Statistical Mechanics of Economic Systems

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ABSTRACT

Short summary of the contents of your thesis.

To someone special

ACKNOWLEDGEMENTS

Put your acknowledgements here.

DECLARATION

Put your declaration here.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACRONYMS

DRY Don't Repeat Yourself

API Application Programming Interface

UML Unified Modeling Language

1

INTRODUCTION

1.1 MAIN RESULTS

Part I

THEORY

2

STATISTICAL MECHANICS AND INFERENCE

2.1 A SECTION

3

ECONOMICS AND THE GENERAL EQUILIBRIUM THEORY

In this chapter we will discuss some of the economic theories behind the work of the thesis. Mainly, we will introduce General Equilibrium Theory and some of its main results and statements. We will then discuss how it relates to Statistical Physics and some of its criticism in the Economics literature.

General Equilibrium Theory is a mature and consolidated field of Economics [1, 5, 6] which aims to characterize the existence and properties of equilibria in certain market settings. Economic systems are assumed to frequently have actors with opposing goals: the owner of a good wants to sell it for the highest possible price, while its potential buyers would like to purchase it for as low as possible. Fishermen would like to catch as many fishes as possible as long as their peers care to not also overdo it otherwise they may extinguish the oceans. In this way, one expects economies and markets to converge to a certain steady state and among other things, General Equilibrium Theory characterizes these steady states in a rigorous manner. In this sense, it's also a theory in Microeconomic, because it explains macrobehavior from the incentives of microscopic agents.

3.1 A BRIEF EXPOSITION

The exposition in this chapter is mostly adapted and simplified from [5] and will be considerably more formal than the rest of this thesis. This is due to the way the discipline is commonly studied.

In General Equilibrium Theory, an economy is defined through the following components: we assume J consumers, N firms and M goods. Each consumer has a consumption set X_j which contains all possible consumption bundles $x_j = (x_j^1, \dots x_j^M)$ that the consumer has access to, ie, each bundle x_j is a $M-$dimensional vector with nonnegative entries (we are assuming he cannot consume a negative amount of a good). X_j is limited by "physical" contraints, such as no access to water or bread, but not monetary constraints, which will arise later.

The consumer also has an utility function $U_j(x)$ that takes every element $x_j \in X_j$ to a real number, representing how much the consumer values each bundle of his consumption set. This allows us to define define a preference relationship over the elements in X_j (ie, if the consumer prefers bundle x to x'), which is **complete**[1] and **transitive**[2], two standard requirements in Economics for rational behavior.

Finally, the consumer is also endowed with an initial bundle of goods $\omega_j = (\omega_j^1, \dots, \omega_j^M)$, $\omega_j^\mu \geq 0$ which will define his budget given a set of prices for the goods and will constraint his choices on X_j.

Each firm i has a production set Ξ_i of technologies $\xi_i = (\xi_i^1, \dots, \xi_i^M)$ which it is able to operate. Unlike consumption bundles, which are final allocations and therefore must be nonnegative, technologies can

1 For every $x, x' \in X_j$, either $U_j(x) \geq U_j(x')$ or $U_j(x) \leq U_j(x')$.

2 For every $x, y, z \in X_j$, if $U_j(x) \geq U_j(y)$ and $U_j(y) \geq U_j(z)$, then $U_j(x) \geq U_j(z)$.

be any real number: the negative entries are inputs and the positive entries are outputs that the firm can operate. Ξ_i is also limited only by "physical" constraints, not by monetary constraints. A firm that has $\xi_i = (-1, 2)$ in its production set is able to transform one unit of good 1 into two units of good 2. It won't necessarily be able to transform two units of good 1 into four units of good 2, for that it must also have $\xi_i' = (-2, 4)$ in Ξ_i. It might be the case, for example, that companies get more efficient with production and therefore it might have $\xi_i'' = (-2, 6)$ in its production set.

In General Equilibrium Theory, an economy is formally defined as the tuple

$$E = \left(\{(X_j, U_j)\}_{j=1}^{J}, \{\Xi_i\}_{i=1}^{N}, \{\omega_j\}_{j=1}^{J} \right). \tag{1}$$

One of the theory's assumptions is that the economy described is **complete**, that is, every agent can exchange every good with no transaction costs and complete information about the firm's technologies, other consumer's consumption, etc. Also, a good μ contains all the possible information that a consumer would take into account when making his choice. That is, among the space of goods we could have "umbrella" and "chocolate", or we could also have "an umbrella on August 13th, 2016 in Sao Paulo with 50% chance of rain" and "an umbrella on December 12th, 2016 in Chicago with 90% chance of rain".

It's assumed that agents are **price-takers**, that is, they are unable to affect the market prices and therefore take them as a given. The prices of the goods are given by a $M-$dimensional vector $p = (p_1, \ldots, p_M)$, where each price is a strictly positive quantity, ie, $p_\mu > 0$ for all μ. This assumes that goods have global prices, which is consistent with the completeness assumption: there is no reason why the market prices should be different for certain consumers or firms if they have complete knowledge and no transaction costs.

With a price vector p defined, we say the consumer j has a budget $B_j = p \cdot \omega_j$, which is the monetary value of his initial endowment. Any bundle he chooses to purchase will cost him $p \cdot x_j$. His objective, therefore, is to find the best bundle x_j he is able to afford, that is:

$$\max_{x_j \in X_j} U(x_j), \quad \text{s.t. } p \cdot x_j \leq p \cdot \omega_j \tag{2}$$

The firms, on the other hand, have an operating profit for each technology given by $p \cdot \xi_i$, which is how much money they earn by selling their outputs ($\xi_i^\mu > 0$) minus how much they spend purchasing the inputs ($\xi_i^\mu < 0$). Their objective is to maximize their profits, that is:

$$\max_{\xi_i \in \Xi_i} p \cdot \xi_i \tag{3}$$

With these ingredients laid out, we define an **allocation** of the economy as a set of specific choices for consumption bundles and technologies, ie, an allocation a of an economy E is

$$a = (x_1, \ldots, x_J, \xi_1, \ldots, \xi_N),\ x_j \in X_j,\ \xi_i \in \Xi_i \tag{4}$$

The economy is closed, and therefore all that is produced must come from the initial endowments and be consumed by the consumers. We therefore say an allocation is **feasible** if it satisfies **market clearing** for all the goods:

$$\sum_{j=1}^{J} x_j^\mu = \sum_{j=1}^{J} \omega_j^\mu + \sum_{i=1}^{N} \xi_i^\mu, \quad \forall\, \mu \in \{1, \ldots, M\} \tag{5}$$

This is a strong condition which constraints many quantities in the economy. In particular, if we multiply both sides of the equation by p^μ and sum then in μ we get, in vector notation,

$$\sum_{j=1}^{J} p \cdot (x_j - \omega_j) = \sum_{i=1}^{N} p \cdot \xi_i \tag{6}$$

The left handside is the leftover money the consumers have after making their choice of consumption, also called the value of excess demand, whereas the right handside is the firms aggregate profit, also known as the value of excess suply. Because we assume that the consumer may not spend more than his budget, the value of each consumer's individual excess demand has to be non positive. Simultaneously, if we assume that the firms always have $\xi_i = 0$ in their production set, ie, we assume that they can always opt to not produce at all and leave the market, then the value of excess supply for each firm has to be non negative. Because they must be equal, we conclude that in an economy for which market clearing holds, the consumer spends all his available budget and the firms all have zero profit, a result known as **Walras' Law**.

Given a set of possible feasible allocations $\{a_k\}$, we may wonder if there is any allocation we desire most over the other. This of course depends on the criteria we use to judge them: we may like allocations with less inequality, with the most aggregate utility, with the smallest minimum utility, etc. Economist opt to use one particular condition which is called **Pareto optmality**.

Intuitively, a **Pareto optimal** (or **Pareto efficient** allocation is one that you can't make a consumer better without making another consumer worse off. The idea is that, a non Pareto optimal allocation has some waste in it: one could change the consumption bundles in order to increase some utilities and no other consumer would complain. Because firms have zero profit in feasible allocations, they wouldn't mind the change.

More formally, a feasible allocation $a = (x, \xi)$ is said to be **Pareto optimal** if there is no other allocation that **Pareto dominates** it, that is, no allocation $a' = (x', \xi')$ such that $U(x'_j) \geq U(x_j)$ for all j and $U(x'_j) > U(x_j)$ for at least one j.

The Pareto optimality concept therefore defines a socially desirable outcome in a "non-controversial" way, by definition no agent in the economy would have a problem with policies or actions taken to make it more Pareto efficient. However, it says nothing about equality: an allocation in which one consumer has all the goods and no other consumer has any goods is Pareto optimal.

We finally arrive to the concept of equilibrium in an economy. A **Walrasian equilibrium** (or competitive equilibrium or simply equilibrium) in an economy E is an allocation (x^*, ξ^*) and a price vector p such that

1. Every firm i maximizes it's profits in its production set Ξ_i, that is
$$p \cdot \xi_i^* \geq p \cdot \xi_i, \quad \forall \xi_i \in \Xi_i, \quad \forall iin\{1, \ldots, N\} \tag{7}$$

2. Every consumer j maximizes his utility in his consumption set X_j, that is
$$U(x_j^*) \geq U(x_j), \quad \forall x_j \in X_j, \quad \forall j \in \{1, \ldots, J\} \tag{8}$$

3. The allocation (x^*, ξ^*) is feasible, that is,
$$\sum_{i=j}^{J} x_j^* = \sum_{j=1}^{J} \omega_j + \sum_{i=1}^{N} \xi_i^* \tag{9}$$

The Walrasian equilibrium is essentially a pair allocation - prices in each all the optimization problems are solved at once. Although we have made no mention of dynamics in this economy, it's considered an equilibrium because all agents are as satisfied as possible with their allocation given the prices, which we have assumed to be global and unchangeable by any agent's action. This is not exactly a definition of equilibrium as used in Physics, but we will discuss this point later. For now, we point out that a Walrasian equilibrium is in some sense stable.

We have thus defined two desirable properties of an allocation: efficiency and equilibrium. The fundamental results of General Equilibrium Theory are the **welfare theorems**, which define the conditions in which an equilibrium is Pareto optimal and vice versa.

The **First Fundamental Welfare Theorem** asserts that if the consumers have a utility function continuous on X_j[3], then all Walrasian

3 The actual theorem asserts a weaker condition, that the preferences be locally non-satiated, that is, for every $x \in X$, there is an $x' \in X$ such that $\|x - x'\| < \varepsilon$ and x' is preferred to x.

equilibria are Pareto optimal. This result is simple yet useful, because it tells us that if our economy is in equilibrium, we don't have to care about checking if it's efficient. The violation is also important: if a given economy we are studying is in an inefficient equilibrium, then it must be that one of the theorem's condition was violated. This sheds light in where to look for market failures. We remind the reader, however, that some extra strong assumptions were made for the economies described by this therorem, namely, completeness of market and global prices that no single agent is capable of influencing.

The **Second Fundamental Welfare Theorem** requires extra assumptions: it afirms that if an economy satisfies the condition of the first fundamental theorem, the utility functions U_j plus all sets X_j and Y_i are convex and if we are able to redistribute the initial endowments at will while keeping the total amount $\sum_{j=1}^{J} \omega_j$ constant, then for every Pareto efficient allocation there exists a wealth allocation ω and price vector p^* such that (x^*, ξ^*, p^*) is a Walrasian equilibrium.

The second theorem is considerably more interesting than the first one: any Pareto optimal allocation we would like in an economy can be an equilibrium given the appropriate price vector and a possible wealth transfer, albeit under a stronger set of conditions. The wealth transfer may sound like an undesirable condition, but it can be achieved with redistribution policies.

3.2 CRITICISM TO GENERAL EQUILIBRIUM

[10] [9] [8] [5] [7, 3] [11, 4, 12]

4

THE RANDOM LINEAR ECONOMY MODEL

In this chapter we will discuss the Random Linear Economy model [2] presented by Andrea De Martino, Matteo Marsili and Isaac Pérez Castillo which will be the basis of some of the applications discussed in the rest of this thesis.

An economy in this model is defined by M goods, N firms, with a technological density parameter of $n = N/M$, and one representative consumer. The consumer has an initial random wealth $x_0 = (x_0^1, \dots, x_0^M)$, $x_0^\mu \geq 0$ drawn independently from an exponential distribution, and wishes to improve its welfare in the market according to a separable utility function $U(x) = \sum_{\mu=1}^{M} u(x^\mu)$.

Each firm in the market has a random technology $\xi_i = (\xi_i^1, \dots, \xi_i^\mu)$, where $\xi_i^\mu < 0$ represents an input and $\xi_i^\mu > 0$ represents an output. Each firm i only has one technology, and its only decision is the scale s_i at which it operates this technology, ie, each company represents a transformation in the space of goods given by the M dimensional vector $s_i\xi_i$. The elements ξ_i^μ are drawn from a $\mathcal{N}(0, M^{-1})$ normal distribution, normalized so that $\sum_{\mu=1}^{M} \xi_i^\mu = -\epsilon$, that is, all technologies are a little inefficient. This is too guarantee that there's no way to combine two (or more) technologies and produce infinite goods.

This economy is closed and therefore we must have a market clearing condition: the N dimensional production scale vector s has to be such that

$$x = x_0 + \sum_{i=1}^{N} s_i \xi_i \tag{10}$$

ie, all the inputs the firms use have to come from the consumer's initial endowment. As with traditional General Equilibrium scenarios, the market clearing condition makes it so that solving the consumer maximum utility problem, $x^* = \arg\max_x U(x)$, simultaneously solves the firms maximum profits problem, $s_i^* = \arg\max_{s_i} s_i p_i \cdot \xi_i$, with the prices being set also by the first order condition of the consumer's maximization problem, $p_\mu = \frac{\partial U}{\partial x_\mu}$. In brief, market clearing makes so that the firms production and the market prices are set to satisfy consumer's desired demand, and no actor in the market has an incentive to deviate from this equilibrium.

Market clearing also carries a strong restriction. If we multiply both sides of the equation (10) by p, we get

$$p \cdot (x - x_0) = \sum_i s_i p \cdot \xi_i, \tag{11}$$

The left side of the above equation has to always be smaller or equal to zero, because of the budget condition. But the right hand side has to be always greater or equal to zero, because this term represents the sum of the individual firms' profits and they can always choose $s_i = 0$ if a firm is losing money. Therefore, we must have that both sides are equal to zero, and the consequence is that the agent completely spends all his available budget (ie, $p \cdot x = p \cdot x_0$, he has not "leftover" cash after choosing x) and that the firms either have zero profit ($p \cdot \xi_i = 0$) or leave the market ($s_i = 0$).

One of the important implications of equation (11) is that we may not have more than M firms active at any given equilibrium realization. This is because $p \cdot \xi_i = 0$ is an equation on p_μ with M variables, and if we have more than M equations, this system has no solution.

The model has some very interesting properties which are described at length in [2]. In particular, it's possible to analytically calulate the distribution probabilities of x and s (and therefore of p) and see that all macroscopic quantities derived from these two quantities depend on the number of firms per good $n = N/M$. The model displays a regime change at $n = 2$, ie, two random technologies per good. When $n < 2$, the market is competitive and the fraction of active firms $\phi = \sum_i \mathbb{I}(s_i > 0)/N$ is around $\phi = 0.5$. Because each firm has on average half the goods as inputs and half as outputs, when $n < 2$ you don't have enough firms to span the whole M dimensional space in order to be able to fine tune the quantities desired for all the goods.

When $n > 2$, however, there are many firms to choose from and statistically it's possible to choose M linear independent firms that span the whole good space. In this regime, the market becomes monopolistic and ϕ assymptotically goes to zero with n, due to a saturation of active firms on M.

The change in the model economy's GDP also reflects the qualitative change in allocation. The authors define the gross product for the model as the total value of goods produced, that is, the sum of $(x_\mu - x_0^\mu)p_\mu$ for all goods μ that are produced, ie, $x_\mu > x_0^\mu$. However, the market clearing condition (11) makes the value of goods produced equal to the value of goods used as input, so we calculate the GDP Y by averaging over the absolute value of all trades:

$$Y = \frac{\sum_{\mu=1}^{M} |x_\mu - x_0^\mu| p_\mu}{2\sum_{\mu=0}^{M} p_\mu}, \tag{12}$$

where the denominator also includes a normalization for the prices.

What is shown in [2] is that in the competitive regime when $n < 2$, a new firm will have a significant positive effect on Y, while in the monopolistic regime $n > 2$ a new firm will have negligible impact on the gross product. We will revisit this result later in this paper.

The Random Linear Economies model is particularly suitable for further analysis because it's a General Equilibrium setting with few ingredients, but the introduction of stochastic elements offers a non-trivial phase transition which is not observed in similar "simple" economic models in the literature.

Part II

APPLICATIONS

5

INEFFICIENT CONSUMER IN A GENERAL EQUILIBRIUM SETTING

6

INPUT-OUTPUT OF RANDOM ECONOMIES AND REAL WORLD DATA

7

WHEN DOES INEQUALITY FREEZE AN ECONOMY?

8

CONCLUSION

BIBLIOGRAPHY

[1] Kenneth J Arrow and Gerard Debreu. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, 1954.

[2] Andrea De Martino, Matteo Marsili, and Isaac Perez Castillo. Statistical Mechanics Analysis of the Equilibria of Linear Economies. *J. Stat. Mech.*, 2004.

[3] Herbert Gintis. The Dynamics of General Equilibrium. *The Economic Journal*, 117(523):1280–1309, 2007.

[4] Daniel Kahneman and Amos Tversky. Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, 1979.

[5] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory*. The MIT Press, 1995.

[6] Lionel W. McKenzie. *Classical General Equilibrium Theory*. Oxford University Press, 2002.

[7] Tim Roughgarden. Computing equilibria: a computational complexity perspective. *Economic Theory*, 42, 2010.

[8] L.J. Savage. *The Foundations of Statistics*. Dover Publications, 1972.

[9] J. A. Schumpeter. *Business cycles*. Cambridge Univ Press, 1939.

[10] Stefan Thurner, Peter Klimek, and Rudolf Hanel. Schumpeterian Economic Dynamics as a Quantifiable Model of Evolution. *New Journal of Physics*, 12, 2010.

[11] Amos Tversky and Daniel Kahneman. Judgment under uncertainty: Heuristics and biases. *Science*, 185, 1974.

[12] Amos Tversky and Daniel Kahneman. The framing of decisions and the psychology of choice. *Science*, 211, 1981.