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Statistical Mechanics of Economic Systems

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RESUMO

Nesta tese, exploramos o potencial de ser usar técnicas de Mecânica Estatística para o estudo de sistemas econômicos, mostrando como tal abordagem pode contribuir significativamente ao permitir o estudo de sistemas complexos que exibem comportamentos ricos como transições de fase, criticalidade e fases vítreas, não encontradas normalmente em modelos econômicos tradicionais. Exemplificamos este potencial através de três problemas específicos: *(i)* um framework de Mecânica Estatística para lidar com consumidores irracionais, no qual a racionalidade é controlada pela temperatura do sistema, que define o tamanho dos desvios do estado de máxima utilidade. Mostramos que um consumidor irracional aumenta a atividade econômica ao mesmo tempo que diminui seu próprio bem estar; *(ii)* uma anáise usando Teoria da Informação de matrizes Input-Output de economias reais, mostrando que os métodos de agregação utilizados para construí-las provavelmente subestima a dependência das cadeias de produção em certos setores cruciais, com consequências importantes para a analíse destes dados; *(iii)* um modelo em que agentes com uma riqueza inicial distributida como lei de potências trocam aleatoriamente objetos com preços distintos. Mostramos que esta desigualdade inicial gera uma desigualdade ainda maior em dinheiro livre, reduzindo a liquidez total na economia e diminuindo a quantidade de trocas. Discutimos as consequências dos resultados destes três problemas, bem como sua relevância na perspectiva geral em Economia.

ABSTRACT

In this thesis, we explore the potential of employing Statistical Mechanics techniques to study economic systems, showing how such an approach could greatly contribute by allowing the study of very complex systems, exhibiting rich behavior such as phase transitions, criticality and glassy phases, which are not found in the usual economic models. We exemplify this potential via three specific problems: *(i)* a Statistical Mechanics framework for dealing with irrational consumers, in which the rationality is set by a parameter akin to a temperature which controls deviations from the maximum of his utility function. We show that an irrational consumer increases the economic activity while decreasing his own utility; *(ii)* an analysis using Information Theory of real world Input-Output matrices, showing that the aggregation methods used to build them most likely underestimated the dependency of the production chain on a few crucial sectors, having important consequences for the analysis of these data; *(iii)* a zero intelligence model in which agents with a power law distributed initial wealth randomly trade goods of different prices. We show that this initial inequality generates a higher inequality in free cash, reducing the overall liquidity in the economy and slowing down the number of trades. We discuss the insights obtained with these three problems, along with their relevance for the larger picture in Economics.

CONTENTS

1

INTRODUCTION

> The wide perspective opening up, if we think of applying this science to the statistics of living beings, human society, sociology and so on, instead of only to mechanical bodies, can here only be hinted at in a few words. [14]

Describing the behavior of a large system composed of a large number of small parts for which we know the rules of individual behavior is the main theme of statistical mechanics. It is successful in this regard because at large system sizes the variation in individual behavior, for which we despite knowing the rules we can not measure precisely, is irrelevant: the aggregate quantities are robust and predictable. However, there is no reason to believe this applications are limited to the traditional physics jurisdiction of gases, condensed matter, etc. Any system large enough that its uncertainties can be aggregated into robust properties can be studied using the tools of statistical mechanics: proteins composed of hundreds of aminoacids have a very large number of ways to fold themselves in three dimension, but the rules of attraction for individual aminoacids makes it possible to calculate the probability of certain folding configurations [22]; The opinion of an individual is impossible to predict and model precisely, but the voting patterns or the opinion on moral issues of a nation comprised of millions of unpredictable individuals can be modelled with good accuracy [42, 89]; An aggregation of neurons with local firing rules can generate a neural network capable of storing and remembering patterns [46].

This large range of applications was made more explicit when E.T. Jaynes showed that the methods of statistical physics can be derived not only as a consequence of thermodynamics and physical laws but as the solution of an inference problem following Shannon's Information Theory [49]. A century later Boltzmann's prescience turned out to be accurate.

In Economics, traditional theory usually describes the behavior of economic actors such as consumers and firms as the result of rigorous mathematical deduction from a few starting axioms of behavior. From there, one typically deduces the properties of an economy by treating this behavior as the average representative of a larger set of actors. However, this is only valid when interactions among actors is very negligible. When economic actors interact, as they often do, the possibilities are much richer than the representative agent approach allows [18]. A Statistical Mechanics approach could greatly contribute to the field of Economics since it allows for the study of very complex systems, exhibiting rich behavior such as phase transitions, criticality and glassy phases, which are not found in the usual economic models.

The aim of this thesis is to make the case that the intersection of these two fields has a very large potential, being essentially two domains of knowledge that study very similar problems with different dressings: the properties and characteristics of interacting systems.

1.1 RESULTS OF THIS THESIS

The second part of the thesis concerns the three problems worked during this PhD: *(i)* an approach for consumers that do not strictly maximize their utility in the Random Linear Economy model, *(ii)* comparison of the Input-Output tables for real world countries and the ones obtained in the random economies and *(iii)* the impact of inequality when randomly trading goods of different prices. We now briefly describe each of them.

1.1.1 *Inefficient consumer in a general equilibrium setting*

In Chapter 5 we extend the Random Linear Economy model introduced in Chapter 4 by considering the cases where the consumer does not strictly maximizes his utility when choosing a bundle of M goods $x = (x_1, \ldots, x_M)$, starting from his endowment $x_0 = (x_1^0, \ldots, x_M^0)$, in a market composed of N firms each with a random technology $\xi_i = (\xi_i^1, \ldots, \xi_i^M)$ and scalar scale of production $s_i \geq 0$. In the original model, the consumer's choice x is given by the Gibbs distribution at the zero temperature limit, that is,

$$x^* = \arg\max_x \lim_{\beta \to \infty} \frac{1}{Z} e^{\beta U(x)} \delta(x - x_0 - \sum_{i=1}^{N} s_i \xi_i) \tag{1}$$

We make the case that the best way to model a suboptimal utility choice is by removing the zero temperature limit, or $\beta \to \infty$, and adjust how much the consumer deviates from "rational" behavior by changing β. By lifting this simple restriction we obtain nontrivial behavior, as shown on Figure 1. On the left, we show the agents utility as a function of the number of technologies per good $n = N/M$ for different values of $\log \beta$. When β is large (around 10^2), the agent optimizes efficiently enough that his utility always increases with n. However, when β gets lower and the representative consumer starts making worse choices (according to his utility function), his expected utility in the market *decreases* with n instead of increasing.

This result corroborates an increasing number of empirical results from Behavioral Economics where the amount of choice a consumer faces may decrease the quality

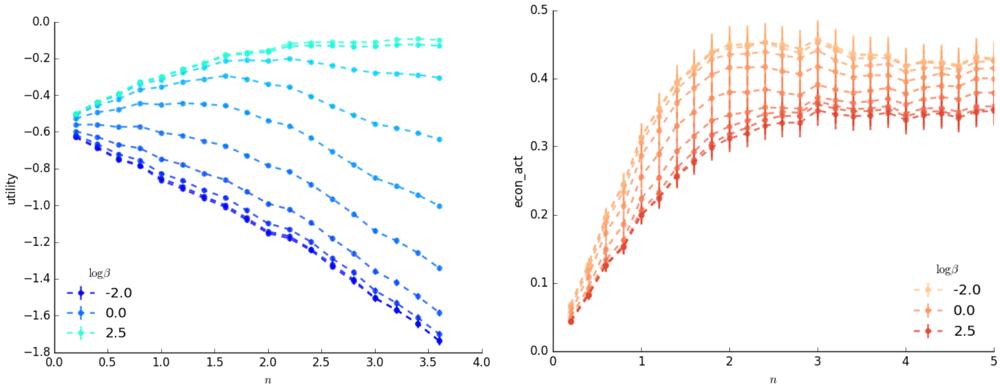

Figure 1.: **(Left)** Average consumer utility per good as a function of the number of technologies per good n, for several degrees of inefficiency β (when $\beta \to \infty$, the consumer strictly maximizes his utility). A large n means there are more trades available for the consumer and a rational consumer should always increase his utility when faced with a growing number of possibilities. This is not the case for a consumer that chooses inefficiently. **(Right)** Average economic activity (volume of goods exchanged per good) as a function of n, for several degrees of inefficiency. An inefficient agent may choose poorly when faced with many choices, but he trades a larger quantity of goods.

of his choices [47, 76]. What is more interesting is that the economic activity in the market, in this case measured by the density of goods being exchanged, *increases* with the inefficiency in consumer choice, because he deviates a lot more from x_0 than if he were strictly maximizing his utility. In this stylized economy, markets with agents that make bad decisions have unhappier agents but are more active and, considering economy activity is usually correlated with wealth, richer.

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1.1.2 *Input-Output of random economies and real world data*

In Chapter 6 we collect several years of the Input-Output tables for ten real world economies, which are matrices showing how much the industries of each sector of the economy produce and use as input of every good in the economy. The goods are also divided in the same sectors as the industries, in the case of the US at three levels of aggregation: detailed level, with 389 sectors, aggregated with 71 and summary with 15, going from ”Mining” in the summary level to ”Coal mining”, ”Iron, gold, silver mining”, etc, in the detailed level, and at two levels of aggregation for the EU countries: 64 and 10 levels.

From these Input-Ouput tables we are able to build the Direct Requirement matrices, in which each element represents many dollars of a good are needed to produce a dollar of another good. These matrices represent directed, weighted graphs with the goods as nodes, and by definition the indegrees (sum of all edges that point to the node) are equal to one. The outdegrees, however, are variable, and they indicate the dependency of the production network on a specific good. Acemoglu et al [2] show that the heavier tailed the outdegree distribution is, the more susceptible to shocks is an economy.

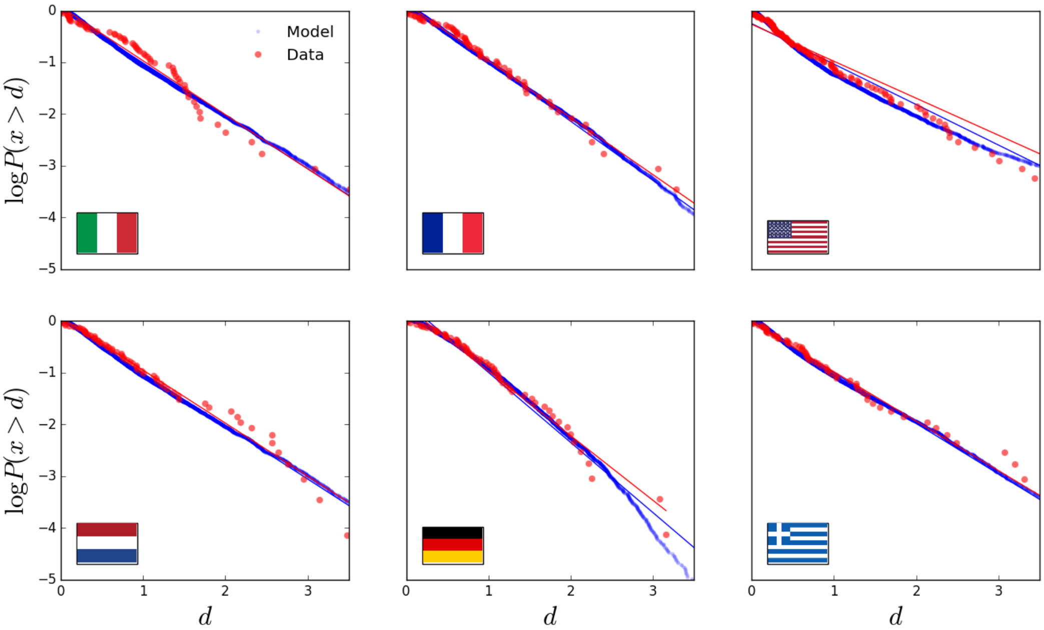

Figure 2.: Log of the counter cumulative degree distribution for six of the ten countries analyzed, compared with the closest degree distribution generated by the Random Linear Economy model with $M = 100$, $\beta \to \infty$ for different values of n. Both datasets are plotted along with their regression, which is very close to an exponential distribution.

We show that the outdegree (or simply degree) distribution of ten countries we have analysed at the aggregated level (64 sectors for the EU, 71 for the US) is very close to an exponential distribution, whereas data at the detailed level of aggregation for the US has much heavier tails. Given that these degrees are random variables with a fixed average, Information Theory tells us that in the absence of any extra constraints the distribution that maximizes entropy is an exponential distribution. This allows us to conjecture that the aggregation process has washed out the structural information of the economy.

We check this hypothesis by generating the Input-Output matrices, and consequently the Direct Requirement matrices, for artificial economies generated by the model introduced in Chapter 4 and showing that their aggregation produces the same patterns we observe in real world data. Furthermore, we show that different methods of aggregation yield different results, and the one employed in real world data is closer to random than to one that takes input and output correlations into account. This has a direct consequence on Acemoglu et al's conclusion for the structural fragility of the US economy: distribution tails that were taken into account may have been a simple artifact of the aggregation process, and the real distribution may be heavier tailed than what they calculated. If this is the case, the predictions made on [2] concerning the fragility of the U.S. economy to random shocks may have been underestimated.

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1.1.3 *When does inequality freeze an economy?*

In Chapter 7 we show the statistical effects of inequality in a simple trading economy where we have N agents with a initial capital that is power law distributed $P(c_i > c) \sim c^{-\beta}$, and agents trade M goods each with its own price $\pi_1, \ldots, \pi_M$, such that the leftover cash of an agent is its capital c_i minus the price of the goods he owns. At each trading step, one of the M goods is chosen at random and its owner tries to sell it to another agent, which automatically accepts it if he has enough cash to do so.

The parameters are set in such a way that even the most expensive good is affordable to the poorest agent. Despite that, in the stationary state we show that this zero intelligence dynamic divides the agents into "classes" where an agent can afford goods up to a certain price and none more expensive than this threshold. If we define an agent's leftover cash as his liquidity in the market, given a unequal distribution of capital the economy's liquidity concentrates into fewer agents, and as the inequality parameter β goes to one, the economy freezes completely as only the richest agents carry any meaningful trade.

These results are summed up by the top panel of Figure 3. There we plot the Gini coefficient for the cash as a function of the Gini coefficient for the capital. The Gini coefficient is a measure of inequality in a distribution and goes from zero to one: zero means perfect equality, all data in the dataset are the same. One means perfect inequality: only one point is positive and the rest is all zero. In Figure 3, we show that liquidity in the model is always much more concentrated than the initial capital, converging to perfect inequality when $\beta \to 1$. This has an arresting effect where all trade in the economy stops. This result is tested versus empirical data in the bottom panel of Figure 3, where we compare the velocity of money, as defined by the US Federal Reserver Bank, as a function of the inequality for the model and the historical US data, showing that there is good corroboration that inequality indeed decreases the velocity of money.

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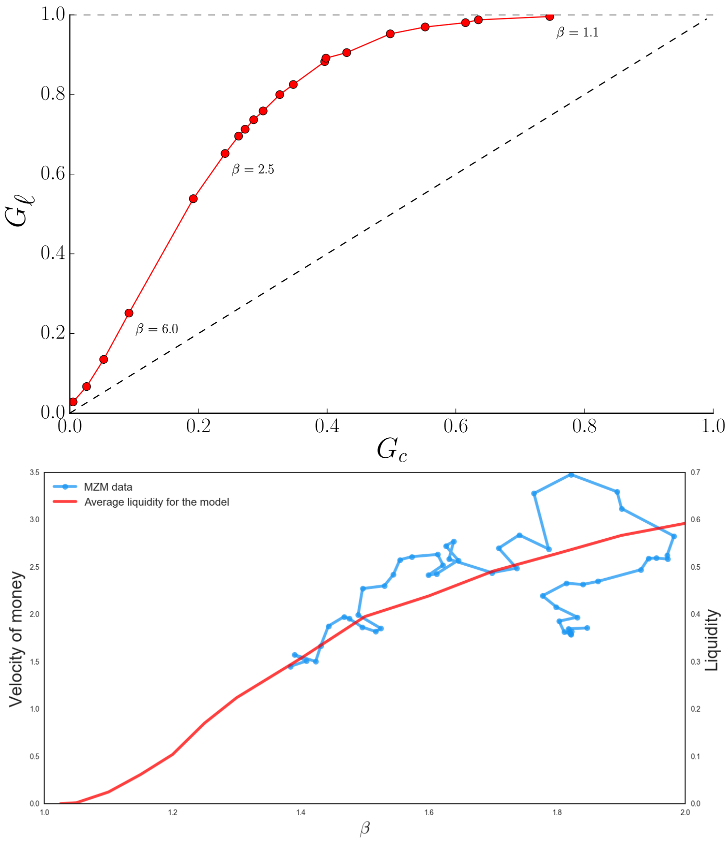

Figure 3.: **(Top)** Gini coefficient G_ℓ of the cash distribution (liquid capital) in the stationary state as a function of the Gini G_c of the capital distribution, for numerical simulations described in Chapter 7. Cash, or liquidity, is always more unequally distributed than capital, resulting in perfect inequality (when only a countable number of agents have non zero liquidity) when β approaches one. **(Bottom)** Velocity of money, defined for the model by equation (119), as a function of the level of inequality represented by the power law parameter β, with comparison to the historical US data. Our model corroborates the fact that as inequality increases, the number of money units that are exchanged in a time interval decreases.

1.2 ORGANIZATION OF THIS THESIS

The first part of the thesis will concern the theoretical context of this work, along with some examples in the literature: in Chapter 2 we will go over a brief introduction of General Equilibrium theory and mainstream Economics research in general. The main aim of the chapter is to give readers of a Physics background the necessary context to understand the standard protocol in Economics when studying a problem, what are some important questions in the field, what techniques are usually used, etc. Our goal is that after reading the chapter, the reader will be convinced of the proximity between the two fields, and how the methods differ despite their aims being essentially the same.

In Chapter 3, we will make the case of why Statistical Mechanics is a suitable tool for exploring and studying Economics. Although the Physics minded reader most likely does not need much convincing, it is still useful for those that never had any exposition to a general, information theoretical approach to Statistical Mechanics as introduced by Jaynes [49]. We hope that a reader with a background in Economics (and who hopefully did not stop reading the thesis after Chapter 2) will agree with us that the methods presented therein offer an alternative to the usual methods in Economics. In this chapter we will also present a number of previous work that have already explored this intersection.

We end the first part by introducing the Random Linear Economy model in Chapter 4, which consists of a simple market model where companies have random technologies and one consumer wishes to improve his utility by trading his current goods in the market, which is exactly the sort of problem General Equilibrium Theory describes. Despite its simplicity, the model exhibits a rich behavior due to the stochasticity introduced in the technologies and its solution is arrived at by using techniques from the Physics of disordered systems. Its introduction merits a dedicated chapter because not only it serves as the basis for the work developed on Chapter 5 and parts of Chapter 6, but also because it is a good representative of the ideas put forward in Chapters 2 and 3.

The second part of this thesis, Chapters 5, 6 and 7, are the three problems we have mentioned in the last section.

2

ECONOMICS AND THE GENERAL EQUILIBRIUM THEORY

In this chapter we will give a background of the relevant economic theory for this thesis. We aim to formally introduce a central theory in economics which is General Equilibrium Theory. Along with it, we hope to expose the reader with Physics background to the usual concerns of Economics. We also discuss the common criticisms about General Equilibrium Theory, along with more recent work to solve some of the problems discussed.

General Equilibrium Theory is a mature field of Economics [9, 57, 58] which aims to characterize the existence and properties of equilibria in certain market settings. Economic systems are frequently assumed to have actors with opposing goals: the owner of a good wants to sell it for the highest possible price, while its potential buyers would like to purchase it for as low as possible. Fishermen would like to catch as many fishes as possible and expect their peers to not also overdo it otherwise they may extinguish the oceans. Businessmen would like to price their products in such a way that as many people as possible buy it for as high as possible. A couple in vacation would like to buy a plane ticket to Paris as cheap as possible as long as the layover is not longer than a couple of hours. One expects that economies and markets composed of many such actors trying to get their way will converge to a certain steady state. The main goal of General Equilibrium Theory is trying to characterize these equilibrium states in a rigorous manner. In this sense, it's a microeconomic theory, because it explains macrobehavior from the incentives (or micromotives [78]) of agents.

While the work in this thesis doesn't strictly aim at studying economies as they are described in General Equilibrium Theory, its purposes and questions are very similar to the ones physicists usually ask when studying a complex system: namely, the characterization of its equilibrium state. It's important, therefore, to have a deeper understanding of how this is done in economics, what are its main concerns and assumptions, if only to realize the two field have very similar goals.

2.1 A BRIEF EXPOSITION

The exposition in this chapter is mostly adapted and simplified from Mas Colell's seminal book on Microeconomic Theory [57] and will be considerably more formal than the rest of this thesis due to the way the discipline is commonly studied.

In General Equilibrium Theory, an economy is defined through the following components: we assume there are J consumers, N firms and M goods in a market. Each consumer has a consumption set X_j which contains all possible consumption bundles $x_j = (x_j^1, \ldots x_j^M)$ that the consumer has access to, i.e., each bundle x_j is a $M-$dimensional vector with nonnegative entries (we are assuming he cannot consume a negative amount of a good). X_j is limited by "physical" constraints, such as no access to water, iron or bread, but not monetary constraints, which we will deal with later.

The consumer also has an utility function $U_j(x)$ that takes every element $x_j \in X_j$ to a real number, representing how much the consumer values each bundle of his consumption set. This allows us to define define a preference relationship over the elements in X_j (i.e., if the consumer prefers bundle x to x'), which is complete[1] and transitive[2], two standard requirements in Economics for rational behavior.

Finally, the consumer is also endowed with an initial bundle of goods $\omega_j = (\omega_j^1, \ldots, \omega_j^M)$, $\omega_j^\mu \geq 0$ which will define his budget given a set of prices for the goods and will constraint his choices on X_j. His initial budget is defined in terms of a bundle of goods for two reasons: first, we want the economy as a whole to be unaffected by the scales of prices, so if all prices are multiplied by a constant factor the consumer's purchasing power remains the same. Second, the theory described *general* equilibriums, which are assumed to be valid for closed markets. All goods used as input have to come from somewhere inside the economy, and the consumer acts as the provider of raw material.

Each firm i has a production set Ξ_i of technologies $\xi_i = (\xi_i^1, \ldots, \xi_i^M)$ that it is able to operate. Unlike consumption bundles, which are final allocations and therefore must be nonnegative, technologies can be any real number: the negative entries are inputs and the positive entries are outputs in the technology's operation. Ξ_i is also limited only by "physical" constraints, not by monetary constraints. A firm that has $\xi_i = (-1, 2)$ in its production set is able to transform one unit of good 1 into two units of good 2. It won't necessarily be able to transform two units of good 1 into four units of good 2, for that it must also have $\xi_i' = (-2, 4)$ in Ξ_i. It might be the case, for example, that companies get more efficient with production and therefore it might have $\xi_i'' = (-2, 6)$ in its production set.

In General Equilibrium Theory, an economy is formally defined as the tuple

$$E = \left(\{(X_j, U_j)\}_{j=1}^J, \{\Xi_i\}_{i=1}^N, \{\omega_j\}_{j=1}^J \right). \tag{2}$$

One of the theory's assumptions is that the economy described is **complete**, which means every agent can exchange every good with no transaction costs and complete information about the firm's technologies, other consumer's consumption, etc. Also, a

1 For every $x, x' \in X_j$, either $U_j(x) \geq U_j(x')$ or $U_j(x) \leq U_j(x')$.

2 For every $x, y, z \in X_j$, if $U_j(x) \geq U_j(y)$ and $U_j(y) \geq U_j(z)$, then $U_j(x) \geq U_j(z)$.

good μ contains all the possible information that a consumer would take into account when making his choice. That is, among the space of goods we could have "umbrella" and "chocolate", or we could also have "an umbrella on August 13th, 2016 in São Paulo with 50% chance of rain" and "an umbrella on December 12th, 2016 in Chicago with 90% chance of rain". There is no hidden information, so the consumer is never in doubt of whether the good or company is actually reliable or any such considerations.

It's also assumed that agents are **price-takers**, that is, they are unable to affect the market prices and therefore have to take them as a given. The prices of the goods are given by a $M-$dimensional vector $p = (p_1, \ldots, p_M)$, where each price is a strictly positive quantity, i.e., $p_\mu > 0$ for all μ. This carries the restriction that goods have global prices, which is consistent with the completeness assumption: there is no reason why the market prices should be different for certain consumers or firms if they have complete knowledge and no transaction costs.

With a price vector p defined, we say the consumer j has a budget $B_j = p \cdot \omega_j$, which is the monetary value of his initial endowment. Any bundle he chooses to purchase will cost him $p \cdot x_j$. His objective, therefore, is to find the best bundle x_j he is able to afford, that is:

$$\max_{x_j \in X_j} U(x_j), \quad \text{s.t. } p \cdot x_j \leq p \cdot \omega_j \tag{3}$$

The firms, on the other hand, have an operating profit for each technology given by $p \cdot \xi_i$, which is how much money they earn by selling their outputs ($\xi_i^\mu > 0$) minus how much they spend purchasing the inputs ($\xi_i^\mu < 0$). Their objective is to maximize their profits, that is:

$$\max_{\xi_i \in \Xi_i} p \cdot \xi_i \tag{4}$$

With these ingredients laid out, we define an **allocation** of the economy as a set of specific choices for consumption bundles and technologies, i.e., an allocation a of an economy E is

$$a = (x_1, \ldots, x_J, \xi_1, \ldots, \xi_N),\ x_j \in X_j,\ \xi_i \in \Xi_i \tag{5}$$

We are considering closed economies and therefore all that is produced must come from the initial endowments and end up part of the consumers' final bundles. We therefore say an allocation is **feasible** if it satisfies **market clearing** for all the goods:

$$\sum_{j=1}^{J} x_j^\mu = \sum_{j=1}^{J} \omega_j^\mu + \sum_{i=1}^{N} \xi_i^\mu, \quad \forall \mu \in \{1, \ldots, M\} \tag{6}$$

This is a strong condition which couples many quantities in the economy. In particular, if we multiply both sides of the equation by p^{μ} and sum over μ we get, in vector notation,

$$\sum_{j=1}^{J} p \cdot (x_j - \omega_j) = \sum_{i=1}^{N} p \cdot \xi_i \tag{7}$$

The left-handside is the leftover money the consumers have after making their choice of consumption, also called the value of excess demand, whereas the right-handside is the firms aggregate profit, also known as the value of excess supply. Because we assume that the consumer may not spend more than his budget, the value of each consumer's individual excess demand has to be non positive. Simultaneously, if we assume that firms always have $\xi_i = 0$ in their production set, i.e., we assume that they can always opt to not produce at all and leave the market, then the value of excess supply for each firm has to be non negative. Because they must be equal, we conclude that in an economy for which market clearing holds, the consumer spends all his available budget and firms all get zero profit, a result known as **Walras' Law**.

Given a set of feasible allocations $\{a_k\}$, we may wonder if there is any allocation we desire most over the others. This of course depends on the criteria we use to judge them: we may like allocations with less inequality, with the most aggregate utility, with the smallest minimum utility, etc. Economists opt to use one particular condition which is called **Pareto optmality**.

Intuitively, a **Pareto optimal** (or **Pareto efficient**) allocation is one in which you can't make a consumer better without making another consumer worse off. The idea is that, a non Pareto optimal allocation has some waste in it: one could change the consumption bundles in order to increase some utilities and no other consumer would complain. Because firms have zero profit in feasible allocations, they wouldn't mind the change.

More formally, a feasible allocation $a = (x, \xi)$ is said to be **Pareto optimal** if there is no other allocation that **Pareto dominates** it, that is, no allocation $a' = (x', \xi')$ such that $U(x'_j) \geq U(x_j)$ for all j and $U(x'_j) > U(x_j)$ for at least one j.

The Pareto optimality concept therefore defines a socially desirable outcome in a "non-controversial" way, by definition no agent in the economy would have a problem with policies or actions taken to make it more Pareto efficient. However, it says nothing about equality: an allocation in which one consumer has all the goods and no other consumer has any goods is Pareto optimal.

We finally arrive at the concept of equilibrium in an economy. A **Walrasian equilibrium** (or competitive equilibrium or simply equilibrium) in an economy E is an allocation (x^*, ξ^*) and a price vector p such that

1. Every firm i maximizes its profits in its production set Ξ_i, that is

$$p \cdot \xi_i^* \geq p \cdot \xi_i, \quad \forall \xi_i \in \Xi_i, \quad \forall i \in \{1, \ldots, N\} \tag{8}$$

2. Every consumer j maximizes his utility in his consumption set X_j, that is

$$U(x_j^*) \geq U(x_j), \quad \forall x_j \in X_j, \quad \forall j \in \{1, \ldots, J\} \tag{9}$$

3. The allocation (x^*, ξ^*) is feasible, that is,

$$\sum_{i=j}^{J} x_j^* = \sum_{j=1}^{J} \omega_j + \sum_{i=1}^{N} \xi_i^* \tag{10}$$

The Walrasian equilibrium is essentially a pair allocation - prices such that all optimization problems are solved at once. Although we have not mentioned any dynamics in this economy, it's considered an equilibrium because all agents are as satisfied as possible with their allocation given the prices, which we have assumed to be global and unchangeable by any agent's action. This is not exactly a definition of equilibrium as used in Physics, but we will discuss this point later. For now, we point out that a Walrasian equilibrium is in some sense stable.

We have thus defined two desirable properties of an allocation: efficiency and equilibrium. The fundamental results of General Equilibrium Theory are the **Welfare Theorems**, which define the conditions for which an equilibrium is Pareto optimal and for when a specific Pareto optimal allocation is an Walrasian equilibrium.

The **First Fundamental Welfare Theorem** asserts that if the consumers have a continuous utility function on X_j[3], then all Walrasian equilibria are Pareto optimal. This result is simple yet useful, because it tells us that if our economy is in equilibrium, we don't have to care about checking if it's efficient. The violation is also important: if a given economy we are studying is in an inefficient equilibrium, then it must be that one of the theorem's condition was violated. This sheds light in where to look for market failures. We remind the reader, however, that some extra strong assumptions were made for the economies described by this theorem, namely, completeness of market and global prices that no single agent is capable of influencing.

The **Second Fundamental Welfare Theorem** requires extra assumptions: it affirms that if an economy satisfies the conditions of the first fundamental theorem, the utility functions U_j and all sets X_j, Y_i are convex and if we are able to redistribute the initial endowments at will, while keeping the total amount $\sum_{j=1}^{J} \omega_j$ constant, then for every Pareto efficient allocation there exists a wealth allocation ω and price vector p^* such that (x^*, ξ^*, p^*) is a Walrasian equilibrium.

The second theorem is considerably more interesting than the first one: any Pareto optimal allocation we would like in an economy can be an equilibrium given the appropriate price vector and a possible wealth transfer, albeit under a stronger set of conditions. It serves both as a benchmark, because we know what we can expect from

3 The actual theorem asserts a weaker condition, that the preferences be locally nonsatiated, that is, for every $x \in X$, there is an $x' \in X$ such that $\|x - x'\| < \varepsilon$ and x' is preferred to x.

markets at their "optimal conditions", but also as a warning: we can only guarantee that an efficient allocation will be an equilibrium under a very strong set of requirements.

2.2 LIMITATIONS OF GENERAL EQUILIBRIUM

A conspicuous element was missing from the exposition above: there are no rules for the dynamics of the economies described above. The prices are taken as a given, as are the consumer and firm choices. What happens if a firm closes? What happens if a new firm appears? The equilibrium is simply "recalculated" and the economy moves to the new one?

Indeed, this is a long standing criticism to General Equilibrium Theory. Walras proposed it as a process of **tatônnement**[4]: a central figure, known as the Walrasian auctioneer, suggests a price and asks all the firms and consumers how much would they like to produce and buy at these given prices, but without any transaction taking place at out of equilibrium prices. The auctioneer updates the prices in the direction of diminishing excess demand or supply, a gradient descent process, until equilibrium is reached, at which point transactions finally take place. The auctioneer must also have a way of guaranteeing that agents will be price takers: it must either be able to monitor and enforce all transactions or buy and sell arbitrary amounts of every good at their equilibrium prices.

It is clear that such process is very convoluted. Chiefly, this auctioneer figure doesn't exist in most decentralized markets: goods are traded at agreed prices by both parts, which do not wait until their transaction is authorized by some central authority. Even if there were auctioneers, such authority would require an infinitely large computational capability to compute the excess demand and supply of every consumer and firm and for every good in a modern economy [11, 69]. Worst of all, even if there was such central figure with such an arbitrary large amount of computing power, not all price updating dynamics are guaranteed to converge [44]. Finally, even if it converges, we have no assurance that it will converge in finite time.

These are all well known and acknowledged shortcomings of the theory. Some areas of Economics, such as the Schumpeterian Economics [75], eschew the ideal of an static equilibrium altogether, studying the pattern of changes from certain evolutionary rules instead. These ideas are also popular amongst physicists [86].

However, the usefulness of General Equilibrium Theory in Economics stands not from practical applications, but as a benchmarking tool for real world policies. Their conclusions are mathematically precise and correct. So when faced with an economic equilibrium that is not Pareto efficient, then by definition it must be because one of the Second Fundamental Welfare Theorem conditions was violated, and therefore one knows where to look for inefficiencies to try and fix it.

4 From "trial and error" in french

Decades after the introduction of the Welfare Theorems, the field of Applied General Equilibrium arose as a way to compute equilibria and explore them for policy decisions [79, 80]. It used real world data to calibrate production and utility functions along with iterative schemes to calculate the equilibrium prices [72]. From there, it could be used to explore the impact of policy changes in more complex settings.

More recently, these limitations have been tackled by the Dynamic Stochastic General Equilibrium (DSGE) models, that have origin in the work by Kydland and Prescott [55]. In that seminal paper, the authors calculate the evolution in time of macroeconomic variables such as economic output over time by treating it as the trajectory in time of a microeconomic general equilibrium problem with stochastic elements. This has led to a new class of economic models based on this approach of using dynamical systems with stochastic terms and inferring the parameters from real world data [27, 83], which are employed today by institutions such as the European Central Bank for policy analysis [82].

However, what all of theses models have in common is that they all look for equilibria in the Classical Mechanics sense of Physics. Generally speaking, an equilibrium state in Economics is when all incentives cancel each other out and no actor has the desire or the possibility of deviating from the current configuration, which is an approach that heavily draws from Classical Mechanics in Physics and dynamical systems in Mathematics. In fact, Stephen Smale even wrote a paper on the dynamics of General Equilibrium [81] trying to tackle some of the open dynamical questions discussed earlier, employing modern mathematical theory of dynamical systems.

This Classical Mechanics approach has limitations: due to the need of having one exact solution for the equilibrium configuration, economic models usually employ some standard simplifications to make problems tractable. The most common of which is the representative agent, in which a single agent represents all consumers, another represents all firms, all the government policies, etc, and his objective functions are considered as the average of an heterogeneous population, with the different parts of the economy interacting through these ”average demands”.

Trimming down a problem to its bare essential features isn't exactly unknown in Physics. However, one must be careful not to change the problem completely when doing so. In Statistical Physics, it is widely known that it is precisely the interaction between a large number of particles that generate rich phenomena such as phase transitions, in which the average quantities of a system go through a discontinuous transition. From that point of view, it seems a waste of opportunity to treat the demand function for all consumers as a continuous and well behaved function. This has been pointed out both by economists [52, 53] and physicists [18, 15] and probably the most common criticism of Economic Theory is that it has done a poor job in predicting major crisis as recently as the 2007 - 2008 burst of the housing bubble in the US, which generated a new wave of criticism for the lack of foresight [16]. Using interacting agents allows for models in which crisis and transitions come from endogenous rules of interaction instead of arbitrary adhoc modelling.

But perhaps the biggest difference in the approaches is that, in Statistical Mechanics, an equilibrium is defined as an ensemble of possible configurations that appear with certain probabilities, and instead of trying to find the exact numbers that equilibrate the system, one looks for the average of macroscopic quantities. This allows for the interpretation of fluctuations as a natural phenomena, due to the nature of the uncertainty involved in inferring a complex system, as we will describe in the next chapter. Then, one can truly define the behavior of microscopic interactions, as opposed to representative agents, and say something about the aggregate.

Finally, because fluctuations around a minimum are a fundamental part of the Statistical Mechanics framework, they allow us to have a principled way of modeling agents that don't necessarily act in an optimal way. Ever since the work of Tversky and Kahneman [87, 50, 88], economic mainstream has paid more and more attention to the empirical evidence that real life decisions are not always optimal. While this has been known for a long time, traditional Economic Theory has always treated optimality in choice as a good approximation to real life scenarios, akin to disregarding air drag in a Classical Mechanics problem. However, the amount of evidence that human beings consistently make suboptimal choices has increased the need for alternative approaches. In Economic Theory, suboptimal behavior is usually introduced as a stochastic shock in the consumer optimization problem or through other adhoc heuristics. A Statistical Mechanics approach can contribute to this topic by offering a natural way of dealing with these fluctuations, as we will show in Chapters 3 and 5. Later, in Chapter 7 we will show that we can get valuable insights into statistical properties of the economy if we assume no rationality at all – an agent that chooses at random.

3

STATISTICAL MECHANICS AND INFERENCE

In this Chapter we aim at answering the question of why Statistical Mechanics is suitable to study economic systems. The essence of the answer is that, among other things, the techniques of Physics allows one to find the minimum energy configuration (or at least very good approximations) and the expectation value of certain quantities of interest in rather complicated settings, which is precisely what Economics could mostly benefit of.

To the reader familiar with Statistical Mechanics, we draw attention to the fact that we present here the canonical ensemble not as a derivation from Thermodynamics, but as a systematic inference of a system's observables using Information Theory. Instead of assuming a system connected to a heat bath at a fixed temperature and employing the first postulate of Statistical Physics, we follow the work of Jaynes [49] and show that the Gibbs distribution is the solution for an inference problem with limited information.

3.1 STATISTICAL MECHANICS AS AN INFERENCE PROBLEM

Suppose we have an interacting system composed of N particles, each with its own state x_i, which can be its position, velocity, orientation, decision of whether to buy a Mac or a PC, political affiliation, etc. The whole system can be fully characterized by the configuration vector $x = (x_1, \ldots, x_N)$ and we assume all the information we have about this system is the expected value of function $H(x)$, often called in physics settings the energy function. There are many questions we can ask about the system, for example what is the probability of this system being at a configuration x, or what is the expected value of another quantity $G(x)$. These questions are inference problems, and through Information Theory we can find out what is the best way we can answer them.

As proposed by Shannon [77], when faced with a choice of several probability functions that describe some data or phenomena, one should always opt for that which makes the least assumptions, given the constraints of the problem. This amounts to finding the probability distribution $p(x)$ that maximizes the **Shannon entropy**

$$S[p] = -\int dx P(x) \log P(x) \tag{11}$$

subject to the constraints imposed by observation. In our case, the constraint is that the energy function $H(x)$ has an average value $\langle H(x) \rangle = \int dx\, P(x) H(x) = E$ and we also have to impose the constraint that $P(x)$ is a probability distribution and therefore must be normalized, i.e., $\int dx\, P(x) = 1$. This means that to find $P(x)$ for our system, we must find $P(x)$ that maximizes the Lagrangian

$$\mathcal{L}[P] = -\int dx P(x) \log P(x) + \alpha \left(\int dx\, P(x) - 1 \right) + \beta \left(\int dx\, P(x) H(x) - E \right), \tag{12}$$

where α and β are the Lagrange multipliers of this maximization problem.

We assume that at the maximum P^* a small perturbation $P(x) + \delta P(x)$ does not alter the Shannon entropy. If we assume that all $\delta P(x)$ are independent (i.e., $\delta P(x)$ and $\delta P(x')$ are not correlated for every x, x' in the support of the distribution), then for every x this becomes a regular maximization problem. For every x, we must solve that $\frac{\delta \mathcal{L}(P)}{\delta P}(x)$ is equal to zero, that is:

$$\frac{\partial}{\partial P(x)} \left\{ -P(x) \log P(x) + \alpha \left(P(x) - 1 \right) - \beta \left(P(x) H(x) - E \right) \right\} = 0, \quad \forall x \tag{13}$$

Solving this equation we have that for every value of x

$$-\log P(x) - 1 + \alpha + \beta H(x) = 0 \Rightarrow \tag{14}$$

$$\Rightarrow P(x) = e^{-1+\alpha-\beta H(x)} \tag{15}$$

The Lagrange multipliers must be set so that the constraints are satisfied. For α we have that $e^{\alpha-1}$ must normalize the probability distribution, i.e.

$$\int dx e^{-1+\alpha-\beta H(x)} = 1 \Rightarrow \tag{16}$$

$$e^{1-\alpha} = \int dx e^{-\beta H(x)} = Z \tag{17}$$

This normalization term is the sum over all the configurations and is called the **partition function**. For β we must have that

$$\int dx H(x) \frac{e^{-\beta H(x)}}{Z} = -\frac{\partial}{\partial \beta} \log Z = E \tag{18}$$

This means β must be such that the average energy of the system is equal to the observed average E. However, suppose we have not actually observed E, all we know is that it is fixed to some value. Then, because E is given by the above equation for which the only degree of freedom is a Lagrange multiplier, all the possible values it can take are given by varying β from 0 to ∞. We have finally arrived at the maximum entropy distribution for our inference problem, which is the **Gibbs distribution**

$$P(x|\beta) = \frac{1}{Z(\beta)} e^{-\beta H(x)} \tag{19}$$

The extreme cases for β give us an intuition on how the Gibbs distribution behaves. For $\beta = 0$, $P(x) = \frac{1}{Z}$, for all values of x: in this limit all configurations are equally likely, regardless of their energy $H(x)$. In the opposite case, when $\beta \to \infty$, Z becomes more and more concentrated around its maximum point, where $E(x)$ is minimum, and eventually $P(x)$ collapses to a delta function around the minimum energy configuration, also known as the **ground state** (it can also have an equal mass in several points in the case of multiple minima). Therefore, in the full β spectrum, the Gibbs distribution starts completely uniform in the space of all configurations and slowly coalesces around the minimum energy values. If we assume $H(x)$ is bounded, then for every finite value of β, the system has a finite probability of being in any configuration (what is known as ergodicity). Given a system described by the Gibbs distribution, the average value of another desired observable $G(x)$ is given by

$$g = \langle G(x) \rangle = \int dx G(X) \frac{1}{Z} e^{-\beta H(x)} \tag{20}$$

Though we have called $H(x)$ the energy function for customary reasons, this function is in principle any arbitrary function of the system configuration that has a well defined average value. For most systems of interest we can always decompose it as a sum of small scale interactions, that is, we can write

$$H(x) = \sum_a H_a(x_a), \tag{21}$$

where a represent minimal cliques, usually pairwise, where we can reduce the interactions in the system to microscopic interactions. In this way, the behavior of macroscopic quantities such as the average energy or any other observable we are interested depends on the sum of a large amount of simple interactions.

Besides allowing for more realistic modelling, interactions have a side effect which is very well known to physicists but that don't appear frequently in economic debates: the existence of **phase transitions**. Instead of asking questions about the specific details of one equilibrium configuration, in Physics one usually explore the parameter space looking for sharp transitions in the behavior of average quantities such as average utility per agent, average number of goods traded, etc. Phase transitions are of special interest because they represent some of the most interesting phenomena a system can present, and indeed, many popular questions in Economics, such as business cycles, crisis, altruistic cooperation, etc, can be framed in terms of phase transitions.

We note that despite this being the standard theory for the canonical ensemble in Statistical Physics, we have not made so far any Thermodynamical (or any other "physical") assumptions. We have been describing generic systems where we simply applied the tools of Information Theory for the inference of a random variable for which we have limited information. There's nothing that limits us to using the Gibbs distribution only for gases in which the molecules interact according to the laws of Physics. This is the fundamental reason why Statistical Mechanics is so successful at explaining such a varied wealth of phenomena: despite being first developed via physical laws, its results are general.

The only real difference when dealing with a thermodynamical system is that when we plug the Gibbs equation back into the Shannon entropy we have

$$S[P_G] = \log Z + \beta E, \tag{22}$$

which is still general, but we can now use one of the Maxwell's relations to give β a physical interpretation:

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \beta \tag{23}$$

Therefore in physical systems β is identified as the inverse temperature and when $T \to \infty$, the system is equally likely to assume any possible configuration. Likewise, when $T = 0$, the system is frozen at one of the ground states.

3.1.1 *A simple example*

We make explicit the general nature of the Gibbs distribution deduced in this section by a simple example[1] of a random variable x that can take three values: -1, 0 or 1. We know it's average is $\langle x \rangle = m$. What is the best inference we can make for it's probability distribution $P(x)$? The Gibbs distribution is

$$P(x) = \frac{e^{-\lambda x}}{Z(\lambda)}, \tag{24}$$

where $Z(\lambda)$ is given by:

$$Z(\lambda) = \sum_{x \in \{-1,0,1\}} e^{-\lambda x} = 1 + 2\cosh\lambda \tag{25}$$

1 This example was taken from the course notes of Nestor Caticha.

And λ is given by

$$m = -\frac{\partial}{\partial \lambda} \log Z(\lambda) = -\frac{2 \sinh \lambda}{1 + 2 \cosh \lambda} \tag{26}$$

Writing $u = e^{-\lambda}$ and writing the hyperbolic functions as $2 \cosh \lambda = e^{\lambda} + e^{-\lambda}$ and $2 \sinh \lambda = e^{\lambda} - e^{-\lambda}$ we have

$$\frac{u - u^{-1}}{1 + u + u^{-1}} = m \Rightarrow m + (m+1)u + (m-1)u^{-1} = 0 \tag{27}$$

Multiplying both sides of the equation by u, we have the second order equation $(m-1)u^2 + mu + (m+1) = 0$, for which the (positive) solution is

$$u = \frac{-m - \sqrt{m^2 - 4(m^2 - 1)}}{2(m-1)} \tag{28}$$

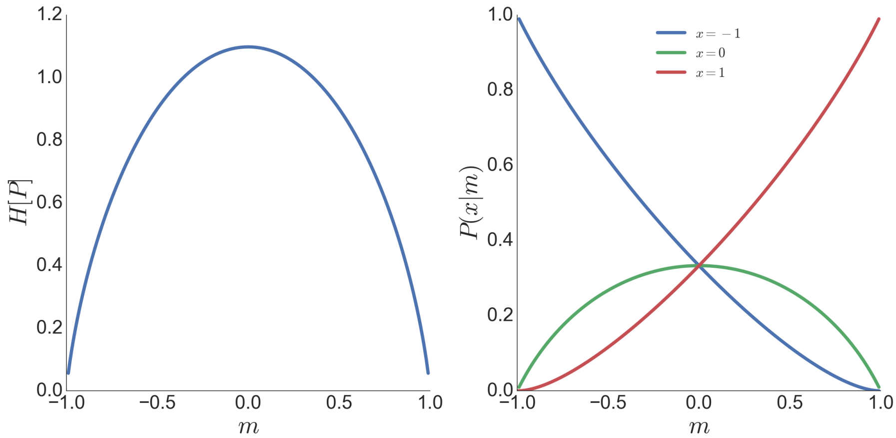

Figure 4.: **(Left)** Entropy $H[P]$ for the Gibbs distribution of a random variable x that takes three values, -1, 0 and 1 as a function of its known average $\langle x \rangle = m$. **(Right)** Probability distribution $P_G(x|m)$ as a function of m for each of the three values.

And finally we find that $\lambda = -\log u$. We plot on Figure 4 the entropy for the Gibbs distribution as a function of m and the probability $P(x|m)$ for the three values. We see that as expected entropy is maximal when the three states are equally likely, and when $m = \pm 1$ the variable is fully identified, so the entropy goes to zero.

3.1.2 *Optimization Problems*

The Gibbs distribution offers a natural way of solving maximization problems: given a system that we know has energy function $H(x)$, its ground state is simply the distribution of states x at zero temperature, or at $\beta \to \infty$. Likewise, we can find any quantity of interest $G(x)$ at the maximum by computing $\langle G(x) \rangle$ in that limit.

This is certainly not the only way one can find solutions to optimization problems, however, framing it as a Statistical Physics problem has a couple of benefits, namely, if we accept approximate solutions, we can find them very close to the optimal in a time orders of magnitude smaller. Usually this is done using general purpose Monte Carlo algorithms to sample from the Gibbs distribution at a certain β value and slowly increasing β up (i.e., decreasing the system's temperature) until convergence, a technique known as **simulated annealing**. For well behaved convex functions there are certainly more efficient optimization algorithms, but Monte Carlo techniques are very robust and allow us to add constraints and interactions in the energy function without having to adopt an alternative maximization procedure.

As an example from [60], consider a conference planner that would like to distribute N scientists in two available hotels. Scientists either like or dislike each other. We represent this by a positive interaction constant $J_{ij} = 1$ if i and j like each other and $J_{ij} = -1$ if i and j don't like each other. Scientists would then prefer to stay in hotels with their friends and not be in the same hotel with scientists they don't like. If we represent the hotel that a scientist is by $s_i = \pm 1$, the planner has to optimize for each scientist i his utility

$$u_i(\vec{s}) = \sum_{j=1, j\neq i}^{N} J_{ij} s_i s_j \tag{29}$$

And then his problem is to find the configuration $\vec{s}$ which maximizes the total utility for all scientists, i.e.

$$U(\vec{s}) = \sum_i u_i(\vec{s}) = \sum_{i,j} J_{ij} s_i s_j. \tag{30}$$

For as little as 3 scientists, the problem can be frustrated: if all $J_{ij} = -1$ or if two are positive and one is negative, there are multiple ground states. For the general case, a brute force solution would require a search over 2^N configurations, which is unfeasible even for conferences with $N = 100$ participants. However, with a Monte Carlo simulation we can find close approximations much more quickly.

This scenario, of course, is the Sherrington-Kirkpatrick model of a simple **spin glass**. Indeed, the theory of spin glasses are a very successful case in which complex systems with non regular patterns of interaction (also called **disordered systems**) can be studied, often solved analytically and exhibit very rich and interesting behavior.

These scenarios are not limited to physical systems: in portfolio optimization theory, one usually assumes that given a pool of possible financial assets, each with expected return R_i and a covariance matrix C for all assets, then given a fixed total assumed risk there is only one portfolio composition that maximizes the return and that a rational agent should adopt [31]. However, if one adds in the portfolio composition problem the requirement that any buy or sell operation carries a fee proportional to the value of the asset, then the optimization problem becomes glassy, presenting an exponential number of solutions to the number of available assets [43, 41]. These aren't suboptimal solutions due to high temperature, but optimal portfolios a perfectly rational agent would choose.

3.2 IN ECONOMICS

The utility of employing Statistical Mechanics to better model economic problems has not gone unnoticed by economists, even though it's usage is far from mainstream. It is most frequently used in interaction based models, in areas such as Game Theory, which arose precisely to deal with situations in which the decision of one agent affects the payoff of another, and rising microeconomic fields such as Social Interactions [73].

Probably the most influential work to show how a large interacting population can lead to unexpected outcomes is Schelling work on segregation [74], where two types of agents live in a city and all agents prefer to live in neighborhoods where their type is slightly more common than the other. This search for optimality will lead to complete segregation of agents, despite everyone preferring to live in mixed areas. In Schelling's words: "there is no simple correspondence of individual incentive to collective results". For a Statistical Mechanics treatment of Schelling's segregation model, we refer the reader to [29].

One of the first major proposals of connecting Statistical Mechanics with Economics came from Santa Fe Institute's seminal work *The economy as an evolving complex system* [4, 10], which influenced physicists and economists alike.

Later, in [21] and [20], Brock and Durlauf describe an interacting model where each agent i chooses between two binary actions $\omega_i = -1$ or $\omega_i = 1$ and his utility function has three terms: a private, deterministic term $u(\omega_i)$, one that interacts with other agents via a $J\omega_i\omega_j$ utility interaction and a random shock $\epsilon(\omega_i)$ whose distribution is given by the probability that $\epsilon(\omega_i = 1)$ is larger than $\epsilon(\omega_i = -1)$, given by a logistic distribution

$$P(\epsilon(1) - \epsilon(-1) > x) = \frac{1}{1 + e^{-\beta x}} \tag{31}$$

In this setup the probability of agent i choosing ω_i is given by

$$P(\omega_i) = \frac{1}{Z} e^{\beta u(\omega_i) + J\omega_i \left\langle \frac{1}{N} \sum_{j \neq i} \omega_j \right\rangle} \tag{32}$$

where Z is the normalization term. Writing $h = (u(1) - u(-1))/2$, then in equilibrium the expected value for the individual choice $m_i = m = \langle \omega_i \rangle$ is given by the implicit solution.

$$m = \tanh(\beta h + \beta J m) \tag{33}$$

This is, of course, the solution for the mean field Ising model, which is exactly what the distribution probability (32) represents. It is known that below a certain critical temperature T_c there are three solutions: $m = 0$ or $m = \pm m_0$, where m_0 can be found numerically. Above T_c, the only solution for equation (33) is $m = 0$.

What is of note for this model is: *(i)* how immediately useful framing an Economics problem into Statistical Mechanics can be. Economic problems can be modeled directly as well known systems, such as the Curie-Weiss model above. In this case, we now know that this economic interaction has three possible outcomes: when shocks to the consumer's utility are small, there are two possible rational behaviours: everyone chooses on average m_0 or $-m_0$. Otherwise, with large shocks, choice is essentially random. *(ii)* How the current "classical equilibrium" mindset requires contrived choices for parameters. The Gibbs distribution was arrived at by assuming an specific family of shocks. By treating it as an inference problem, we arrived at the Gibbs distribution and the rich phenomenology that comes with by first principles.

Aside from problems that deal directly with the effects of interaction, Statistical Mechanics has also been used in Macroeconomics. In particular, Masanao Aoki studied many of its subject matters such as policy effectiveness, price stickness, business cycles and labor market by using stochastic processess [5, 6, 8]. He also repeatedly called attention to the fact that many economic settings are non self averaging, and the usual representative agent approach to Macroeconomics fails to grasp the full picture [7].

3.2.1 *Statistical Equilibrium of Markets*

Besides the work of Masanao Aoki, Duncan Foley has also proposed a simple framework inspired by Statistical Mechanics for approaching market equilibrium which he calls **Statistical Equilibrium of Markets** [37, 38], in which agents do not wait for a central authority to give them a price reference and instead make exchanges in a decentralized way. The only information an agent knows is whether he is willing to carry out a certain trade or not. With this simple rule, we are able to construct a model market with very interesting phenomenology.

Specifically, the model is composed of M goods and N agents which can be of K different types ($K < N$). A transaction in this model is a vector $x = (x_1, \ldots, x_M)$, where an entry $x_\mu \in \mathbb{R}$ represents a good to be acquired in the trade, if $x_\mu > 0$ or traded away, if $x_\mu < 0$. Each type k of agent has an offer set A^k of transactions he is willing to make, which can be thought of single transactions or the results of a set of several trades. The nature of these sets is that they compose all transactions an agent would accept, regardless of feasibility. Therefore, one would expect that transactions of the type $x_\mu \geq 0$ for all goods $\mu = 1, \ldots, N$ are in the offer set for all groups k, because obviously no agent would shy away from free goods.

The offer set is the core ingredient of this model, because it contains all the information about what the consumer is willing to trade for, without assuming much in terms of rationality: instead of knowing the optimal product bundle given all information available in the market, the consumer merely has to know if he likes a trade or not, a much better assumption. We can also write the offer set generated by an utility function in a straightforward manner: given an initial endowment ω, the offer set A_u for an utility function $u(y)$ is the set of all trades x such that $u(\omega + x) > u(\omega)$. Companies and technologies are also included as agents, and the offer set of a company includes all the inputs it needs to operate its technology and all the resulting outputs. We assume that money is also a good, and therefore a company sets its prices defining how much money it is willing to get from the goods it offers.

A market transaction is a matrix X composed of transaction vectors for all agents, $X = (x_1, \ldots, x_N)$. Given a large enough market, we can also represent this matrix as frequencies: $h_k(x|X)$ is the frequency that transaction x is carried by agents of type k in X, where it must hold that $\sum_{x \in A_k} h_k(x|X) = 1$. If N_k is the number of agents of type k and $w_k = N_k/N$ is the proportion of type k in the population, then we define the average excess demand vector for a transaction X as

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i = \sum_{k=1}^{K} w_k \sum_{A_k} x h_k(x|X) \tag{34}$$

The main question of this model is: what type of transactions are most likely to be carried out in this market? We assume we know nothing more about the market except the offer sets, and we require for consistency that the average excess demand is zero, that is, we would like that on average our market clears and that all transactions have a counterpart. Framing it this way, we end up with the problem that we want to find the distributions $h_k(x)$ best suited for our model given that they sum (or integrate) up to one and satisfy condition (34). This is an inference problem for which the solution is

$$h_k(x) = \frac{1}{Z_k} e^{-\pi \cdot x}, \tag{35}$$

where $Z_k = \sum_{x \in A_k} e^{-\pi \cdot x}$ is the partition function for group k and π are the Lagrange multipliers for the average demand restriction, which Foley dubbed the entropic prices. Because the vector π guarantees the market clearing condition, they are considered the equilibrium prices that emerged from the market.

A practical application of the above setup is applying it to the labor market [38]. We can consider labor and money as the only two goods in the economy, which consists of two types of agents: workers, which have in their offer set either being unemployed or provide one unit of labor given that they receive at least more than a reservation wage, and firms, which demand one unit of labor and are willing to pay up to a certain amount. One of the consequences of this framing is that unemployment arises as a consequence of statistical fluctuation, showing it is present even in "efficient" markets. Furthermore, Foley shows that in this (decidedly stylized) scenario an employer subsidy to wages (i.e., giving them money to hire people) is more effective at reducing unemployment and increasing average salaries than a lump sum transfer to workers (complementing their wages with an extra).

One of the most interesting aspects of this Statistical Mechanics view on markets appears when one tries to understand what would a Walrasian equilibrium look like in this market. A configuration where all agents trade at the same prices, where agents with similar utility functions end up with the exact same consumption bundles, etc, would have zero entropy, which would require an enormous amount of information to arrive at from an initial configuration. This information reduction process is personified in the auctioneer figure, which, according to Foley, is as an impossible figure as the Maxwell demon, costlessly ordering an otherwise highly disordered system.

3.3 THE ROLE OF DYNAMICS

We end this chapter by touching briefly on the dynamical side of Statistical Physics. We have thus far described the role of inference in equilibrium configurations, where a global function for all the agents is maximized. However, there are other types of systems studied by Statistical Physics, such as stochastic dynamics, where we do not assume the system is at equilibrium, but explicitly write the probabilistic evolution rules for every particle. For example, the conference scenario described above could be replaced by one where we merely assume each scientist has a probability of changing hotels proportional to the difference in utility from being in one hotel or the other.

These methods can be equivalent to the approach described thus far if the dynamical rules converge to an equilibrium configuration, in which case they offer a different perspective for modelling economic situations, as is the case of the system we will study in Chapter 7. Even if they surely converge to equilibrium though, glassy systems may take a very long time to do so, staying stuck in configurations which are seemingly stable but have higher energy than the ground state, a phenomenon known as **metastability**. This happens when the energy landscape of the system is very rugged and displays many local minima. This rugged landscape is frequently present when dealing with asymmetric interactions, as is the case of actual glasses and of some General Equilibrium dynamics, as we have mentioned in the last chapter. In these situations even if we have a theoretical proof of convergence, it may be useless for practical purposes because it is never reached.

However, it may be the case that the system never reaches an equilibrium. Furthermore, the behaviour of a particle may not even depend on a local energy function, as is the case in the Minority Game [23]: N agents have to decide whether or not to go to a bar (or purchase a stock). Agents would like to go instead of staying home, however the bar is enjoyable only if it's not too crowded: if more than $L > N/2$ agents choose to go, agents prefer to stay home instead. However, if all agents decide simultaneously, they cannot check to see if the bar is full or not before going, and must make a decision only based on the knowledge of past attendances. It is called the Minority Game because it's advantageous to stay in the minority as the majority will always have made the worse choice.

There are no deterministic strategies which solve the Minority Game, because if one existed all agents would adopt it and therefore it would stop being optimal. However, the agents can adopt mixed strategies to, on average, have a good payoff. This is well known from Game Theory, but the time aspect of the problem allow for agents with finite memory and a portfolio of strategies to be introduced [24, 25].

Besides the minority game, some other examples are of note. Non-equilibrium dynamics are specially suitable for modelling Schumpeterian Economics, an area that mainly concerns with business cycles and "creative destruction" where innovation displace and remove old business from market, both ideas proposed by Joseph Schumpeter. Examples of Statistical Physics applied to schumpeterian dynamics can be found in [86], where the authors show in a stylized economy that depending on the innovation rate, the economy will be either in a state of constant change akin to pure noise for high innovation rates, in a frozen configuration for low innovation rates or, more interestingly, in metastable states for intermediate innovation rates, staying still for long periods of time and then completely reshuffling.

In this thesis, we focus on systems at equilibrium, even in Chapter 7 where we begin by describing the dynamical rules for the economy and then study its steady state in terms of dynamical quantities. However, it's worth noting that this is not the only class of systems that can be studied with Statistical Physics, and out of equilibrium problems can also be useful in Economics. As time constrains our choices, following such interesting direction is left as an exercise for the reader.

4

THE RANDOM LINEAR ECONOMY MODEL

In this chapter we will present in detail and discuss the Random Linear Economy model [30] developed by Andrea De Martino, Matteo Marsili and Isaac Pérez Castillo which will be the basis for some of the applications discussed in the second part of this thesis.

There are some reasons why we chose to work with this model in particular: first, it presents a General Equilibrium Model which has few ingredients but displays a rich behavior, including phase transitions which depend on the number of firms in the market. Secondly, it is analytically solvable using Statistical Mechanics techniques, such as using the replica method to calculate the partition function. Therefore, it was ideal for trying new venues of exploration without the difficulty imposed in trying to prove general results.

4.1 THE MODEL INGREDIENTS

A model economy is composed by two types of actors: consumers and firms. We assume N firms and one single representative consumer with utility function $U(x)$ and initial endowment x_0. As we mentioned on Chapter 2, this is a common approximation when doing equilibria calculations in Economics due to its simplicity: if we have J consumers with independent utility functions U_j (i.e., U_j never depends on x_k, $k \neq j$) and initial endowments ω_j, then either we do not allow wealth transfers of ω_j and the optimization problem becomes very complicated, or we allow the central authority to carry out wealth transfers prior to allocation, and then the the demands generated by the consumers in this scenario is equivalent to that of a single representative consumer with utility function $U_R = \sum_{j=1}^{J} U_j$ and wealth $\omega_R = \sum_{j=1}^{J} \omega_j$.

The representative consumer assumption receives considerable criticism [52], chiefly because disregarding interaction among agents (via the utility of one depending on the decisions of the others) washes out the possibility of interactions and the wide range of important and interesting phenomena that in the statistical physics community we know to be generated precisely by these interactions [18], whereas the representative agent is a mean field approximation for consumers.

The representative consumer is used in this model precisely because it generates an energy function which is convex and therefore has a well defined, unique minimum. Besides, the resulting partition function can be calculated analytically in the zero temperature limit. The interactions are at the supply side, which we will describe next, and are enough to generate rich behavior even with the representative consumer approximation.

The consumer and N firms will trade M goods, with a density parameter given by $n = N/M$. As with General Equilibrium settings, we assume the consumer has an initial wealth $x_0 = (x_0^1, \ldots, x_0^M)$, $x_0^\mu \geq 0$, and wishes to improve its welfare in the market by using his endowment x_0 to purchase a consumption bundle x according to a separable utility function $U(x) = \sum_{\mu=1}^{M} u(x^\mu)$. His initial endowment, however, is random, with each x_0^μ drawn independently from an exponential distribution with unitary scale, i.e.,

$$P(x_0^\mu) = e^{-x_0^\mu} \tag{36}$$

This particular form of distribution for the initial endowments is chosen so the consumer has a large imbalance of initial goods for which he needs the market to improve on. As before, the aim of the consumer in this economy is to solve the utility maximization problem

$$x^* = \arg\max_x U(x) \text{ s. t. } p \cdot x \leq p \cdot x_0 \tag{37}$$

We will treat the particular case of the consumer's utility being a separable function $u(x_\mu) = \log x_\mu$, although any concave function would exhibit qualitatively similar behavior. The logarithm is a common choice for the consumer's utility function because it satisfies some of the properties desired for the consumer behavior in Economics. The often called the Inada conditions are: first, the consumer is **loss averse**, which means that he will always prefer a guaranteed amount a of any good to a lottery in which he can win $a + \delta$ with probability 0.5 and $a - \delta$ with probability 0.5, for any $\delta > 0$. He is loss averse because the disutility losing δ is larger than the utility of gaining δ. We don't have any stochastic payoff such as lotteries in the model, but the principle holds for two goods: if he has $\bar{x} - \delta$ of good μ and $\bar{x} + \delta$ of good ν, he will try to find a company that trades this excess of good ν so he can average both goods and may even do so at a loss (i.e., he ends up with $\bar{x} - \varepsilon$ for both goods, for some $\varepsilon < \delta$). Furthermore, with separable utility as chosen, there are no complementary or substitute goods, i.e., goods for which the consumer prefers to have more (or less) of one if he has another. Finally, because $u(0) = -\infty$, the consumer will always try to obtain a little bit of every good, even if at a great cost, because nothing is worse than having none of a particular good.

Each firm, on the other hand, has an M-dimensional random technology $\xi_i = (\xi_i^1, \ldots, \xi_i^M)$, where $\xi_i^\mu < 0$ represents an input and $\xi_i^\mu > 0$ represents an output.

The production set of each firm is the space of all vectors which are proportional to ξ_i, that is, $\Xi_i = s\xi_i$, $s \geq 0$. Each firm i only has one technology and its only decision is the scale s_i at which it operates this technology. Once chosen the scale s_i, a company will consume $s_i\xi_i^-$ goods and produce $s_i\xi_i^+$ goods, where $\xi_i^\pm$ are the positive and negative entries of the ξ_i vector.

The elements ξ_i^μ are initially independently drawn from a normal distribution with zero mean and Δ/M variance, where $\Delta > 0$ but are then normalized so that the sum over all the goods for a company is fixed at a negative value and all technologies are a little inefficient. We must have then:

$$P(\xi_i^\mu) = \mathcal{N}(\xi_i^\mu | 0, \Delta M^{-1}) \tag{38}$$

$$\sum_{\mu=1}^{M} \xi_i^\mu = -\epsilon \tag{39}$$

We normalize the technologies to be inefficient so that we don't have a combination of firms producing infinite goods, i.e., firm i and j can produce infinite amounts of certain goods by each feeding its output to be used as the other's input. This would make the economy violate the second law of Thermodynamics, which is surely an undesirable property for any practical model of a closed economy.

The objective of each company in the market is the same as before: each firm i tries independently to choose it's production scale s_i as to maximize it's profits:

$$s_i^* = \arg\max_{s_i > 0} p \cdot (s_i \xi_i) \tag{40}$$

Other underlying assumptions of General Equilibrium Theory are valid here: we assume a complete market, where each agent knows the offer and demand of all other agents, there is no transaction costs and a good is uniquely defined. Also, agents are price-takers, which means that they have no power over the prices and must accept them as given.

We also treat the economy as closed and therefore it must satisfy the market clearing condition. Because we have just one consumption bundle, then the N dimensional production scale vector s has to be such that

$$x = x_0 + \sum_{i=1}^{N} s_i \xi_i \tag{41}$$

ie, all the inputs the firms use have to come from the consumer's initial endowment. If we sum over all the goods we have a strict condition on the final conversions from x_0 to x:

$$\sum_{\mu=1}^{M} x_\mu - x_0^\mu = -\epsilon \sum_{i=1}^{N} s_i \tag{42}$$

Because market clearing hold and agents are price takers, we can also derive the strong restriction on profits discussed before. If we multiply both sides of the equation (41) by p, we get

$$p \cdot (x - x_0) = \sum_i s_i p \cdot \xi_i, \tag{43}$$

The left side of the above equation has to be less or equal to zero as a consequence of the budget condition. But the right hand side has to be always greater or equal to zero, because this term represents the sum of the individual firms' profits and if a firm is losing money they can always choose to set $s_i = 0$ and leave the market. Therefore, we must have that both sides are equal to zero, and the consequence is that the agent completely spends all his available budget (i.e., $p \cdot x = p \cdot x_0$, he has not "leftover" cash after choosing x, this is Walras's law for this model) and that the firms either have zero profit ($p \cdot \xi_i = 0$) or leave the market ($s_i = 0$).

One of the important implications of equation (43) for the Random Linear Economy model is that we may have no more than M firms active at any given realization of equilibrium. If the right hand side of equation (43) has to be zero, then for every firm either $s_i = 0$ or $p \cdot \xi_i = 0$. If ϕ is the fraction of firms active in the market, that is

$$\phi = \frac{\sum_{i=1}^{N} \mathbb{I}(s_i > 0)}{N}, \tag{44}$$

then all of them have $p \cdot \xi_i = 0$. Because the price is the same for all of them, we have ϕN equations of this type, and M unknowns. For this system to have a non-trivial solution (ie, $p_\mu > 0$ for all μ), it must be that $\phi N \leq M$, which implies that

$$\phi \leq \frac{1}{n} \tag{45}$$

Having a single representative consumer (or many consumers but with wealth transfers) has two important consequences: first, the price vector is entirely determined by the consumer's demand. This is a consequence of the first order condition for the maximization problem. By taking the derivative of equation (37) with the proper Lagrange multiplier we get

$$\frac{\partial U(x)}{\partial x_\mu} - \lambda p_\mu = 0 \Rightarrow p_\mu = \frac{1}{\lambda x_\mu} \tag{46}$$

Furthermore, the market clearing condition binds the optimization problem of the consumer and the firms. If we substitute equation (41) in the consumer's utility, we get:

$$s^* = \underset{s:s_i \geq 0}{\arg\max}\, U(x_0 + \sum_{i=1}^{N} s_i \xi_i) \tag{47}$$

We can easily check that the zero profit condition is preserved with this solution. If s_i is in s^*, the solution for the consumer's maximization problem, then either $s_i = 0$ or $s_i > 0$. If $s_i = 0$, the condition is satisfied. Otherwise, if $s_i > 0$, it means that the constraint $s_i \geq 0$ was not enforced and the derivative at s_i must be zero. We then have

$$0 = \frac{\partial U}{\partial s_i} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s_i} = p \cdot \xi_i \tag{48}$$

Our problem is now considerably reduced: to find the equilibria in this model economy all we have to do is to solve the maximization problem in equation (47).

4.2 THE ROLE OF STATISTICAL MECHANICS

If we were to employ standard convex optimization techniques to solve (47), we would be able to find the solution for a specific realization of x_0 and ξ given a fixed N and M. But if we were to calculate quantities of interest such as consumer utility, average good consumption, average good price, price deviation among goods, number of active firms, etc, these would all be random variables which depend on the realization of endowments and technologies.

This is, of course, a well known setup in Statistical Mechanics. We solve this by treating the case where the system size is very large, so that these average quantities converge to a single value. This isn't always the case, but holds for the so called *self-averaging* systems. In these systems, these average quantities for large systems converge to an average over the realizations for smaller systems.

The general approach to finding the equilibrium properties of a physical system is is to calculate the partition function for a specific realization of ξ, x_0:

$$Z(\beta|\xi, x_0) = \int dx e^{\beta U(x|x_0,\xi)}, \tag{49}$$

where β is the inverse value of the temperature. From this, we can calculate the average value of the utility function by taking the derivative of $\log Z$:

$$\langle U \rangle (\beta|\xi, x_0) = \int_0^\infty dx \frac{e^{\beta U(x|\xi,x_0)}}{Z(\beta|\xi, x_0}U(x|\xi, x_0) = \frac{\partial}{\partial \beta}\log Z(\beta|\xi, x_0) \tag{50}$$

The maximum value for the utility $U(x)$ is equivalent to the average value on the ground state[1], ie:

$$\max_x U(x|\xi, x_0) = \lim_{\beta\to\infty}\langle U \rangle(\beta|\xi, x_0) \tag{51}$$

However, we are still calculating the maximum as a function of the samples x_0 and ξ. In order to get the average behavior, which holds for a large system, we must average

1 This is true because $U(x)$ is convex and therefore has only one maximum.

the utility over the disorder. Writing it all explicitly, we finally get the solution to equation (47):

$$\max_x U(x) = \int d\xi dx_0 P(\xi) P(x_0) \lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \log \int dx e^{\beta U(x|x_0,\xi)} \tag{52}$$

The analytical calculation of the expression above is considerably involved and makes use of a method commonly known as **replica method** in the Statistical Physics community. We leave the lengthy calculation for Appendix A. The solution of this calculation is given by

$$\lim_{N\to\infty} \frac{1}{N} \max_x U(x) = \max_\theta h(\Omega, \kappa, p, \sigma, \chi, \hat{\chi}), \tag{53}$$

where $\theta = (\Omega, \kappa, \sigma, \chi, \hat{\chi})$ are order parameters that appear during the calculation and h is given by:

$$\begin{aligned} h(\Omega, \kappa, p, \sigma, \chi, \hat{\chi}) = & \left\langle \max_s \left[-\frac{\hat{\chi}}{2} s^2 + (t\sigma - \epsilon p) s \right] \right\rangle_t + \\ & + \frac{1}{2} \left(\Omega\hat{\chi} + \frac{\kappa p}{n} \right) - \frac{1}{2n\Delta} \chi\sigma^2 - \frac{1}{2n} \chi p^2 + \\ & + \frac{1}{n} \left\langle \max_x \left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega} t)^2}{2\chi} \right] \right\rangle_{t, x_0}, \end{aligned} \tag{54}$$

where t is a normal random variable with zero mean and unitary variance.

The zero temperature limit makes the terms involving s and x revert to the two original optimization problems of the firms and consumer, respectively. However, the extra terms couple those two maximization issues. Because we are in the thermodynamic limit where $M, N \to \infty$ with $n = N/M$ fixed, the maximization solution is a distribution over the N productions scales and M good quantities. The solution for s^* is obtained by solving $\frac{\partial}{\partial s} \left[-\frac{\hat{\chi}}{2} s^2 + (t\sigma - \epsilon p) s \right] = 0$. Keeping in mind the constraint that $s \geq 0$, we have

$$s^*(t) = \begin{cases} \frac{(t\sigma - \epsilon p)}{\hat{\chi}}, & \text{if } t \geq \frac{\epsilon p}{\sigma} \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

The probability distribution of s has a mass on $s = 0$, which has probability equal to the case where $t < \epsilon p/\sigma$, and a Gaussian probability distribution for $t \geq \epsilon p/\sigma$. We can write it in closed form by integrating over t:

$$P(s) = \int_{-\infty}^{\infty} dt \frac{1}{2\pi} e^{-\frac{t^2}{2}} \delta(s - s^*) \tag{56}$$

We use the identity that for the Dirac delta function $\delta(x-a) = |f'(a)|\delta(f(x))$ given any function $f(x)$ with a simple root in a. Choosing $f(s) = \frac{s\hat{\chi}+\epsilon p}{\sigma} - t$ we have

$$P(s) = (1-\phi)\delta(s) + \Theta(s)\frac{\hat{\chi}}{\sqrt{2\pi}\sigma}e^{-\frac{(\hat{\chi}s+\epsilon p)^2}{2\sigma^2}}, \tag{57}$$

where $\phi = \frac{1}{2}\operatorname{erfc}\left(\frac{\epsilon p}{\sqrt{2}\sigma}\right)$ is the fraction of active firms ($s > 0$) in the economy.

In the same manner, the solution of the consumer's maximization problem on x is given by the implicit equation obtained when solving $\frac{\partial}{\partial x}\left[U(x) - \frac{(x-x_0+\kappa+\sqrt{n\Delta\Omega}t)^2}{2\chi}\right] = 0$:

$$x^* = x : U'(x) = \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)}{\chi} \tag{58}$$

Using $f(x) = \frac{x-x_0-\chi u'(x)+\kappa}{\sqrt{n\Delta\Omega}} + t$ for the same replacement as in (57) we have

$$P(x) = \frac{1-\chi U''(x)}{\sqrt{2\pi n\Delta\Omega}}e^{-\frac{(x-x_0-\chi U'(x)+\kappa)^2}{2n\Delta\Omega}} \tag{59}$$

To find the maximum value of h given the order parameters $\Omega, \kappa, \sigma, \chi, \hat{\chi}$, we must again take the derivative with respect to each of these parameters. These give us the so called **saddle point equations** which, despite being used to find the maximum of h and therefore the expected value of the utility in a large system, also give us insights into the behavior of quantities of interest such as the good prices. The saddle point equations for equation (54) are (see Appendix A for details):

$$p = \left\langle U'(x^*)\right\rangle_{t,x_0} \tag{60}$$

$$\Omega = \left\langle (s^*)^2\right\rangle_t \tag{61}$$

$$\sigma = \sqrt{\Delta\left(\langle U'(x^*)^2\rangle_{t,x_0} - \langle U'(x^*)\rangle^2_{t,x_0}\right)} \tag{62}$$

$$\chi = \frac{n\Delta}{\sigma}\langle ts^*\rangle_t \tag{63}$$

$$\hat{\chi} = \sqrt{\frac{\Delta}{n\Omega}}\left\langle U'(x^*)t\right\rangle_{t,x_0} \tag{64}$$

$$\kappa = p\chi + n\epsilon\langle s^*\rangle_t \tag{65}$$

The equations above allow for some sanity checking: if we take the expected value of equation (58) with respect to t and x_0 we have

$$\langle \chi U'(x^*)\rangle_{t,x_0} = \langle x\rangle_{t,x_0} - \langle x_0\rangle_{x_0} + \kappa + \langle\sqrt{n\Delta\Omega}t\rangle_{t,x_0} \tag{66}$$

But the last term is the average of t which is zero, and if we replace $\kappa = p\chi + n\epsilon \langle s^* \rangle_t$ and $\langle \chi U'(x^*) \rangle_{t,x_0}$ by χp we have

$$\langle x \rangle_{t,x_0} = \langle x_0 \rangle_{x_0} - n\epsilon \langle s^* \rangle_t \,, \tag{67}$$

which is the thermodynamic limit version of equation 42.

4.3 REGIME CHANGE AT $n = 2$

One of the most interesting aspects of this seemingly simple model it that it exhibits two very different economic behaviors, one when $n < 2$ and the other when $n > 2$: when $n < 2$, the economy is competitive, half of the firms in the market are active and operating but are not efficient enough that the consumer is completely satisfied, so new firms entering improve the consumer's utility and increase the economy's GDP. When $n > 2$, the market is saturated, monopolistic (most firms are inactive) and new firms entering have a negligible impact over the economy.

This behavior has a simple geometric explanation which can be shown without the results obtained above. If we write the initial endowments as $x_0 = \bar{x}_0 + \delta x_0$, such that $\bar{x}_0$ is the average (which has an expected value of one) and $\sum_\mu \delta x_0^\mu = 0$, then the consumer picks[2] s as to minimize the dispersion vector δx_0 as much as possible. Therefore, firms that have $\xi_i \cdot \delta x_0 < 0$ will have a positive production scale, whereas firms that have $\xi_i \cdot \delta x_0 > 0$ will increase the dispersion of the initial goods, reducing the consumer's utility and therefore won't get used, having $s_i = 0$.

At the $\epsilon \to 0$ limit, each component ξ_i^μ is drawn from a normal distribution with zero mean, so the plane $\xi_i \delta x_0 = 0$ divides the set of firms in half, because each firm has equal probability of having its independent (when $\epsilon = 0$) technologies drawn in such a way that $\xi_i \cdot \delta x_0 < 0$, no matter what vector is δx_0 as long as it's not zero. This means at most $N/2$ are active at any given moment. However, we know from equation (45) that there can be at most M firms active in the economy. Therefore, in the limit of $\epsilon \to 0$, we must have $\phi = 1/2$ up until $N/2 = M$, which implies a transition at $n = 2$.

To further characterize this transition, we show the results of numerical simulations for the fraction of active firms ϕ as a function of n, for $\epsilon = 0.01$ in Figure 5. We see that, indeed, there is a collapse in the fraction of active firms when $n = 2$, as predicted, when the number of active firms has reached its maximum and new firms either are immediately out of the market or replace old ones entirely, whereas when $n < 2$ new firms enter the market and with probability half operate among the incumbent ones. This effect can be seen even further on Figure 6, where we plot the average scale of production $\langle s \rangle$ for active firms, the spread reduction $\xi \cdot \delta x_0$ and $\langle x \rangle$ as a function of n. The interesting result is that when $n < 2$, each new technology actually increases

2 Remember that market clearing couples the consumer's maximization problem with the firms'

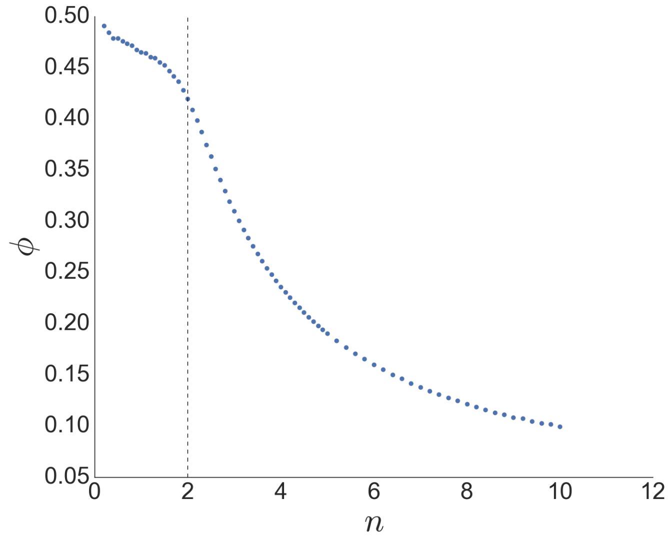

Figure 5.: Fraction of active firms ϕ as a function of n, for $\epsilon = 0.005$ and $M = 100$. At around $n = 2$, the number of active firms collapses.

the average scale of production for all other firms, because it offers a new conversion possibility all firms can take advantage of. In the other hand, in the monopolistic regime, each new firm decreases the average scale of production because at the threshold, new firms only offer increasingly more specialized conversion rates that suit best the consumer's demand. However, the reduction in spread $\xi \cdot \delta x_0$ decreases roughly at the same rate, indicating that beyond $n = 2$, new firms are simply replacing older, less efficient ones.

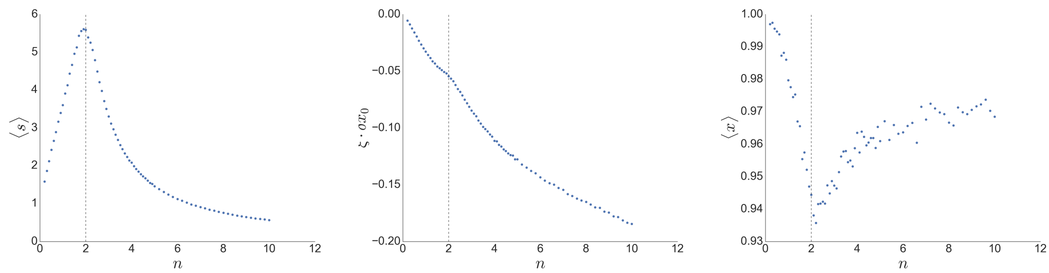

Figure 6.: **(Left)** Average scale of production s **(Middle)** Reduction in spread $\xi \cdot \delta x_0$ **(Right)** Average final bundle $\langle x \rangle$, all as a function of the technological density $n = \frac{N}{M}$. The plots show the regime change at $n = 2$: when $n < 2$, in the competitive regime, each firm increases their production when a new technology enters the market, whereas the consumer is willing to sacrifice a bit more of his total amount of goods to increase his utility. At $n > 2$, the monopolistic regime, the consumer doesn't sacrifice any more of his total amount of goods, and new firms are more efficient and decrease the average scale of all others. These results are from numerical simulations averaged over 1500 replicas, with $M = 100$ and $\epsilon = 0.005$.

The graph for $\langle x \rangle$ in Figure 6 shows that the consumer too has a different behavior in the two regimes. In the competitive setting, firms are suboptimal and the consumer is able to maximize his utility only by sacrificing an increasingly larger amount of his initial endowment. Beyond $n = 2$ the technologies are good enough that the consumer loses less when choosing an optimal bundle. In terms of utility, it increases rapidly with n up until $n = 2$ and it stalls a bit afterwards, showing the saturation of the economy, as shown in figure 7.

We can also define the gross product (GDP) for the model as the total value of goods produced, that is, the sum of $(x_\mu - x_0^\mu) p_\mu$ for all goods μ that are produced, i.e., $x_\mu > x_0^\mu$. Since market clearing condition (43) makes the value of goods produced equal to the value of goods used as input, we calculate the GDP Y by averaging over the absolute value of all trades:

$$Y = \frac{\sum_{\mu=1}^{M} |x_\mu - x_0^\mu| p_\mu}{2 \sum_{\mu=1}^{M} p_\mu}, \tag{68}$$

where the denominator also includes a normalization for the prices. The numerical results for the GDP are shown in figure 7, and, like the utility, it increases linearly with n up until $n = 2$, where it quickly stalls, another indicator that the economy enters a mature regime at this point. The utility and the GDP as a function of n will be specially relevant in Chapter 5, when we will relax the zero temperature constraint.

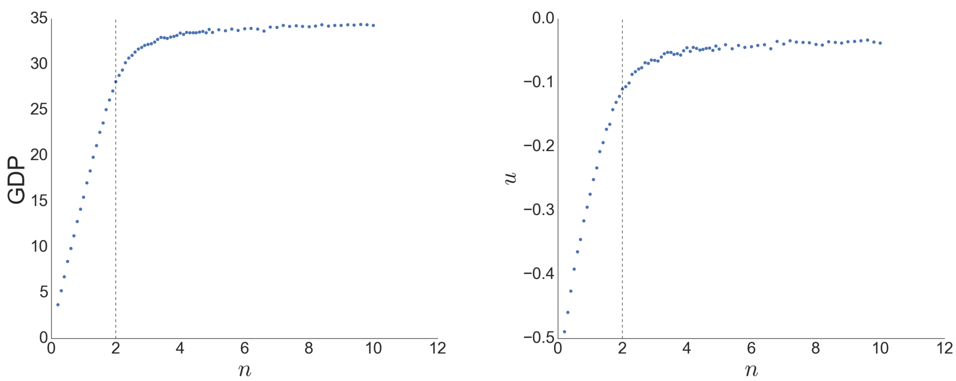

Figure 7.: **(Left)** Average GDP as defined by equation (68) **(Right)** Average consumer utility per good, both as a function of the technological density n. Both increase rapidly as a function of new firms in the competitive regime $n < 2$ and saturate in the monopolistic regime $n > 2$.

The Random Linear Economies model is particularly suitable for further analysis because it's a General Equilibrium setting with few ingredients, but the introduction of stochastic elements offers a nontrivial regime change which is not observed in similar "simple" economic models in the literature and are not solvable through standard economic maximization methods, as discussed in Chapter 3.

For these reasons we used it as a basis for two of the applications presented in this thesis: in Chapter 5 we will discuss how relaxing the $\beta \to \infty$ limit to arbitrary β is a principled way of modeling an irrational consumer. On Chapter 6 we will build Input-Output matrices for the model and compare to real world data.

5

INEFFICIENT CONSUMER IN A GENERAL EQUILIBRIUM SETTING

In this chapter, we introduce an inefficient consumer within a General Equilibrium scenario. As we have described in Chapter 2, standard Microeconomic Theory supposes that an agent faced with a choice of goods or services to purchase will maximize an expected utility function given his constraints [57]. It also predicts an equilibrium state resulting from these agents simultaneously and independently trying to optimize their own choices.

Over the last decade these assumptions have been under intense scrutiny [17, 53]. Among other factors, the 2007-2008 mortgage crisis, for which the predictions made by mainstream theories were ineffective in preventing, and the growing empirical body of evidence being gathered in the field of Behavioral Economics have shown that perfect rationality may be a very poor proxy for the observed behavior of economic agents.

In fact, it has been shown that assuming zero intelligence agents also produces realistic results [45, 84] due to general statistical properties that may emerge in interacting systems. As argued by the authors in [84], treating economic agents as having full rationality and perfect knowledge is clearly a significant assumption, so why not model the other end of the spectrum? It has been shown in [1] that, for an economy in which trades occur in a random fashion, the wealth distribution at equilibrium is a Gibbs distribution. This is very close to empirical observations for several countries, at least in the bulk of the distribution. By adding a rate of return on capital, power law distributions are obtained [19], what is again consistent with empirical data for top earners.

As we have argued in Chapter 3, Statistical Mechanics provides a robust framework, based on maximum entropy principles, for identifying and calculating macroscopic quantities relevant for the description of interacting systems. Within this framework, the temperature of a system represents how large observed deviations from the ground state are likely to be. A zero temperature system is always at the configuration which minimizes its energy function, while a system at infinite temperature will be at any configuration allowed by its constraints with equal probability.

These considerations suggest a way to model consumers that may turn to be more realistic: the degree of rationality may be represented as a parameter within a Statistical Mechanics model with energy given by the negative utility. As we have discussed in Chapter 3, a similar approach has already appeared in the contexts of Game Theory [21, 20, 13] and General Equilibrium [37, 38].

In this chapter, we present the equilibrium state for the Random Linear Economy model described in Chapter 4 at several different values of the temperature parameter β and discuss the consequences of non zero temperature to some relevant macroeconomic quantities, namely, the consumer's average utility and the GDP. We will show that when we remove the $\beta \to \infty$ limit, the uniqueness in firm production at equilibrium vanishes and it's challenging to define prices. We offer an alternative method for arriving at finite temperature Gibbs distribution based on [56] which preserves the price vector. Finally, we show that the utility behavior at positive temperatures is consistent with findings from Behavioral Economics, specifically, the fact that more options sometimes lead to worse decisions by costumers [47, 76].

On a technical note, although deviations from maximum utility are here named, in accordance with Economics literature, "suboptimal" or "inefficient", it is known in statistical inference and reinforcement learning that a "softmax" decision rule can actually be beneficial in exploit-explore situations where the agent has incomplete knowledge about his available options and must, at each moment, decide whether to exploit known but possibly suboptimal actions or to explore risky but potentially more rewarding ones [28]. In these situations, it is shown that the optimal strategy for choosing options is to adopt a "softmax" decision rule with a certain temperature parameter. Maybe the irrational consumer we describe in this chapter isn't so irrational after all.

5.1 THE IRRATIONAL CONSUMER

We now turn to the question of how we can treat the consumer as an agent that makes suboptimal choices while making as little extra assumptions as possible for his behavior. We have argued in Chapter 3 that this type of question is answered in Statistical Physics as the inference problem of finding the probability distribution for x that has maximum entropy given the constraints that $P(x)$ must be a probability distribution and $U(x)$ must have a well defined average $\bar{U}$ [49]. The solution for this entropy maximization problem is the Gibbs distribution:

$$P(x) = \frac{1}{Z(\beta)} e^{\beta U(x)}, \tag{69}$$

where $Z = \int_0^\infty dx\, e^{\beta U(x)}$ is the normalization term and β is the Lagrange multiplier that solves

$$\frac{1}{Z(\beta)} \int dx U(x) e^{\beta U(x)} = \bar{U} \tag{70}$$

As it is typical in Statistical Mechanics we can instead treat it as a parameter that controls how much we allow $\bar{U}(\beta)$ to deviate from its maximum value, found when $\beta \to \infty$. It's clear that in this limit the distribution collapses to a Dirac delta and the inference reduces to the classical consumer problem. The parameter β provides a structured way to treat irrationality or uncertainty about the consumer decision. Moreover, we are certain that we are not making any additional assumptions about his behavior.

Because market clearing still holds, we can plug equation (41) directly into the Gibbs measure. The probability of the consumer choosing a bundle x is then transformed to the probability of having the firms set a production vector s, and is given by:

$$P(s|\xi, x_0) = \frac{1}{Z} e^{\beta U(x_0 + \sum_i s_i \xi_i)} \Theta(x_0 + \sum_i s_i \xi_i) \tag{71}$$

Although this distribution is algebraically untractable except in the $\beta \to \infty$ limit, we can find the expected values of macroeconomic variables for the Random Linear Economy model by sampling this distribution using a Metropolis-Hastings Monte Carlo dynamic and averaging over the disorder (ξ, x_0). We first find the average fraction of active firms $\langle \phi \rangle_{\xi, x_0}$ at equilibrium as a function of β. The results are shown in Figure 8. We see that the inefficiency parameter β has changes the "two regimes" scenario of Figure 7, which is reproduced when $\beta > 10^5$. For lower values, the monopolistic regime vanishes, and after a certain point the fraction of active firms actually increases with n.

More interesting, however, is the behavior of other macroeconomic variables of the model. The utility at the $\beta \to \infty$ limit is always maximized by the consumer given his options, and therefore has to be an increasing function in n: a larger set of firms imply a wider set of choices, which the consumer can either take if it increases his utility or stay at his current bundle if it doesn't. When we allow for suboptimal choice this is not always the case: the consumer can now change his final bundle due to "entropic" reasons, and it's not guaranteed that his expected utility will increase in this case. We compute the average utility per good $\langle u \rangle$ for the consumer as a function of n:

$$\langle u \rangle = \frac{1}{M} \langle U \rangle_{\xi, x_0} = \frac{1}{M} \int dx \int d\xi dx_0 U(x) \frac{1}{Z} e^{\beta U(x)} \Theta(x) \tag{72}$$

As it can be seen in Figure 9, for low temperatures $\langle u \rangle$ behaves as expected. However, when we increase inefficiency, the average utility of the consumer starts decreasing instead of increasing with the number of firms available per good. This is quite interesting because it corroborates empirical evidence from Behavioral Economics suggesting that choice can be excessive [47, 76]: as the number of options available to consumers increases, without perfect knowledge and full rationally, the probability of them making mistakes also increases.

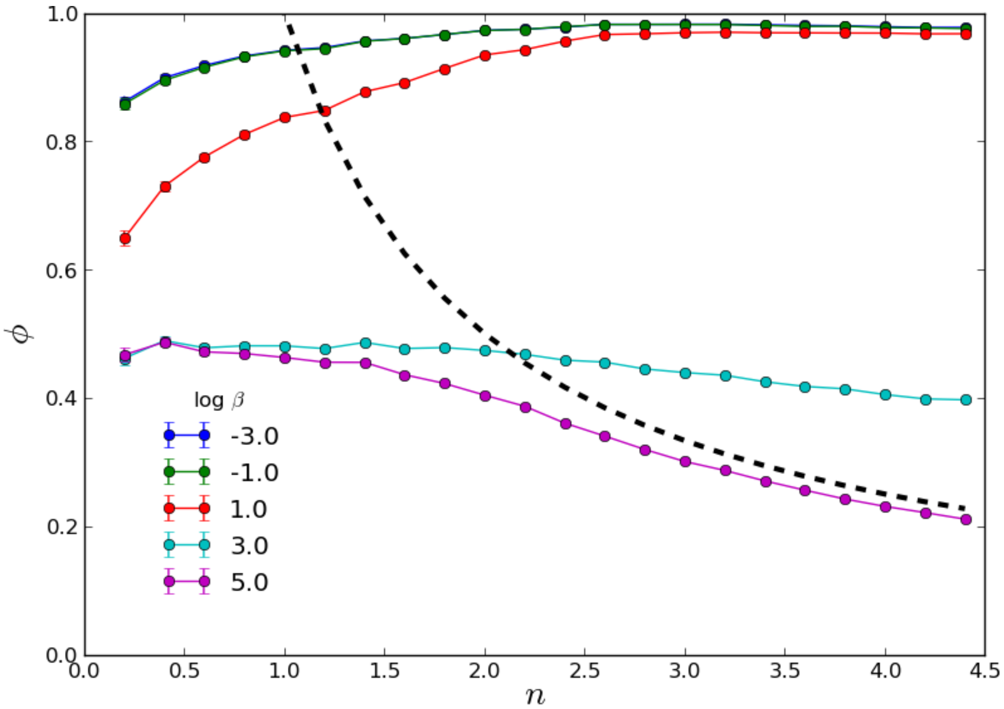

Figure 8.: Fraction of active firms ϕ as a function of the technological density n, for different values of β. The limit of active firms $\phi = 1/n$ is shown as the dashed line. When β is high, around 10^5, the fraction of active firms behaves as the original model shown in Chapter 4 and is always below the $1/n$ threshold. However, when β decreases, the two regimes vanish and after a certain point the number of active firms increases with n. For all β below a certain point, the limit $\phi < 1/n$ is violated and the economy no long supports a global price vector.

We would like also to calculate the GDP of this economy, but we have not defined the price vectors yet in this framework. For now we use the absolute amount of goods traded as a proxy for economic activity, i.e., the economic activity Λ is given by

$$\Lambda = \frac{1}{M} \sum_{\mu=1}^{M} \left| x_\mu - x_0^\mu \right| \tag{73}$$

We see in the simulation results on the right panel of Figure 9 that the economic activity always increases as as function of the technological density n, but the rate of increase in Λ also increases with the inefficiency, as opposed to the utility. This is a very peculiar behavior: suboptimal choice makes the amount of goods traded in the economy increase as a whole, but the consumer's utility falls because more exchanges are made inefficiently.

These results, however, have not come without drawbacks. Whereas originally the prices were uniquely defined by the first order condition of the consumer's optimization problem, in this new framework there is no such restraint: given a probability

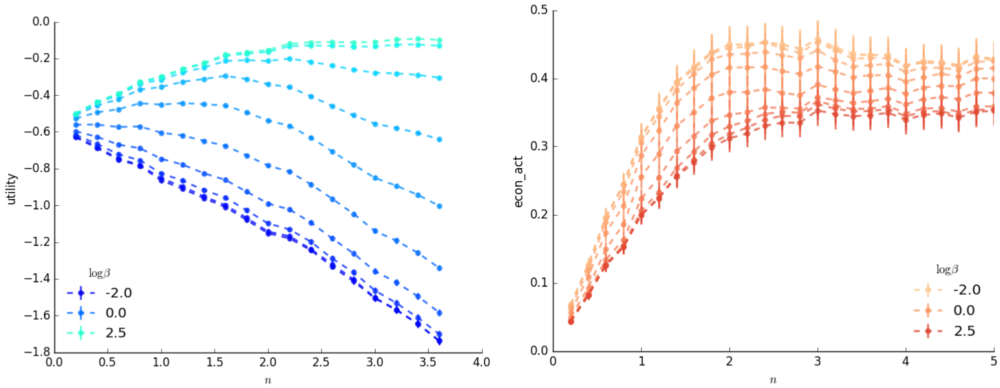

Figure 9.: **(Left)** Average consumer's utility per good as a function of the technological density n, for different values of β. For high values of β, an increase in the number of options available always increases the consumer's utility, just like the rational case presented on Chapter 4. However, after a certain degree of inefficiency the utility decreases as one increases the number of choices. **(Right)** Average economic activity per good for the economy as a function of n. Unlike the utility, the economic activity always increases with the number of choices, increasing even further when the consumer is inefficient.

distribution $P(x|\xi, x_0)$ finding the resulting price vector is an ill defined problem. In fact, any vector p that satisfies

$$p \cdot \xi_i = 0, \text{ for all } i \text{ if } s_i > 0 \tag{74}$$

is a candidate for global price. However, we have N_a equations of this type, one for each active firm, and M unknown variables p_μ. It's straightforward to conclude that there is a threshold $N_a = M$ for which the above system of equations has a solution. If we assume that all firm technologies ξ_i are linearly independent, then for any N_a above M, there's no price vector p that simultaneously satisfies the zero profit constraint for all firms. In terms of ϕ, this means that for $\phi > 1/n$ there is no global price vector p that supports the economic activity observed. The curve $\phi = 1/n$ is plotted as the dashed black curve in Figure 8 and we see that only when the consumer is very close to perfect rationality the market stays in this region for all values of n. For higher inefficiency, there is always a threshold where no global prices exist. This does not mean that there are no market transactions taking place, only that prices must be defined locally, and we have no way of defining this more precisely without additional assumptions.

5.2 UNOBSERVED UTILITY

We can solve this indefinition of prices by altering the setup so that the equilibrium is still obtained from a maximization problem of the consumer. In [56] Marsili et al show that a complex system that maximizes an objective function $U(\vec{s})$ that can be decomposed as

$$U(s) = u(s_v) + v(s_h|s_v), \tag{75}$$

where $u(s_v)$ is the known component and $v(s_h|s_v)$ is the hidden component which we assume is unobservable. If $v(s_h|s_v)$ is not heavy tailed, then the observed part s_v is Gibbs distributed:

$$P(s_v) = \frac{1}{Z} e^{\beta u(s_v)}, \tag{76}$$

where β is a function of the number of known and unknown variables.

We can use this framework to arrive at the Gibbs distribution in the Random Economies through a stochastic perturbation in the consumer's utility function that we will treat as unobserved. We call U_t the "true" utility function and write it as

$$U_t(x) = U(x) + h \cdot x, \tag{77}$$

where h is a M dimensional vector where all entries are independent exponential random variables with scale λ. This term represents unknown quantities about the consumer's utility which we do not have access to, such as shocks in preference which change daily. We point out that the true utility function U_t is still a concave function and therefore the equilibrium is still unique and well behaved.

As shown in [56], the resulting probability distribution for x when the consumer is maximizing the utility distribution $U_t(x)$ is a Gibbs distribution for $U(x)$. This allows us to treat it as a framework where the consumer does not choose the "optimal" for the utility $U(x)$, but we still have a well defined price vector which is the result of the first order conditions from the maximization problem. For the utility $U(x) = \sum_\mu \log(x_\mu)$ the price of a good is given by $p_\mu = \partial_\mu U_t(x) = \frac{1}{x_\mu} + h_\mu$.

The results for the fraction of active firms ϕ are the same as the results obtained in Chapter 4, because it still remains a maximization problem with the same constraints and with a concave utility function, equivalent to a convex energy function which has a single, replica symmetric solution for the ground state. Therefore, we still have a competitive regime for $n < 2$ where around half of the firms are active, which collapses to a monopolistic regime when $n > 2$ where few firms are active and ϕ reaches zero asymptotically.

However, the behavior of the macroeconomic quantities described above, average utility per good $\langle u \rangle$ and GDP Y, which we are now able to calculate using the good prices, are similar to the inefficient consumer framework described above as can be seen on figure 10, where we plot the results of the maximization process for the true utility for several values of the scale λ for the exponential distribution of h_μ.

As in the previous case, after a certain value of λ the observed utility starts decreasing with n, while the GDP has an increased rate of growth with n for larger lambdas.

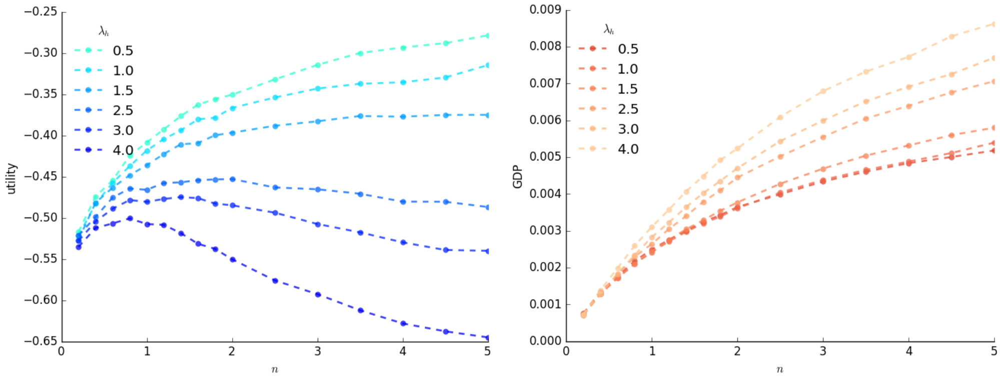

Figure 10.: **(Left)** Average consumer's observed utility per good as a function of the technological density n in the unobserved utility framework, for different values of β. Just like in the first scenario, the utility behaves as expected for very small perturbations, but large perturbations make it decrease with the number of choices available. **(Right)** Average GDP for the economy as a function of n. Again in this case, just like the original scenario, GDP rises even further when the consumer is inefficient and has a lower utility.

The downside of this approach, of course, is that we had to introduce some extra assumptions about the consumer's utility. However, as it is pointed out in [56], the Gibbs shape for the known part of the function is dependent only on very general conditions for the unknown part, and therefore the results we have obtained are qualitatively robust to different choices in the perturbed utility.

5.3 CONCLUSION

In this chapter we have shown two simple ways of treating inefficient or irrational consumer behavior in a general equilibrium setting, via the Random Linear Economies model presented above. Both cases, however, are applicable to any General Equilibrium setting. We have shown that when the consumer does not choose the bundle that strictly maximizes his utility, the behavior of macroeconomic quantities such as average utility and GDP have diverging slopes as a function of the number of options available on his consumption set.

Crucially, as inefficiency in choice increases the consumer finds itself being overwhelmed with the number of choices possible and chooses poorly. This corroborates some well known results in behavioral economics [47, 76] and it's very encouraging that we can reproduce these empirical behaviors in this simple setup.

We also observe that GDP growth increases, as opposed to the utility, with the increase in choice inefficiency. A higher number of options may result in poorer choices, but it also leads to increased economic activity. From a country perspective, therefore, it's desirable that its consumers are not completely rational agents. Less efficiency in choice leads to an increase in GDP and therefore in tax revenues. If one thinks of country development as an evolutionary process, there is would be selective pressure for higher inefficiency, and countries with perfectly rational consumers would disappear due to lower economic activity. However, if the average utility is a proxy for individual fitness, then this would be similar to a "emergence of cooperation" scenario usually dealt in Economics, where a rational consumer would thrive in a country with low choice efficiency.

This is certainly too deep a conclusion to be arrived from just this simple model. However, the results may be robust for other equilibrium scenarios, and we hope this work sets the stage for future research on this topic.

6

INPUT-OUTPUT OF RANDOM ECONOMIES AND REAL WORLD DATA

In this chapter, we will discuss the issue of aggregation by focusing on input-output statistics. The production network of an economy has been a subject of intense recent study. One long standing issue concerns the origin of macro-economic fluctuations: the traditional view of the GDP as the sum of many weakly dependent random shocks clashes with the observation that GDP fluctuations are markedly non-gaussian in the tails [33]. Ref. [39] invoked the extreme heterogeneity of firm sizes, suggesting that shocks in a single large firm can account for the observed distribution. Acemoglu *et al.* [2] instead noticed that propagation of shocks across the input-output production network can also reproduce non-gaussian fluctuations in GDP.

We collected extensive data on the production network of a wide range of countries for different years. A central quantity in input-output economics is the direct requirement matrix D (see [2, 63]), a square matrix with order equal to the number of goods M available in the data. Each element $D_{\mu\nu}$ of the direct requirement matrix is the amount of dollars (or euros) needed of good μ to produce one dollar of good ν in the economy. The matrix generates a weighted, directed graph of dependencies between goods, which allows one to quantify the cascade effect of isolated shocks in the economy, i.e., how much does a 50% decrease in oil prices affect the production capabilities of goods that directly use oil as an input, and how this shock spreads to goods that are produced using outputs of oil intensive industries.

The sum of the elements of the direct requirement matrix along a column yields the *degree* of the good in the corresponding row, which quantifies its level of dependence on other inputs. The first result we find is that, in a wide range of economies, degrees follow an exponential distribution (*universality*). This is suggestive, as this is the maximum entropy distribution consistent with sole knowledge of that the average degree is fixed to one, by row normalisation. Furthermore we show that such a maximum entropy distribution arises from aggregation, which is established both by studying a dataset where the input-output network is known at a finer resolution and by studying aggregation in a model of large random economies. The convergence to the exponential distribution depends on how the aggregation is carried out and this has important consequences on the estimate of aggregated fluctuations: classification methods currently employed by the economic agencies may lead to underestimated aggregate fluctuations.

6.1 INPUT-OUTPUT ECONOMICS: DEFINITIONS AND STYLISED FACTS

The data we will focus our analysis on in this chapter are the Input-Output tables published by the Bureau of Economic Analysis (BEA) for the United States and by the Eurostat for the European Union. The Input-Output tables are part of a country's national accounts, and they flow of intermediate and final goods between different producing sectors in an economy. More specifically, in these tables, all the industries and commodities of a country are aggregated into sectors, such as "Agriculture, hunting and related services", "Financial services, except insurance and pension funding", etc. The tables then describe how much of each type of good these industries consume (i.e., use as an Input) for their operations, and how much they producte (i.e., have as Output).

The classification system employed for categorizing the different sectors of the economy, the NACE[1] for the EU and the NAICS[2] for North America, is the same for both industries and goods, and are each defined in different levels of aggregation: the 2007 US data, for example, is available at the detailed, aggregated and summary levels, with 389, 71 and 15 different sectors respectively. In the NAICS, the sector "Mining" in the summary level of aggregation (with the whole economy split into 15 sectors) breaks down to "Oil and gas extraction", "Mining, except oil and gas" and "Support activities for mining" at the aggregated level. Then, in the detailed level, is further broken down into 8 categories, some of which are "Coal mining", "Iron, gold, silver, and other metal ore mining", etc. The EU data has the same characteristics, except it is only available at the 64 and 10 sector aggregation levels.

The main raw data available online [62, 32] are two matrices named Make (or Supply) and Use tables, which we will refer to as M and U respectively. M_i^μ is the amount of euros (or dollars) that the industries of sector i produce of good μ and U_i^μ is the monetary amount of good μ that industries in sector i use for their production. In a simplified version of the economy that would disregard, among other things, wages, capital devaluation and taxes, the profits of a sector i would be given simply by $\sum_\mu (M_i^\mu - U_i^\mu)$.

For the European Union, we used the workbooks of nine countries: Austria, Denmark, Finland, France, Germany, Greece, Italy, Netherlands, Spain. Other countries considered were Ireland, Portugal, Sweden and the United Kingdom, but the data for these countries has a significant part omitted for confidentiality reasons, so these countries were discarded. For every country there were two tables of output (the "Use" tables) available for each year: one at the seller's prices and one at the purchaser's prices. The latter was used for all purposes.

1 Acronym for *Nomenclature statistique des activités économiques dans la Communauté européenne*
2 *North American Industry Classification System*

We are particularly interested in the interdependence between the goods, and for that purpose we follow the BEA handbook and construct a product-by-product Direct Requirements table [63], which will give us how many dollars we need of one good to produce one dollar of another. We first build the product-by-industry Direct Requirement matrix by dividing each company's usage of goods by its total outputs, that is:

$$B_i^\mu = \frac{U_i^\mu}{\sum_{\nu=1}^M M_i^\nu} \tag{78}$$

The matrix B_i^μ gives us how many dollars of product μ we need to produce one dollar of industry i's output. We then define the Market Share matrix, which is simply the fraction of a given good that a company produces:

$$W_i^\mu = \frac{M_i^\mu}{\sum_{j=1}^N M_j^\mu} \tag{79}$$

Given these two matrices, we sum over the firms to define the product by product Direct Requirement matrix, given by $D = BW$, that is:

$$D_{\mu\nu} = \sum_{i=1}^N \frac{U_i^\mu}{\sum_{\eta=1}^M M_i^\eta} \frac{M_i^\nu}{\sum_{j=1}^N M_j^\nu} \tag{80}$$

Each element $D_{\mu\nu}$ of the direct requirement matrix is a sum of the firms direct requirements of μ weighted by their market share on ν. This gives us how many dollars of good μ are used in the economy for the production of one dollar of good ν.

This matrix therefore defines a directed weighted graph on the goods, with the incoming edges of μ being its inputs and the outgoing edges its usage by other products in the economy. Therefore it's natural to characterize goods by their indegrees, the weighted sum of the incoming edges, and the outdegree, the weighted sum of the outgoing edges. That is:

$$d'_\nu = \sum_{\mu=1}^M D_{\mu\nu}, \quad d_\mu = \sum_{\nu=1}^M D_{\mu\nu} \tag{81}$$

The indegree d'_ν is how many dollars are used to produce one dollar of a certain good. This defines the profitability of the good, minus salary and wages, so we expect that for every good $d'_\nu \leq 1$. In a perfect competitive market, one would have $d'_\nu = 1$.

The outdegree d_μ is a more interesting quantity. It gives us the total amount of dollars of good μ used in the economy (given that each good requires \$1 to be produced). In a sense, high outdegree goods are "structural" and very necessary for production, while low outdegree goods are less so. Note that this does not mean that they are unimportant. Retail services, for example, have outdegree equal or close to zero in all data analysed in this paper. This is because no firm uses retail stores services as an input to its production, all of the sector's output is used as consumption in the economy, clearly it does not mean it's an irrelevant sector. It does mean, however, that a sudden closure of half of the retail stores in an economy would certainly have a much smaller effect than the equivalent shock in other goods such as energy and financial services. In loose terms, this suggests that high outdegree sectors are more systemically important than low degree ones.

In [2] Acemoglu *et al* used the US direct requirement tables produced in a model economy to quantify how susceptible the American economy is to independent normally distributed random shocks. In that paper, they challenged the standard notion in Macroeconomics that these shocks are self averaging and cancel themselves out. Instead, what was shown is that, in the context of their production model, independent shocks do not average out when the outdegree distribution has heavy tails. Therefore aggregate fluctuations remain large even in the limit of large economies. Before addressing these issues, we first discuss the degree distribution that emerges from empirical data.

6.1.1 *Universality of the degree distribution*

In order to relate to their results, we follow [2] and normalize all the indegrees to unity so we are able to compare the data with a Random Economy model's competitive equilibrium. We therefore only look at the outdegrees, from now on, that we simply call degrees [3].

We plot on figure 11 the counter cumulative distribution of the degrees $P(d < x)$ in a semi-log scale for six out of fourteen OECD countries for which we obtained the Input-Output data. Also plotted is the respective linear regression, in which the slopes are a very good fit close to the degree average $\overline{d} = 1$, suggesting that most of the distributions have an exponential shape. The two main outliers from the exponential law in our data, the US and Germany, are shown in the figure.

To better illustrate this universal feature of the distributions, we derive two main quantities for each country (we refer the reader to Appendix B for more details on the statistical measures used throughout the chapter): The first is the p-value in a Kolmogorov-Smirnov test against an exponential distribution.When p is very small the hypothesis that the degrees come from an exponential distribution should be rejected.

The second quantity we use is derived from the Bayesian Information Criterion (BIC) [51]. This compares different models from which the data may have been generated, by computing the difference ΔBIC in the log-likelihoods corrected by the BIC term which accounts for the different complexity of the models. Specifically, in Figure 12, we compare the exponential distribution with a Gaussian distribution. This choice derives from the fact that one may naively expect that, upon aggregation, degrees be-

3 We found qualitatively similar results even without this normalization

have as sums of random variables and hence should ultimately converge to Gaussian variables.

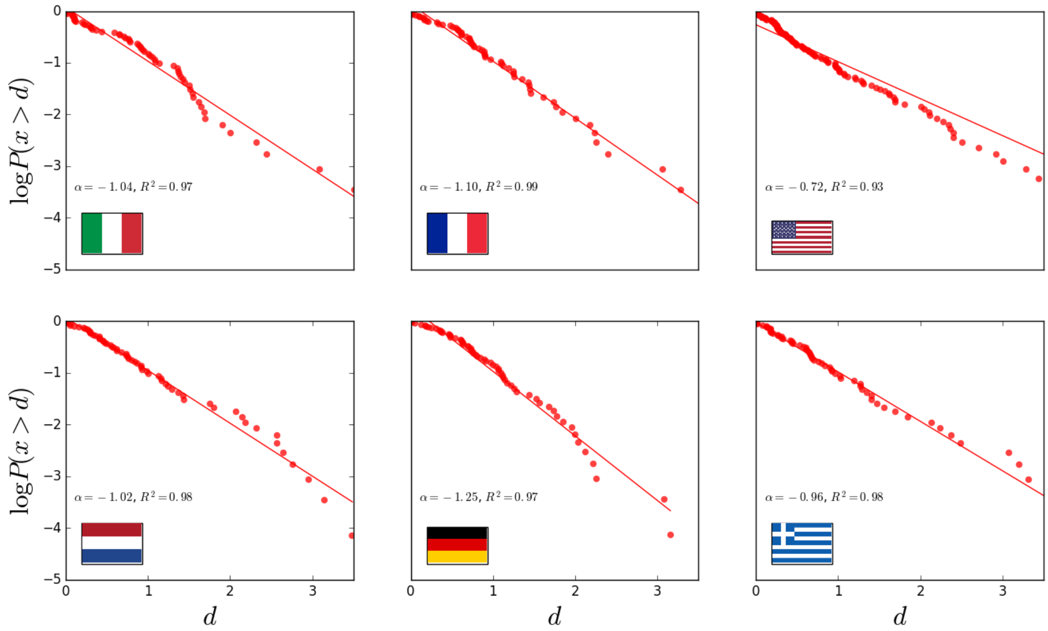

Figure 11.: Outdegree distribution of several countries. The data shown is the counter cumulative in semilog scale, along with the linear fit. The largest outliers of our data, the US and Germany, are shown here.

Figure 12 depicts p versus ΔBIC for all the countries studied. We see that the bulk of the EU countries lie together in a small cluster of the graph, indicating that the exponential hypothesis is a very good fit for them. This is further confirmed by putting the finer resolution data for the United States (represented by the US300 point) in the same plane: the disaggregated distribution is a very strong outlier compared to the coarser data.

A more careful analysis at the "exponential bulk" shows that the main outliers are the US data, that exhibits a distribution bending upwards in the semi-log plot of Figure 11, and Germany with a degree distribution bending downward. This motivated us to carry out the BIC test between the exponential distribution hypothesis and a stretched exponential defined as

$$P(d) = A(b,\theta)e^{-bx^{\theta}}, \qquad b,\theta > 0 \tag{82}$$

The result of this BIC test is shown in Figure 13, where the shaded region is where the exponential ($\theta = 1$) is preferred to the stretched exponential ($\theta \neq 1$). We see that all countries analysed lie in this region, with the exception of the US – that is best fitted with $\theta < 1$ – and Germany, Spain and Finland – that are best fitted with $\theta > 1$.

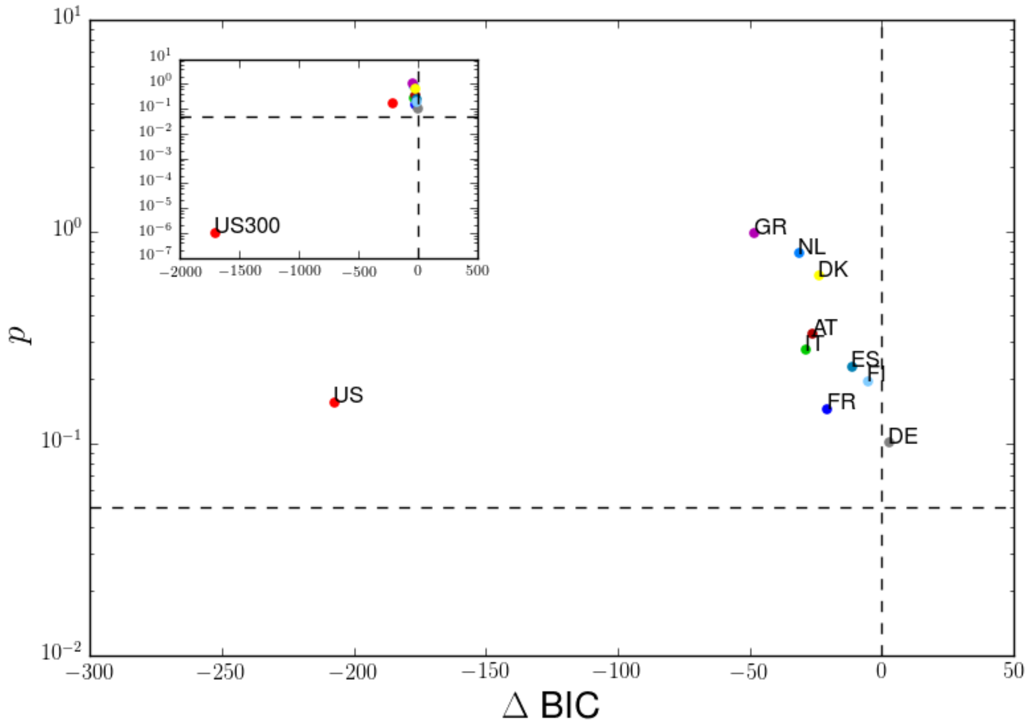

Figure 12.: Countries sorted by ΔBIC of gaussian vs exponential (x axis) and probabilities of KS distance against exponential (y axis). Inset: same plot with the US $M = 300$ level of disaggregation included. Dashed lines indicate ΔBIC$= 0$ and $p = 0.05$.

An exponential degree distribution across most of the countries under study is of major relevance because the average degree of these datasets is set to one by the normalisation requirement $d'_\mu = 1$, and the distribution of maximal entropy consistent with this constraint is the exponential distribution. Since this is the least informative distribution given the constraints we have, this suggests that all the details of the input-output economics at the microscopic scale have been washed out in the aggregation process.

Of course, the fact that this is the least informative doesn't mean there is no information available. One can learn important things about the economies in the aggregated dataset by looking at the relative importance of the different sectors. For example, among the high degree products, "Financial services", "Legal and accounting services" and "Chemicals and chemical products" are the highest ranking ones, whereas, "Imputed rents of owner-occupied dwellings" and "Retail trade services" have zero degree in all countries. One can also look at the structural differences between countries: in France, for example, products of "Security and investigation services; office administration; office support and other business support services" are twice as used as in Italy (degree of 3 vs 1.69), whereas in Italy, "Electricity, gas, steam and airconditioning" are twice as depended upon as in France (degree 4 vs 2). Yet the fact that the distribution of degree takes a form consistent with maximum entropy, suggests that the distribution carries no specific information on the economy. We shall now argue that this information is actually "lost in aggregation" by looking at how the degree distribution evolves at different level of aggregation.

6.2 LOSS OF INFORMATION VIA AGGREGATION

The above results show that aggregated data have a less informative distribution of degrees. Indeed, for the most disaggregated data available – the 2007 US economy with $M = 382$ – we find that the distribution of degrees is further away from the exponential distribution with respect to the US data with $M = 64$ sectors (see point US300 in Fig. 13). At the other extreme, if at a certain level of aggregation the distribution is exponential, when we aggregate further, we expect the distribution to converge to a Gaussian limit, because the degree at the coarser level is the sum of the degrees at the finer one. This is appropriate if the degrees are weakly dependent random variables, which in turn depends on how the aggregation process is carried out, i.e. which sectors are put together at the finer scale. To answer the question of whether the loss of information depends on the manner in which the aggregation is carried out we will artificially aggregate the goods in the Use and Make tables described in Section 6.1 by collapsing pairs of goods into a single one, ie, if we collapse μ_1 and μ_2 into μ the new Use matrix would be N by $M - 1$ and would have $U_{i\mu} = U_{i\mu_1} + U_{i\mu_2}$. We would like to compare aggregation in two extreme cases *(i) random*: we choose μ_1, μ_2 randomly and *(ii) ranked*: we choose the two most correlated goods via Spearman rank of the $M - U$ matrix, so that goods that highly correlate in the usage as inputs - outputs by the industries will be aggregated together first. In loose terms, as suggested by the previous argument, we expect a faster loss of information (i.e. convergence to the exponential distribution) when the aggregation is carried out without taking into account the structure of the data. As we'll discuss, this has practical consequences, because a standardised aggregation method that has to be applied to many countries, such as the one used in EU, may determine a faster convergence to an uninformative distribution than a method that is tailored to fewer countries (the one used by the US, Canada and Mexico).

6.2.1 *Empirical data: the case of the US*

We take the most disaggregated data available, the 2007 US economy with $M = 382$ sectors, and aggregate according to the two methods described above. The results in the (p, θ) plane are shown in figure 13. We observe that we indeed obtain very different behavior depending on how the aggregation is carried out. Interestingly, the artificial method closer to the BEA classification system is random aggregation,

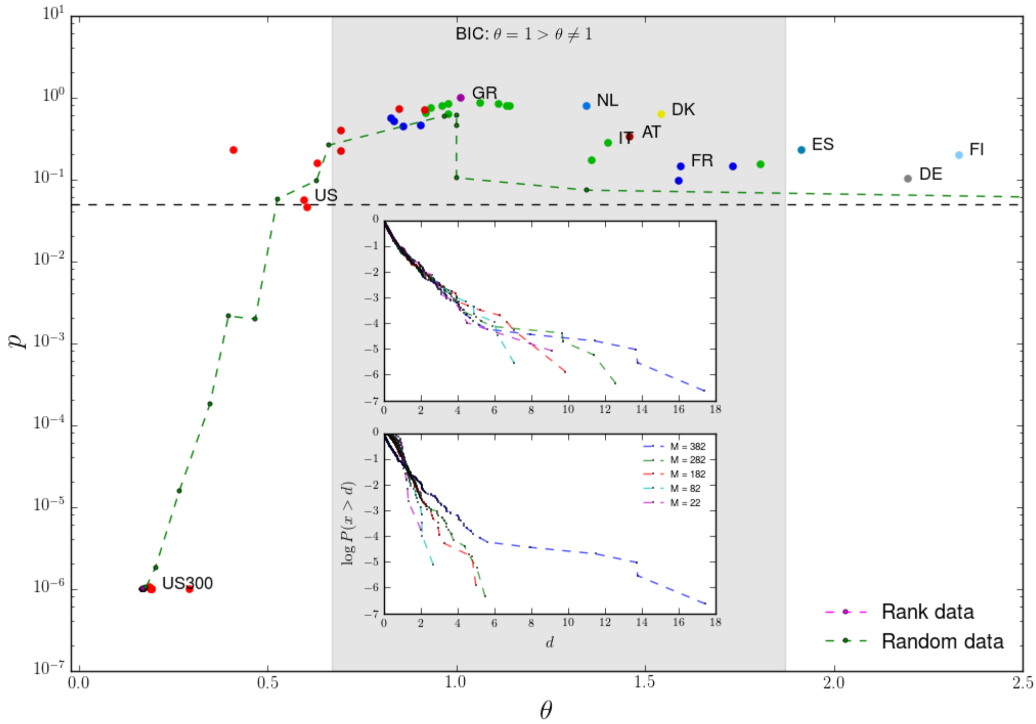

Figure 13.: Countries in the (p, θ) plane, along with the two artificial aggregation processes carried out on the US300 data. The shaded area is the region where the BIC favors exponential distribution ($\theta = 1$) as opposed to stretched ($\theta \neq 1$). *Insets*: Evolution of the degree distribution under rank (top) and random (bottom) aggregation.

while the ranked aggregation never converges to the exponential distribution. This is because it creates a very dense "supergood" with positive entries in all of its Make and Use tables that has a very high degree, while the rest of the goods are left with sparse entries in the tables. Nonetheless, ranked aggregation preserves the maximum degree observed in the economy, as shown in the insets of figure 13.

6.2.2 *Random economies*

We now turn to the question of whether one can reproduce the maximum entropy degree distribution with a simple model and what is the effect of aggregation in those artificial economies. For that, we will build Input-Ouput matrices for economies in the Random Linear Economy model described in chapter 4.

The Use and Make tables can be defined for the Random Economies as a function of the price vector p, the scales of production s and the technologies ξ. The total amount of a good μ a firm i produces is given by $p_\mu s_i \xi_i^\mu$ for all goods μ that have positive ξ_i^μ.

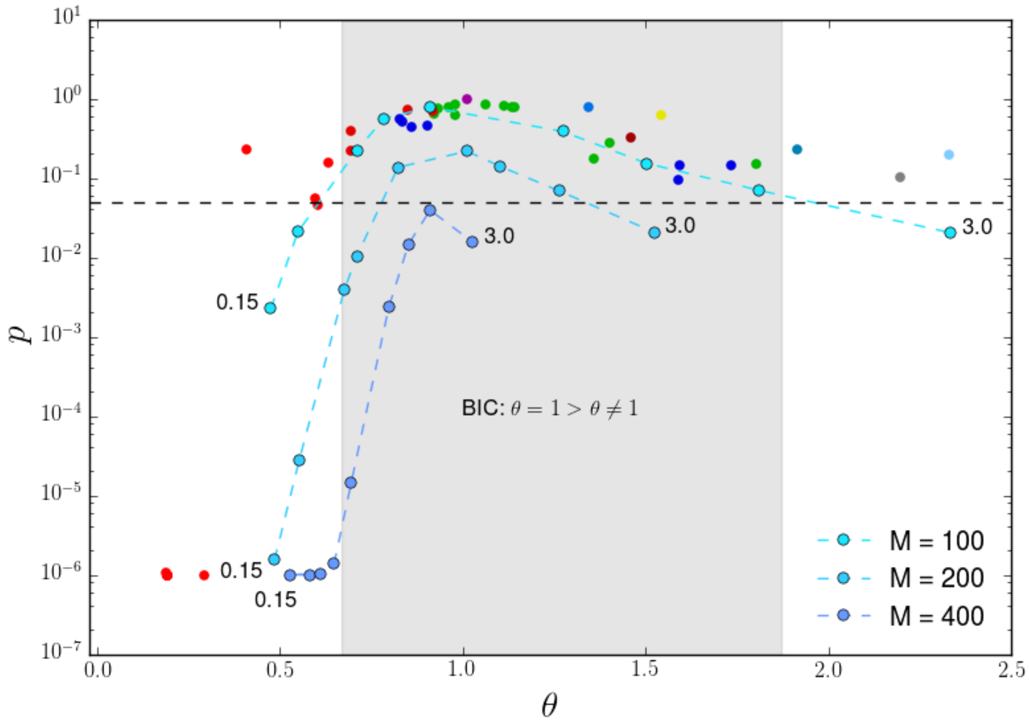

Figure 14.: Comparison of the Random Economies with real data in the (p, θ) plane. The model is shown for three quantity of goods M, each curve represents the model results with the number of firms per good $n = N/M$ varying from 0.15 to 3. We observe that the model parameters are able to represent the range of economies well, however these results are not scale dependent, not converging to a well defined limit when we increase M keeping n constant.

Likewise, the amount used is given by the negative terms in ξ_i. We define them the Use and Make tables as:

$$U_{i\mu} = p_\mu s_i \left| \xi_i^{\mu,-} \right|, \quad M_{i\mu} = p_\mu s_i \xi_i^{\mu,+}, \tag{83}$$

where $\xi_i^{\mu,+} = \xi_i^\mu$ if $\xi_i^\mu > 0$ and 0 otherwise. Likewise, $\xi_i^{\mu,-} = \xi_i^\mu$ only if $\xi_i^\mu < 0$. One thing to note is that, unlike real Input-Output data, in the model a good is never used as an input to itself. Therefore either $M_{i\mu}$ is positive or $U_{i\mu}$, never both and $D_{\mu\mu} = 0$ for all goods, unlike the data where the diagonal terms of the D are usually large. Yet this holds only at the microscopic level: for the model, $D_{\mu,\mu}$ can be nonzero after aggregation.

We observe in numerical simulations that the properties of the model's direct requirement matrices are similar to the real world data. For $M = 100$, the degree distributions of the random economies are remarkably similar to the ones given by real data, if we vary N (and therefore n), as shown in Figure 14. When the repertoire of technologies is relatively small (small N) compared to the number of goods, the model exhibits a broad distribution of degrees reminiscent of the US data whereas when N increases the distribution becomes less heavy tailed. However, as can be seen in Figure 14, these degree distributions are dependent on the parameters M, N and ϵ in a complicated manner and do not converge to a well defined limit as M diverges with $n = N/M$ and ϵ finite.

Yet the behavior of the model under aggregation is quite similar to that observed in real data, as seen in Figure 15 and degree distribution converges to the same cluster of noninformative exponential distributions of the EU / US aggregated data.

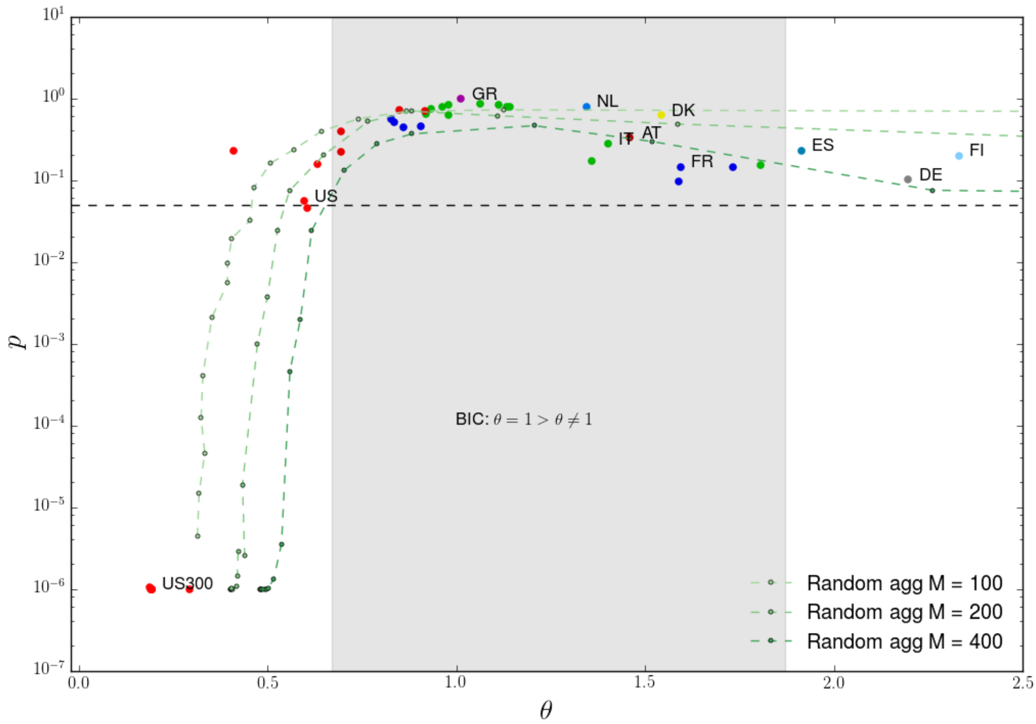

Figure 15.: Comparison of the randomly aggregated Random Economies with real data in the (p, θ) plane. The model always starts at one single realization of the Make and Use tables (as opposed to averaged over the disorder) for $n = 0.15$ and randomly aggregated.

6.2.3 *Consequences of aggregation*

The above results provide good evidence that the method by which the BEA aggregates the sectors in the US economy is closer to a random aggregation that to a ranked one. The Eurostat has no data publicly available for a higher level of disaggregation, yet comparing the degree distributions at the $M \sim 64$ sectors level suggests that EU aggregation is also performed in an effectively random manner. In this section we explore the consequences of this washing out of information.

The direct requirement graph allows us to characterize the structural susceptibility of the economy to shocks and perturbations: the distribution of degrees give us a notion of how asymetric are the dependencies between goods and what fraction of the production capabilities depend on a single (or a few) goods. One of the ways aggregation can lose information is if it doesn't take the good degrees into account when selecting pairs to merge. Then, it will likely happen that high degree goods will be aggregated with low degree goods, possibly creating a good that has an degree in between the original ones[4]. This will gradually be responsible for eliminating the heavy tail of the distribution, masking the very high dependencies which are the most important for determining shock volatility. We test this hypothesis by calculating the normalized standard deviation of the ranks bundled together at each step: we define the rank of a good as its position in an ordered list of degrees, 1 being the lowest degree good and M the highest degree good, then given a new good $\mu' = \sum_{k=1}^{K} \mu_k$ and given by $r(\mu)$, then we the standard deviation of the aggregation as

$$\frac{\sigma(\mu')}{M} = \frac{1}{M}\sqrt{\sum_{k=1}^{M} \left(r(\mu_k) - \bar{r}\right)^2} \tag{84}$$

The results of $\frac{\sigma}{M}$ for each bundle of goods aggregated at each step using the three methods are shown in Figure 16. One sees that, as expected, this spread in the degrees being aggregated is highest when the random method is used and lowest when rank based is used, with the actual category based classification in between.

The degrees, however, are a first order measure for the spread of perturbations which does not take cascade effects into account: the perturbation of a good will impact the production of its dependencies which may also have very high degree, amplifying the initial shock. In [2], the authors show that the norm of a quantity similar to the Bonacich centrality vector [48] is a lower bound for the aggregated fluctuations generated by random, independent shocks acting on the whole economy. This vector is given by

$$\chi = \frac{\alpha}{M}(\mathbb{I} - (1-\alpha)D)^{-1} \cdot \mathbb{1}, \tag{85}$$

where $\mathbb{I}$ is the identity matrix, $\mathbb{1}$ is a vector with all entries equal to one and $\alpha \in (0,1)$. The argument used by Acemoglu *et al* in [2] is that in a balanced structure in which random independent shocks in the whole network average themselves out, as M increases the norm $\|\chi\|(M)$ of the vector should decrease proportionally to $\sqrt{M}$, but if one calculates the χ both for the $M = 382$ and $M = 71$ aggregation levels of the BEA data, the decrease is approximately $n^{\frac{1}{8}}$, considerably slower than a balanced

4 One must keep in mind that because aggregation is linear on the Use and Make tables, not in the Direct Requirement table, it's not true that the good resulting from the aggregation of a set of goods will have degree equal to the average of this set.

economy. This is interpreted as an evidence that the US economy is more susceptible than expected to shocks.

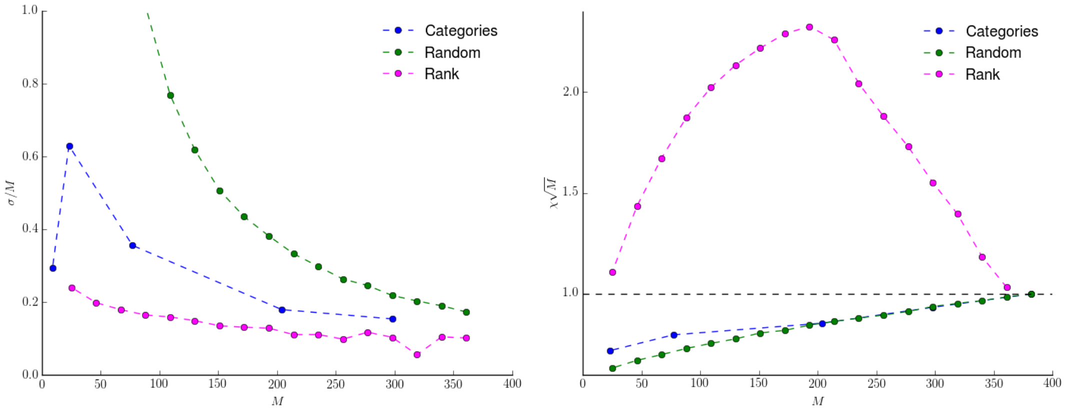

Figure 16.: **(Left)** Scaled standard deviation σ/M for the degree rank of the goods that are considered a single good at each aggregation step. The random aggregation has the highest average standard deviation, while the rank based has the lowest. This further corroborates the idea that random (and category) aggregation washes out high dependencies by merging them with low ones. **(Right)** Normalized average Bonacich centrality $\chi\sqrt{M}$ [2]. This quantity should be constant if aggregated shocks have a limited effect (because χ decreases with $\sqrt{M}$) Here we see that networks with random aggregation have a different susceptibility to shocks than rank based ones, in particular, random (and category based) aggregation has χ decreasing slower than $\sqrt{M}$, exacerbating the effects of shocks.

We calculate the same quantity for the three aggregation types and plot the normalized (by $M = 382$) $\|\chi\|\sqrt{M}$ on figure 16. This quantity should be constant if $\|\chi\|$ decreases proportionally to $\sqrt{M}$, as expected in a balanced structure, and increasing with M if it falls slower than that. We see on the graph that random aggregation, again, generates a slightly unbalanced structure, but the behaviour under rank aggregation is very different, first increasing rapidly with M (i.e., what would indicate a very susceptible structure), but then it falls equally fast. We interpret this result as a clear evidence that the direct requirement graph properties are not simply a characteristic of the economy itself, but very dependent on the method used for aggregation.

The important takeaway from this analysis is that sector classification changes the whole structure of Direct Requirement matrices. If the current classification methods serve a specific purpose not related to the input-output structure, then when analysing the Input-Output tables one must take into account that systemic risks like the ones discussed in [2] are aggregated away and underrepresented.

However, if the purpose of building the Input-Output tables is to make an assessment of which sectors are structural and which ones are not, then the way the classification hierarchy is built must change. The current one is, for a lack of better word, "thematic". Food is always aggregated together, so are the byproducts of mining and so are services. These do not take into account the fact that strawberries and eggs may have wildly different centralities and degrees in the input-output network. The Spearman rank method we used is a highly artificial way to aggregate, but it still preserves the high degree of dependence of the US economy in certain crucial sectors.

6.3 CONCLUSION

The economic woes of the last decade brought to surface the importance of identifying firms, and sectors, that are highly structural and from which a sizeable portion of the production depends, and properly taking steps to make the economy less vulnerable. However, we have shown here that if one is not careful when organizing and classifying economic data, this information can be lost.

Our results also show that at an intermediate scale of aggregation, economies exhibit universal statistical properties that suggest that a Statistical Mechanics approach may be feasible. In particular, the study of the aggregation properties of models of large economies can provide valuable hints on building a theory of macroeconomic behaviour based on microeconomic interactions.

Going back to the question of what is the purpose of the current classification scheme. From the BEA handbook we read

> The I-O tables are used to study changes in the structure of the U.S. economy and to assess the impact of specified events on economic activity.

However, both the North American Industry Classification System (NAICS), used by the BEA to construct the US data, and the NACE, used by Eurostat, have as main purpose and advantage the usage of a uniform classification for all the countries and their respective statistical agencies in North America and the European Union. It is not surprising, therefore, that the effects of their aggregation are closer to a random process than to a method that takes the underlying data structure into account.

7

WHEN DOES INEQUALITY FREEZE AN ECONOMY?

The debate on wealth inequality has a long history dating back at least to the work of Kutznets [54] on the U-shaped relationship of inequality on development, where economic growth at first increases wealth inequality and then after a certain point it decreases. Much research has focused on this relation between inequality and growth [65]), and inequality has also been suggested to be positively correlated with a number of indicators of social disfunction, from infant mortality and health to social mobility and crime [91].

The subject has regained much interest recently, in view of the claim that inequality has reached the same levels as in the beginning of the 20th century [67]. Saez and Zucman [70] corroborate these findings, studying the evolution of the distribution of wealth in the US economy over the last century, and they find an increasing concentration of wealth in the hands of the 0.01% of the richest. Figure 17 shows that the data in Saez and Zucman [70] is consistent with a power law distribution $P\{w_i > x\} \sim x^{-\beta}$, with a good agreement down to the 10% of the richest. The exponent β has been steadily decreasing in the last 30 years, reaching the same levels it attained at the beginning of the 20th century ($\beta = 1.43 \pm 0.01$ in 1917).

One of the most robust empirical stylised facts in economics, since the work of Pareto[64], is the observation of a broad distribution of wealth which approximately follows a power law. What is interesting about it is that such a power law distribution of wealth does not require sophisticated assumptions on the rationality of players as we have dealt so far in this thesis, but it can be reproduced by a plethora of simple models [3, 19, 92, 40, 71], in which it emerges as a typical behaviour within quite general settings. This relates to the often made criticism that the standard approach of Economics, aimed at explaining global behaviour in terms of perfectly rational actors, has largely failed [68, 16, 53]. Yet, persistent statistical regularities in empirical data suggest that a less ambitious goal of explaining economic phenomena as emergent statistical properties of a large interacting system may be possible, without requiring much from agents' rationality [45, 84].

Taking inspiration from such independence from economic rationality, our goal is to study inequality via a zero intelligence agent model, in which agents with different capital trade goods randomly in a market. Rather than focusing on the determinants

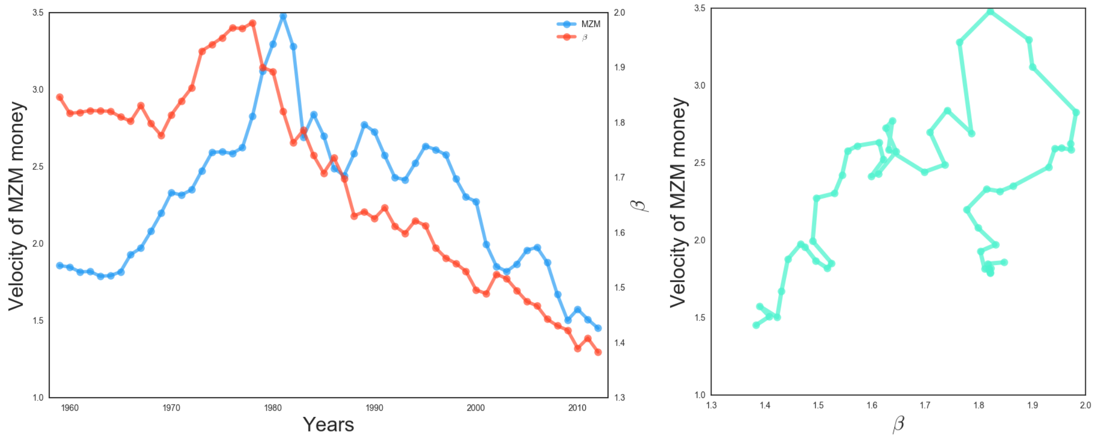

Figure 17.: **(Left)** Velocity of money of MZM stocks (left y-axis) and Pareto exponent β of the wealth distribution (right y-axis) as a function of time. Both time series refer to the US, the data on the money velocity is retrieved from [34], the data on the wealth distribution is taken from [70]. **(Right)** MZM velocity of money as a function of β, for the same data.

of inequality, we focus on a specific consequence of inequality: its impact on liquidity. There are a number of reasons why this is relevant. First of all, the efficiency of a market economy essentially resides on its ability to allow agents to exchange goods. A direct measure of efficiency is the number of possible exchanges that can be realised or equivalently the probability that a random exchange can take place. This probability quantifies the "fluidity" of exchanges and we shall call it **liquidity** in what follows. This is the primary measure of efficiency that we shall focus on.

Secondly, liquidity, as intended here, has been the primary concern of monetary polices such as Quantitative Easing aimed at contrasting deflation and the slowing down of the economy, in the aftermath of the 2008 financial crisis. A quantitative measure of liquidity is provided by the **velocity of money** [35], measured as the ratio between the nominal Gross Domestic Product and the money stock[1] and it quantifies how often a unit of currency changes hand within the economy. As Figure 17 shows, the velocity of money has been steadily declining in the last decades, with a clear correlation to the level of inequality. The results in this chapter suggests that the velocity decline and the increasing level of inequality occurring together is not a coincidence. Rather the former is a consequence of the latter.

Without clear yardsticks marking levels of inequality that seriously hamper the functioning of an economy, the debate on inequality runs the risk of remaining at a qualitative or ideological level. The main finding of the work presented in this chapter is that, in the simplified setting of the model presented, there is a sharp threshold beyond which inequality cripples the economy. More precisely, when the power law

1 The data reported in this chapter concerns the MZM (money with zero maturity), the broadest definition of money stock that includes all money market funds. We refer to [34] for further details.

exponent of the wealth distribution approaches one, liquidity vanishes and the economy halts because all available (liquid) financial resources concentrate in the hands of few agents. This provides a precise, quantitative measure of when inequality becomes too much.

The main goal of this work is thus to isolate the relation between inequality and liquidity in the simplest possible model that allows us to draw sharp and robust conclusions. Specifically, the model is based on a simplified trading dynamics in which agents with a Pareto distributed wealth randomly trade goods of different prices. Agents receive offers to buy goods and each such transaction is executed if it is compatible with the budget constraint of the buying agent. This reflects a situation where, at those prices, agents are indifferent between all feasible allocations. The model is in the spirit of random exchange models, as for example the Statistical Equilibrium of Markets described in Chapter 3 and in other works [36, 92], but our emphasis is not on whether the equilibrium can be reached or not. In fact we show that the dynamics converges to a steady state, which corresponds to a maximally entropic state where all feasible allocations occur with the same probability. Rather we focus on the allocation of cash in the resulting stationary state and on the liquidity of the economy, defined as the fraction of attempted exchanges that are successful. Since the wealth distribution is fixed, the causal link between inequality and liquidity is clear in the simplified setting we consider.

In a nutshell, within our model the freezing of the economy occurs because when inequality in the wealth distribution increases, financial resources (i.e. cash) concentrate more and more in the hands of few agents (the wealthiest), leaving the vast majority without the financial means to trade. This ultimately suppresses the probability of successful exchanges, i.e. liquidity.

7.1 THE MODEL

We introduce an inequality trading model consisting of an economy with N agents, each with wealth c_i, $i = 1, \dots, N$. There are M objects to be traded among the agents in the economy and each object $m = 1, \dots, M$ has a price π_m. A given allocation of goods among the agents is described by an $N \times M$ allocation matrix $\mathcal{A}$ with entries $a_{i,m} = 1$ if agent i owns good m and zero otherwise, but agents can only own baskets of goods they can afford, i.e. whose total value does not exceed their wealth. The wealth of an agent not invested in goods corresponds to the cash (liquid capital) ℓ_i that agent i has available for trading, i.e.,

$$c_i - \sum_{m=1}^{M} a_{i,m}\pi_m = \ell_i \geq 0, \qquad i = 1, \dots, N, \tag{86}$$

Therefore the set of feasible allocations – those for which $\ell_i \geq 0$ for all i – is only a small fraction of the M^N possible realizations of the allocation matrices $\mathcal{A}$.

The dynamics are similar to those typical of zero-intelligent agent-based models in economics and are as following: starting from a feasible allocation matrix $\mathcal{A}$, at each time step a good m is picked uniformly at random among all goods. Its owner then attempts to sell it to another agent i also drawn uniformly at random among the other $N-1$ agents. If agent i has enough cash to buy the product m, that is if $\ell_i \geq \pi_m$, the transaction is successful and his cash decreases by π_m while the cash of the seller increases by π_m, otherwise the transaction fails, with no possibility of an object being divided. The total capital c_i of agents does not change over time, so c_i and the good prices π_m are quenched parameters of the model. We reemphasize that there are no decisions to be made by the agents, they have no utility function to maximize and simply accept trades when selected if they have enough cash.

A crucial property of these dynamical rules is that the stochastic transition matrix $W(\mathcal{A} \to \mathcal{A}')$ is symmetric between any two feasible configurations $\mathcal{A}$ and $\mathcal{A}'$: $W(\mathcal{A} \to \mathcal{A}') = W(\mathcal{A}' \to \mathcal{A})$ and any feasible allocation $\mathcal{A}$ can be reached from any other feasible allocation $\mathcal{A}'$ by a sequence of trades. This implies that the dynamic satisfies the detailed balance condition:

$$W(\mathcal{A} \to \mathcal{A}')P(\mathcal{A}) = W(\mathcal{A}' \to \mathcal{A})P(\mathcal{A}'), \quad \forall \mathcal{A}, \mathcal{A}' \tag{87}$$

with a stationary distribution over the space of feasible configurations that is uniform, i.e., $P(\mathcal{A}) = \frac{1}{A}$, where A is the number of feasible allocations. This is a consequence of the symmetric transition rates, and would be the same for every trading rule that has $W(\mathcal{A} \to \mathcal{A}') = W(\mathcal{A}' \to \mathcal{A})$. In fact, the current trading rules employed in this model are a particular case of a general rule for which we first select a subset of n agents, $2 \leq n \leq N$, then we pick a random good from these n agents and try to trade it with the remaining $n-1$ agents, automatically accepting if the chosen buyer has enough cash to purchase it. This rule may sound cryptic, but it's familiar in the particular cases of $n = N$, which is the current one described in the model, and of $n = 2$, in which we first pick two consumers and then try to exchange a random good among themselves. All of these rules generate the same stationary distribution.

For the distribution of agents capital, we focus on realisations where the wealth c_i is drawn from a Pareto distribution $P\{c_i > c\} \sim c^{-\beta}$, for $c > c_{\min}$ for each agent i, which is compatible with many empirical observations of real world wealth distribution. The parameter β will be the main quantity to be explored in this work because it regulates the different levels of inequality among the agents. To compare different economies, the ratio between the total wealth $C = \sum_i c_i$ and the total value of all objects $\Pi = \sum_m \pi_m$ will be kept fixed. Because C is a random variable with potentially very high variance, the number of goods M is realization dependent, we will populate the economy with as many items as needed to keep the ratio C/Π constant, which will be described in details late. In order to have feasible allocations, it must hold that $C > \Pi$.

For the goods, we are going to limit the analysis to cases where the M objects are divided into a small number of K classes with M_k objects per class ($k = 1, \ldots, K$), where objects belonging to class k have the same price $\pi_{(k)}$. If $z_{i,k}$ is the number of object of class k that agent i owns, then the budget condition given by equation (86) takes the form

$$c_i = \sum_{k=1}^{K} z_{i,k}\pi_{(k)} + \ell_i \tag{88}$$

As with standard Statistical Mechanics systems, we will throughout this work assume that the economy is very large, that is, $N, M \to \infty$ but with the ratio C/Π constant. This will allow us to calculate the probability distribution of goods $P(z_i^k|c_i)$ for each agent in an exact manner in the large system (thermodynamic) limit. In the next section we will solve the master equation and find the distribution of goods as a function of capital c_i. As it will be shown, the main result of this model is that the flow of goods among agents becomes more and more congested as inequality increases until it halts completely when the Pareto exponent β tends to one.

7.2 THE CASE OF ONE TYPE OF GOOD

We begin our analysis by the simplest case in which there is only one type of good being traded, i.e., $K = 1$ with prices $\pi_k = \pi$, but the results shown will extend for the general setting. A formal approach to this problem would consist in writing the complete master equation that describes the evolution for the probability $P(z_1, \ldots, z_N)$ of finding the economy in a state where each agent $i = 1, \ldots, N$ has a specific number z_i of goods. We would then take the sum over all values of z_j for $j \neq i$ to derive the master equation for a single agent with wealth c_i. However, due to the mean-field like nature of the interactions, we can directly write the detailed balance conditions for a single marginal $P_i(z)$, which will be solvable with a few approximations that exploit the large system size. The detailed balance equation for a single agent is

$$\Delta P_i(z) = P_i(z+1 \to z)P_i(z+1) + P_i(z-1 \to z)P_i(z-1) \tag{89}$$
$$- \left(P_i(z \to z+1)P_i(z) + P_i(z \to z-1)P_i(z)\right) \tag{90}$$

However, there are further steps we can take to simplify this equation: we are looking for the stationary distribution of $P_i(z)$ knowing that all the feasible allocations $\mathcal{A}$ are equiprobable and the probability rates are symmetric, $W(\mathcal{A} \to \mathcal{A}') = W(\mathcal{A}' \to \mathcal{A})$. We therefore can solve the detailed balance equation using a stronger condition in which, not only it has to hold that the probability of coming to a state z has to be equal to the probability of leaving it, as equation (89) states, but instead requiring that the probability of going from a specific state z to another z' is equal to the probability of going from z' to z. If we take $z' = z + 1$, we have

$$P_i(z+1)P_i(z+1 \to z) = P_i(z)P(z \to z+1) \tag{91}$$

The transition probabilities are given as follows: $P_i(z+1 \to z)$ is the probability of selling a good, which is composed of two independent events: *(i)* a good of agent i is chosen as the good to be traded, which happens with probability $\frac{z+1}{M}$ and *(ii)* the transaction is successful, ie, the chosen recipient is able to afford the good. This happens with probability p^s, whose form we will write later. Therefore $P_i(z+1 \to z)$ is given by $\frac{z+1}{M}p^s$. $P_i(z \to z+1)$, on the other hand, is the probability that agent i buys a good, which happens if a good he doesn't own is chosen, which has probability $\frac{M-z}{M}$, he is picked as buyer, with probability $\frac{1}{N-1}$ and has enough cash to purchase it. This means the probability is truncated by a term $1 - \delta_{z,m_i}$, where $m_i = \lfloor c_i/\pi \rfloor$ is the maximum number of goods which agent i can buy with his wealth c_i. If $z = m_i$ then he cannot afford the item and the probability is zero.

We put it all together and make two large size approximations: $\frac{1}{N-1} \approx \frac{1}{N}$ and $\frac{M-z}{M} \approx 1$, that is, we used the fact that $N \gg 1$ and $z \ll M$. The first assumption holds by definition, but the latter breaks down if $\beta < 1$, as we will discuss later. We finally have the simplified detailed balance equation for agent i

$$P_i(z+1)\frac{z+1}{M}p^s = P_i(z)\frac{1}{N}\left(1 - \delta_{z,m_i}\right), \qquad z = 0, 1, \ldots, m_i \tag{92}$$

The probability of success p^s for the trade is given by

$$p^s = 1 - \frac{1}{N-1}\sum_{j \neq i} P(z_j = m_j | z_i = z) \tag{93}$$

$$\cong 1 - \frac{1}{N}\sum_j P_j(m_j) \qquad (N, M \gg 1) \tag{94}$$

where the last relation holds because when $N, M \gg 1$ the dependence on z and on agent i becomes negligible. This is important because it implies that for large N the variables z_i can be considered as independent, i.e., $P(z_1, \ldots, z_N) = \prod_i P_i(z_i)$, and the problem can be reduced to that of computing properties from the marginals $P_i(z)$. The probability p^s will be the main dynamical quantity of interest in this model, as an indicator of market activity: when p^s is close to 1, all trades succeed and we consider the market liquid, with goods trading hand frequently. When $p^s \to 0$, most transactions fail and the market becomes frozen.

We can plug an Ansatz in equation (92) and see that it is solved by a truncated Poisson distribution with parameter $\lambda = M/(Np^s)$:

$$P_i(z) = \frac{1}{Z_i}\left(\frac{\lambda^z}{z!}\right)\Theta\left(m_i - z\right), \tag{95}$$

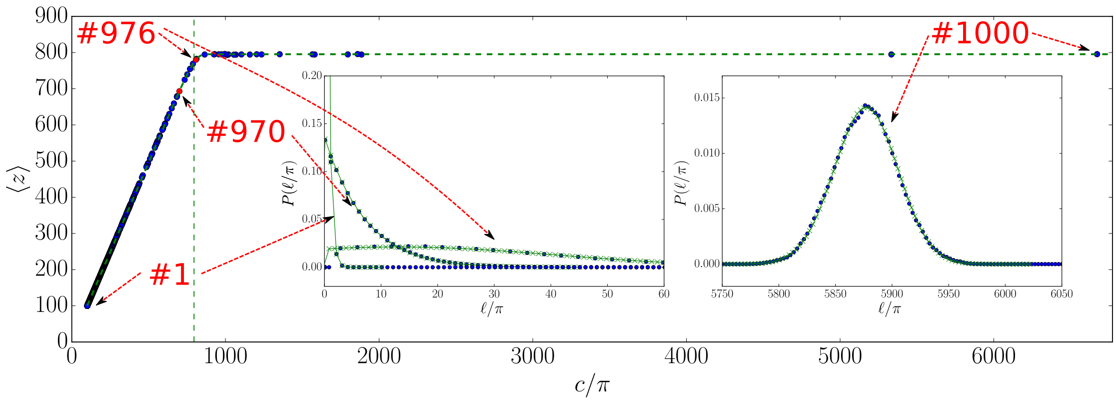

Figure 18.: **(Main)** Average number of owned goods in an economy with a single type of good, $N = 10^3$ agents, $\beta = 1.8$, $M \approx 2.10^5$ and $C/\Pi = 1.1$. Points $\{(\langle z \rangle_i, c_i)\}_{i=1}^N$ denote the average composition of capital for different agents obtained in Monte Carlo simulations, compared with the analytical solution obtained from the Master Equation (green dashed line) given by equation (95). The vertical dashed line at $c^{(1)} \simeq 7.98 = M/Np_1^s$ indicates the analytically predicted value of the crossover wealth that separates the two classes of agents. **(Insets)** Cash distributions $P_i(\ell)$ for indicated agents. Agent #1 is the poorest and has very high probability of owning few (around $z_i = 3$) goods, which quickly goes to zero at $z_i = 5$ goods. Agents #970 and #976 are near the class threshold and have reasonable probability of having several goods. Agent #1000 is the richest of the simulation and has two orders of magnitude more goods, with no probability of having less than a hundred times more than agent #976.

where Z_i is a normalization factor that can be determined by $\sum_z P_i(z) = 1$. The value of p^s, or equivalently of λ, can be found only in certain approximations of equation (94), which we will show later.

From the stationary distribution (95), we see that the most likely value of z for an agent with $m_i = m$ is given by

$$z^*(m) = \arg\max_z P(z) = \begin{cases} m, & \text{if } m \le \lambda \\ \lambda, & \text{if } m \ge \lambda \end{cases}. \tag{96}$$

This provides a natural distinction between cash-poor agents – those with $m \le \lambda$ – that often cannot afford to buy any other object, and cash-rich ones – those with $m > \lambda$ – who typically have enough cash to buy further objects. We can run computer simulations for the model and compare the goods distribution after equilibrium against theoretical predictions, the results are shown in figure 18. We indeed observe these two types of agents in the simulations, which agree very well with the prediction. The inset shows the cash distribution $P_i(\ell/\pi)$ (where $\ell/\pi = c_i/\pi - z$ represents the number of goods they are able to buy) for some representative agents. While cash-poor agents have a cash distribution peaked at 0, the wealthiest agents have cash in abundance. When $\lambda \gg 1$, the distribution $P_i(z)$ is sharply peaked around $z^{\text{mode}}(m)$ so that its average is $\langle z \rangle \simeq z^{\text{mode}}(m)$. Then the separation between the two classes becomes rather sharp, as it is the case for Figure 18.

In terms of wealth, we can use the threshold shown in equation (96) to define the threshold wealth $c^{(1)}$ for which the poor are defined as those with $c_i < c^{(1)}$ whereas the rich ones have $c_i > c^{(1)}$. This threshold wealth is given by the initial capital an agent requires to have $m = \lambda$. Because all goods have the same price, this is simply given by $\lambda\pi$. So the threshold wealth for the two classes $c^{(1)}$ is given by

$$c^{(1)} = \lambda\pi = M\pi/(Np^s) \tag{97}$$

We can compute p^s by using equation (94), $p^s = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(z = m_i)$, and approximating the probability to be on a threshold $P_i(z = m_i)$ by

$$P_i(z = m_i) = \begin{cases} \left(1 - \frac{m_i}{\lambda}\right) & \text{for } m_i \ll \lambda \\ 0 & \text{for } m_i > \lambda \end{cases} \tag{98}$$

The first case can be understood by noting that in the limit $m_i \ll \lambda$ we have the approximation

$$P_i(z = m_i) = \frac{\lambda^{m_i}\frac{1}{m_i!}}{\sum_{x=0}^{m_i}\lambda^x\frac{1}{x!}} = \frac{1}{1 + \frac{m_i}{\lambda} + \frac{m_i(m_i-1)}{\lambda^2} + \dots} \simeq \left(1 - \frac{m_i}{\lambda}\right), \tag{99}$$

Assuming this approximation to be valid for all $m_i < \lambda$ is clearly a bad assumption for all agents with m_i close to λ. However the wealth is power law distributed and so the weight of agents with $m_i \sim \lambda$ is negligible in the sum over all agents in equation (99). The accuracy of this approximation increases when the exponent of the power law β decreases and the mass of agents with capitals around λ vanquishes.

We use equation (98) and write $m_i \approx \frac{c_i}{\pi}$ to rewrite equation (94):

$$p^s = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(z = m_i) \simeq 1 - \frac{1}{N}\sum_{i=1}^{N}\left(1 - \frac{c_i}{\lambda\pi}\right)\Theta\left(c_i - \lambda\pi\right) \tag{100}$$

Because we have a very large number of agents, we are able to transform the sum over the agents into an integral over the capital c, with density equal to the capital distribution, ie, we replace

$$\frac{1}{N}\sum_i c_i \to \int_1^{\infty} c\beta c^{-\beta-1}dc \tag{101}$$

Thus when we truncate by $c_i \leq \lambda\pi = c^{(1)}$ we get

$$p^s = 1 - \int_1^{c^{(1)}=\lambda\pi} dc\,\beta c^{-\beta-1}\left(1 - \frac{c}{\lambda\pi}\right). \tag{102}$$

This is an implicit expression for p^s, since it appears on the left hand side of the equation and also inside the integral, because $\lambda = \frac{M}{Np^s}$, which makes it intractable to solve analytically.

However, we can get good insights on the behavior of p^s by again exploiting the fact that we are assuming a very large system and take $N, M \to \infty$. In this limit, λ can be replaced by its expected value on the realizations, i.e., for finite system sizes, M/N is a random variable that depends on the realization of the capital distribution due to the fixed constant Π/C, but in the limit we can replace M by it's expected value, Π/π and N by $C/\langle c \rangle$, where $\langle c \rangle$ is the expected capital per agent, which is given by the average of the beta distribution, $\langle c \rangle = \beta/(\beta - 1)$. For finite N, this is only a reasonable approximation if $\beta \gg 1$, and breaks down in the limit $\beta \to 1^+$ due to the infinite variance of the capital distribution, but it should be accurate for all $\beta > 1$ in the limit $N \to \infty$. Using the approximation on (97) and replacing $\lambda = c^{(1)}/\pi$, we have:

$$\frac{M}{N\lambda} \to \left\langle \frac{M}{N} \right\rangle \frac{\pi}{c^{(1)}} = \frac{\Pi}{C} \frac{\beta}{\beta - 1} \frac{1}{c^{(1)}} \tag{103}$$

So the equation for p^s is, by the definition of λ:

$$p^s = \frac{M}{N\lambda} = \frac{\Pi}{C} \frac{\langle c \rangle}{c^{(1)}}, \tag{104}$$

which gives us p^s but as a function of $c^{(1)}$, which we still don't know. But now that this is independent of p^s, we can put this expression back into (102) and carry out the integration to get an analytic form for $c^{(1)}$:

$$\frac{\Pi}{C} \frac{\langle c \rangle}{c^{(1)}} = c^{(1)^{-\beta}} \left(\frac{1}{1 - \beta} \right) - \frac{\beta}{1 - \beta} \frac{1}{c^{(1)}}, \tag{105}$$

Solving for $c^{(1)}$ we have

$$c^{(1)} = \left[\beta \left(1 - \frac{\Pi}{C} \right) \right]^{1/(1-\beta)}. \tag{106}$$

And replacing this back on equation (104) gives p_s as a function of the intensive variables for the economy

$$p^s = \frac{\Pi}{C} \frac{\beta}{\beta - 1} \frac{1}{\left[\beta \left(1 - \frac{\Pi}{C} \right) \right]^{1/(1-\beta)}}. \tag{107}$$

When the inequality in wealth becomes too large, in the limit $\beta \to 1^+$, $\langle c \rangle$ diverges, but within this approximation the threshold wealth $c^{(1)}$ diverges much faster. More precisely, we note that $\Pi/C < 1$, so that $\beta(1 - \Pi/C) \sim (1 - \Pi/C)$ is a number smaller than 1 (yet positive). From equation (106), we have $c^{(1)} \sim (1 - \Pi/C)^{-1/(\beta - 1)} \to \infty$ and therefore the liquidity p^s vanishes as $\beta \to 1^+$. We now arrive at the main result of this work: when the distribution of capital gets too unequal, the probability of successful transactions vanishes and the economy freezes.

For finite N, this approximation breaks down when β gets too close to or smaller than one, because $\langle c \rangle$ is ill-defined and in equation (104) it should be replaced with

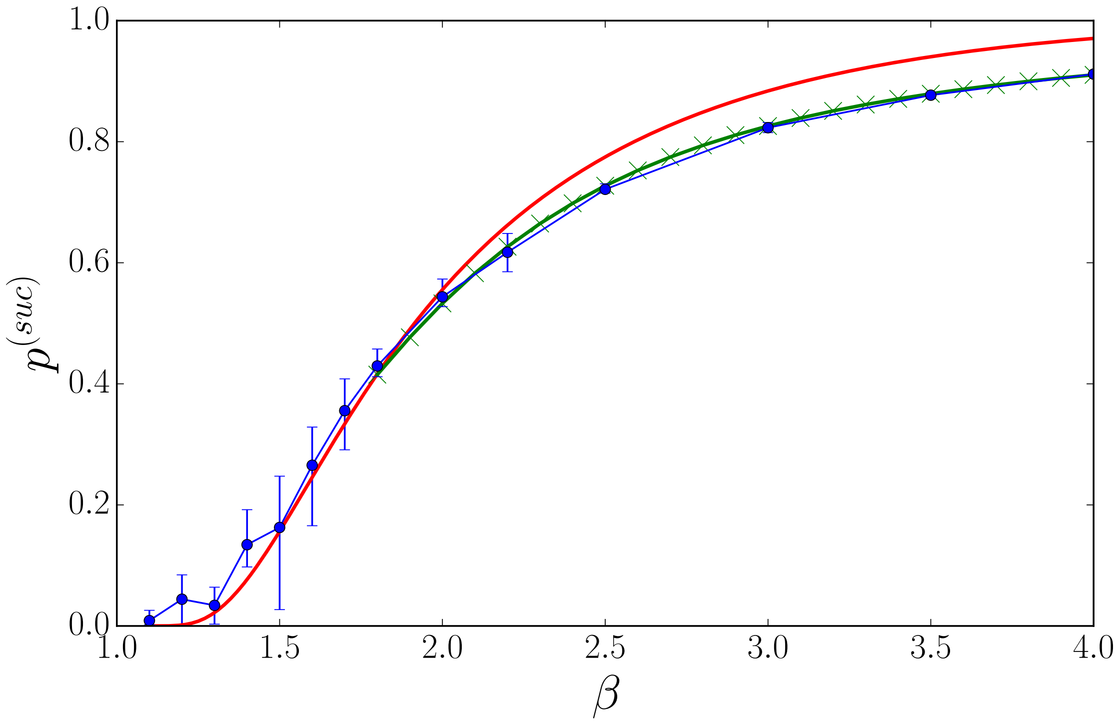

Figure 19.: Comparison between numerical simulations and analytical estimates for the success probability of transaction p^s for one class of goods, as a function of the Pareto exponent β. The blue solid circles are the result of Monte Carlo simulations performed for $N = 10^5$ agents and averaged over 5 realizations, with the error bars indicating the min and max value of p^s over all realizations. The red lines are the analytic estimates according to equation (107). The green crossed lines correspond to numerically iterating equation (100) for a population composed of $N = 64$ "agents". For lower β the capital distribution becomes too broad and this solution becomes unfeasible.

$1/N \sum_i c_i$, which strongly fluctuates between realizations and depends on N. An estimate of p^s for finite N and $\beta < 1$ can be obtained by observing that the wealth $c^{(1)}$ that marks the separation between the two classes cannot be larger than the wealth $c_{\max}$ of the wealthiest agent. By extreme value theory, the latter is given by $c_{\max} \sim N^{1/\beta}$, with $a > 0$. Therefore the solution is characterised by $c^{(1)} = \pi\lambda \sim c_{\max} \sim N^{1/\beta}$. Furthermore, for $\beta < 1$ the average wealth is dominated by the wealthiest few, i.e. $\langle c \rangle \sim N^{1/\beta - 1}$ and therefore $p^s \sim N^{1/\beta-1}/c^{(1)} \sim N^{-1}$. In other words, in this limit the cash-rich class is composed of a finite number of agents, who hold almost all the cash of the economy. In regimes such as this, not only the wealthiest few individuals own a finite fraction of the whole economy's wealth, as observed in [19], but they also drain all the financial resources in the economy.

7.3 the case of K types of goods

The analysis presented in the last section carries over to the general case in which K classes of goods are considered, starting from the full Master Equation for the joint probability of the ownership vectors $\vec{z}_i = (z_{i,1} \ldots, z_{i,K})$ for all agents $i = 1, \ldots, N$. For the same reasons as before, the problem can be reduced to that of computing the marginal distribution $P_i(\vec{z}_i)$ of a single agent. The main complication is that the maximum number $m_{i,k}$ of goods of class k that agent i can get now depends on how many of the other goods agent i owns, i.e. $m_{i,k}(z_i^{(k)}) = \lfloor (c_i - \sum_{k'(\neq k)} z_{i,k'}\pi_{(k')})/\pi_k \rfloor$, where $z_i^{(k)} = \{z_{i,k'}\}_{k'(\neq k)}$.

Again, as with the single good case, because all transition rates are symmetric we can write the detailed balance condition in a stricter manner: the probability of going from one specific state to another has to be the same as doing the reverse trajectory. In this case, however, all the exchanges are confined to a dimension in the K-dimensional space of ownership, ie, an agent can go from z to either $z + \hat{e}_k$ or $z - \hat{e}_k$, where $\hat{e}_k$ is the vector with all zero components and with a k^{th} component equal to one, but not to $z + \hat{e}_k - \hat{e}_{k'}$. Therefore, the stationary distribution $P_i(z)$ has to satisfy the strict detailed balance condition for all k.

We write the probability of agent i going from z to $z + \hat{e}_k$ as $P_i(z \to z + \hat{e}_k) = \frac{M_k - z_k}{M}\frac{1}{N}\left(1 - \delta_{z_k, m_{i,k}(z_{(k)})}\right)$, the exact analogous of the single good case. Likewise, $P(z + \hat{e}_k \to z) = \frac{z_k}{M} p_k^s$, where p_s^k is the probability that a sale of an object of type k is successful.

Putting it all together with the the approximations for the $N, M \to \infty$ limit, $\frac{1}{N-1} \approx \frac{1}{N}$ and $\frac{M_k - z_k}{M} \approx \frac{M_k}{M}$, assuming $z_k \ll M_k$, we have the detailed balance condition for goods of type k:

$$P_i(\vec{z} + \hat{e}_k)\frac{z_k + 1}{M} p_k^s = P_i(\vec{z})\frac{M_k}{M}\frac{1}{N}\left(1 - \delta_{z_k, m_{i,k}(z_{(k)})}\right) \tag{108}$$

It can easily be checked that a solution to this set of equations is given by a product of Poisson distributions with parameters $\lambda_k = M_k/(N p_k^s)$, with the constraint given by equation (86)

$$P_i(z_1, ..., z_K) = \frac{1}{Z_i}\left(\prod_{k=1}^{K} \frac{\lambda_k^{z_k}}{z_k!}\right)\Theta\left(c_i - \sum_k^K z_k \pi_{(k)}\right), \tag{109}$$

where Z_i is a normalization factor obeying $\sum_{z_1} ... \sum_{z_K} P_i(z_1, ..., z_K) = 1$. Each probability of success p_k^s is given by

$$p_k^s = 1 - \frac{1}{N}\sum_{i=1}^{N} P\left(z_{i,k} = m_{i,k}(z_i^{(k)})\right) \tag{110}$$

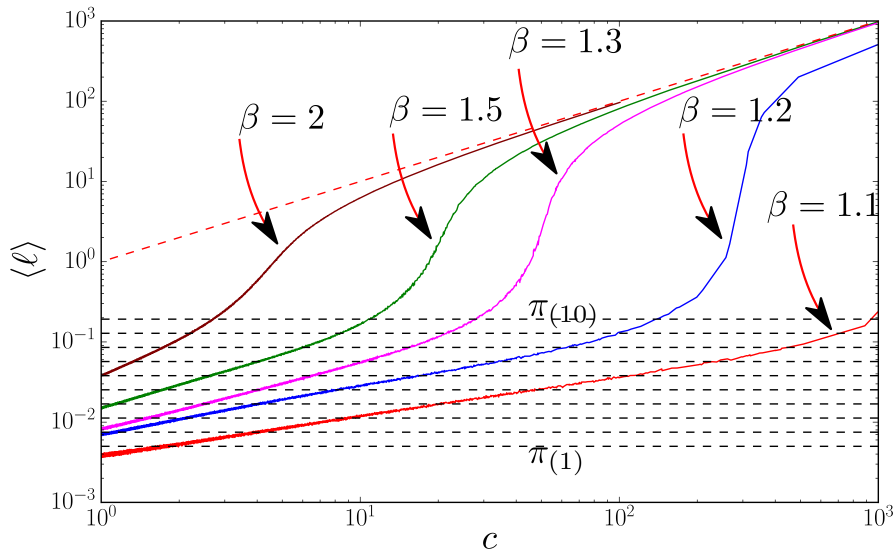

Figure 20.: Time averaged cash $\langle \ell_i \rangle$ as a function of wealth c_i, from $\beta = 1.1$ to $\beta = 2$ for a system of $N = 10^5$ agents exchanging $K = 10$ classes of goods ($\pi_{(k)} = \pi_{(1)} g^{k-1}$ with $g = 1.5$, $\pi_{(1)} = 0.005$, $M_k \pi_{(k)} = \Pi/K$ and $C/\Pi = 1.2$). The dashed lines indicate the different prices of goods. Agents with $\langle \ell_i \rangle$ below the price of a good typically have not enough cash to buy it. Cash is proportional to wealth for large levels of wealth (see the upper straight red dashed line).

When the total number of objects per agent is large for any class k, we expect that $\lambda_1, ..., \lambda_K \gg 1$, and then the average values of $z_{i,k}$ are close to their most likely values, as in the single good case. This means that, as with the single good case, and agent with wealth $c_i < \lambda_k \pi_k = c^{(k)}$ will be saturated with goods of type $k' \leq k$ and won't be able to afford goods of type $k'' \geq k$.

The consequence is that the population of agents now splits into K classes, defined by the intervals $c_i \in [c^{(k-1)}, c^{(k)}]$, each filled with objects cheaper than k and unable to purchase more expensive ones. This structure into classes can be seen in the computer simulations of Figure 20, where we present the average cash $\langle \ell_i \rangle$ of agents as a function of their initial wealth c_i. The horizontal lines denote the prices $\pi_{(k)}$ of the different objects, and the intersections with the horizontal lines define the thresholds $c^{(k)}$.

To see the effect of inequality in the trade activity, we must again find an analytical expression for the liquidities p^s, which are given by the following expression

$$p_k^s = 1 - \frac{1}{N} \sum_{i=1}^{N} P\left(z_{i,k} = m_{i,k}(z_i^{(k)})\right) = 1 - \frac{1}{N} \sum_{i=1}^{N} P_i(\text{not accepting good type k}) \quad (111)$$

An analytic derivation for the p_k^s and $c^{(k)}$ can be obtained only in the limit in which prices are well separated (i.e. $\pi_{(k+1)} \gg \pi_{(k)}$) and the total values of good of any class is approximately constant (we use $M_k \pi_{(k)} = \Pi/K = \text{const}$), because in this limit

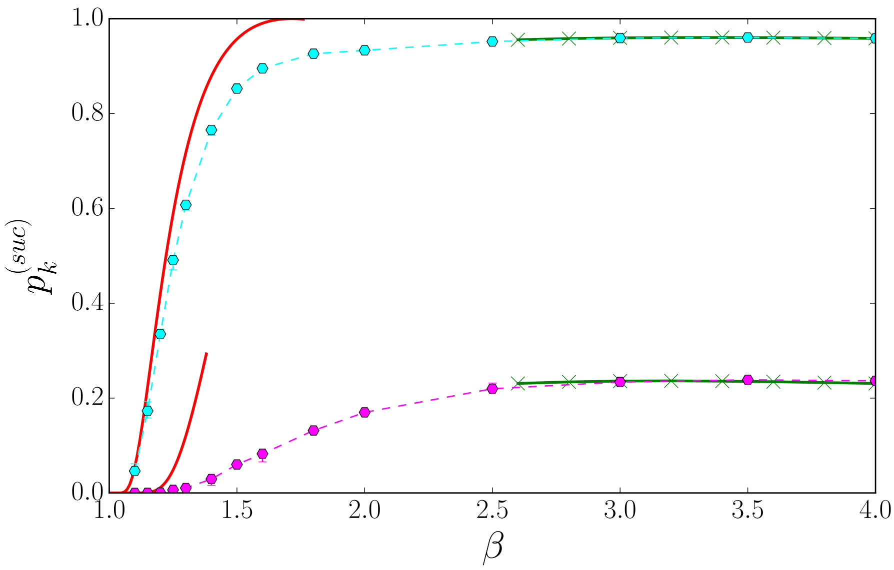

Figure 21.: Comparison between numerical simulations and analytical estimates of the success probability of transaction p_k^s for two classes of goods as a function of the Pareto exponent β. The blue solid circles are the result of Monte Carlo simulations performed for $N = 10^5$ agents and averaged over 5 realizations, with the error bars indicating the min and max value of p_k^s over all realizations. The red lines are the analytic estimates according to equation (116). The green crossed lines correspond to numerically solving the analytical solution (109) for a population composed of $N = 64$ agents.

we expect to find a sharp separation of the population of agents into classes. When the prices are separated by an order of magnitude, then $M_1 \gg M_2 \gg \ldots \gg M_K$, which implies that the market is flooded with objects of the class 1, which constantly change hands and essentially follow the laws found in the single type of object case. On top of this dense gas of objects of class 1, we can consider objects of class 2 as a perturbation (they are picked M_2/M_1 times less often). On the time scale of the dynamics of objects of type 2, the distribution of cash is such that all agents with a wealth less than $c^{(1)} = \pi_{(1)}\lambda_1$ have their budget saturated by objects of type 1 and typically do not have enough cash to buy objects of type 2 nor more expensive ones. Likewise, there is a class of agents with $c^{(1)} < c_i \leq c^{(2)}$ that will manage to afford goods of types 1 and 2, but will hardly ever hold goods more expensive that $\pi_{(2)}$, and so on. In this scenario we can write the probability of not accepting a good of type k for an agent in the same linear fashion as in equation (98) for $K = 1$:

$$P_i(\text{not accepting good type k}) = \begin{cases} 1 & \text{for } m_i \leq \lambda_{k-1} \\ \left(1 - \frac{m_i}{\lambda_k}\right) & \text{for } \lambda_{k-1} < m_i < \lambda_k \ , \\ 0 & \text{for } m_i \geq \lambda_k \end{cases} \tag{112}$$

Then p^s_k becomes a simple integral by employing the same steps taken on equations (100)-(102).

$$p^s_k \simeq 1 - \int_1^{c^{(k-1)}} dc\, \beta c^{-\beta-1} - \int_{c^{(k-1)}}^{c^{(k)}} dc\, \beta c^{-\beta-1} \left(1 - \frac{c}{c^{(k)}}\right) \tag{113}$$

In the K goods case we now again replace $\frac{M}{N}$ by its average $\frac{\Pi}{C}\frac{\beta}{\beta-1}\frac{1}{\pi}$ to find, from the definition of λ_k:

$$p^s_k = \frac{M_k}{N\lambda_k} = \frac{\Pi}{KC}\frac{\langle c\rangle}{c^{(k)}} \tag{114}$$

Plugging this back again on the integral above, we get the general recurrence relation for k goods.

$$c^{(k)} = \left(\beta \left(c^{(k-1)}\right)^{1-\beta} - \beta\frac{\Pi}{KC}\right)^{\frac{1}{1-\beta}} . \tag{115}$$

Iterating, we write it as a function of the intensive variables

$$c^{(k)} = \left[\beta^k - \left(\frac{\beta - \beta^{k+1}}{1-\beta}\right)\frac{\Pi}{KC}\right]^{\frac{1}{1-\beta}} , \tag{116}$$

And finally p^s_k as a function of the intensive variables:

$$p^s_k = \frac{M_k}{N\lambda_k} \simeq \frac{\Pi}{KC}\frac{\langle c\rangle}{c^{(k)}} . \tag{117}$$

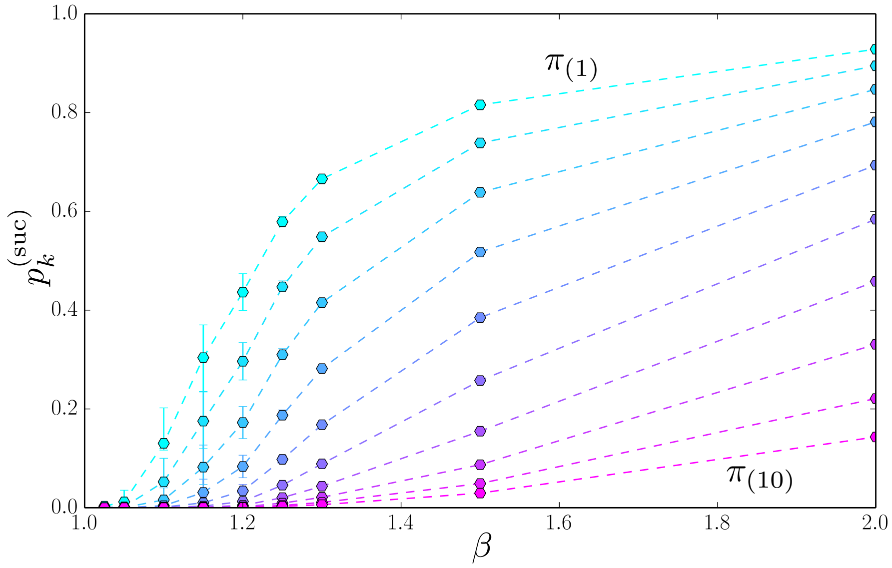

Figure 22.: Liquidity of goods $\{p_k^s\}_{k=1}^K$ as a function of the inequality exponent β for a system of $N = 10^5$ agents exchanging $K = 10$ classes of goods, with parameters $\pi_{(k)} = \pi_{(1)} g^{k-1}$ with $g = 1.5$, $\pi_{(1)} = 0.005$, $M_k \pi_{(k)} = \Pi/K$ and $C/\Pi = 1.2$. Note that all success rates p_k^s vanish when $\beta \to 1^+$. The curves are ordered from the cheapest good (top) to the most expensive (bottom). The markers are the result of numerical simulations, with error bars indicating the minimum and maximum values obtained by averaging over 5 realizations of the wealth allocations.

In the limit $\beta \to 1^+$ of large inequality, close inspection[2] of equation (116) shows that $c^{(k)} \to \infty, \forall k$, which implies that all agents become cash-starved except for the wealthiest few. Since $p_k^s \sim \langle c \rangle / c^{(k)}$, this implies that all markets freeze: $p_k^s \to 0, \forall k$. The arrest of the flow of goods appears to be extremely robust against all choices of the parameter $\pi_{(k)}$, as p_1^s is an upper bound for the other success rates of transactions p_k^s. These conclusions are fully consistent with the results of extensive numerical simulations, as illustrated in figure 22, in which we simulate an economy with $K = 10$ classes of goods (see figure caption for details) and different values of β. As expected, for a fixed value of the Pareto exponent β the success rate increases as the goods become cheaper, as they are easier to trade. It also shows that trades of all classes of goods halt as β tends to unity, which is when wealth inequality becomes too large, independently of their price.

An alternative way to interpret the freezing of the economy is to compare the cash and capital inequalities via their Gini coefficients. The Gini coefficient is a measure

2 Note that the term in square brackets is smaller than one, when $\beta \to 1^+$.

of inequality in a distribution: it is a function of the relative mean of the absolute difference among all the elements of the distribution, that is

$$G(\{x_i\}) = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} \|x_i - x_j\|}{2N \sum_{i=1}^{N} x_i} \tag{118}$$

The normalization term is so that the Gini coefficient has a support on $[0, 1]$ independent of the distribution, and it is invariant to scalings of the type $x_i \to \lambda x_i$. The measure is most commonly used in Economics, for measuring income inequality among different countries, and a Gini coefficient of 0 means perfect equality, where every point in a distribution is the same, whereas a Gini coefficient close to 1 means perfect inequality: only one point of the (presumed large) distribution is nonzero.

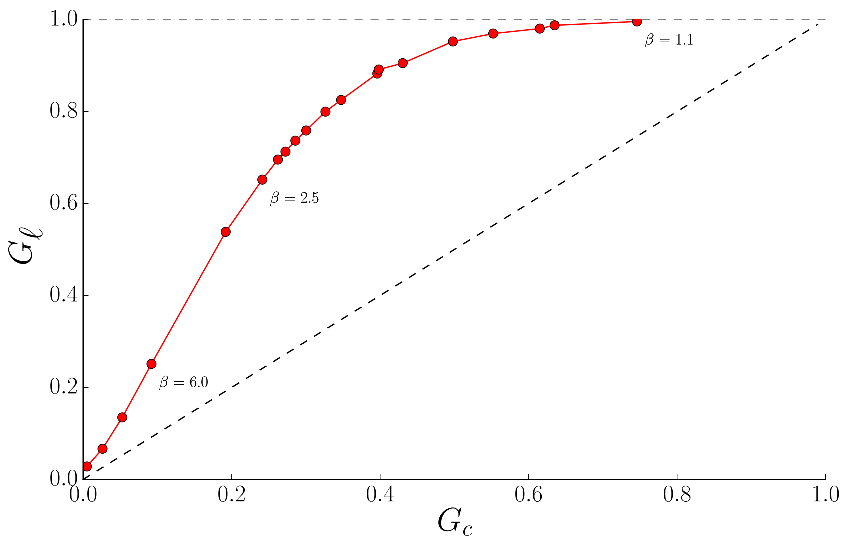

Figure 23.: Gini coefficient G_ℓ of the cash distribution (liquid capital) in the stationary state as a function of the Gini G_c of the wealth distribution, calculated through the numeral simulations of figure 22. The dashed line indicates proportionality between cash and wealth, in which case the inequality in both is the same.

For our economy, we plot on figure 23 for various values of β the Gini coefficient for cash G_ℓ as a function of the Gini coefficient for the capital G_c in the $K = 10$ system of figure 22. The dashed line is the case where a certain inequality of capital implies the exact same inequality of cash. We see that the liquidity over concentrates, being much more unequal than the original capital distribution, approaching perfect inequality, i.e., concentrating in the hands of few agents much faster than the capital.

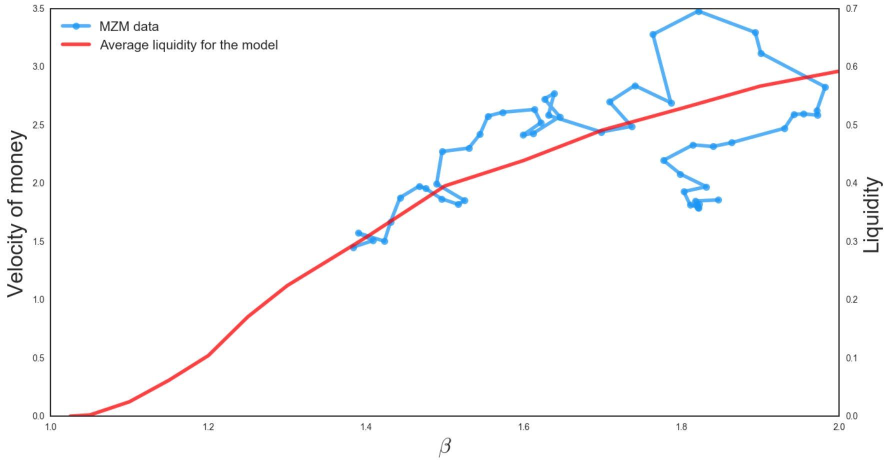

Figure 24.: Comparison of MZM money velocity from the US data to average liquidity (as defined in equation (119)) calculated in the simulations of figure 22.

Note finally that p_k^s quantifies liquidity in terms of goods. In order to have an equivalent measure in terms of cash that can be compared to the velocity of money described in the Introduction, we average $\pi_{(k)} p_k^s$ over all goods

$$\bar{p}^s = \frac{1}{\Pi} \sum_{k=1}^{K} M_k \pi_{(k)} p_k^s. \tag{119}$$

This quantifies the frequency with which a unit of cash changes hand in our model economy as a result of a successful transaction. It's behaviour as a function of β for the same parameters of the economy in Figure 22 is shown on Figure 24. Even though we cannot make a one to one comparison with real world data, we see that the freezing we observe in the model due to high inequality is, at least qualitatively, corroborated by the data.

7.4 CONCLUSIONS

We have introduced in this chapter a zero-intelligence trading dynamics in which agents have a Pareto distributed wealth and randomly trade goods with different prices. We have shown that this dynamics leads to a uniform distribution in the space of the allocations that are compatible with the budget constraints and when the inequality in the distribution of wealth increases, the economy converges to an equilibrium where typically (i.e. with probability very close to one) the less wealthy agents have less and less cash available, as their budget becomes saturated by objects of the cheapest type. At the same time this class of cash-starved agents takes up a larger and larger fraction of the economy, thereby leading to a complete halt of the economy when the distribution of wealth becomes so broad that its expected average diverges (i.e. when $\beta \to 1^+$). In these cases, a finite number of the wealthiest agents own almost all the cash of the economy.

The model presented is intentionally simple, so as to highlight a robust and quantifiable link between inequality and liquidity. In particular, the model neglects important aspects such as *i)* agents' incentives and preferential trading, *ii)* endogenous price dynamics and *iii)* credit. It is worth discussing each of these issues in order to address whether the inclusion of some of these factors would revert our finding that inequality and liquidity are negatively related.

First, our model assumes that all exchanges that are compatible with budget constraints will take place, but in a more realistic setting only exchanges that increase each party's utility should take place, just as we have approached in the rest of this thesis. Yet if the economy freezes in the case where agents would accept all exchanges that are compatible with their budget, it should also freeze when only a subset of these exchanges are feasible. The model also assumes that all agents trade with the same frequency whereas one might expect that rich agents trade more frequently than poorer ones. Could liquidity be restored if trading patterns exhibit some level of homophily, with rich people trading more often and preferentially with rich people?

First we note that both these effects are already present in our simple setting. Agents with higher wealth are selected more frequently as sellers as they own a larger share of the objects. In spite of the fact that buyers are chosen at random, successful trades occur more frequently when the buyer is wealthy. So, in the trades actually observed the wealthier do trade more frequently than the less wealthy, and preferentially with other wealthy agents. Furthermore, if agents are allowed to trade only with agents having a similar wealth (e.g. with the q agents immediately wealthier or less wealthy) it is easy to show that detailed balance still holds with the same uniform distribution on allocations. As long as all the states are accessible, the stationary probability distribution remains the same: the dynamics would change and thus p_k^s would too change, in particular for goods more expensive than $\pi_{(1)}$, the seller is typically cash-rich and thus its neighbours are too. This can induce to have a liquidity of expensive goods higher than that of cheaper ones. However in the limit $\beta \to 1^+$, it is still true that cash concentrates in the hands of a vanishing fraction of agents, and there is still a freeze of the economy. Therefore, the model conclusions are robust with respect to a wide range of changes in its basic setting that would account for more realistic trading patterns.

Secondly, it is reasonable to expect that prices will adjust – i.e. deflate – as a result of a diminished demand caused by the lack of liquidity. Within the model, the inclusion of price adjustment, occurring on a slower time-scale than trading activity, would reduce the ratio Π/C (between total value of goods and total wealth), but it would also change the wealth distribution. Since the freezing phase transition occurs irrespective of the ratio Π/C, the first effect, though it might alleviate the problem, would not change the main conclusion. The second effect would make it more compelling, because cash would not depreciate as prices do, so deflation would leave wealthy agents – who hold most of the cash – even richer compared to the cash deprived agents, that would suffer the most from deflation. So while price adjustment apparently increases liquidity, this may promote further inequality, which would curtail liquidity further.

Finally, can the liquidity freeze be avoided by allowing agents to borrow? Access to credit will hardly improve the situation. Allowing agents to borrow using goods as collaterals is equivalent to doubling the wealth of cash-starved agents, provided that any good can be used only once as a collateral, and that goods bought with credit cannot themselves be used as collaterals. This would at most blur the crossover between cash-rich agents and cash-starved ones, as intermediate agents would sometimes use credit. This does not change the main conclusion that inequality and liquidity are inversely related and that the economy would halt when $\beta \to 1^+$. This is in line with the results in [92] and for similar reasons. Credit may mitigate illiquidity in the short term, but cash deprived agents would have to borrow from wealthier ones. With positive interest rates, this would make inequality even larger in the long run. Credit is therefore likely to make things worse, in line with the arguments[3] in [66].

Therefore, even though the model presented here can be enriched in many ways, it's not clear what would revert the relation between inequality and liquidity.

Corroborating the present model with empirical data goes beyond the scope of this work, yet we remark that our findings are consistent with recent economic trends, as shown in Figure 17. For example, it is worth observing that, alongside with increasing levels of inequality, trade has slowed down after the 2008 crisis[4]. More generally, avoiding deflation, or promoting inflation, has been a major target of monetary policies after 2008, which one could take as an indirect evidence of the slowing down of the economy. Furthermore, the fact that inequality hampers liquidity and hence promotes demand for credit suggests that the boom in credit market before 2008 and the increasing levels of inequality might not have been a coincidence.

An interesting side note is that the concentration of capital in the top agents goes hand in hand with a flow of cash to the top. Indeed, in the model an injection of extra capital in the lower part of the wealth pyramid –the so-called *helicopter money* policy– is necessarily followed by a flow of this extra cash to the top, via many intermediate agents, thus generating many transactions on the way. This *trickle up* dynamics should be contrasted with the usual idea of the *trickle down* policy, which advocates injections of money to the top in order to boost investment. In this respect, it is tempting to relate our findings to the recent debate on Quantitative Easing measures, and in par-

3 Piketty [66] observes that when the rate of return on capital exceeds the growth rate of the economy (which is zero in our setting), wealth concentrates more in the hands of the rich.

4 The *U.S. Trade Overview, 2013* of the International Trade Administration observes that "Historically, exports have grown as a share of U.S. GDP. However, in 2013 exports contributed to 13.5% of U.S. GDP, a slight drop from 2012"' (see `http://trade.gov/mas/ian/tradestatistics/index.asp#P11`). A similar slowing down can be observed at the global level, in the UNCTAD *Trade and Development Report, 2015*, page 7 (see `http://unctad.org/en/pages/PublicationWebflyer.aspx?publicationid=1358`).

ticular to the proposal that the (European) central bank should finance households (or small businesses) rather than financial institutions in order to stimulate the economy and raise inflation [59, 26]. Clearly, our results support the helicopter money policy, because injecting cash at the top does not disengages the economy from a liquidity stall.

Extending our minimal model to take into account the endogenous dynamics of the wealth distribution and of prices, accounting for investment and credit, is an interesting avenue of future research, for which the present work sets the stage. In particular, this could shed light on understanding the conditions under which the positive feedback between returns on investment and inequality, that lies at the very core of the dynamics which has produced ever increasing levels of inequality according to [67, 66, 70], sets in.

8

CONCLUSION

The main objective of this thesis was illustrating the similarity of objectives in Statistical Mechanics and Economics, and how the interdisciplinary field arising from applying methods of the first into the latter is promising and worth delving into. In the earlier part of this thesis, we made the case that study of Economics is built on models characterized by the solution of maximization problems, for which Statistical Mechanics offer a wealth of tools suitable for exploration, even in the case of complex systems with a large number of interacting agents. Most importantly, we showed that this is due not to naively applying Physics laws to economic system, but because the foundation of Statistical Physics are insensitive to subject matter. This allows us to drawn upon the body of knowledge built over the last decades that deals with the rich phenomena of phase transitions, spin glasses, critically, universality, etc, to study complex systems of all sorts, including economic ones.

We then presented three specific problems in which we show how this interdisciplinary application can be fruitful. In the problem of the inefficient consumer, we have shown how Statistical Mechanics offer a principled framework for dealing with "irrational" behavior, by treating it as a problem with non zero temperature. In our specific application, we have shown that irrational consumers may increase GDP without an increase in the average utility. We are confident that this same procedure may be applied for most equilibrium situations in economics, and the question this work leaves is: what sort of unexpected behavior can we observe in traditional economic models involving agents with varying rationality?

In the problem of the Input-Output matrices, we have shown how the proper employment of Information Theory let us identify when aggregation of real world data destroyed the information contained in it, allowing for wrong conclusions to be drawn from it. The cautionary tale of the Input-Output matrices serve as a lesson that aggregation methods matter, a lesson applicable to all economic statistics. If one is not aware of this danger, the analysis of economic data may be biased. On the other hand, aware of this pitfall of aggregation, we may develop better methods that preserve crucial data structure.

Finally, in the problem of random trading with inequality, we have shown that one can observe relevant features in zero intelligence based models, offering a perspective of what are the ”entropic” stationary states of real world trade dynamics. We have shown that spontaneous trades tend to concentrate the wealth, which in turns leads to reduce the liquidity of a market. This is relevant because it sheds light into the importance of offsetting this ”entropic” wealth concentration. Being a minimal model, there is a large amount of possibilities for further work in this direction by studying the effects of additional ingredients from real economies, such as return on available capital, but also one can use the model to test strategies that offset the increase of the inequality, as for example government taxation on purchases, wealth transfer programs, etc. This sort of analysis would deepen the discussion on social policies and provide a testing ground for planners.

Admittedly, the problems approached in this thesis merely scratch the surface of what can be done in the crossroads. It’s our belief that Statistical Physics and Economics have an very large potential, capable of shifting the status quo of economics just as other fields, such as psychology, with the works of Tversky and Kahneman, and mathematics, with the work in Game Theory of von Neumann and Nash, have done in the past. This thesis would serve its purpose if it inspires the reader that this is a path worth pursuing.

A

CALCULATION OF THE PARTITION FUNCTION FOR THE RANDOM LINEAR ECONOMY MODEL

In this appendix we provide details of the calculation for the partition function described in section 4.2 and originally presented in [30]. Although the analytical form for the maximization will not be used in the problems addressed in this thesis, we believe it may be instructive to the reader that is not familiar with the replica method. The level of detail employed here is not published anywhere, and thus we consider it a pedagogical contribution of this thesis.

To solve the maximization problem we need to calculate the following integrals (from equation (52)):

$$\max_x U(x) = \int d\xi dx_0 P(\xi) P(x_0) \lim_{\beta\to\infty} \log \overbrace{\int dx\, e^{\beta U(x|\xi,x_0)}}^{Z(\beta|\xi,x_0)} \tag{120}$$

We also know from (41) that $x = x_0 + \sum_{i=1}^{N} s_i \xi_i$, so we insert this constraint as an integral in $Z(\beta)$:

$$Z(\beta|\xi, x_0) = \int_0^\infty ds \int_0^\infty dx\, e^{\beta U(x)} \delta\left(x - x_0 - \sum_i s_i \xi_i\right) \tag{121}$$

Carrying out the integration in equation (120) is extremely hard, mainly because the log function in the integrand prevents us to factorize the integrals from the coupling created by ξ_i^μ and the market clearing condition. This is a recurrent problem when calculating free energies energy of disordered systems in Statistical Mechanics, which was solved by a clever and extremely useful technique to deal with the logarithm function, the so called **replica method** [61], which consists in writing $\log Z$ as:

$$\log Z = \lim_{r\to 0} \frac{Z^r - 1}{r} \tag{122}$$

The identity above is still exact, but the clever part of the method is exchanging the $\lim_{r\to 0}$ term with the rest of the integrals, treating r like an integer throught the whole calculation: Z^r is written as a product of independent partition functions, i.e., $Z^r = Z_1 Z_2 \dots Z_r$, each integrated over their own dynamical variables x^a and s^a, $a =$

$1, \ldots, r$. These multiple independent systems are the *replicas* that give the method its name. This may seem strange at first, but the beauty of the replica method is that this change of operations (between the limit and the integrals) works very well and has recently been proved to be correct [85].

The general strategy of the full calculation will be as follows: we exchange the order of integrations and limits until we have

$$\lim_{N\to\infty} \frac{1}{N} \max_x U(x) = \lim_{r\to\infty} \frac{1}{N} \lim_{\beta\to\infty} \int d\xi dx_0 P(\xi) P(x_0) \frac{Z(\beta|\xi, x_0)^r - 1}{r} \tag{123}$$

The $\frac{1}{N}$ factor was added to avoid the divergence of $U(x)$ in the limit $N \to \infty$[1]. The term $\int d\xi dx_0 P(\xi) P(x_0) \frac{Z(\beta|\xi,x_0)^r - 1}{r}$ is the partition function averaged over the disorder. Because only $Z(\beta|\xi, x_0)$ depends on ξ and x_0, we write it as

$$\int d\xi dx_0 P(\xi) P(x_0) \frac{Z(\beta|\xi, x_0)^r - 1}{r} = \frac{\langle Z^r \rangle_{\xi, x_0} - 1}{r} \tag{124}$$

Where $\langle \cdot \rangle_{\xi, x_0}$ indicates the average over the disorder. Arriving at a final expression for the term $\langle Z^r \rangle_{\xi, x_0}$ is the bulk of the work in calculating $\max U(x)$, but the goal is to write it in the form

$$\langle Z^r \rangle_{\xi, x_0} = \int d\theta \, e^{\beta N r h(\theta)}, \tag{125}$$

where θ is a vector of order parameters. Because we assume $N \to \infty$, the integral is dominated by it's maximal value, θ^*. We then take the series expansion around θ^* and keep the first two terms, i.e.,

$$\int d\theta \, e^{\beta N r h(\theta)} = e^{\beta N r h(\theta^*)} \approx 1 + \beta N r h(\theta^*) \tag{126}$$

Finally, we plug this approximation into equation (123) to get

$$\lim_{N\to\infty} \frac{1}{N} \max_x U(x) = h(\theta^*) \tag{127}$$

With this strategy in mind, we now begin calculating $\langle Z^r \rangle_{\xi, x_0}$ proper. Using the replica assumption, we expand Z^r as

1 The divergence actually comes from the limit $M \to \infty$ because $U(x)$ is linear in M, but because we assume $n = N/M$ fixed, scaling on N is equivalent to scaling on M

$$Z^r = \prod_{a=1}^{r} \int_0^\infty ds^a \int_0^\infty dx^a e^{\beta U(x^r)} \delta\left(x^a - x_0 - \sum_i s_i^a \xi_i\right) = \tag{128}$$

$$= \int_0^\infty ds^1 \int_0^\infty dx^1 e^{\beta U(x^1)} \delta\left(x^1 - x_0 - \sum_i s_i^1 \xi_i\right) \times \tag{129}$$

$$\times \ldots \times \tag{130}$$

$$\times \int_0^\infty ds^r \int_0^\infty dx^r e^{\beta U(x^r)} \delta\left(x^r - x_0 - \sum_i s_i^r \xi_i\right)$$

Gathering all the terms together:

$$Z^r = \int_0^\infty \prod_{a=1}^{r} dx_a \int_0^\infty \prod_{a=1}^{r} ds_a e^{\beta \sum_a U(x_a)} \prod_{a=1}^{r} \prod_{\mu=1}^{M} \delta\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right) \tag{131}$$

We write explicitely the distributions for x_0 and ξ:

$$P(x_0) = \prod_\mu e^{-x_0^\mu} \tag{132}$$

and

$$P(\xi_i) = \frac{1}{P_\xi} \prod_{\mu=1}^{M} \frac{1}{\sqrt{2\pi M^{-1}\Delta^2}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta^2}} \delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right), \tag{133}$$

where P_ξ is the normalization term given by

$$P_{\xi_i} = \int_0^\infty \prod_{\mu=1}^{M} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} \delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right) \tag{134}$$

The integral over x_0 we can leave to the end because they are already factored and involve no other terms except the initial endowments. The integrals over ξ_i^μ, however, are coupled due to the normalization term. Because the δ integral is not feasible due to the couplings, in order to calculate P_{ξ_i}, and throughout this appendix, we will make use of an important identity for the Dirac delta function, in which we replace it by it's Fourier transform:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \tag{135}$$

With this identity we are able to integrate all the terms in the normalization term P_ξ:

$$P_{\xi_i} = \int_0^\infty \prod_{\mu=1}^{M} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} \int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik(\sum_\mu \xi_i^\mu + \epsilon)} = \tag{136}$$

$$\int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik\epsilon} \prod_{\mu=1}^{M} \int_0^\infty d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta} + ik\xi_i^\mu} \tag{137}$$

We also use another useful identity to solve the Gaussian integral in ξ_i^μ. The integral of e^{-ax^2+bx} can be easily done if we complete the square:

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \tag{138}$$

This identity is useful in both directions: sometimes we would like to carry out an integration, and then we go from the left hand side to the right hand side. And sometimes, we would like to linearize a squared term (b in this case), and we go from the right hand side to the left hand side, gaining an integral in the process. We now use it to integrate equation (136):

$$P_{\xi_i} = \int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik\epsilon} e^{-M\frac{M^{-1}\Delta k^2}{2}} = \frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}} \tag{139}$$

Going back to Z^r:

$$\int d\xi P(\xi) Z^r = \int_{-\infty}^{\infty} \prod_{\mu=1}^{M} \prod_{i=1}^{N} \frac{1}{P_{\xi_i}} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} \delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right) \times \tag{140}$$
$$\times \int_0^\infty \prod_{a=1}^{r} dx_a \int_0^\infty \prod_{a=1}^{r} ds_a e^{\beta \sum_a U(x_a)} \prod_{a=1}^{r} \prod_{\mu=1}^{M} \delta\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right)$$

We again use the Fourier transform identity for the δ terms:

$$\delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right) = \int_{-\infty}^{\infty} \frac{1}{2\pi} d\hat{z}_i \, e^{i\hat{z}_i\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right)} \tag{141}$$

$$\delta\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right) = \int_{-\infty}^{\infty} \frac{1}{2\pi} d\hat{x}_\mu^a \, e^{i\hat{x}_\mu^a\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right)} \tag{142}$$

Writing only the terms involving ξ_i^μ from equation (140), we take the integral over $d\xi_i^\mu$. For each pair i, μ we have

$$\int_{-\infty}^{\infty} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} e^{i\hat{z}_i \xi_i^\mu} e^{-\sum_a i\hat{x}_\mu^a s_i^a \xi_i^\mu} = e^{-\frac{\Delta}{2M}\left(\hat{z}_i - \sum_a \hat{x}_\mu^a s_i^a\right)^2} \tag{143}$$

Again, in the above equation we have used the Gaussian integral identity of equation (138) and the normalization term was cancelled.

Plugging the product $\prod_{i,\mu} e^{-\frac{\Delta}{2M}\left(\hat{z}_i - \sum_a \hat{x}^a_\mu s^a_i\right)^2}$ back on equation (140) we end up with:

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{2\pi} d\hat{z}_i \int_{-\infty}^{\infty} \prod_{a=1}^{r} \frac{1}{2\pi} \prod_{\mu=1}^{M} d\hat{x}^a_\mu \int_0^\infty dx^a \int_0^\infty ds^a \frac{1}{\left[\frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}}\right]^N} \times \tag{144}$$

$$\times \exp\left[\beta \sum_a U(x_a) + i\epsilon \sum_{i=1}^{N} \hat{z}_i + i \sum_{a=1}^{r} \sum_{\mu=1}^{M} \hat{x}^a_\mu \left(x^a_\mu - x^\mu_0\right) - \frac{\Delta}{2M} \sum_{i=1}^{N} \sum_{\mu=1}^{M} \left(\hat{z}_i - \sum_{a=1}^{r} \hat{x}^a_\mu s^a_i\right)^2\right] \tag{145}$$

We now have hit another wall in integrating these expressions: some of the variables we are integrating on are coupled via the $\left(\hat{z}_i - \sum_{a=1}^r \hat{x}^a_\mu s^a_i\right)^2$ term. This means we are not able to integrate over, for example, s^a_i and s^b_i independently. To get around this, we introduce new variables which allows us to factor the exponential:

$$\omega_{ab} = \frac{1}{N} \sum_{i=1}^{N} s^a_i s^b_i \quad \text{and} \quad k_a = \frac{1}{N} \sum_{i=1}^{N} \hat{z}_i s^a_i \tag{146}$$

To substitute these terms in the equation above, we multiply it again by a delta term and integrate over it, then replace by its Fourier transform:

$$1 = \int dk_a \delta\left(k_a - \frac{1}{N} \sum_{i=1}^{N} s^a_i\right) = \int dk_a d\hat{k}_a \frac{N}{2\pi i} e^{\hat{k}_a [N k_a - \sum_i s^a_i]} \tag{147}$$

$$1 = \int d\omega_{ab} \delta\left(\omega_{ab} - \sum_{i=1}^{N} s^a_i s^b_i\right) = \int d\omega_{ab} d\hat{\omega}_{ab} \frac{N}{4\pi i} e^{\frac{1}{2}\hat{\omega}_{ab} [N\omega_{ab} - \sum_i s^a_i s^b_i]} \tag{148}$$

A few extra steps were taken in the above passage: first, we used the identity $\delta(x) = \alpha\delta(\alpha x)$ to write $\delta(k_a - \frac{1}{N}\sum_i s^a_i) = N\delta(Nk_a - \sum_i s^a_i)$. This change is useful because both terms are of order N and this will allow us to write $\langle Z^r \rangle_{\xi,x_0}$ in the form of (125). The second step taken was to carry out a change of variable in the integration, $\hat{k}_a \to i\hat{k}_a$ and $\hat{\omega}_{ab} \to \frac{i}{2}\hat{\omega}_{ab}$.

For simplicity, we will now omit the integration limits when the integral is $\int_{-\infty}^{\infty}$. Replacing the new variables in (144):

$$\langle Z^r \rangle_{\xi,x_0} = \int d\omega_{ab} d\hat{\omega}_{ab} \frac{N}{4\pi i} e^{N\hat{\omega}_{ab}\omega_{ab}} e^{-\hat{\omega}_{ab}\sum_i s_i^a s_i^b} \int dk_a d\hat{k}_a \frac{N}{2\pi i} e^{N\hat{k}_a k_a} e^{-\hat{k}_a \sum_i s_i^a} \times$$
$$\times \int \prod_{i=1}^{N} \frac{1}{2\pi} d\hat{z}_i \int \prod_{a=1}^{r} \frac{1}{2\pi} \prod_{\mu=1}^{M} d\hat{x}_\mu^a \int_0^\infty dx^a \int_0^\infty ds^a \frac{1}{\left[\frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}}\right]^N} \times \tag{149}$$
$$\times\, e^{\left[\beta \sum_a U(x_a) + i\epsilon \sum_{i=1}^N \hat{z}_i + i \sum_{a=1}^r \sum_{\mu=1}^M \hat{x}_\mu^a \left(x_\mu^a - x_0^\mu\right) - \frac{\Delta}{2M} \sum_{\mu=1}^M \left(\sum_i \hat{z}_i - 2N \sum_a k_a \hat{x}_\mu^a + N \sum_{a,b} \omega_{ab} \hat{x}_\mu^a \hat{x}_\mu^b\right)\right]}$$

The sums over i are now completely factorized, which allows us to replace $\sum_i s_i^a$ by Ns^a and again we get the N factor to put in evidence. We write the integral over $\omega, \hat{\omega}, k$ e $\hat{k}$ as

$$\langle Z^r \rangle_{\xi,x_0} = \int \prod_{a,b=1}^{r} N \frac{d\omega_{ab} d\hat{\omega}_{ab}}{4\pi i} \int \prod_{a=1}^{r} N \frac{dk_a d\hat{k}_a}{2\pi i} e^{Nh(\omega,\hat{\omega},k,\hat{k})}, \tag{150}$$

which is what we wanted initially. When we take the limit of $N \to \infty$, the integral will be dominated by the maximum value of h, which we divide in three terms, $h = g_1 + g_2 + g_3$:

$$g_1 = -\sum_{a,b=1}^{r} \frac{1}{2} \hat{\omega}_{ab} \omega_{ab} - \sum_{a=1}^{r} \hat{k}_a k_a \tag{151}$$

$$g_2 = \log \int \frac{d\hat{z}}{2\pi} \int_0^\infty \prod_{a=1}^{r} \exp\left[\frac{1}{2} \sum_{a,b} \hat{\omega}_{ab} s_a s_b + \hat{z} \sum_{a=1}^{r} \hat{k}_a s_a + i\epsilon\hat{z} - \frac{\Delta}{2}\hat{z}^2\right] - \log \frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}} \tag{152}$$

$$g_3 = \frac{1}{N} \sum_\mu \log \int \prod_a \frac{d\hat{x}_a}{2\pi} \int_0^\infty \prod_a dx^a e^{\beta \sum_a U(x^a) + i \sum_a \hat{x}^a (x^a - x_0^\mu) - \frac{n\Delta}{2} \sum_{a,b} \hat{x}^a \hat{x}^b \omega_{ab} + n\Delta \sum_a \hat{x}^a k_a} \tag{153}$$

To find the maximum of h we must solve the following system of equations

$$\begin{aligned} \frac{\partial h}{\partial \omega_{ab}} &= 0, \qquad & \frac{\partial h}{\partial \hat{\omega}_{ab}} &= 0 \\ \frac{\partial h}{\partial k_a} &= 0, \qquad & \frac{\partial h}{\partial \hat{k}_a} &= 0 \end{aligned} \tag{154}$$

These are the saddle points for the replica method. Although we are actually calculating the maximum value of $U(x)$, the saddle point equations give us important information on how the order parameters relate to each other.

At this point we take another important approximation to these calculations. The r^2 order parameters ω_{ab} are the overlap between two replicas of the system, a and b, i.e., how similar are the s vectors in two independent copies of our economy. Because $U(x)$ is a convex function, we know that the maximum of $U(x)$ exists and is unique. Therefore, we expect every replica to converge to the same equilibrium value of s^a in the limit $\beta \to \infty$, and in this case, we cannot distinguish between ω_{ab} for any two pairs of replica a and b. We assume, then, there are only two possible values for ω_{ab}. Either $a = b$ and $\omega_{aa} = \langle s^2 \rangle = \Omega$, the variance of s, or $a \neq b$ and $\omega_{ab} = \omega$, the overlap of two different systems. This is the so called replica symmetric approximation, which is exact in this case because we know there is only one equilibrium in the zero temperature limit.

Writing this explicitly, we have that ω_{ab} and k_a are given by

$$\begin{aligned} \omega_{ab} &= \Omega\delta_{ab} + \omega(1-\delta_{ab}) \\ \hat{\omega}_{ab} &= \hat{\Omega}\delta_{ab} + \hat{\omega}(1-\delta_{ab}) \\ k_a &= k \\ \hat{k}_a &= \hat{k} \end{aligned} \tag{155}$$

Replacing these new values in equations (151) - (153) and taking the limit $r \to 0$ we get

$$\lim_{r\to 0} \frac{1}{r} g_1 = -\frac{1}{2}\left(\Omega\hat{\Omega} - \omega\hat{\omega}\right) - k\hat{k} \tag{156}$$

$$\lim_{r\to 0} \frac{1}{r} g_2 = \int dt \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \log \int_0^\infty ds\, e^{\frac{\hat{\Omega}-\hat{\omega}}{2}s^2 + \left[t\left(\frac{\hat{k}^2}{\Delta}+\hat{\omega}\right)^{\frac{1}{2}} + i\hat{k}\frac{\epsilon}{\Delta}\right]s} \tag{157}$$

$$\lim_{r\to 0} \frac{1}{r} g_3 = \int dt \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \log \int_0^\infty dx\, e^{\beta U(x) - \frac{\left(x - x_0 + \sqrt{n\Delta\omega}t - in\Delta k\right)^2}{2n\Delta(\Omega-\omega)} - \frac{1}{2}\log[2\pi n\Delta(\Omega-\omega)]} \tag{158}$$

In the equations above, t is a Gaussian random variable with zero mean and unit variance, which arises from using the identity (138) to factorize $\left(\sum_a s^a\right)^2$, gaining an integral in the process.

We now have an order parameter vector $\theta = (\Omega, \hat{\Omega}, \omega, \hat{\omega}, k, \hat{k})$ and we wish to find the values θ^* that maximizes $h(\theta)$. However, we have some conditions on this solution, in particular we know that it must be well defined for $\beta \to \infty$. Again, because we know that in this limit all replicas must have the same equilibrium, then it must hold that the overlap between replicas must vanish, i.e.,

$$\lim_{\beta\to\infty} \Omega - \omega = \frac{1}{2N} \sum_{i=1}^{N} \left(s_i^a - s_i^b\right)^2 = 0 \tag{159}$$

But this would imply that some terms in g_3 would diverge in the zero temperature limit, so we have to rescale the order parameters for them to remain finite in this limit. We define the new parameters, which are always finite:

$$\chi = n\beta\Delta(\Omega - \omega), \quad \hat{\chi} = -\frac{\hat{\Omega} - \hat{\omega}}{\beta}, \quad \kappa = -in\Delta k, \tag{160}$$

$$\hat{\kappa} = \frac{i\hat{k}}{\beta\Delta}, \quad \hat{\gamma} = \frac{\hat{\omega}}{\beta^2} \tag{161}$$

The function h then becomes

$$\begin{aligned} h = \frac{1}{2}\left(\Omega\hat{\chi} - \frac{\hat{\gamma}\chi}{n\Delta}\right) - \frac{1}{n}\kappa\hat{\kappa} + \frac{1}{\beta}\left\langle \log \int_0^\infty ds\, e^{\beta\left[-\frac{\hat{\chi}}{2}s^2 + (t\sqrt{\hat{\gamma} - \Delta\hat{\kappa}^2} + \hat{\kappa}\epsilon)s\right]} \right\rangle_t + \\ + \frac{1}{n\beta}\left\langle \log \int_0^\infty dx\, e^{\beta\left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)^2}{2\chi}\right]} \right\rangle_{t,x_0} \end{aligned} \tag{162}$$

We then finally take the limit $\beta \to \infty$ and again use the saddle point method to solve the integrals on x and s, which means they are dominated by their maximum value:

$$\begin{aligned} h(\beta \to \infty) = \left\langle \max_s \left[-\frac{\hat{\chi}}{2}s^2 + (t\sqrt{\hat{\gamma} - \hat{\kappa}^2\Delta} + \hat{\kappa}\epsilon)s \right] \right\rangle_t + \frac{1}{2}\left(\Omega\hat{\chi} - \frac{\hat{\gamma}\chi}{n\Delta}\right) - \frac{1}{n}\kappa\hat{\kappa} + \\ + \frac{1}{n}\left\langle \max_x \left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Omega\Delta}t)^2}{2\chi} \right] \right\rangle_{t,x_o} \end{aligned} \tag{163}$$

Replacing in equation (163) the variables x and s by their maximum values x^* and s^* and taking the derivatives on the order parameters we finally have the saddle point equations for $h(\theta)$:

$$\frac{\partial h}{\partial \Omega} = \frac{\hat{\chi}}{2} - \frac{1}{2\chi}\sqrt{\frac{\Delta}{n\Omega}} \left\langle (x^* - x_0 + \kappa + t\sqrt{n\Omega\Delta})t \right\rangle_{t,x_0} = 0 \tag{164}$$

$$\frac{\partial h}{\partial \kappa} = -\frac{1}{n}\hat{\kappa} - \frac{1}{n\chi}\left\langle x^* - x_0 + \kappa + t\sqrt{n\Omega\Delta} \right\rangle_{t,x_0} = 0 \tag{165}$$

$$\frac{\partial h}{\partial \hat{\kappa}} = -\frac{\hat{\kappa}\Delta}{\sqrt{\hat{\gamma} - \hat{\kappa}^2\Delta}} \left\langle ts^* \right\rangle_t + \epsilon \left\langle s^* \right\rangle_t - \frac{\kappa}{n} = 0 \tag{166}$$

$$\frac{\partial h}{\partial \hat{\gamma}} = \frac{1}{2\sqrt{\hat{\gamma} - \hat{\kappa}^2\Delta}} \left\langle ts^* \right\rangle_t - \frac{\chi}{2n\Delta} = 0 \tag{167}$$

$$\frac{\partial h}{\partial \chi} = -\frac{\hat{\gamma}}{2n\Delta} + \frac{\left\langle (x^* - x_0 + \kappa + t\sqrt{n\Omega\Delta})^2 \right\rangle_{t,x_0}}{2n\chi^2} = 0 \tag{168}$$

$$\frac{\partial h}{\partial \hat{\chi}} = -\frac{1}{2}\left\langle (s^*)^2 \right\rangle_t + \frac{1}{2}\Omega = 0 \tag{169}$$

We can find x^* by solving $\frac{\partial}{\partial x}\left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Omega\Delta}t)^2}{2\chi}\right] = 0$, resulting in the implicit equation

$$x^* = x : U'(x^*) = \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)}{\chi} \tag{170}$$

We can replace this in the equations (164) - (169) to obtain some useful relations. The equation (165) becomes, for $x = x^*$

$$\hat{\kappa} = -\left\langle U'(x^*) \right\rangle_{t,x_0} \tag{171}$$

This allows us to identify $\hat{\kappa} = -p$ due to the price equation derived from the first order conditions of the consumer's maximization problems. Equation (169) allows us to write

$$\Omega = \left\langle (s^*)^2 \right\rangle_t \tag{172}$$

The remaining parameters are found through simple algebraic manipulations. With Ω and p defined, one immediately obtains $\hat{\chi}$

$$\hat{\chi} = \sqrt{\frac{\Delta}{n\Omega}} \left\langle U'(x^*)t \right\rangle_{t,x_0} \tag{173}$$

And from (168) we have

$$\hat{\gamma} = \Delta \left\langle U'(x^*)^2 \right\rangle_{t,x_0} \tag{174}$$

With that, $U'(x)$ variance is written as

$$\sigma = \sqrt{\hat{\gamma} - \hat{\kappa}^2\Delta} = \sqrt{\Delta\left(\langle U'(x^*)^2\rangle_{t,x_0} - \langle U'(x^*)\rangle_{t,x_0}^2\right)} \tag{175}$$

Finally, we have χ via equation (167)

$$\chi = \frac{n\Delta}{\sigma} \langle ts^* \rangle_t \tag{176}$$

And κ via equation (166)

$$\kappa = p\chi + n\epsilon \langle s^* \rangle_t \tag{177}$$

Thus we have now arrived at equations (60) - (65).

B

DEFINITIONS

In this Appendix we briefly explain some of the statistical measures used throughout the thesis. While they can easily be found in other sources, we consider it useful for the reader to have them consolidated in a single source.

B.1 SPEARMAN RANK CORRELATION

The **Spearman Rank Correlation** is a statistical measure of dependence between two random variables. Unlike the Pearson correlation, the most commonly employed metric for dependence between two variables, the Spearman rank is not a linear metric. Instead, it measures the monoticity between samples of two random variables, i.e., how much one can be described as a monotonic function of another.

More precisely, let $x = \{x_1, \ldots, x_N\}$ and $y = \{y_1, \ldots, y_N\}$ be two datasets, and $r(x_i)$ is the rank of x_i in the order from lowest to highest (that is, $r(x_i) = 1$ for the smallest x_i and $r(x_i) = N$ for the highest. Then the Spearman Rank Correlation is the Pearson correlation between the set of ranks $r(x) = \{r(x_1), \ldots, r(x_N)\}$ and $r(y) = \{r(y_1), \ldots, r(y_N)\}$, i.e.:

$$r_s = \frac{\mathrm{cov}(r(x), r(y))}{\sigma_{r(x)}\sigma_{r(y)}} \tag{178}$$

If there are no ties in the ranks, i.e., if there is not two elements identical in both samples, then the Spearman rank correlation is given by

$$r_s = 1 - \frac{6\sum_i (r(x_i) - r(y_i))^2}{n(n^2-1)} \tag{179}$$

A Spearman correlation of 1 implies that for every pair i, j in the datasets, $(x_i - x_j)(y_i - y_j) > 0$. Likewise, a Spearman correlation of -1 implies that $(x_i - x_j)(y_j - y_j) < 0$.

One of the main advantages of Spearman rank correlation is its sensivity to outliers. If an element of x is much larger than the rest of the set, in a linear measure like Pearson correlation it would distort the measure by a term proportional to its deviation from the mean. In the Spearman rank calculation, no matter how large the outlier, it is still only one rank above the second largest data point.

B.2 KOLMOGOROV-SMIRNOV DISTANCE

When facing a sample of points, one often would like to know how likely it is that these points were sampled from a specific probability distribution. Alternatively, one would like to know how likely it is that two different samples were drawn from the same distribution. One of the most popular methods employed for this task is the **Kolmogorov-Smirnov distance**[90]. Given the empirical cumulative distribution of the dataset $x = \{x_1, \ldots, x_N\}$ given by

$$F_d(x) = \frac{1}{N} \sum_{i=1}^{N} \Theta(x - x_i), \tag{180}$$

where $\Theta(x)$ is the Heaviside function.

If we want to compare it to a theoretical cumulative distribution $F(x)$, the KS distance is the maximum distance between these two distributions, that is,

$$D_d = \sup_x |F(x) - F_d(x)|. \tag{181}$$

From the point of view of Classical Statistics, the most relevant measure for a model selection trial using the Kolmogorov-Smirnov distance, however, is not the distance itself but its associated p-value. Given a random sample drawn from $F(x)$, the p-value is the probability that this sample will have a KS distance equal to D_d. It can be shown that the scaled distance $D_d\sqrt{N}$ has a cumulative distribution similar to the Kolmogorov distribuiton:

$$P(K < x) = 1 - 2\sum_{k=1}^{\infty}(-1)^{k-1}e^{-2k^2x^2}, \tag{182}$$

Thefore p is the probability that a scaled distance $D_d\sqrt{(N)}$ sampled from the Kolmogorov distribution has a value equal or above its measured empirical value, i.e.,

$$p = P(K > D_d\sqrt{N}) = 1 - P(K < D_d\sqrt{N}) \tag{183}$$

Which is the probability used throughout Chapter 6.

B.3 BAYESIAN INFORMATION CRITERION (BIC)

In the theory of Bayesian model selection, when deciding between two models H_1 and H_2, with parameters θ_1 and θ_2 respectively, that explain the data x one should compare their posterior probabilities:

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)}. \tag{184}$$

The equality above is just Bayes theorem. The likelihood $p(x|H_i)$ of the data given the hypothesis is calculate by integrating over all hypothesis parameters, i.e.,

$$p(x|H_i) = \int d\theta_i p(x|\theta_i)p(\theta_i|H_i), \tag{185}$$

where $p(x|\theta)$ is the likelihood of the data given a specific choice of parameters θ and $p(\theta|H_i)$ is the prior probability on the parameters given the model.

If we have no prior information on which model is more likely, then we simply use $p(H) = 1/2$ as prior. The dependency on $p(x)$ can be eliminated by calculating the so called *Bayes factor* instead of the posteriors:

$$K = \frac{p(x|H_1)}{p(x|H_2)} = \frac{\int d\theta_1 p(x|\theta_1)p(\theta_1|H_1)}{\int d\theta_2 p(x|\theta_2)p(\theta_2|H_2)} \tag{186}$$

A Bayes factor $K > 1$ means that H_1 is more likely to be the model that generated the data, likewise $K < 1$ means H_2 is more likely. How much more likely depends, of course, on the magnitude of the ratio.

In the case of real unbounded variables, however, defining the prior $p(\theta|H)$ is not trivial. One workaround to this problem is to assume uniform prior but do a saddle point approximation on the integrals of equation (186) [12]. One then gets the **Bayesian Information Criterion score**:

$$\mathrm{BIC} = -2\log p(x|\theta^*) + k\log(M), \tag{187}$$

where θ^* are the values that maximize $p(x|\theta)$ (i.e., the maximum likelihood parameters) and k is the dimensionality of this vector (i.e., number of parameters). This is essentially a maximum likehood which is penalized by the number of parameters. The model with lowest BIC is the one that should be preferred, and the magnitude of the difference tells us how strongly should we prefer one over another [51].

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