random technologies per good. When $n < 2$, the market is competitive and the fraction of active firms $\phi = \sum_i \mathbb{I}(s_i > 0)/N$ is around $\phi = 0.5$. Because each firm has on average half the goods as inputs and half as outputs, when $n < 2$ you don't have enough firms to span the whole M dimensional space in order to be able to fine tune the quantities desired for all the goods.

When $n > 2$, however, there are many firms to choose from and statistically it's possible to choose M linear independent firms that span the whole good space. In this regime, the market becomes monopolistic and ϕ assymptotically goes to zero with n, due to a saturation of active firms on M.

The change in the model economy's GDP also reflects the qualitative change in allocation. The authors define the gross product for the model as the total value of goods produced, that is, the sum of $(x_\mu - x_0^\mu)p_\mu$ for all goods μ that are produced, ie, $x_\mu > x_0^\mu$. However, the market clearing condition (15) makes the value of goods produced equal to the value of goods used as input, so we calculate the GDP Y by averaging over the absolute value of all trades:

$$Y = \frac{\sum_{\mu=1}^{M} |x_\mu - x_0^\mu| p_\mu}{2\sum_{\mu=0}^{M} p_\mu}, \tag{27}$$

where the denominator also includes a normalization for the prices.

What is shown in [4] is that in the competitive regime when $n < 2$, a new firm will have a significant positive effect on Y, while in the monopolistic regime $n > 2$ a new firm will have negligible impact on the gross product. We will revisit this result later in this paper.

The Random Linear Economies model is particularly suitable for further analysis because it's a General Equilibrium setting with few ingredients, but the introduction of stochastic elements offers a nontrivial phase transition which is not observed in similar “simple” economic models in the literature.

Part II

APPLICATIONS

5

INEFFICIENT CONSUMER IN A GENERAL EQUILIBRIUM SETTING

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INPUT-OUTPUT OF RANDOM ECONOMIES AND REAL WORLD DATA

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WHEN DOES INEQUALITY FREEZE AN ECONOMY?

8

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A

CALCULATION OF THE PARTITION FUNCTION FOR THE RANDOM LINEAR ECONOMY MODEL

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