Statistical Mechanics of Economic Systems

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ABSTRACT

Short summary of the contents of your thesis.

To someone special

ACKNOWLEDGEMENTS

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DECLARATION

Put your declaration here.

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ECONOMICS AND THE GENERAL EQUILIBRIUM THEORY

In this chapter we will discuss some of the economic theories behind the work of the thesis. Mainly, we will introduce General Equilibrium Theory and some of its main results and statements. We will then discuss how it relates to Statistical Physics and some of its criticism in the Economics literature.

General Equilibrium Theory is a mature and consolidated field of Economics [1, 8, 9] which aims to characterize the existence and properties of equilibria in certain market settings. Economic systems are assumed to frequently have actors with opposing goals: the owner of a good wants to sell it for the highest possible price, while its potential buyers would like to purchase it for as low as possible. Fishermen would like to catch as many fishes as possible as long as their peers care to not also overdo it otherwise they may extinguish the oceans. In this way, one expects economies and markets to converge to a certain steady state and among other things, General Equilibrium Theory characterizes these steady states in a rigorous manner. In this sense, it's also a theory in Microeconomic, because it explains macrobehavior from the incentives of microscopic agents.

While the work in this thesis does not aim to strictly expand economies as they are describe in General Equilibrium Theory, it's purposes and questions are very similar to the ones physicists usually go for when studying an economic system: namely, the characterization of it's equilibrium state. It's important, therefore, to have a deeper understanding of how it's done in economics, what are the main concerns and assumptions.

3.1 A BRIEF EXPOSITION

The exposition in this chapter is mostly adapted and simplified from [8] and will be considerably more formal than the rest of this thesis. This is due to the way the discipline is commonly studied.

In General Equilibrium Theory, an economy is defined through the following components: we assume J consumers, N firms and M goods. Each consumer has a consumption set X_j which contains all possible consumption bundles $x_j = (x_j^1, \dots x_j^M)$ that the consumer has access to, ie, each bundle x_j is a $M-$dimensional vector with nonnegative entries (we are assuming he cannot consume a negative amount of a good). X_j is limited by "physical" contraints, such as no access to water or bread, but not monetary constraints, which will arise later.

The consumer also has an utility function $U_j(x)$ that takes every element $x_j \in X_j$ to a real number, representing how much the consumer values each bundle of his consumption set. This allows us to define define a preference relationship over the elements in X_j (ie, if the consumer prefers bundle x to x'), which is **complete**[1] and **transitive**[2], two standard requirements in Economics for rational behavior.

Finally, the consumer is also endowed with an initial bundle of goods $\omega_j = (\omega_j^1, \dots, \omega_j^M)$, $\omega_j^\mu \geq 0$ which will define his budget given a set of prices for the goods and will constraint his choices on X_j.

Each firm i has a production set Ξ_i of technologies $\xi_i = (\xi_i^1, \dots, \xi_i^M)$ which it is able to operate. Unlike consumption bundles, which are final allocations and therefore must be nonnegative, technologies can be any real number: the negative entries are inputs and the positive entries are outputs that the firm can operate. Ξ_i is also limited only by "physical" constraints, not by monetary constraints. A firm that has $\xi_i = (-1, 2)$ in its production set is able to transform one unit of good 1 into two units of good 2. It won't necessarily be able to transform two units of good 1 into four units of good 2, for that it must also have $\xi_i' = (-2, 4)$ in Ξ_i. It might be the case, for example, that companies get more efficient with production and therefore it might have $\xi_i'' = (-2, 6)$ in its production set.

In General Equilibrium Theory, an economy is formally defined as the tuple

$$E = \left(\{(X_j, U_j)\}_{j=1}^J, \{\Xi_i\}_{i=1}^N, \{\omega_j\}_{j=1}^J \right). \tag{1}$$

One of the theory's assumptions is that the economy described is **complete**, that is, every agent can exchange every good with no transaction costs and complete information about the firm's technologies, other consumer's consumption, etc. Also, a good μ contains all the possible information that a consumer would take into account when making his choice. That is, among the space of goods we could have "umbrella" and "chocolate", or we could also have "an umbrella on August 13th, 2016 in Sao Paulo with 50% chance of rain" and "an umbrella on December 12th, 2016 in Chicago with 90% chance of rain".

It's assumed that agents are **price-takers**, that is, they are unable to affect the market prices and therefore take them as a given. The prices of the goods are given by a $M-$dimensional vector $p = (p_1, \dots, p_M)$, where each price is a strictly positive quantity, ie, $p_\mu > 0$ for all μ. This assumes that goods have global prices, which is consistent with the completeness assumption: there is no reason why the market prices should be different for certain consumers or firms if they have complete knowledge and no transaction costs.

With a price vector p defined, we say the consumer j has a budget $B_j = p \cdot \omega_j$, which is the monetary value of his initial endowment. Any bundle he chooses to purchase will cost him $p \cdot x_j$. His objective, therefore, is to find the best bundle x_j he is able to afford, that is:

$$\max_{x_j \in X_j} U(x_j), \quad \text{s.t. } p \cdot x_j \leq p \cdot \omega_j \tag{2}$$

1 For every $x, x' \in X_j$, either $U_j(x) \geq U_j(x')$ or $U_j(x) \leq U_j(x')$.
2 For every $x, y, z \in X_j$, if $U_j(x) \geq U_j(y)$ and $U_j(y) \geq U_j(z)$, then $U_j(x) \geq U_j(z)$.

The firms, on the other hand, have an operating profit for each technology given by $p \cdot \xi_i$, which is how much money they earn by selling their outputs ($\xi_i^\mu > 0$) minus how much they spend purchasing the inputs ($\xi_i^\mu < 0$). Their objective is to maximize their profits, that is:

$$\max_{\xi_i \in \Xi_i} p \cdot \xi_i \tag{3}$$

With these ingredients laid out, we define an **allocation** of the economy as a set of specific choices for consumption bundles and technologies, ie, an allocation a of an economy E is

$$a = (x_1, \ldots, x_J, \xi_1, \ldots, \xi_N),\ x_j \in X_j,\ \xi_i \in \Xi_i \tag{4}$$

The economy is closed, and therefore all that is produced must come from the initial endowments and be consumed by the consumers. We therefore say an allocation is **feasible** if it satisfies **market clearing** for all the goods:

$$\sum_{j=1}^{J} x_j^\mu = \sum_{j=1}^{J} \omega_j^\mu + \sum_{i=1}^{N} \xi_i^\mu, \quad \forall\, \mu \in \{1, \ldots, M\} \tag{5}$$

This is a strong condition which constraints many quantities in the economy. In particular, if we multiply both sides of the equation by p^μ and sum then in μ we get, in vector notation,

$$\sum_{j=1}^{J} p \cdot (x_j - \omega_j) = \sum_{i=1}^{N} p \cdot \xi_i \tag{6}$$

The left handside is the leftover money the consumers have after making their choice of consumption, also called the value of excess demand, whereas the right handside is the firms aggregate profit, also known as the value of excess suply. Because we assume that the consumer may not spend more than his budget, the value of each consumer's individual excess demand has to be non positive. Simultaneously, if we assume that the firms always have $\xi_i = 0$ in their production set, ie, we assume that they can always opt to not produce at all and leave the market, then the value of excess supply for each firm has to be non negative. Because they must be equal, we conclude that in an economy for which market clearing holds, the consumer spends all his available budget and the firms all have zero profit, a result known as **Walras' Law**.

Given a set of possible feasible allocations $\{a_k\}$, we may wonder if there is any allocation we desire most over the other. This of course depends on the criteria we use to judge them: we may like allocations with less inequality, with the most aggregate utility, with the smallest minimum utility, etc. Economist opt to use one particular condition which is called **Pareto optmality**.

Intuitively, a **Pareto optimal** (or **Pareto efficient** allocation is one that you can't make a consumer better without making another consumer worse off. The idea is that, a non Pareto optimal allocation has some waste in it: one could change the consumption bundles in order to increase some utilities and no other consumer would complain. Because firms have zero profit in feasible allocations, they wouldn't mind the change.

More formally, a feasible allocation $a = (x, \xi)$ is said to be **Pareto optimal** if there is no other allocation that **Pareto dominates** it, that is, no allocation $a' = (x', \xi')$ such that $U(x'_j) \geq U(x_j)$ for all j and $U(x'_j) > U(x_j)$ for at least one j.

The Pareto optimality concept therefore defines a socially desireble outcome in a "non-controversial" way, by definition no agent in the economy would have a problem with policies or actions taken to make it more Pareto efficient. However, it says nothing about equality: an allocation in which one consumer has all the goods and no other consumer has any goods is Pareto optimal.

We finally arrive to the concept of equilibrium in an economy. A **Walrasian equilibrium** (or competitive equilibrium or simply equilibrium) in an economy E is an allocation (x^*, ξ^*) and a price vector p such that

1. Every firm i maximizes it's profits in its production set Ξ_i, that is
$$p \cdot \xi_i^* \geq p \cdot \xi_i, \quad \forall \xi_i \in \Xi_i, \quad \forall iin\{1, \ldots, N\} \tag{7}$$

2. Every consumer j maximizes his utility in his consumption set X_j, that is
$$U(x_j^*) \geq U(x_j), \quad \forall x_j \in X_j, \quad \forall j \in \{1, \ldots, J\} \tag{8}$$

3. The allocation (x^*, ξ^*) is feasible, that is,
$$\sum_{i=j}^{J} x_j^* = \sum_{j=1}^{J} \omega_j + \sum_{i=1}^{N} \xi_i^* \tag{9}$$

The Walrasian equilibrium is essentially a pair allocation - prices in each all the optimization problems are solved at once. Although we have made no mention of dynamics in this economy, it's considered an equilibrium because all agents are as satisfied as possible with their allocation given the prices, which we have assumed to be global and unchangeable by any agent's action. This is not exactly a definition of equilibrium as used in Physics, but we will discuss this point later. For now, we point out that a Walrasian equilibrium is in some sense stable.

One

We have thus defined two desirable properties of an allocation: efficiency and equilibrium. The fundamental results of General Equilibrium Theory are the **welfare theorems**, which define the conditions in which an equilibrium is Pareto optimal and vice versa.

The **First Fundamental Welfare Theorem** asserts that if the consumers have a utility function continuous on X_j[3], then all Walrasian equilibria are Pareto optimal. This result is simple yet useful, because it tells us that if our economy is in equilibrium, we don't have to care about checking if it's efficient. The violation is also important: if a given economy we are studying is in an inefficient equilibrium, then it must be that one of the theorem's condition was violated. This sheds light in where to look for market failures. We remind the reader, however, that some extra strong assumptions were made for the economies described by this therorem, namely, completeness of market and global prices that no single agent is capable of influencing.

The **Second Fundamental Welfare Theorem** requires extra assumptions: it afirms that if an economy satisfies the condition of the first fundamental theorem, the utility functions U_j plus all sets X_j and Y_i are convex and if we are able to redistribute the initial endowments at will while keeping the total amount $\sum_{j=1}^{J} \omega_j$ constant, then for every Pareto efficient allocation there exists a wealth allocation ω and price vector p^* such that (x^*, ξ^*, p^*) is a Walrasian equilibrium.

The second theorem is considerably more interesting than the first one: any Pareto optimal allocation we would like in an economy can be an equilibrium given the appropriate price vector and a possible wealth transfer, albeit under a stronger set of conditions.

3.2 THE DYNAMICS OF GENERAL EQUILIBRIUM

A conspicuous element was missing from the exposition above: there are no rules for the dynamics of the economies described above. The prices are taken as a given, as are the consumer and firm choices. What happens if a firm closes? What happens if a new firm appears? The equilibrium simply "recalculates" and the economy moves to the new one?

Indeed, this is a long standing criticism of General Equilibrium Theory. Walras proposed it as a process of **tatônnement**[4]: a central figure, known as the Walrasian auctioneer, sugests a price and asks all the firms and consumers how much would they like to produce and buy at these given prices, but without any transaction taking place at the out of equilibrium prices. The auctioneer updates the prices in the direction of diminishing excess demand or supply, a "gradient descent" of sorts, until equilibrium is reached.

However, this process is quite indetermined. Chiefly, this auctioneer figure doesn't exist in most decentralized markets: goods are traded at agreed prices by both parts, which do not wait until their transaction

3 The actual theorem asserts a weaker condition, that the preferences be locally nonsatiated, that is, for every $x \in X$, there is an $x' \in X$ such that $\|x - x'\| < \varepsilon$ and x' is preferred to x.

4 From "trial and error" in french

is authorized by some central authority. Even if they did, such an auctioneer would require an infinitely large computational capability to compute the excess demand and supply of every consumer and firm and for every good in a modern economy [2, 11]. Worst of all, even if there was such central figure with such an arbitrary large amount of computing power, the price updating dynamic is not guaranteed to converge [5]. Finally, even if it converges, we have no assurance that it will converge in finite time.

[14] [13] [12] [8]

[15, 6, 16]

4

THE RANDOM LINEAR ECONOMY MODEL

In this chapter we will present in detail and discuss the Random Linear Economy model [4] developed by Andrea De Martino, Matteo Marsili and Isaac Pérez Castillo which will be the basis for some of the applications discussed in the second part of this thesis.

There are some reasons why we chose to work with this model in particular: first, it presents a General Equilibrium Model which has few ingredients but already displays a rich behavior, including phase transitions which depend on the number of firms in the market. Secondly, it is analytically solvable using statistical mechanics techniques, such as using the replica trick to calculate the partition function. Therefore, it was ideal for trying new venues of exploration without the difficulty imposed in trying to prove general phenomena.

4.1 THE MODEL INGREDIENTS

An economy in the model is, like the General Equilibrium setting, composed by two distinct actors: consumers and firms. We assume N firms and one single representative consumer with utility function $U(x)$ and initial endowment x_0. This is a common approximation when doing equilibria calculation in Economics due to the simplicity: if we have J consumers with independent utility functions U_j (ie, U_j never depends on x_k, $k \neq j$) and initial endowments ω_j, then either we do not allow wealth transfers of ω_j and the optimization problem becomes very complicated, or we allow the central authority to carry out wealth transfers prior to allocation, and then the the demands generated by the consumers in this scenario is equivalent to that of a single representative consumer with utility function $U_R = \sum_{j=1}^{J} U_j$ and wealth $\omega_R = \sum_{j=1}^{J} \omega_j$.

The representative consumer assumption receives considerable criticism [7], chiefly because disregarding interaction among agents (via the utility of one depending on the decisions of the others) washes out the possibility of interactions and the wide range of important and interesting phenomena that in the statistical physics community we know to be generated precisely by these interactions [3], whereas the representative agent is a mean field approximation for consumers.

That said, the representative consumer is used in this model precisely because it generates an energy function which is convex and has a well defined, unique minimum and the resulting partition function can be calculated analytically in the zero temperature limit, while at the same time generating interesting behavior.

The consumer and the N firms will trade M goods, with a technological density parameter given by $n = N/M$. We assume as before that the consumer has an initial wealth $x_0 = (x_0^1, \ldots, x_0^M)$, $x_0^\mu \geq 0$, and wishes to improve its welfare in the market by using his endowment x_0 to purchase a consumption bundle x according to a separable utility function $U(x) = \sum_{\mu=1}^{M} u(x^\mu)$. His initial endowment, however, is assumed to be random, each x_0^μ drawn independently from a exponential distribution with unitary scale, ie,

$$P(x_0^\mu) = e^{-x_0^\mu} \tag{10}$$

As before, the aim of the consumer in this economy is to solve the maximization problem

$$x^* = \arg\max_x U(x) \text{ s. t. } p \cdot x \leq p \cdot x_0 \tag{11}$$

In most of the analysis done in this thesis we will treat the particular case of the consumer's separable utility function as $u(x_\mu) = \log x_\mu$, although any concave function would work exhibit similar qualitative behavior. The logarithm is a common choice for the consumer's utility function because it satisfies some of the usual properties desired for the consumer behavior in economics: first, the consumer is **loss averse**, which means that he will always prefer a guaranteed amount a of any good to a lottery in which he can win $a+\delta$ with probability 0.5 and $a-\delta$ with probability 0.5, for any $\delta > 0$. He is loss averse because the disutility losing δ is larger than the utility of gaining δ. In our case, we don't have lotteries, but the principle holds for two goods: if he has $\bar{x}+\delta$ of good μ and $\bar{x}+\delta$ of good ν, he will try to find a company that trades this excess of good ν so he can average both goods and in fact, may even do so at a loss (ie, he ends up with $\bar{x}-\varepsilon$ for both goods, for some $\varepsilon < \delta$). Also, with the separable utility as chosen, there are not complementary or substitute goods, ie, goods for which the consumer prefers to have more (or less) of one if he has another. Finally, because $u(0) = -\infty$, the consumer will always try to obtain a little bit of every good, even if at a great cost, because nothing is worse than having none of a particular good.

The firms on the other hand have each an M-dimensional random technology $\xi_i = (\xi_i^1, \ldots, \xi_i^M)$, where $\xi_i^\mu < 0$ represents an input and $\xi_i^\mu > 0$ represents an output. The production set of each firm is the space of all vectors which are proportional to ξ_i, that is, $\Xi_i = s\xi_i$, $s \geq 0$. This means that each firm i only has one technology and its only decision is the scale s_i at which it operates this technology. Once chosen the scale s_i, a company will consume $s_i\xi_i^-$ goods and produce $s_i\xi_i^+$ goods, where $\xi_i^\pm$ are the positive and negative entries of the ξ_i vector.

The elements ξ_i^μ are independently drawn from a normal distribution with zero mean and Δ/M variance, where $\Delta > 0$, and are normalized so that the sum over all the goods for a company is fixed at a negative value and all technologies are a little inefficient. We must have then:

$$P(x_i^\mu) = \mathcal{N}(x_i^\mu | 0, \Delta M^{-1}), \quad \sum_{\mu=1}^{M} \xi_i^\mu = -\epsilon \tag{12}$$

We normalize the technologies to be inefficient so that we don't have a combination of firms producing infinite goods, ie, firm i and j can produce infinite amounts of certain goods by each feeding its output to be used as the other's input.

The objective of each company in the market is the same as before: each firm i tries independently to choose it's production scale s_i as to maximize it's profits:

$$s_i^* = \arg\max_{s_i > 0} p \cdot (s_i \xi_i) \tag{13}$$

Other underlying assumptions of General Equilibrium Theory are valid here: we assume a complete market, where each agent knows the offer and demand of all other agents, there is no transaction costs and a good is uniquely defined. Also, agents are price-takers, which means that they have no power over the prices and must accept them as given.

We also treat the economy as closed and therefore it must satisfy the market clearing condition. Because we have just one consumption bundle, then the N dimensional production scale vector s has to be such that

$$x = x_0 + \sum_{i=1}^{N} s_i \xi_i \tag{14}$$

ie, all the inputs the firms use have to come from the consumer's initial endowment.

Because market clearing hold and agents are price takers, we can also derive the strong restriction on profits discussed before. If we multiply both sides of the equation (14) by p, we get

$$p \cdot (x - x_0) = \sum_i s_i p \cdot \xi_i, \tag{15}$$

The left side of the above equation has to always be smaller or equal to zero, because of the budget condition. But the right hand side has to be always greater or equal to zero, because this term represents the sum of the individual firms' profits and if a firm is losing money they can always choose to set $s_i = 0$ and leave the market. Therefore, we must have that both sides are equal to zero, and the consequence is that the agent completely spends all his available budget (ie, $p \cdot x = p \cdot x_0$, he has not "leftover" cash after choosing x) and that the firms either have zero profit ($p \cdot \xi_i = 0$) or leave the market ($s_i = 0$).

One of the important implications of equation (15) for the Random Linear Economy model is that we may not have more than M firms active at any given realization of equilibrium. If the right hand side of equation (15) has to be zero, then for every firm either $s_i = 0$ or $p \cdot \xi_i = 0$. If ϕ is the fraction of firms active in the market, that is

$$\phi = \frac{\sum_{i=1}^{N} \mathbb{I}(s_i > 0)}{N}, \tag{16}$$

then all of them have $p \cdot \xi_i = 0$. Because the price is the same for all of them, we have ϕN equations of this type, and M unknowns. For this system to have a non-trivial solution (ie, $p_\mu > 0$ for all μ), it must be that $\phi N \leq M$, which implies that

$$\phi leq \frac{1}{n} \tag{17}$$

Having a single representative consumer (or many consumers but with wealth transfers) has two important consequences: first, the price vector is entirely determined by the consumer's demand. This is a result of the first order condition for the maximization problem. By taking the derivative of equation (11) with the proper Lagrange multiplier we get

$$\frac{\partial U(x)}{\partial x_\mu} - \lambda p_\mu = 0 \Rightarrow p_\mu = \frac{1}{\lambda x_\mu} \tag{18}$$

Furthermore, the market clearing condition binds the optimization problem of the consumer and the firms. If we substitute equation (14) in the consumer's utility, we get:

$$s^* = \arg\max_{s:s_i \geq 0} U(x_0 + \sum_{i=1}^{N} s_i \xi_i) \tag{19}$$

We can easily check that the zero profit condition is preserved with this solution. If s_i is in s^*, the solution for the consumer's maximization problem, then either $s_i = 0$ or $s_i > 0$. If $s_i = 0$, the condition is satisfied. Otherwise, if $s_i > 0$, it means that the constraint $s_i \geq 0$ was not enforced and the derivative at s_i must be zero. We then have

$$0 = \frac{\partial U}{\partial s_i} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s_i} = p \cdot xi_i \tag{20}$$

Our problem is now considerably reduced: to find the equilibria in this model economy all we have to do is solve the maximization problem in equation (19).

4.2 THE ROLE OF STATISTICAL MECHANICS

If we were to employ standard convex optimization techniques to solve to solve (19), we would be able to find the solution for a specific realization of x_0 and ξ given a fixed N and M. But if we were to calculate quantities of interest such as consumer utility, average good consumption, average good price, price deviation among goods, number of active firms, etc, these would all be random variables which depend on the realization of endowments and technologies.

This is, of course, a well known behavior in statistical mechanics. We solve this by treating the case where the system size is very large, so that these average quantities converge to a single value. This isn't always the case, but holds for the so called *self-averaging* systems. In these systems, these average quantities for large systems converge to an average over the realizations for smallers systems.

The general approach to finding the equilibrium properties of a physical system is is to calculate the partition function for a specific realization of ξ, x_0:

$$Z(\beta|\xi, x_0) = \int dx e^{\beta U(x|x_0,\xi)}, \tag{21}$$

where β is the inverse value of the temperature. From this, we can calculate the average value of the utility function by taking the derivative of $\log Z$:

$$\langle U \rangle (\beta|\xi, x_0) = \int_0^\infty dx \frac{e^{\beta U(x|\xi,x_0)}}{Z(\beta|\xi, x_0} U(x|\xi, x_0) = \frac{\partial}{\partial \beta} \log Z(\beta|\xi, x_0) \tag{22}$$

The maximum value for the utility $U(x)$ is equivalent to the average value on the ground state[1], ie:

$$\max_x U(x|\xi, x_0) = \lim_{\beta\to\infty} \langle U \rangle (\beta|\xi, x_0) \tag{23}$$

However, we are still calculating the maximum as a function of the samples x_0 and ξ. In order to get the average behavior, which holds for a large system, we must average the utility over the disorder. Assembling all pieces together, we finally get the solution to equation (19):

$$\max_x U(x) = \int d\xi dx_0 P(\xi) P(x_0) \lim_{\beta\to\infty} \frac{\partial}{\partial \beta} \log \int dx e^{\beta U(x|x_0,\xi)} \tag{24}$$

The explicit calculation of the expression above is considerably involved and makes use of a method commonly known as **replica trick** in the statistical physics community. We proceed with the detailed calculation on Appendix A. The solution of this calculation is given by

$$\lim_{N\to\infty} \frac{1}{N} \max_x U(x) = \max_\theta h(\Omega, \kappa, p, \sigma, \chi, \hat{\chi}), \tag{25}$$

where $\theta = (\Omega, \kappa, \sigma, \chi, \hat{\chi})$ are order parameters that arise during the calculation and h is given by:

$$\begin{aligned} h(\Omega, \kappa, p, \sigma, \chi, \hat{\chi}) = &\left\langle \max_s \left[-\frac{\hat{\chi}}{2} s^2 + (t\sigma - \epsilon p)s \right] \right\rangle_t + \\ &+ \frac{1}{2}\left(\Omega\hat{\chi} + \frac{\kappa p}{n} \right) - \frac{1}{2n\Delta}\chi\sigma^2 - \frac{1}{2n}\chi p^2 + \\ &+ \frac{1}{n} \left\langle \max_x \left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega} t)^2}{2\chi} \right] \right\rangle_{t,x_o}, \end{aligned} \tag{26}$$

1 This is true because $U(x)$ is convex and therefore has only one maximum.

where t is a normal random variable with zero mean and unitary variance.

4.3 RESULTS

The model has some very interesting properties which are described at length in [4]. In particular, it's possible to analytically calulate the distribution probabilities of x and s (and therefore of p) and see that all macroscopic quantities derived from these two quantities depend on the number of firms per good $n = N/M$. The model displays a regime change at $n = 2$, ie, two random technologies per good. When $n < 2$, the market is competitive and the fraction of active firms $\phi = \sum_i \mathbb{I}(s_i > 0)/N$ is around $\phi = 0.5$. Because each firm has on average half the goods as inputs and half as outputs, when $n < 2$ you don't have enough firms to span the whole M dimensional space in order to be able to fine tune the quantities desired for all the goods.

When $n > 2$, however, there are many firms to choose from and statistically it's possible to choose M linear independent firms that span the whole good space. In this regime, the market becomes monopolistic and ϕ assymptotically goes to zero with n, due to a saturation of active firms on M.

The change in the model economy's GDP also reflects the qualitative change in allocation. The authors define the gross product for the model as the total value of goods produced, that is, the sum of $(x_\mu - x_0^\mu)p_\mu$ for all goods μ that are produced, ie, $x_\mu > x_0^\mu$. However, the market clearing condition (15) makes the value of goods produced equal to the value of goods used as input, so we calculate the GDP Y by averaging over the absolute value of all trades:

$$Y = \frac{\sum_{\mu=1}^{M} |x_\mu - x_0^\mu| p_\mu}{2\sum_{\mu=0}^{M} p_\mu}, \tag{27}$$

where the denominator also includes a normalization for the prices.

What is shown in [4] is that in the competitive regime when $n < 2$, a new firm will have a significant positive effect on Y, while in the monopolistic regime $n > 2$ a new firm will have negligible impact on the gross product. We will revisit this result later in this paper.

The Random Linear Economies model is particularly suitable for further analysis because it's a General Equilibrium setting with few ingredients, but the introduction of stochastic elements offers a nontrivial phase transition which is not observed in similar "simple" economic models in the literature.

Part II

APPLICATIONS

5

INEFFICIENT CONSUMER IN A GENERAL EQUILIBRIUM SETTING

6

INPUT-OUTPUT OF RANDOM ECONOMIES AND REAL WORLD DATA

7

WHEN DOES INEQUALITY FREEZE AN ECONOMY?

7.1 THE MODEL

The inequality trading model consists of an economy with N agents, each with wealth c_i with $i = 1, \dots, N$. Agents are allowed to trade among themselves M objects, with each object $m = 1, \dots, M$ having a price π_m. A given allocation of goods among the agents is described by an $N \times M$ allocation matrix $\mathcal{A}$ with entries $a_{i,m} = 1$ if agent i owns good m and zero otherwise. Agents can only own baskets of goods that they can afford, i.e. whose total value does not exceed their wealth. The wealth not invested in goods

$$c_i - \sum_{m=1}^{M} a_{i,m}\pi_m = \ell_i \geq 0, \qquad i = 1, \dots, N, \tag{28}$$

corresponds to the cash (liquid capital) that agent i has available for trading. Therefore the set of feasible allocations – those for which $\ell_i \geq 0$ for all i – is only a small fraction of the M^N conceivable allocation matrices $\mathcal{A}$.

Starting from a feasible allocation matrix $\mathcal{A}$, we introduce a random trading dynamics in which a good m is picked uniformly at random among all goods. Its owner then attempts to sell it to another agent i drawn uniformly at random among the other agents. If agent i has enough cash to buy the product m, that is if $\ell_i \geq \pi_m$, the transaction is successful and his/her cash decreases by π_m while the cash of the seller increases by π_m, otherwise the transaction fails, with no possibility of an object being divided. Notice that the total capital c_i of agents does not change over time, so c_i and the prices π_m are parameters of the model. The entries of the allocation matrix, and consequently the cash, are dynamical variables, which evolve over time according to this dynamics. This model belongs to the class of zero-intelligent agent-based models, because agents do not try to maximize any utility function and simply trade the goods they have at random.

A crucial property of the dynamic described above is that the stochastic transition matrix $W(\mathcal{A} \to \mathcal{A}')$ is symmetric between any two feasible configurations $\mathcal{A}$ and $\mathcal{A}'$: $W(\mathcal{A} \to \mathcal{A}') = W(\mathcal{A}' \to \mathcal{A})$. Note that any feasible allocation $\mathcal{A}$ can be reached from any other feasible allocation $\mathcal{A}'$ by a sequence of trades. This implies that the dynamic satisfies the detailed balance condition:

$$W(\mathcal{A} \to \mathcal{A}')P(\mathcal{A}) = W(\mathcal{A}' \to \mathcal{A})P(\mathcal{A}'), \quad \forall \mathcal{A}, \mathcal{A}' \tag{29}$$

with a stationary distribution over the space of feasible configurations that is uniform, ie, $P(\mathcal{A}) = \text{const}$. This is consequence of the symmetric transition rates, and would be the same for every trading rule that has $W(\mathcal{A} \to \mathcal{A}') = W(\mathcal{A}' \to \mathcal{A})$. In fact, the current trading rules employed in this model are a particular case of a general rule for which we first select a subset of n agents, $2 \leq n \leq N$, then we pick a random good from these n agents and try to trade it with the remaining $n-1$ agents, automatically accepting if the chosen buyer has enough cash to purchase it. This rule may sound cryptic, but it's common in the particular cases of $n = N$, which is the current one described in the model, and for $n = 2$, in which we first pick two consumers and then try to exchange a random good among themselves. All of these rules generate the same stationary distribution.

The analysis presented here is restricted to symmetric rates because otherwise, even if we held detailed balance, we would have to explicitly find the probability density over the configurations. Since the resulting density would be non-uniform, it would be more difficult to link dynamical observables (rate of money transfer, etc.) to static variables (number of neighbouring configuration to a given configuration). We do not explore these cases.

These rules are not the only ones providing symmetric rates, or even that this property is necessary for interesting dynamics. We simply point out that the detail of the choice of the dynamical rule is not crucial, as long as we as we have a simple zero-intelligent dynamics which generates symmetric transition rates. We now give a few examples of rules that are either identical to those above, or do not yield symmetrical probabilities of transfer:

For the intial agent's capital, we choose to focus on realisations where the wealth c_i is drawn from a Pareto distribution $P\{c_i > c\} \sim c^{-\beta}$, for $c > c_{\min}$ for each agent i, which is compatible with many empirical observations of real world wealth distribution. We allow β to vary so that we are able to explore different levels of inequality, and compare different economies in which the ratio between the total wealth $C = \sum_i c_i$ and the total value of all objects $\Pi = \sum_m \pi_m$ is kept fixed. We use $C > \Pi$ so as to have feasible allocations.

For the goods, we are going to limit the analysis to cases where the M objects are divided into a small number of K classes with M_k objects per class ($k = 1, \ldots, K$), objects belonging to class k have the same price $\pi_{(k)}$. If $z_{i,k}$ is the number of object of class k that agent i owns, then (28) takes the form

$$c_i = \sum_{k=1}^{K} z_{i,k}\pi_{(k)} + \ell_i \tag{30}$$

In the next section we will solve the master equation and find the distribution of goods as a function of capital c_i. As it will be shown, the main result of this model is that the flow of goods among agents becomes more and more congested as inequality increases until it halts completely when the Pareto exponent β tends to one.

7.2 THE SOLUTION TO THE MASTER EQUATION

A formal approach to this problem consists in writing the complete Master Equation that describes the evolution of the probability $P(z_1, \ldots, z_N)$ to find the economy in a state where each agent $i = 1, \ldots, N$ has a definite number z_i of goods. Taking the sum over all values of z_j for $j \neq i$, one can derive the Master Equation for a single agent with wealth c_i. The corresponding marginal distribution $P_i(z)$ in the stationary state can be derived from the detailed balance condition

$$P_i(z+1)\frac{z+1}{M}p^s = P_i(z)\frac{1}{N}\left(1-\delta_{z,m_i}\right), \qquad z = 0, 1, \ldots, m_i \tag{31}$$

where $m_i = \lfloor c_i/\pi \rfloor$ is the maximum number of goods which agent i can buy with wealth c_i and p^s is the probability that a transaction where agent i sells one good (i.e. $z+1 \to z$) is successful. Equation (31) says that, in the stationary state, the probability that agent i has z objects and buys a new object is equal to the probability to find agent i with $z+1$ objects, selling successfully one of them. The factor $1-\delta_{z,m_i}$ enforces the condition that agent i can afford at most m_i goods and it implies that $P_i(z) = 0$ for $z > m_i$. Exchanges are successful if the buyer j does not already have a saturated budget $z_j = m_j$. So the probability p^s is also given by

$$p^s = 1 - \frac{1}{N-1}\sum_{j\neq i} P\{z_j = m_j | z_i = z\} \tag{32}$$

$$\cong 1 - \frac{1}{N}\sum_j P_j(m_j) \qquad (N, M \gg 1) \tag{33}$$

where the last relation holds because when $N, M \gg 1$ the dependence on z becomes negligible. This is important, because it implies that for N large the variables z_i can be considered as independent, i.e. $P(z_1, \ldots, z_N) = \prod_i P_i(z_i)$, and the problem can be reduced to that of computing the marginals $P_i(z_i)$ self-consistently.

The solution of equation (31) can be written as a truncated Poisson distribution with parameter $\lambda = M/(Np^s)$:

$$P_i(z) = \frac{1}{Z_i}\left[\frac{\lambda^z}{z!}\right]\Theta\left(m_i - z\right), \tag{34}$$

where Z_i is a normalization factor that can be determined by $\sum_z P_i(z) = 1$. Finally, the value of p^s – or equivalently of λ – can be found self-consistently, by solving equation (33).

Notice that the most likely value of z for an agent with $m_i = m$ is given by

$$z^*(m) = \arg\max_z P(z) = \begin{cases} m, & \text{if } m \leq \lambda \\ \lambda, & \text{if } \lambda \leq m \end{cases}. \tag{35}$$

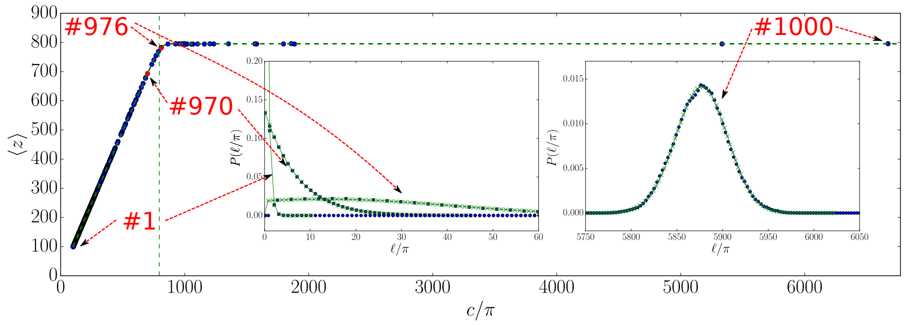

Figure 1.: Capital composition in an economy with a single type of good, $N = 10^3$ agents, $\beta = 1.8$, $M \approx 2.10^5$ and $C/\Pi = 1.1$. Points $\{(\langle z \rangle_i, c_i)\}_{i=1}^N$ denote the average composition of capital for different agents obtained in Monte Carlo simulations. This is compared with the analytical solution obtained from the Master Equation (green dashed line) given by equation (34). The vertical dashed line at $c^{(1)} \simeq 7.98 = M/Np_1^s$ indicates the analytically predicted value of the crossover wealth that separates the two classes of agents. Insets: cash distributions $P_i(\ell)$ of the indicated agents.

This provides a natural distinction between cash-poor agents – those with $m \leq \lambda$ – that often cannot afford to buy further objects, and cash-rich ones – those with $m > \lambda$ – who typically have enough cash to buy further objects. The origin of this behaviour can be understood in the simplest setting where $K = 1$, i.e. all goods have the same price $\pi_m = \pi_{(1)} = \pi$ (we are going to omit the subscript (1) in this case). Figure 1 shows the capital composition $\{(\langle z \rangle_i, c_i)\}_{i=1}^N$ for all agents in the stationary state, where $\langle z \rangle_i$ is the average number of goods owned by agent i. These two classes are separated by a sharp crossover region. The inset of Figure 1 shows the cash distribution $P_i(\ell/\pi)$ (where $\ell/\pi = c_i/\pi - z$ represents the number of goods they are able to buy) for some representative agents. While cash-poor agents have a cash distribution peaked at 0, the wealthiest agents have cash in abundance.

In terms of wealth, the poor are defined as those with $c_i < c^{(1)}$ whereas the rich ones have $c_i > c^{(1)}$, where the threshold wealth is given by $c^{(1)} = \lambda\pi = M\pi/(Np^s)$. Notice that when $\lambda \gg 1$, a condition that occurs when the economy is nearly frozen ($p^s \ll 1$), the distribution $P_i(z)$ is sharply peaked around $z^{\text{mode}}(m)$ so that its average is $\langle z \rangle \simeq z^{\text{mode}}(m)$. Then the separation between the two classes becomes rather sharp, as in Figure 1.

7.2.1 *Derivation of p_s and $c^{(1)}$*

We can also derive an estimate of p^s in the limit $N \to \infty$, for $\beta > 1$. Indeed, we have $P_i(z = m_i) \simeq 1 - \frac{m_i}{\lambda} + O(\lambda^{-2})$ for $\lambda \gg m_i$, so a rough estimate of $P_j(m_j)$ is given by $P_j(m_j) \simeq \max\{0, 1 - m_j/\lambda\}$. Taking the average over agents, as in Eq. (33), and assuming a distribution density of wealth $\rho(c) = \beta c^{-\beta-1}$ for $c \geq 1$ and $\rho(c) = 0$ for $c < 1$, one finds

$$c^{(1)} \simeq \left[\beta\left(1 - \frac{\Pi}{C}\right)\right]^{1/(1-\beta)}, \tag{36}$$

$$p^s = \frac{M}{N\lambda} \simeq \frac{\Pi}{C}\frac{\langle c\rangle}{c^{(1)}}. \tag{37}$$

Here $\langle c\rangle = \beta/(\beta-1)$ is the expected value of the wealth. Notice that $\langle c\rangle$ diverges as $\beta \to 1^+$, but also that within this approximation the threshold wealth $c^{(1)}$ diverges much faster, with an essential singularity. More precisely, we note that $\Pi/C < 1$, so that $\beta(1-\Pi/C) \sim (1-\Pi/C)$ is a number smaller than 1 (yet positive). From Eq. (36), we have $c^{(1)} \sim (1-\Pi/C)^{-1/(\beta-1)} \to \infty$. Therefore the liquidity p^s vanishes as $\beta \to 1^+$.

For finite N, this approximation breaks down when β gets too close to or smaller than one. Also, $\langle c\rangle$ is ill-defined and in equation (37) it should be replaced with $1/N\sum_i c_i$, which strongly fluctuates between realizations and depends on N. An estimate of p^s for finite N and $\beta < 1$ can be obtained by observing that the wealth $c^{(1)}$ that marks the separation between the two classes cannot be larger than the wealth $c_{\max}$ of the wealthiest agent. By extreme value theory, the latter is given by $c_{\max} \sim N^{1/\beta}$, with $a > 0$. Therefore the solution is characterised by $c^{(1)} = \pi\lambda \sim c_{\max} \sim N^{1/\beta}$. Furthermore, for $\beta < 1$ the average wealth is dominated by the wealthiest few, i.e. $\langle c\rangle \sim N^{1/\beta-1}$ and therefore $p^s \sim N^{1/\beta-1}/c^{(1)} \sim N^{-1}$. In other words, in this limit the cash-rich class is composed of a finite number of agents, who hold almost all the cash of the economy. Figure 2 (left) shows that the rough analytical estimate of equation (37) is in good agreement with Monte Carlo simulations.

We can compute p^s using

$$p^s = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(z = m_i) \tag{38}$$

approximating the probability to be on a threshold $P_i(z = m_i)$ by

$$P_i(z = m_i) = \begin{cases} (1 - \frac{m_i}{\lambda}) & \text{for } m_i \ll \lambda \\ 0 & \text{for } m_i > \lambda \end{cases}. \tag{39}$$

The first case can be understood by noting that

$$P_i(z = m_i) = \frac{\lambda^{m_i}\frac{1}{m_i!}}{\sum_{x=0}^{m_i}\lambda^x\frac{1}{x!}} = \frac{1}{1 + \frac{m_i}{\lambda} + \frac{m_i(m_i-1)}{\lambda^2} + \ldots} \simeq \left(1 - \frac{m_i}{\lambda}\right), \tag{40}$$

where the approximation is valid in the limit $m_i \ll \lambda$. Assuming this approximation to be valid in all the range $m_i < \lambda$ is clearly a bad assumption for all agents with m_i close to λ. However the wealth is power law distributed, and so the weight of agents with $m_i \sim \lambda$ is negligible in the sum over all agents, Eq. (38). The accuracy of this approximation increases when the exponent of the power law β decreases.

Then p^s can be computed using

$$p^s = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(z = m_i) \simeq 1 - \int_1^{c^{(1)}=\lambda\pi} dc\, \beta c^{-\beta-1}\left(1 - \frac{c}{\lambda\pi}\right). \quad (41)$$

This is an implicit expression for p^s, since it appears on the l.h.s. of the equation and also on the r.h.s. (because $\lambda = \frac{M}{Np^s}$).

When $\beta > 1$ this expression can be expressed to be realization-independent, using

$$p^s = \frac{M}{N\lambda} = \frac{\Pi}{C}\frac{\langle c\rangle}{c^{(1)}}, \quad (42)$$

where $\langle c\rangle = \beta/(\beta-1)$ is the expected value of the wealth per agent. We also use the fact that we fill in the system a number M of goods in such a way to have a fixed ratio Π/C. Performing the integral on the r.h.s of Eq. (41) gives an equation for $c^{(1)}$:

$$\frac{\Pi}{C}\frac{\langle c\rangle}{c^{(1)}} = c^{(1)^{-\beta}}\left(\frac{1}{1-\beta}\right) - \frac{\beta}{1-\beta}\frac{1}{c^{(1)}}, \quad (43)$$

that simplifies into:

$$c^{(1)} = \left[\beta\left(1 - \frac{\Pi}{C}\right)\right]^{1/(1-\beta)}. \quad (44)$$

The analysis carries forward to the general case in which K classes of goods are considered, starting from the full Master Equation for the joint probability of the ownership vectors $\vec{z}_i = (z_{i,1}\ldots, z_{i,K})$ for all agents $i = 1,\ldots,N$. For the same reasons as before, the problem can be reduced to that of computing the marginal distribution $P_i(\vec{z}_i)$ of a single agent. The main complication is that the maximum number $m_{i,k}$ of goods of class k that agent i can get now depends on how many of the other goods agent i owns, i.e. $m_{i,k}(z_i^{(k)}) = \lfloor(c_i - \sum_{k'(\neq k)} z_{i,k'}\pi_{(k')})/\pi_k\rfloor$, where $z_i^{(k)} = \{z_{i,k'}\}_{k'(\neq k)}$. The detailed balance condition is

$$P_i(\vec{z}+\hat{e}_k)\frac{z_k+1}{M}p_k^s = P_i(\vec{z})\frac{M_k}{M}\frac{1}{N}\left(1 - \delta_{z_k, m_{i,k}(z_{(k)})}\right) \quad (45)$$

This detailed balance condition again yields the stationary state distribution for $N, M \gg 1$. On the left we have the probability that one of the z_k+1 objects of type k of agent i is picked for a successful sale (here $\hat{e}_k$ is the vector with all zero components and with a k^{th} component equal to one, and p_k^s is the probability that a sale of an object of type k is successful). This must balance the probability (on the r.h.s.) that agent i is selected as the buyer of an object of type k, which requires that agent i has less than $m_{i,k}(z_{(k)})$ objects of type k, for the transaction to occur (here M_k/M is the probability that an object of type k is picked at random, and $1/N$ is the probability that agent i is selected as the buyer). It can easily be checked that the solution to this set of equations is given by a product of Poisson distributions with parameters $\lambda_k = M_k/(Np_k^s)$, with the constraint given by equation (28)

$$P_i(z_1, ..., z_K) = \frac{1}{Z_i}\left[\prod_{k=1}^{K} \frac{\lambda_k^{z_k}}{z_k!}\right] \Theta\left(c_i - \sum_{k}^{K} z_k \pi_{(k)}\right), \tag{46}$$

with Z_i a normalization factor obeying $\sum_{z_1} ... \sum_{z_K} P_i(z_1, ..., z_K) = 1$. Here the p_k^s corresponds to the acceptance rates of transactions of goods of class k and are given by

$$p_k^s = 1 - \frac{1}{N}\sum_{i=1}^{N} P\left\{z_{i,k} = m_{i,k}(z_i^{(k)})\right\} \tag{47}$$

As in the case with $K = 1$, the values for the probabilities p_k^s need to be found self-consistently, which can be complicated when K and M are large.

When the total number of objects per agent is large for any class k, we expect that $\lambda_1, ..., \lambda_K \gg 1$, and then the values of $z_{i,k}$ are close to their expected values. This implies that the population of agents splits into K classes, where agents with wealth $c_i \in [c^{(k-1)}, c^{(k)}]$ have their budget saturated with goods of class $k' \leq k$ and cannot afford more expensive objects (here $c^{(k)} = \lambda_k \pi_{(k)}$, $k = 1, \ldots, K$ and $c^{(0)} = c_{\min}$). An estimate for the thresholds $c^{(k)}$ can be derived following the same arguments as for the case of a single type of good, $K = 1$, by observing that when analysing the dynamics of goods of type k, all agents in class $k' < k$ are effectively frozen and can be neglected. Combining this with the conservation of the total number of objects of each kind, we obtain a recurrence relation for $c^{(k)}$.

7.2.1.1 *Derivation of p_k^s and $c^{(k)}$ in the large λ limit for several types of good.*

An analytic derivation for the p_k^s and $c^{(k)}$ can be obtained also for the cases of several goods, but only in the limit in which prices are well separated (i.e. $\pi_{(k+1)} \gg \pi_{(k)}$) and the total values of good of any class is approximately constant (we use $M_k \pi_{(k)} = \Pi/K = \text{const}$). In this limit we expect to find a sharp separation of the population of agents into classes. This is because $M_1 \gg M_2 \gg \ldots \gg M_K$ implies that the market is flooded with objects of the class 1, which constantly change hands and essentially follow the laws found in the single type of object case. On top of this dense gas of objects of class 1, we can consider objects of class 2 as a perturbation (they are picked M_2/M_1 times less often!). On the time scale of the dynamics of objects of type 2, the distribution of cash is such that all agents with a wealth less than $c^{(1)} = \pi_{(1)}\lambda_1$ have their budget saturated by objects of type 1 and typically do not have enough cash to buy objects of type 2 nor more expensive ones. Likewise, there is a class of agents with $c^{(1)} < c_i \leq c^{(2)}$ that will manage to afford goods of types 1 and 2, but will hardly ever hold goods more expensive that $\pi_{(2)}$.

In brief, the economy is segmented into K classes, with class k composed of all agents with $c_i \in [c^{(k)}, c^{(k+1)})$ who can afford objects of class up to k, but who are excluded from markets for more expensive goods, because they rarely have enough cash to buy goods more expensive than $\pi_{(k)}$. This structure into classes can be read off from Figure 4, where we present the average cash of agents, given their cash in a specific case (see caption). The horizontal lines denote the prices $\pi_{(k)}$ of the different objects, and the intersections with the horizontal lines define the thresholds $c^{(k)}$. Agents that have c_i just above $c^{(k)}$ are cash-filled in terms of object of class k, but are cash-starved in terms of objects $\pi_{(k')}, k' > k$.

The liquidities p^s can be given by the following expression

$$p^s_k = 1 - \frac{1}{N}\sum_{i=1}^{N} P\left\{z_{i,k} = m_{i,k}(z_i^{(k)})\right\} = 1 - \frac{1}{N}\sum_{i=1}^{N} P_i(\text{not accepting good type k}) \tag{48}$$

According to the previous discussion of segmentation of the system into K classes, and using the same approximation for this threshold probability discussed in the case of 1 type of good, we assume

$$P_i(\text{not accepting good type k}) = \begin{cases} 1 & \text{for } m_i < \lambda_{k-1} \\ \left(1 - \frac{m_i}{\lambda_k}\right) & \text{for } \lambda_{k-1} < m_i < \lambda_k \\ 0 & \text{for } m_i > \lambda_k \end{cases}, \tag{49}$$

Then

$$p^s_k \simeq 1 - \int_1^{c^{(k-1)}} dc\, \beta c^{-\beta-1} - \int_{c^{(k-1)}}^{c^{(k)}} dc\, \beta c^{-\beta-1}\left(1 - \frac{c}{c^{(k)}}\right) \tag{50}$$

In this case now we have

$$p^s_k = \frac{M_k}{N\lambda_k} = \frac{\Pi}{KC}\frac{\langle c\rangle}{c^{(k)}} \tag{51}$$

With similar calculations to the ones showed for the previous case, one can easily get to the recurrence relation:

$$c^{(k)} = \left[\beta\left(c^{(k-1)}\right)^{1-\beta} - \beta\frac{\Pi}{KC}\right]^{\frac{1}{1-\beta}}. \tag{52}$$

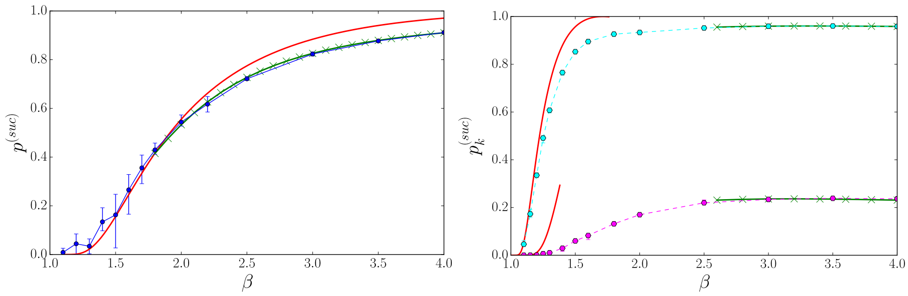

Figure 2.: Success probability of transaction p_k^s as a function of the Pareto exponent β. Comparison between numerical simulations and analytical estimates for one class of goods (left panel) and two classes of goods (right panel). The blue solid circles are the result of Monte Carlo simulations performed for $N = 10^5$ agents and averaged over 5 realizations. Here the error bars indicate the min and max value of p_k^s over all realizations (we used the "adjusted Pareto" law for the right panel, see Appendix **??**). The red lines are the analytic estimates according to Eq. (37) and Eq. (54) for left and right panels, respectively. The green crossed lines correspond to numerically (see Appendix **??**) solving the analytical solution (52) for a population composed of $N = 64$ (kind of) agents.

Iterating, we explicit this into:

$$c^{(k)} = \left[\beta^k - \left(\frac{\beta - \beta^{k+1}}{1-\beta}\right)\frac{\Pi}{KC}\right]^{\frac{1}{1-\beta}}, \tag{53}$$

$$c^{(k)} \simeq \left[\beta^k - \left(\frac{\beta - \beta^{k+1}}{1-\beta}\right)\frac{\Pi}{KC}\right]^{\frac{1}{1-\beta}}, \tag{54}$$

$$p_k^s = \frac{M_k}{N\lambda_k} \simeq \frac{\Pi}{KC}\frac{\langle c\rangle}{c^{(k)}}. \tag{55}$$

In the limit $\beta \to 1^+$ of large inequality, close inspection[1] of equation (54) shows that $c^{(k)} \to \infty, \forall k$, which implies that all agents become cash-starved except for the wealthiest few. Since $p_k^s \sim \langle c\rangle / c^{(k)}$, this implies that all markets freeze: $p_k^s \to 0, \forall k$. The arrest of the flow of goods appears to be extremely robust against all choices of the parameter $\pi_{(k)}$, as p_1^s is an upper bound for the other success rates of transactions p_k^s. These conclusions are fully consistent with the results of extensive numerical simulations (see Figure 2.

We can also compare the liquid and capital concentrations, measured via their Gini coefficients, for various values of β in the system of Fig. 4 ($K = 10, g = 1.5, \pi_{(1)} = 0.001, C/\Pi = 1.2$). In particular, note

1 Note that the term in square brackets is smaller than one, when $\beta \to 1^+$.

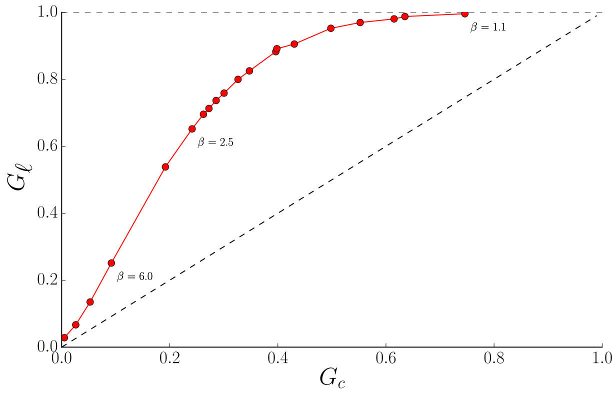

Figure 3.: Gini coefficient G_ℓ of the cash distribution (liquid capital) in the stationary state of the model as a function of the Gini G_c of the wealth distribution. The dashed line indicates proportionality between cash and wealth, in which case the inequality in both is exactly the same. The wealth follows a Pareto distribution with exponent β that tunes the degree of inequality (the higher is β, the more egalitarian the distribution).

that the limit $\beta \to 1^+$ is singular, as G_ℓ reaches one around $\beta = 1.1$, with smaller β yielding also $G_\ell \approx 1$. This is an alternative way to see how the concentration of capital generates an over-concentration of liquidities.

The decrease of p_k^s when inequality increases (i.e. as β decreases) is a consequence of the concentration of cash in the hands of the wealthiest agents. This can be observed in the right panel of Figure 4, which shows the average cash of agents with a given wealth, for different values of β. The freezing of the economy when β decreases occurs because fewer and fewer agents can dispose of enough cash (i.e. have $\ell > \pi_{(k)}$) to buy the different goods (prices $\pi_{(k)}$ correspond to the dashed lines).

Note finally that p_k^s quantifies liquidity in terms of goods. In order to have an equivalent measure in terms of cash that can be compared to the velocity of money, we average $\pi_{(k)} p_k^s$ over all goods

$$\bar{p}^s = \frac{1}{\Pi} \sum_{k=1}^{K} M_k \pi_{(k)} p_k^s. \tag{56}$$

This quantifies the frequency with which a unit of cash changes hand in our model economy, as a result of a successful transaction. It's behaviour as a function of β for the same parameters of the economy in Figure 4 is shown in the right panel of Figure **??**.

7.3 RESULTS

These two observations allow us to trace the origin of the arrest in the economy back to the shrinkage of the *cash-rich class* to a vanishingly

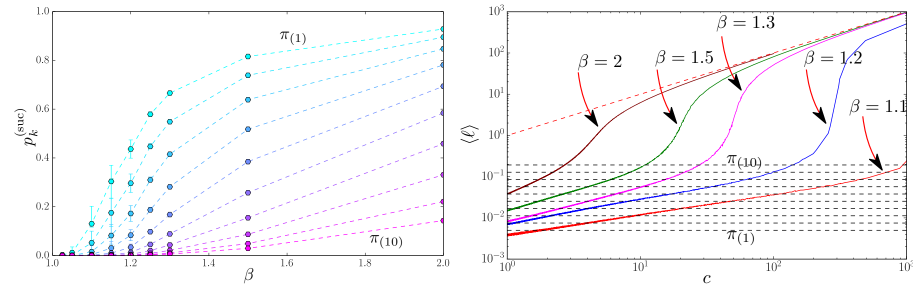

Figure 4.: Left: Liquidity of goods $\{p_k^s\}_{k=1}^K$ as a function of the inequality exponent β for a system of $N = 10^5$ agents exchanging $K = 10$ classes of goods ($\pi_{(k)} = \pi_{(1)} g^{k-1}$ with $g = 1.5$, $\pi_{(1)} = 0.005$, $M_k \pi_{(k)} = \Pi/K$ and $C/\Pi = 1.2$). Note that all success rates p_k^s vanish when $\beta \to 1^+$. The curves are ordered from the cheapest (top) to the most expensive (bottom). The markers are the result of numerical simulations, with error bars indicating the minimum and maximum values obtained by averaging over 5 realizations of the wealth allocations (for more details on the simulations see Appendix **??**). Right: for the same simulations with $K = 10$ classes of goods, we plot the time averaged cash $\langle \ell_i \rangle$ as a function of wealth c_i, from $\beta = 1.1$ to $\beta = 2$. The dashed lines indicate the different prices of goods. Agents with $\langle \ell_i \rangle$ below the price of a good typically have not enough cash to buy it. Cash is proportional to wealth for large levels of wealth (see the upper straight red dashed line).

small fraction of the population, as $\beta \to 1^+$. As we'll see in the next section, when β is smaller than 1 the fraction of agents belonging to this class vanishes as $N \to \infty$. In this regime, not only the wealthiest few individuals own a finite fraction of the whole economy's wealth, as observed in [**?**], but they also drain all the financial resources in the economy.

These findings extend to more complex settings. Figure 4 illustrates this for an economy with $K = 10$ classes of goods (see figure caption for details) and different values of β. In order to visualise the freezing of the flow of goods we introduce the success rate of transactions for goods belonging to class k, denoted as p_k^s. Figure 4 shows that, as expected, for a fixed value of the Pareto exponent β the success rate increases as the goods become cheaper, as they are easier to trade. Secondly it shows that trades of all classes of goods halt as β tends to unity, that is when wealth inequality becomes too large, independently of their price.

8

CONCLUSION

A

CALCULATION OF THE PARTITION FUNCTION FOR THE RANDOM LINEAR ECONOMY MODEL

In this chapter we carry out the calculation for the partition function described in section 4.2 and originally presented in [4] in details. Although the analytical form for the maximization will not be used in the Applications part of this PhD thesis, we believe it's instructive to the reader that is not familiar with the replica method. The level of detail employed here is not published anywhere, and thus we consider an additional contribution of this thesis.

To solve the maximization problem we need to calculate the following integrals (from equation (**??**)):

$$\max_x U(x) = \int d\xi dx_0 P(\xi) P(x_0) \lim_{\beta\to\infty} \log \overbrace{\int dx\, e^{\beta U(x|\xi,x_0)}}^{Z(\beta|\xi,x_0)} \tag{57}$$

We also know from (14) that $x = x_0 + \sum_{i=1}^N s_i \xi_i$, so we insert this constraint as an integral in $Z(\beta)$:

$$Z(\beta|\xi, x_0) = \int_0^\infty ds \int_0^\infty dx\, e^{\beta U(x)} \delta\left(x - x_0 - \sum_i s_i \xi_i\right) \tag{58}$$

Carrying out the integration in equation (64) is extremely hard, mainly because the log function in the integrand prevents us to factorize the integrals from the coupling created by ξ_i^μ and the market clearing condition. This is a recurrent problem when calculating the energy for disordered systems in statistical mechanics, which was solved by a clever and extremely useful technique to deal with the logarithm function, the so called **replica method** [10], which consists in writing $\log Z$ as:

$$\log Z = \lim_{r\to 0} \frac{Z^r - 1}{r} \tag{59}$$

The identity above is still exact, but the clever part of the method is exchanging the $\lim_{r\to 0}$ term with the rest of the integrals, treating r like an integer throught the whole calculation: Z^r is written as a product of independent partition functions, ie, $Z^r = Z_1 Z_2 \dots Z_r$, each integrated over their own dynamical variables x^a and s^a, $a = 1, \dots, r$. These multiple independent systems are the *replicas* that give the method its name. This may seem strange at first, but the beauty of the replica method is that this change of operations (between the limit and the integrals), though not rigorously proved, works very well.

The general strategy of the full calculation will be as follows: we exchange the order of integrations and limits until we have

$$\lim_{N\to\infty} \frac{1}{N} \max_x U(x) = \lim_{r\to\infty} \frac{1}{N} \lim_{\beta\to\infty} \int d\xi dx_0 P(\xi) P(x_0) \frac{Z(\beta|\xi, x_0)^r - 1}{r} \tag{60}$$

The $\frac{1}{N}$ factor was added to avoid the divergence of $U(x)$ in the limit $N \to \infty$[1]. The term $\int d\xi dx_0 P(\xi)P(x_0)\frac{Z(\beta|\xi,x_0)^r-1}{r}$ is the partition function average over the disorder. Because only $Z(\beta|\xi, x_0)$ depends on ξ and x_0, we write it as

$$\int d\xi dx_0 P(\xi)P(x_0)\frac{Z(\beta|\xi,x_0)^r-1}{r} = \frac{\langle Z^r\rangle_{\xi,x_0}-1}{r} \tag{61}$$

Where $\langle\cdot\rangle_{\xi,x_0}$ indicates the average over the disorder. Arriving at a final expression for the term $\langle Z^r\rangle_{\xi,x_0}$ is the bulk of the work in calculating $\max U(x)$, but the goal is to write it in the form

$$\langle Z^r\rangle_{\xi,x_0} = \int d\theta\, e^{\beta N r h(\theta)}, \tag{62}$$

where θ is a vector of order parameters. Because we assume $N \to \infty$, the integral is dominated by it's maximal value, θ^*. We then take the series expansion around θ^* and keep the first two terms, ie,

$$\int d\theta\, e^{\beta N r h(\theta)} = e^{\beta N r h(\theta^*)} \approx 1 + \beta N r h(\theta^*) \tag{63}$$

Finally, we plug this approximation into equation (64) to get

$$\lim_{N\to\infty}\frac{1}{N}\max_x U(x) = h(\theta^*) \tag{64}$$

With this strategy in mind, we now begin calculating $\langle Z^r\rangle_{\xi,x_0}$ proper. Using the replica assumption, we expand Z^r as

$$Z^r = \prod_{a=1}^{r}\int_0^\infty ds^a \int_0^\infty dx^a e^{\beta U(x^r)}\delta\left(x^a - x_0 - \sum_i s_i^a\xi_i\right) = \tag{65}$$

$$= \int_0^\infty ds^1 \int_0^\infty dx^1 e^{\beta U(x^1)}\delta\left(x^1 - x_0 - \sum_i s_i^1\xi_i\right)\times \tag{66}$$

$$\times\ldots\times \tag{67}$$

$$\times\int_0^\infty ds^r \int_0^\infty dx^r e^{\beta U(x^r)}\delta\left(x^r - x_0 - \sum_i s_i^r\xi_i\right)$$

Gathering all the terms together:

$$Z^r = \int_0^\infty \prod_{a=1}^{r} dx_a \int_0^\infty \prod_{a=1}^{r} ds_a e^{\beta\sum_a U(x_a)} \prod_{a=1}^{r}\prod_{\mu=1}^{M}\delta\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a\xi_i^\mu\right) \tag{68}$$

We write explicitely the distributions for x_0 and ξ:

$$P(x_0) = \prod_\mu e^{-x_0^\mu} \tag{69}$$

1 The divergence actually comes from the limit $M \to \infty$ because $U(x)$ is linear in M, but because we assume $n = N/M$ fixed, scaling on N is equivalent to scaling on M

and

$$P(\xi_i) = \frac{1}{P_\xi} \prod_{\mu=1}^{M} \frac{1}{\sqrt{2\pi M^{-1}\Delta^2}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta^2}} \delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right), \tag{70}$$

where P_ξ is the normalization term given by

$$P_{\xi_i} = \int_0^\infty \prod_{\mu=1}^{M} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} \delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right) \tag{71}$$

The integral over x_0 we can leave to the end because they are already factored and involve no other terms except the initial endowments. The integrals over ξ_i^μ, however, are coupled due to the normalization term. Because the δ integral is not feasible due to the couplings, too calculate P_{ξ_i}, and throughout this appendix, we will make use of an important identity for the Dirac delta function, in which we replace it by it's Fourier transform, ie:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \tag{72}$$

With this identity we are able to integrate all the terms in the normalization term P_ξ:

$$P_{\xi_i} = \int_0^\infty \prod_{\mu=1}^{M} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} \int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik(\sum_\mu \xi_i^\mu + \epsilon)} = \tag{73}$$

$$\int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik\epsilon} \prod_{\mu=1}^{M} \int_0^\infty d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta} + ik\xi_i^\mu} \tag{74}$$

We also use another useful identity to solve the gaussian integral in ξ_i^μ. The integral of e^{-ax^2+bx} can be easily done if we complete the square:

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \tag{75}$$

This identity is useful in both directions: sometimes we would like to carry out an integration, and then we go from the left hand side to the right hand side. And sometimes, we would like to linearize a squared term (b in this case), and we go from the right hand side to the left hand side, gaining an integral in the process. We now use it to integrate equation (73):

$$P_{\xi_i} = \int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik\epsilon} e^{-M\frac{M^{-1}\Delta k^2}{2}} = \frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}} \tag{76}$$

Going back to Z^r:

$$\int d\xi P(\xi) Z^r = \int_{-\infty}^{\infty} \prod_{\mu=1}^{M} \prod_{i=1}^{N} \frac{1}{P_{\xi_i}} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} \delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right) \times$$
$$\times \int_0^\infty \prod_{a=1}^{r} dx_a \int_0^\infty \prod_{a=1}^{r} ds_a e^{\beta \sum_a U(x_a)} \prod_{a=1}^{r} \prod_{\mu=1}^{M} \delta\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right) \tag{77}$$

We again use the Fourier transform identity for the δ terms:

$$\delta\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right) = \int_{-\infty}^{\infty} \frac{1}{2\pi} d\hat{z}_i \, e^{i\hat{z}_i\left(\sum_{\mu=1}^{M} \xi_i^\mu + \epsilon\right)} \tag{78}$$

$$\delta\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right) = \int_{-\infty}^{\infty} \frac{1}{2\pi} d\hat{x}_\mu^a \, e^{i\hat{x}_\mu^a\left(x_\mu^a - x_0^\mu - \sum_{i=1}^{N} s_i^a \xi_i^\mu\right)} \tag{79}$$

Writing only the terms involving ξ_i^μ from equation (77), we take the integral over $d\xi_i^\mu$. For each pair i, μ we have

$$\int_{-\infty}^{\infty} d\xi_i^\mu \frac{1}{\sqrt{2\pi M^{-1}\Delta}} e^{-\frac{(\xi_i^\mu)^2}{2M^{-1}\Delta}} e^{i\hat{z}_i \xi_i^\mu} e^{-\sum_a i\hat{x}_\mu^a s_i^a \xi_i^\mu} = e^{-\frac{\Delta}{2M}\left(\hat{z}_i - \sum_a \hat{x}_\mu^a s_i^a\right)^2} \tag{80}$$

Again, in the above equation we have used the Gaussian integral identity of equation (75) and the normalization term was cancelled.

Plugging the product $\prod_{i,\mu} e^{-\frac{\Delta}{2M}\left(\hat{z}_i - \sum_a \hat{x}_\mu^a s_i^a\right)^2}$ back on equation (77) we end up with:

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{2\pi} d\hat{z}_i \int_{-\infty}^{\infty} \prod_{a=1}^{r} \frac{1}{2\pi} \prod_{\mu=1}^{M} d\hat{x}_\mu^a \int_0^\infty dx^a \int_0^\infty ds^a \frac{1}{\left[\frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}}\right]^N} \times \tag{81}$$

$$\times \exp\left[\beta \sum_a U(x_a) + i\epsilon \sum_{i=1}^{N} \hat{z}_i + i \sum_{a=1}^{r} \sum_{\mu=1}^{M} \hat{x}_\mu^a \left(x_\mu^a - x_0^\mu\right) - \frac{\Delta}{2M} \sum_{i=1}^{N} \sum_{\mu=1}^{M} \left(\hat{z}_i - \sum_{a=1}^{r} \hat{x}_\mu^a s_i^a\right)^2\right] \tag{82}$$

We now have hit another wall in integrating these expressions: some of the variables we are integrating on are coupled via the $\left(\hat{z}_i - \sum_{a=1}^{r} \hat{x}_\mu^a s_i^a\right)^2$ term. This means we are not able to integrate over, for example, s_i^a and s_i^b independently. To get around this, we introduce new variables which allows us to factor the exponential:

$$\omega_{ab} = \frac{1}{N} \sum_{i=1}^{N} s_i^a s_i^b \quad \text{and} \quad k_a = \frac{1}{N} \sum_{i=1}^{N} \hat{z}_i s_i^a \tag{83}$$

To substitute these terms in the equation above, we multiply it again by a delta term and integrate over it, then replace by its Fourier transform, ie:

$$1 = \int dk_a \delta\left(k_a - \frac{1}{N}\sum_{i=1}^{N} s_i^a\right) = \int dk_a d\hat{k}_a \frac{N}{2\pi i} e^{\hat{k}_a[Nk_a - \sum_i s_i^a]} \tag{84}$$

$$1 = \int d\omega_{ab} \delta\left(\omega_{ab} - \sum_{i=1}^{N} s_i^a s_i^b\right) = \int d\omega_{ab} d\hat{\omega}_{ab} \frac{N}{4\pi i} e^{\frac{1}{2}\hat{\omega}_{ab}[N\omega_{ab} - \sum_i s_i^a s_i^b]} \tag{85}$$

A few extra steps were taken in the above passage: first, we used the identity $\delta(x) = \alpha\delta(\alpha x)$ to write $\delta(k_a - \frac{1}{N}\sum_i s_i^a) = N\delta(Nk_a - \sum_i s_i^a)$. This change is useful because both terms are of order N and this will allow us to write $\langle Z^r \rangle_{\xi,x_0}$ in the form of (62). The second step taken was to carry out a change of variable in the integration, $\hat{k}_a \to i\hat{k}_a$ and $\hat{\omega}_{ab} \to \frac{i}{2}\hat{\omega}_{ab}$.

For simplicity, we will now omit the integration limits when the integral is $\int_{-\infty}^{\infty}$. Replacing the new variables in (**??**):

$$\begin{aligned}\langle Z^r \rangle_{\xi,x_0} = &\int d\omega_{ab} d\hat{\omega}_{ab} \frac{N}{4\pi i} e^{N\hat{\omega}_{ab}\omega_{ab}} e^{-\hat{\omega}_{ab}\sum_i s_i^a s_i^b} \int dk_a d\hat{k}_a \frac{N}{2\pi i} e^{N\hat{k}_a k_a} e^{-\hat{k}_a \sum_i s_i^a} \times \\ &\times \int \prod_{i=1}^{N} \frac{1}{2\pi} d\hat{z}_i \int \prod_{a=1}^{r} \frac{1}{2\pi} \prod_{\mu=1}^{M} d\hat{x}_\mu^a \int_0^\infty dx^a \int_0^\infty ds^a \frac{1}{\left[\frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}}\right]^N} \times\end{aligned} \tag{86}$$

$$\times\, e^{\left[\beta \sum_a U(x_a) + i\epsilon \sum_{i=1}^{N} \hat{z}_i + i\sum_{a=1}^{r} \sum_{\mu=1}^{M} \hat{x}_\mu^a \left(x_\mu^a - x_0^\mu\right) - \frac{\Delta}{2M}\sum_{\mu=1}^{M}\left(\sum_i \hat{z}_i - 2N\sum_a k_a \hat{x}_\mu^a + N\sum_{a,b}\right.\right.}$$

The sums over i are now completely factorized, which allows us to replace $\sum_i s_i^a$ by Ns^a and again we get the N factor to put in evidence. We write the integral over $\omega, \hat{\omega}, k$ e $\hat{k}$ as

$$\langle Z^r \rangle_{\xi,x_0} = \int \prod_{a,b=1}^{r} N \frac{d\omega_{ab} d\hat{\omega}_{ab}}{4\pi i} \int \prod_{a=1}^{r} N \frac{dk_a d\hat{k}_a}{2\pi i} e^{Nh(\omega,\hat{\omega},k,\hat{k})}, \tag{87}$$

which is what we wanted initially. When we take the limit of $N \to \infty$, the integral will be dominated by the maximum value of h, which we divide in three terms, $h = g_1 + g_2 + g_3$:

$$g_1 = -\sum_{a,b=1}^{r} \frac{1}{2}\hat{\omega}_{ab}\omega_{ab} - \sum_{a=1}^{r} \hat{k}_a k_a \tag{88}$$

$$g_2 = \log \int \frac{d\hat{z}}{2\pi} \int_0^\infty \prod_{a=1}^{r} \exp\left[\frac{1}{2}\sum_{a,b} \hat{\omega}_{ab} s_a s_b + \hat{z}\sum_{a=1}^{r} \hat{k}_a s_a + i\epsilon\hat{z} - \frac{\Delta}{2}\hat{z}^2\right] - \log \frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{\epsilon^2}{2\Delta}} \tag{89}$$

$$g_3 = \frac{1}{N}\sum_{\mu}\log\int\prod_a\frac{d\hat{x}_a}{2\pi}\int_0^{\infty}\prod_a dx^a e^{\beta\sum_a U(x^a)+i\sum_a\hat{x}^a(x^a-x_0^{\mu})-\frac{n\Delta}{2}\sum_{a,b}\hat{x}^a\hat{x}^b\omega_{ab}+n\Delta\sum_a\hat{x}^a k_a} \tag{90}$$

To find the maximum of h we must solve the following system of equations

$$\begin{aligned}\frac{\partial h}{\partial\omega_{ab}} &= 0, \qquad \frac{\partial h}{\partial\hat{\omega}_{ab}} = 0\\ \frac{\partial h}{\partial k_a} &= 0, \qquad \frac{\partial h}{\partial\hat{k}_a} = 0\end{aligned} \tag{91}$$

These are the saddle points for the replica method. Although we are calculating the maximum value of $U(x)$, the saddle point equations give us important information on how the order parameters relate to each other.

At this point we take another important approximation to these calculations. The r^2 order parameters ω_{ab} are the overlap between two different system replicas, a and b, ie, how similar are the s vectors in two independent copies of our economy. Because $U(x)$ is a convex function, we know that the maximum of $U(x)$ exists and is unique. Therefore, we expect every replica to converge to the same equilibrium value of s^a in the limit $\beta\to\infty$, and in this case, we cannot distinguish between ω_{ab} for any two pairs of replica a and b. We assume, then, there are only two possible values for ω_{ab}. Either $a=b$ and $\omega_{aa}=\langle s^2\rangle=\Omega$, the variance of s, or $a\neq b$ and $\omega_{ab}=\omega$, the overlap of two different systems. This is the so called replica symmetric approximation, which is exact in this case because we know there is only one equilibrium in the zero temperature limit.

Writing this explicitely, we have that ω_{ab} and k_a are given by

$$\begin{aligned}\omega_{ab} &= \Omega\delta_{ab}+\omega(1-\delta_{ab})\\ \hat{\omega}_{ab} &= \hat{\Omega}\delta_{ab}+\hat{\omega}(1-\delta_{ab})\\ k_a &= k\\ \hat{k}_a &= \hat{k}\end{aligned} \tag{92}$$

Replacing these new values in equations (88) - (90) and taking the limit $r\to 0$ we get

$$\lim_{r\to 0}\frac{1}{r}g_1 = -\frac{1}{2}\left(\Omega\hat{\Omega}-\omega\hat{\omega}\right)-k\hat{k} \tag{93}$$

$$\lim_{r\to 0}\frac{1}{r}g_2 = \int dt\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\log\int_0^{\infty}ds\, e^{\frac{\hat{\Omega}-\hat{\omega}}{2}s^2+\left[t\left(\frac{\hat{k}^2}{\Delta}+\hat{\omega}\right)^{\frac{1}{2}}+i\hat{k}\frac{\epsilon}{\Delta}\right]s} \tag{94}$$

$$\lim_{r\to 0}\frac{1}{r}g_3 = \int dt \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \log \int_0^\infty dx\, e^{\beta U(x) - \frac{\left(x - x_0 + \sqrt{n\Delta\omega}t - in\Delta k\right)^2}{2n\Delta(\Omega-\omega)}} - \frac{1}{2}\log[2\pi n\Delta(\Omega-\omega)] \tag{95}$$

In the equations above, t is a gaussian random variable with zero mean and unit variance, which arises from using the identity (**??**) to linearize $\left(\sum_a s^a\right)^2$, gaining an integral in the process.

We now have a order parameter vector $\theta = (\Omega, \hat{\Omega}, \omega, \hat{\omega}, k, \hat{k})$ and we wish to find the values θ^* that maximizes $h(\theta)$. However, we have some conditions on this solution, in particular we know that it must be well defined for $\beta \to \infty$. Again, because we know that in this limit all replicas must have the same equilibrium, then it must hold that the overlap between replicas must vanish, ie,

$$\lim_{\beta\to\infty} \Omega - \omega = \frac{1}{2N}\sum_{i=1}^{N}\left(s_i^a - s_i^b\right)^2 = 0 \tag{96}$$

But this would imply that some terms in g_3 would diverge in the zero temperature limit, so we have to rescale the order parameters for them to remain finite in this limit. We define the new parameters, which are always finite:

$$\chi = n\Delta\beta(\Omega-\omega), \quad \hat{\chi} = -\frac{\hat{\Omega}-\hat{\omega}}{\beta}, \quad \kappa = -in\Delta k, \tag{97}$$

$$\hat{\kappa} = \frac{i\hat{k}}{\Delta\beta}, \quad \hat{\gamma} = \frac{\hat{\omega}}{\beta^2} \tag{98}$$

The function h then becomes

$$h = \frac{1}{2}\left(\Omega\hat{\chi} - \frac{\hat{\gamma}\chi}{n\Delta}\right) - \frac{1}{n}\kappa\hat{\kappa} + \frac{1}{\beta}\left\langle \log\int_0^\infty ds\, e^{\beta\left[-\frac{\hat{\chi}}{2}s^2 + (t\sqrt{\hat{\gamma}-\Delta\hat{\kappa}^2} + \hat{\kappa}\epsilon)s\right]}\right\rangle_t + \\ + \frac{1}{n\beta}\left\langle \log\int_0^\infty dx\, e^{\beta\left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)^2}{2\chi}\right]}\right\rangle_{t,x_0} \tag{99}$$

We then finally take the limit $\beta \to \infty$ and again use the saddle point method to solve the integrals on x and s, which means they are dominated by their maximum value, ie

$$h(\beta\to\infty) = \left\langle \max_s\left[-\frac{\hat{\chi}}{2}s^2 + (t\sqrt{\hat{\gamma}-\Delta\hat{\kappa}^2} + \hat{\kappa}\epsilon)s\right]\right\rangle_t + \frac{1}{2}\left(\Omega\hat{\chi} - \frac{\hat{\gamma}\chi}{n\Delta}\right) - \frac{1}{n}\kappa\hat{\kappa} + \\ + \frac{1}{n}\left\langle \max_x\left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)^2}{2\chi}\right]\right\rangle_{t,x_o} \tag{100}$$

Replacing in equation (100) the variables x and s by their maximum values x^* and s^* and taking the derivatives on the order parameters we finally have the saddle point equations for $h(\theta)$:

$$\frac{\partial h}{\partial \Omega} = \frac{\hat{\chi}}{2} - \frac{1}{2\chi}\sqrt{\frac{\Delta}{n\Omega}}\left\langle (x^* - x_0 + \kappa + t\sqrt{n\Delta\Omega})t \right\rangle_{t,x_0} = 0 \tag{101}$$

$$\frac{\partial h}{\partial \kappa} = -\frac{1}{n}\hat{\kappa} - \frac{1}{n\chi}\left\langle x^* - x_0 + \kappa + t\sqrt{n\Delta\Omega} \right\rangle_{t,x_0} = 0 \tag{102}$$

$$\frac{\partial h}{\partial \hat{\kappa}} = -\frac{\Delta\hat{\kappa}}{\sqrt{\hat{\gamma} - \Delta\hat{\kappa}^2}}\left\langle ts^* \right\rangle_t + \epsilon\left\langle s^* \right\rangle_t - \frac{\kappa}{n} = 0 \tag{103}$$

$$\frac{\partial h}{\partial \hat{\gamma}} = \frac{1}{2\sqrt{\hat{\gamma} - \Delta\hat{\kappa}^2}}\left\langle ts^* \right\rangle_t - \frac{\chi}{2n\Delta} = 0 \tag{104}$$

$$\frac{\partial h}{\partial \chi} = -\frac{\hat{\gamma}}{2n\Delta} + \frac{\left\langle (x^* - x_0 + \kappa + t\sqrt{n\Delta\Omega})^2 \right\rangle_{t,x_0}}{2n\chi^2} = 0 \tag{105}$$

$$\frac{\partial h}{\partial \hat{\chi}} = -\frac{1}{2}\left\langle (s^*)^2 \right\rangle_t + \frac{1}{2}\Omega = 0 \tag{106}$$

We can find x^* by solving $\frac{\partial}{\partial x}\left[U(x) - \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)^2}{2\chi}\right] = 0$, resulting in the implicit equation

$$x^* = x : U'(x^*) = \frac{(x - x_0 + \kappa + \sqrt{n\Delta\Omega}t)}{\chi} \tag{107}$$

We can replace this in the equations (101) - (106) to obtain some useful relations. The equation (102) becomes, for $x = x^*$

$$\hat{\kappa} = -\left\langle U'(x^*) \right\rangle_{t,x_0} \tag{108}$$

This allows us to identify $\hat{\kappa} = -p$ due to the price equation derived from the first order conditions of the consumer's maximization problems (equation (**??**)). Equation (106) allows us to write

$$\Omega = \left\langle (s^*)^2 \right\rangle_t \tag{109}$$

The remaining parameters are found through simple algebraic manipulations. With Ω and p defined, one immediately obtains $\hat{\chi}$

$$\hat{\chi} = \sqrt{\frac{\Delta}{n\Omega}}\left\langle U'(x^*)t \right\rangle_{t,x_0} \tag{110}$$

And from (105) we have

$$\hat{\gamma} = \Delta\left\langle U'(x^*)^2 \right\rangle_{t,x_0} \tag{111}$$

With that, $U'(x)$ variance is written as

$$\sigma = \sqrt{\hat{\gamma} - \Delta\hat{\kappa}^2} = \sqrt{\Delta\left(\left\langle U'(x^*)^2 \right\rangle_{t,x_0} - \left\langle U'(x^*) \right\rangle_{t,x_0}^2\right)} \tag{112}$$

Finally, we have χ via equation (104)

$$\chi = \frac{n\Delta}{\sigma}\left\langle ts^* \right\rangle_t \tag{113}$$

And κ via equation (103)

$$\kappa = p\chi + n\epsilon \left\langle s^* \right\rangle_t \tag{114}$$

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