Choosing How to Choose

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CCM
Todo/questions:

1. Refs to Teddy?
2. Indep stuff
3. imprecise nu and ead?
4. Add a bit more explicit proof of result for maximality stuff, it's not immediate and I need to think it through to get the non-independence version clear in my head!
5. ...?

It is our lot to face decisions when it is uncertain which act from among those available to us will lead to the best outcome. Uncertain about the day's weather, you must choose whether or not to take an umbrella when you leave your house; uncertain about what it would help them most to hear, you must choose what to say to a friend who is going through a bad time; and uncertain what effect different approaches will have, a parent must choose how to raise their child. How are we to make such choices? It is the task of decision theory to provide an answer. And philosophers, economists, and psychologists have met this remit by developing a slew of rival theories of rational decision. Expected utility theory is the most well known and widely used, but there are many alternatives available, and we will meet a good few in the course of this paper.

There are various ways to argue for your preferred decision theory. You might note that it agrees with your intuitive verdict about a specific decision problem that you describe, while its rivals don't. For instance, you might intuitively judge the Allais or Ellsberg preferences rationally permissible, and note that certain risk- or ambiguity-sensitive decision theories permit them, while expected utility theory does not. Or you might note that your favoured theory has a formal feature that you find intuitively desirable, while its rivals lack that feature. For instance, you might intuitively judge the Independence Axiom or the Sure Thing Principle a requirement of rationality, and note that expected utility theory satisfies both, while its risk-sensitive rivals don't.

But there is another approach, and it has the advantage that it avoids such appeals to our intuitive judgments and the stalemates in which they often result. It begins with the observation that a decision theory is an account of rational means-ends reasoning: agnostic about whether your ends are good or bad, desirable or undesirable, benevolent or malign, it purports to tell you the rational way to pick between different possible means to the ends you in fact have.

Granted this, it seems natural to assess a decision theory by asking how well it performs in the role of getting you those ends. The only problem with this approach is that, in order to assess a decision theory or anything else as a means to your ends, we need an account of which means to your ends it is rational to use. And without a decision theory, we don't have that.

Yet all is not lost, for this line of thinking nonetheless furnishes us with a test we can conduct on a theory of decision-making, and while it might not tell in favour of the theory if it passes, it seems to tell against it if it fails. We can ask of the theory: If I were to use you not only to make my normal day-to-day decisions, but also to make the higher-order decision about which decision theory to use, would you recommend yourself? If it would, we call it *self-recommending*; if it wouldn't, we call it *self-undermining*. We claim that no self-undermining decision theory can be correct. This is not to say that a self-recommending theory is thereby adequate—for instance, the theory that says that any available act is rationally permissible is self-recommending, but it is not correct. Nonetheless, we can use this test to winnow the list of candidate decision theories, removing those that fail it.

1 Risk-sensitive decision theories

Expected utility theory rules out as irrational certain natural ways of taking risk into account in decision-making. In particular, as noted above, it rules out the so-called *Allais preferences* (**?**). In response, decision theorists have presented a range of alternatives that permit those preferences (**????**). Let's begin by showing a straightforward way in which any such decision theory is self-undermining.

Here are the four gambles over which the Allais preferences are defined—for reasons that will become clear, we present them as if they are defined over four possible states of the world, $\omega_1, \ldots, \omega_4$:

	ω_1	ω_2	ω_3	ω_4
P	$1/100$	$10/100$	$89/200$	$89/200$
1A	£1m	£1m	£1m	£1m
1B	£0m	£5m	£1m	£1m
2A	£1m	£1m	£0m	£0m
2B	£0m	£5m	£0m	£0m

And the preferences are these: 1A $\succ$ 1B and 2A $\prec$ 2B.

Now, consider the following two decision problems, D_1 and D_2, where the acts available in them are defined over two states of the world, ω'_1, ω'_2:

D_1	ω'_1	ω'_2
p	$1/11$	$10/11$
A	£1m	£1m
B	£0m	£5m

D_2	ω'_1	ω'_2
p	$1/2$	$1/2$
C	£0m	£0m

Suppose first that, faced with D_1, your decision theory tells you to pick A over B, while faced with D_2, it tells you to pick the only available option, C. (We'll return to the case in which your decision theory tells you to pick B over A below.)

Now suppose you are certain you'll face D_1 or D_2, but you're uncertain which. Your credence you'll face D_1 is $11/100$ and your credence you'll face D_2 is $89/100$. And, what's more, you think which decision you face is independent of what the world is like.

A *picking strategy* is a function s that takes a decision problem D, which is just a set of available acts, and returns one of those available acts $\mathsf{s}(D)$ from D. As **?** emphasize, there is an important difference between *picking* and *choosing*. Choosing is a matter of settling on what to do in a decision problem for reasons grounded in your preferences. Picking on the other hand is a matter of settling on what to do even after your reasons have run out, either because you are indifferent between the options that you have not rejected or because you find them incomparable. Picking strategies fully settle what to do in every decision problem. We will understand picking inclusively in what follows. If you choose, then you pick, but not vice versa. That is, if you settle on an option for reasons grounded in your preferences, then you count as both choosing and picking that option. If you settle on an option not for preference-based reasons but simply because you must *do something*, then you count as picking but not choosing that option.

Just as we said above we'd judge decision theories by how good they are as means to our ends, let us first do that for picking strategies. So, the utility of a picking strategy in a decision problem is just the utility of the act it tells you to pick in that problem. To specify the utility of a picking strategy, s, then, we need to specify both the decision problem that you face, D, and the state of the world, ω. The strategy selects a unique option from D, $\mathsf{s}(D)$, and the world determines the utility of that option, $\mathsf{s}(D)(\omega)$.

Now return to our example. There are two picking strategies, s_1 and s_2, available to you when you consider facing either D_1 or D_2:

	D_1	D_2
s_1	A	C
s_2	B	C

If you face D_1 and the world is in state ω_1', then the utility of s_1 is the utility of £1m. The reason: s_1 picks A from D_1 and A gets you £1m at ω_1'. More generally, the utility of s_1 and s_2 in any decision problem and world is given by the utility of the following outcomes:

	ω_1' & D_1	ω_2' & D_1	ω_1' & D_2	ω_2' & D_2
P	$1/11 \times 11/100$ $= 1/100$	$10/11 \times 11/100$ $= 10/100$	$1/2 \times 89/100$ $= 89/200$	$1/2 \times 89/100$ $= 89/200$
s_1	£1m	£1m	£0m	£0m
s_2	£0m	£5m	£0m	£0m

But notice: the choice between s_1 and s_2 is extensionally equivalent to the choice between 2A and 2B in the Allais set up—once we equate ω_1 with ω_1' & D_1, ω_2 with ω_2' & D_1, and so on, it has the same pay-offs with the same probabilities. So, if you have the Allais preferences, and hence prefer 2B to 2A, then you prefer s_2 to s_1. But we specified that, when faced with D_1, your decision theory demands that you choose A over B. That is, it demands that you choose (and hence pick) what s_1 picks and not what s_2 picks. So, your decision theory makes two claims: First, if you're uncertain whether you'll face D_1 or D_2, you prefer to choose according to picking strategy s_2 and consider it irrational to use s_1. Second, when actually faced with D_1, you should pick as s_1 does. In this sense, then, your decision theory is self-undermining. It judges the only picking strategy compatible with its demands to be rationally impermissible. Uncertain which of two decision problems you will face, it says that, when you face whichever you do, you should not choose as it demands.

Now, we specified above that, faced with D_1, your decision theory demands you choose A over B. But what if it demands you choose B over A? In that case, we just replace D_2, the trivial decision problem that contains only C, which pays out £0m in either state of the world, with the trivial decision problem that contains only the act D, which pays out £1m in either state of the world. Then the choice between s_1 and s_2 is extensionally equivalent to the choice between 1A and 1B. So, by the Allais preferences, you'll prefer s_1 to s_2, but, faced with D_1, you'll prefer B to A, which is what s_2 tells you to pick and what s_1 tells you not to pick. So again the decision theory is self-undermining.

And there is no funny business going on here. Learning that you face decision problem D_1 does not tell you anything about which ticket will win, for we assumed that the decision problem is independent of the state of the world. And picking one option or another in D_1 does not tell you anything about which state of the world you're in, for we assumed the acts are independent of the states of the world as well.

Being self-undermining is *prima facie* bad. Your decision theory makes recommendations both about *picking strategies* and which *options* to choose in different decision problems. But these recommendations pull you in different directions. The picking strategy it recommends tells you to do one thing when faced with D_1, and the decision theory itself tells you to do another thing. So there's an instability in the theory's recommendations. The oddity of the situation is akin to one that David **?**, 56 identified in a different context:

> It is as if Consumer Bulletin were to advise you that Consumer Reports was a best buy whereas Consumer Bulletin itself was not acceptable; you could not possibly trust Consumer Bulletin completely thereafter.

Any theory that deems the Allais preferences permissible has this *prima facie* bad-making feature. It is self-undermining in the following sense: (1) there is some precise state of uncertainty you could have about which decision problem you'll face—uncertainty that is represented by a precise probability distribution over the possible decision problems; (2) if you are in that state and apply your decision theory to the question of which picking strategy to use, it rules out any compatible strategy, *i.e.*, any strategy that always avoids picking options that are impermissible according to the original decision theory. It demands that you use some picking strategy that is not compatible with it.

This isn't unique to the Allais preferences. Other failures of the Savage's Sure Thing Principle would generate such a way of being uncertain over which decision problem you'll face where the decision theory undermines its recommendations, at least when the failure generates a preference reversal. Similarly for von Neumann-Morgenstern's Independence Principle: if your decision theory deems action a to be preferable to action b, but when evaluating an action which tosses a biased coin and selects a (or b) if the coin lands Heads and c if the coin lands Tails, the decision theory evaluates the b version to be preferable, then similarly we can generate a way of being uncertain over which decision problem you'll be faced with so that the unique picking strategy which is compatible with the decision theory is itself impermissible according to the theory.

1.1 Is it a problem to be self-undermining?

There are at least two ways to argue that being self-undermining tells against a decision theory. We'll describe them and then consider some responses.

First: for any decision problem (set of available acts), D, a decision theory deems some subset of D to be *rejected* or *rationally impermissible*. Now suppose the decision theory is correct: that is, the acts it declares impermissible are indeed impermissible for the individual. But one of its declarations is that picking strategies that never recommend impermissible acts are themselves impermissible—that's what makes it self-undermining. In that case, you might face a rational dilemma. That is, nature might thrust a sequence of decision problems on you where you rationally *must* do something rationally impermissible. Suppose you face the choice of which picking strategy to use. If you choose a rationally impermissible picking strategy, you (obviously) do something impermissible in that decision problem. If you choose some other picking strategy, then there is some other decision problem where it recommends an impermissible act. If nature thrusts this decision problem on you, then implementing your picking strategy will lead you to pick an impermissible act. So you will have done something impermissible in this latter decision problem. This is a rational dilemma. And it is a particularly pernicious sort of rational dilemma, since it is not generated by any self-referentiality or act-state dependence. If there are no genuine rational dilemmas of this sort, then our initial assumption must be false: the decision theory simply cannot be correct.

Perhaps this is not a decisive mark against a self-undermining theory. It could be impermissible to use any decision theory that gives the correct verdicts if, for example, those theories were so complex and computationally demanding that they lay beyond the agent's cognitive capacities to use reliably—not much good using something that gives the right verdicts every time if you can never discover what verdicts it gives. But that is not the case we are considering here. Indeed, we have assumed that all possible picking strategies are available to the agent. So, in particular, it is within the agent's control to choose a picking strategy that never recommends impermissible acts. This is not beyond the agent's ken. And she can perfectly reliably implement this strategy. That is why we take the outcome of a picking strategy to be the outcome of whichever act it recommends.

It's worth adding that, for those decision theories that permit the Allais preferences, they consider it rationally required of you, at the point when you're uncertain which decision problem you'll face, to pay money if it's possible to thereby to bind yourself to doing something other than what they will recommend when you actually face that decision. Of course, we're used to that in cases of temptation—Ulysses should pay his sailors to bind him to the mast as their ship passes the Sirens—but that's because we think your utilities will change under the pressure of temptation or your probabilities will change in an irrational way or you won't choose rationally on the basis of your utilities and probabilities. Nothing like that is going on here.

Second: Relatedly, for someone who uses the decision theory, it gives contradictory advice at different points. Recall the case above in which you were initially uncertain whether you'd face D_1 or D_2. In that state of uncertainty, you preferred the picking strategy s_2, since the choice between s_1 and s_2 was extensionally equivalent to the choice between 2A and 2B in the Allais set up. And yet, when you face D_1, you prefer to use s_1, for it picks what your decision theory says you should pick. And so which should you actually use? Which of these perspectives is privileged? Standing facing the decision problem D_1, you might ask yourself: my decision theory tells me to choose A and not to choose B, but it also tells me that, if I could have chosen how I would choose, it would have told me to choose B and not to choose A—which should I do?

Of course, we're used to that in cases of act-state dependence. Causal decision theorists say that in Newcomb's problem you should take both boxes, but if you can pay to take a pill to turn yourself into a one-boxer before the prediction is made, then you should. But that's because choosing how to choose, in this case, causes you to face a better decision problem down the line. Nothing like that is going on here. And perhaps we think that, at least when choosing a picking strategy has no causal or evidential impact on which decision problems you will face or the value of the options in those problems, the correct theory of rational choice should not give rise to such dilemmas.

1.2 Responses

1.2.1 Limit the decision theory's scope

Perhaps the defender of a self-undermining decision theory will say it was never their intention that their theory should be used for these higher-order decisions. They are offering a theory of first-order rational choice; not a fully general theory that covers any sort of decision, including these higher-order decisions between picking strategies. But that can't be right, for these theories are presented by their proponents as universal decision theories. In their descriptions, it is not specified that they are to be applied only in specific cases. And, most importantly, the arguments in favour of the axioms that characterize them—which, in many cases, constitute the arguments for the theory itself—never make reference to the sort of decision you'll face: if they are plausible at all, they are plausible as applied to all decisions. These arguments are truly formal in the sense that the content of the decisions in question is not relevant to our evaluation of their plausibility. If the theory is not supposed to apply to higher-order decisions as well as first-order decisions, the defender of the theory must explain why the axioms are true when they govern preferences between first-order acts, but not when they govern preferences between higher-order acts like picking strategies.

1.2.2 Change the way of updating credences

There is one way that the defender of these theories could argue against our claim of self-underminingness: they can argue that we have essentially assumed Bayesian conditionalization and argue against conditionalization.

We have assumed conditionalization because, when evaluating the strategies, we have constructed a probability over the higher-order states of the world—which specify both the first-order state of the world and the decision problem you'll face—by taking probabilities for the first-order states, probabilities for the decision problems, and assuming the decision you face is independent of the first-order state of the world. We then assumed that once you face whichever decision problem, you will use those same first-order probabilities to make your decision. This step has made use of Bayesian conditionalization. Perhaps, if you update your credences in some alternative way, what the decision theory recommends in each decision theory might correspond exactly with the optimal strategy.

It is worth noting that many theories insist that updating on irrelevant information does not change your state of uncertainty, not just traditional Bayesianism. For example, Dempster-Shafer theory models evidential irrelevance and updating in a much different way from the Bayesian one (using Dempster's Rule of Combination). Nonetheless, Dempster-Shafer theory also makes this prediction (**?**, pp. 133-5).

But put that to the side. As pointed out in **?**, sec7, several standard arguments for conditionalization assume that rational agents maximise expected rather than risk-weighted expected utility. Perhaps the argument we have made here can be turned into an argument against Bayesian conditionalization.

We could judge a credal update method by the utility of the action it requires you to select (using one's preferred decision theory) in each decision problem. The credal update method which you evaluate to be optimal will exactly be one where picking in accordance with the decision theory using the updated credences is just the picking strategy that the decision theory evaluates as optimal. The example we have discussed implies that it won't always be conditionalization, because making a decision with the first-order probabilities is not evaluated as the optimal picking strategy, so the update cannot return the original first-order probabilities despite the higher-order probabilities encoding the assumption that they are independent.

This approach is closely related to Brown's (**?**) argument for conditionalization as an update rule, which makes use of Good's (**?**) Value of Information theorem. But the Value of Information Theorem fails for these decision theories (**??**): sometimes you think you'll make better decisions if you ignore the evidence (or don't update your credences). The epistemic variant of this argument is from **?**, and **?** consider how it applies to agents who use Buchak's (**?**) risk-weighted decision theory.

However, as **?** stress, whilst this might provide an argument that the optimal update strategy is this particular way, this does not mean that the risk-sensitive agent is rationally required to adopt such an update procedure, unless one has a principle linking previous evaluations of strategies to rational behaviour.

And this is in general controversial, and doubly so in the case of risk-sensitive theories.

In that paper, **?** also consider "local" versions of the epistemic arguments for update procedures, following **?**, showing that they require the risk-sensitive agent to nonetheless update by conditionalization. **?** argue that this is actually what gives the rational requirements on updating, and so conclude that conditionalization is rationally required (and thus, the Value of Information challenge cannot be avoided this way, cf., (**?**)). The analogous questions can be also asked for the practical utility versions: whilst we are unsure what this results in, it is unlikely that they result in the same recommendations as the optimal update plan.[1] And thus, whilst the agent might evaluate the optimal update plan to be one which avoids the undermining we have identified, it is not the update method which we think that they should actually implement.

1.2.3 Uncertainty about decisions

In our argument that Allais-permitting decision theories are self-undermining, we used the preferences they permit over the Allais gambles to construct a *particular* way that you might be uncertain about what decisions you'll face and showed that, *in that case*, you judge some alternative picking strategy to be preferable.

Moreover, the particular way of being uncertain over decisions you'll be faced with was a rather odd one. We specified that the credence you'll face D_1 is $1/11$ and your credence that you'll face D_2 is $89/100$. We could motivate this uncertainty if, for example, you believe you are facing a nefarious opponent who has set you up: they're going to toss a biased coin then offer you D_1 or D_2 at the specified probabilities, having chosen these number specifically to demonstrate your inconsistent judgements.

But this is not how we're usually uncertain over what decisions we'll face. What does this argument show about someone who is uncertain about what decision problems she'll face in an alternative, and more normal, way?

This is analogous to a certain standard response to the Dutch book arguments: perhaps you just think it's unlikely that you'll ever face such a nefarious bookie. A common response is to argue that the mere existence of a Dutch book already shows you are irrational because it shows your preferences are inconsistent in a particular way—they judge the same decision differently when it is presented in different way, perhaps.[2] The set-up in which an opposing Gambler approaches you, buys and sells bets that you deem fair, and thereby saddles you with a sure loss merely dramatizes this inconsistency. The argument doesn't assume you'll ever actually meet such a person. We might argue similarly that the mere existence of self-underminingness for some way of being uncertain about what decision you'll faced is already a challenge to the decision theory. Perhaps it shows that it is inconsistent in the same way the Dutch book argument shows non-probabilistic credences are inconsistent.

[1] It differs from the epistemic versions because we used an r-proper scoring rule, but this is not what we'd get if we just use pragmatic utility.

[2] Cf. (**??**).

In the case of the Dutch book arguments, **?**, Sec 6.2 argues the mere existence of a set of bets you'll accept individually that, taken together, lead to sure loss isn't sufficient to show you are irrational. Instead, he asks what happens if you are uncertain about which decisions you'll face. Drawing on results from Mark **?** and Ben **?**, he shows that, for very many natural ways of being uncertain about the decisions you'll face, if you have non-probabilistic credences and face whatever decision you'll face with those credences, there are alternative probabilistic credences you might have had instead that guide your choices better.

We might here similarly try to bolster our objection by showing a more general result which would apply to a whole host of natural ways of being uncertain over which decision problem you'll face. Of course, we would have a stronger objection if we could show that for *any* way of being uncertain over the decision problems you face, the theory is self-undermining. In fact, we'll never get something as general as that: after all, if your probability distribution places all of its probability on a single decision problem, then it will think of itself as permissible—and indeed, it will think of anything that disagrees with it as impermissible. But we might hope to be able to show that it is self-undermining for a much broader range of distributions over the possible decision theories than we currently have, thus arguing that for any 'plausible' way of being uncertain over possible decisions, the theory is still undermining.

We don't have any general results in this area, but we do have some suggestive particular cases for a particular Allais-permitting theory. This is John Quiggin's (**??**) rank-dependent utility theory, which is formulated for exogenous, objective probabilities, and Lara Buchak's (**?**) risk-weighted expected utility theory, which is a formally equivalent theory formulated for endogenous, subjective probabilities.

To give your *risk-weighted expected utility* for an act a, defined over the states in Ω, we need a probability function p over Ω, and a risk function $r : [0,1] \to [0,1]$, which takes a probability and skews it—we assume r is continuous, strictly increasing, and $r(0) = 0$ and $r(1) = 1$. Now, suppose a is an act, and let $\mathfrak{U}(a)$ be the random variable that gives the utility of a at a given state. If Ω is finite, the risk-weighted expected utility of a can be written as follows:[3]

$$\mathsf{RExp}_{p,r}[\mathfrak{U}(a)] = \sum_{x \in \{\mathfrak{U}(a)(\omega) | \omega \in \Omega\}} (r(p(\mathfrak{U}(a) \geqslant x)) - r(p(\mathfrak{U}(a) > x))) \times x$$

And, more generally, if the utilities are bounded below by l and above by u,

$$\mathsf{RExp}_{p}[\mathfrak{U}(a)] = \int_{u}^{h} r(p(\mathfrak{U}(a) > x))\, dx$$

Then risk-weighted expected utility theory tells you to maximize risk-weighted expected utility.

Now, suppose:

[3]This is not Buchak's favoured formulation; instead it's closer to the usual formulation of rank dependent expected utility theory; see, for example (**?**, ch.6). See also sec.6.9 for the continuous case, although note that Buchak's theory does not make use of a distinction between gains and losses (see **?**, p59).

(i) there are just two first-order possibilities ω_1 and ω_2,

(ii) your credence function is p, with $p(\omega_1) = 0.1, 0.2, \ldots, 0.8$, or 0.9 and $p(\omega_2) = 1 - p(\omega_1)$;

(iii) your risk function is a power function $r_k(x) = x^k$, with $k = 0.5$, 0.6, ..., 0.8, 0.9, 1.1, 1.2, ..., 1.9, or 2;

(iv) you know you'll face a choice between just two acts, but you don't know which two acts, and you place a measure μ over the possible decision problems that takes the utilities of the two acts at the two possibilities to be independent of one another and all distributed according to a beta distribution $\mathrm{Beta}(\alpha, \beta)$ with $\alpha = 1, 2, 3, 4$, or 5 and $\beta = 1, 2, 3, 4$, or 5.

Then, let s be any picking strategy compatible with REU when coupled with p and r_k. Then there is an alternative picking strategy s* such that REU with credences given by p and μ and risk function r_k strictly prefers s* to s. What's more, s* is not compatible with REU with p and r_k. And indeed, it's possible to find s* so that it is compatible with REU with p and $r_{k^\star}$ for some $k^\star \neq k$. That is, s* is a picking strategy compatible with REU coupled with a different risk-averse risk function. So, uncertain which decision they'll face, someone using REU with this risk function would prefer that, when the uncertainty is resolved and they face a particular decision, they use REU with a slightly different risk function.[4]

2 Expected Utility Theory

In this section, we reassure ourselves that Savage-style expected utility theory is self-recommending; that is, if we assume act-state independence, expected utility theory endorses itself. We will need to be more detailed about the framework in order to present our result.

States Ω is the set of possible states of the world. We'll assume that there are only finitely many.

Uncertainty The agent's uncertain beliefs about the world are represented by a single probability function p over Ω.

Acts $\mathcal{A}$ is a non-empty set, the set of all possible acts.

Utilities $\mathfrak{U}$ is the agent's utility function. It takes each act a in $\mathcal{A}$ and state ω in Ω and returns a utility value $\mathfrak{U}(a)(\omega) \in \Re$. We will assume that utilities are bounded above and below. That is, there are $l, u \in \Re$ such that for all acts, $a \in \mathcal{A}$, $l \leqslant \mathfrak{U}(a)(\omega) \leqslant u$.

Decision problems A decision problem D specifies a non-empty finite set of acts: the acts that are available in that decision problem. The set of all decision problems is $\mathcal{D}$.

(In fact, all of our results will go through if we instead assume that each D in $\mathcal{D}$ is compact, relative to the utility function, that is, $\{\mathfrak{U}(a) \mid a \in D\}$ is a compact

[4]See the Mathematica notebook here for the tools to carry out these calculations [link to notebook from journal page as supplementary material].

subset of $[l,u]^\Omega$. But we will continue to assume that decision problems are finite for ease of presentation.)

Choice function A choice function, c, specifies a non-empty subset of each decision problem, so that $\varnothing \neq \mathsf{c}(D) \subseteq D$, for each D in $\mathcal{D}$. If a is *not* in $\mathsf{c}(D)$ then c deems a impermissible in D. Some authors go further and say that any a in $\mathsf{c}(D)$ is rationally *permissible* (e.g., **?**). But others do not. The latter say instead that, if a is in $\mathsf{c}(D)$, then a is not deemed impermissible, but unless a is the only act in $\mathsf{c}(D)$, it does not follow that a is permissible or positively evaluated in any way (e.g., **?**).

Picking strategy As above, a picking strategy, s, specifies an act from each decision problem, so that $\mathsf{s}(D) \in D$, for each D in $\mathcal{D}$. The set of all picking strategies is $\mathcal{S}$.

A picking strategy picks for a choice function c if it never picks an option which c deems impermissible. That is:

Definition 1. *A picking strategy* s picks for a choice function c, *if* $\mathsf{s}(D) \in \mathsf{c}(D)$, *for all decision problems D.*

Given a probability function p defined over Ω, the expected utility of an action a, by the lights of p, is given by

$$\mathsf{Exp}_p[\mathfrak{U}(a)] := \sum_{\omega \in \Omega} p(\omega)\mathfrak{U}(a)(\omega)$$

If one has (probabilistic) credences given by p, expected utility theory says that one should choose an act in D that maximises $\mathsf{Exp}_p[\mathfrak{U}(a)]$.[5] That is, we define the choice function to which expected utility theory gives rise when coupled with probability function p as follows:

Definition 2 (Expected Utility Theory (EU_p))**.**

$$\mathrm{EU}_p(D) := \{a \in D \mid \mathsf{Exp}_p[\mathfrak{U}(a)] \geqslant \mathsf{Exp}_p[\mathfrak{U}(a')] \textit{ for all } a' \in D\}$$

So a picking strategy s picks for EU_p iff $\mathsf{s}(D)$ maximizes expected utility by the lights of p. Since we have assumed that each D is finite, or compact, $\mathsf{Exp}_p[\mathfrak{U}(a)]$ obtains its maximum in D; so $\mathrm{EU}_p(D) \neq \varnothing$.

Now we want to judge the expected utility of a picking strategy itself—we want to ask whether the picking strategies that always pick an act that maximizes expected utility from any decision problem themselves maximize expected utility when you're uncertain which decision problem you'll face. This requires us to fix not only p, which gives your credences over Ω, but also your credences over the decision problems you might face, given by some probability measure μ over $\mathcal{D}$. We will assume these are independent. So your credences over the joint space, $\Omega \times \mathcal{D}$, are given by the product measure $p \times \mu$. That is, your credence that you are in world ω and will be faced with some decision problem in the (measurable) set of decision problems A is given by $(p \times \mu)(\omega, A) = p(\omega) \times \mu(A)$.

[5] The version we present here is of the sort described by **?**, in which it is assumed that the acts are independent of the states of the world. This assumption is dropped in the evidential decision theory of **?** and the causal decision theory of **???**.

We can now simply apply our notion of expected utility with the credences over $\Omega \times \mathcal{D}$ given by $p \times \mu$. We judge a picking strategy by the utility of the acts it requires us to choose, and so we define $\mathfrak{U}(\mathsf{s})(\omega, D) := \mathfrak{U}(\mathsf{s}(D))(\omega)$. We can then apply our definition of expected utility and get that any picking strategy that picks for EU_p maximizes expected utility by the lights of $p \times \mu$.

Proposition 1. *For any p, μ, if s picks for EU_p, then, for any picking strategy s' in $\mathcal{S}$,*

$$\mathsf{Exp}_{p\times\mu}[\mathfrak{U}(\mathsf{s})] \geqslant \mathsf{Exp}_{p\times\mu}[\mathfrak{U}(\mathsf{s}')].$$

That is, if s picks for EU_p, then $\mathsf{s} \in \mathrm{EU}_{p\times\mu}(\mathcal{S})$.

(Recall: $\mathrm{EU}_{p\times\mu}$ is the choice function to which EU gives rise when coupled with $p \times \mu$, and $\mathcal{S}$ is the set of picking strategies, and so $\mathrm{EU}_{p\times\mu}(\mathcal{S})$ is the set of all picking strategies that maximize expected utility by the lights of $p \times \mu$.)

This shows that expected utility theory is not self-undermining in the way the Allais-permitting decision theories considered in the previous section are self-undermining. Expected utility picking strategies are themselves maximisers of expected utility.

What's more, they are the only picking strategies which maximise expected utility. Or at least, the picking strategies which maximise expected utility are those that look like an EU_p strategy from μ's perspective.

Definition 3. *If c is a choice function and s is a picking strategy, then s μ-*surely picks for c *iff* $\mu\{D \in \mathcal{D} \mid \mathsf{s}(D) \in \mathsf{c}(D)\} = 1$

That is, s μ-surely picks for c just in case μ is certain you'll face a decision problem where what s picks is compatible with c, *i.e.*, $\mathsf{s}(D) \in \mathsf{c}(D)$. That is, μ is sure that s does not pick an option that c rejects.

Proposition 2. *For any p and μ, if s μ-surely picks for EU_p, while s' does not, then*

$$\mathsf{Exp}_{p\times\mu}[\mathfrak{U}(\mathsf{s})] > \mathsf{Exp}_{p\times\mu}[\mathfrak{U}(\mathsf{s}')].$$

So, if s does not μ-surely pick for EU_p, then $\mathsf{s} \notin \mathrm{EU}_{p\times\mu}(\mathcal{S})$.

We thus have that $\mathsf{s} \in \mathrm{EU}_{p\times\mu}(\mathcal{S})$ iff s μ-surely picks for EU_p.

The proofs of all the results in the paper are in the appendix.

It is worth noting that the reasoning that delivers these results only holds when we have assumed that the state of the world is independent of the act chosen, i.e., where we are using Savage's (**?**) version of expected utility theory, in which the states of the world are independent of the acts chosen.

2.1 Decision-State Dependence

As well as assuming act-state independence, we've also assumed decision-state independence: that is, we've assumed that, from the point of view of your credences over $\Omega \times \mathcal{D}$, the decision you face and the state of the world are independent of one another, given by $b = p \times \mu$. But, in fact, analogues of Propositions 1 and 2 hold even if we don't assume this.

In any decision problem, we must bring one's probability b up to speed on the problem that you face (by conditionalizing on that information), and then use expected utility theory with this updated credence function to determine what to select.

Definition 4. *If b is a probability over $\Omega \times \mathcal{D}$; we specify a choice function:* $\mathrm{EU}_{b(\cdot|-)}$ *as* $\mathrm{EU}_{b(\cdot|-)}(D) = \mathrm{EU}_{b(\cdot|D)}(D)$*, when this is well-defined.*

check specification and link to appendix

We say that s b-surely picks for $\mathrm{EU}_{b(\cdot|-)}$ *iff*

$$b\{\langle \omega, D\rangle \mid \omega \in \Omega,\, \mathsf{s}(D) \in EU_{b(\cdot|-)}(D)\} = 1$$

Check this notation throughout

We can then show:

Proposition 3. $\mathsf{s} \in \mathrm{EU}_b(\mathcal{S})$ *iff* s *b-surely picks for* $\mathrm{EU}_{b(\cdot|-)}$.

3 Decision theories for imprecise credences

There is another range of decision theories that diverge from expected utility theories: those theories which accommodate ambiguity and imprecision. In the decision theories considered so far, we represent an individual as assigning precise credences to the various states of the world. But some think we do better to model individuals as having imprecise credences instead (**??**). There are many ways to do this, but one of the most well-known represents an individual's doxastic state not by a single credence function, which assigns to each state of the world a single numerical measure of their confidence in that state, but by a set of such functions. We call this set your *credal set*. It is a set $\mathbb{P}$ of probability measures over the states of the world, Ω.[6]

Many decision theories have been proposed for an agent whose uncertain beliefs are represented in this way. We discuss three prominent ones: Γ-Maximin, E-Admissibility and Maximality.

that ok?

3.1 Γ-maximin

To illustrate Γ-Maximin, consider an example that is often used to motivate it, namely, the Ellsberg paradox (**?**):

> An urn contains 90 balls. You know that 30 of them are red, and the remaining 60 are black and yellow, but you don't know how many are black and how many are yellow. I am about to draw a ball from the urn.

If the states of the world are *Red* (I draw a red ball), *Black* (I draw black), and *Yellow* (I draw yellow), you might naturally take your credal set to be

$$\mathbb{P} = \{p \mid p(Red) = 1/3 \;\&\; p(Black) + p(Yellow) = 2/3\}.$$

[6]It is standard in the imprecise probability literature to reserve the term "credal set" for convex sets of probability measures. We do not assume convexity here.

Now consider the following two possible decision problems, D_1 and D_2:

D_1	*Red*	*Black*	*Yellow*
$\mathbb{P}$	$1/3$	x	$2/3 - x$
1A	£10	£0	£0
1B	£1	£11	£1

D_2	*Red*	*Black*	*Yellow*
$\mathbb{P}$	$1/3$	x	$2/3 - x$
2A	£11	£1	£11
2B	£0	£10	£10

Faced with these decisions, people often report the Ellsberg preferences: they will choose 1A from D_1, and 2B from D_2.[7] And indeed that is exactly what Γ-Maximin demands. It says that, faced with a particular decision problem, you should pick one of the acts whose minimum expected utility by the lights of the probability functions in $\mathbb{P}$ is maximal: in D_1, 1A uniquely maximizes minimum expected utility; and in D_2, 2B does that.

Definition 5 (Γ-Maximin$_{\mathbb{P}}$ ($\Gamma_{\mathbb{P}}$))**.**

$$\Gamma_{\mathbb{P}}(D) = \left\{ a \in D \mid (\forall a' \in D) \left[\min_{p \in \mathbb{P}} \mathsf{Exp}_p[\mathfrak{U}(a')] \leqslant \min_{p \in \mathbb{P}} \mathsf{Exp}_p[\mathfrak{U}(a)] \right] \right\}$$

(This should only be applied when these minima exist, e.g., when $\mathbb{P}$ is a closed set.)

So s picks for $\Gamma_{\mathbb{P}}$ iff $\mathsf{s}(D) \in \Gamma_{\mathbb{P}}(D)$ for every $D \in \mathcal{D}$. In this case, the only picking strategy that picks for $\Gamma_{\mathbb{P}}$ must pick 1A from D_1 and 2B from D_2. We call this strategy s_E; the strategy corresponding to the Ellsberg preferences. Such a strategy is incompatible with expected utility theory: it does not pick for EU_p for any $p \in \mathbb{P}$.[8] Indeed, this fact accounts for Ellsberg's use of the case: like Allais, he wished to provide an example of intuitively rational preferences that could not be captured by expected utility theory.

Now we will use the theory itself to judge picking strategies. To do this, we need to describe the agents uncertainty not only over what the world is like, but also which decision problem she'll be faced with. Suppose that in fact you've got precise probabilities over what decision you'll be faced with, and you think that 50% you'll be faced with D_1 and 50% you'll be faced with D_2. So we are representing your uncertainty as a set, $\mathbb{B}$, of (higher-order) probabilities over both Ω and $\mathcal{D}$, each of which makes the state of the world independent of the decision you're faced with. That is, your credal set is given by $\mathbb{B} = \{p \times \mu^* \mid p \in \mathbb{P}\}$, where μ^* is this probability over $\mathcal{D}$, and $\mathbb{P}$ is the credal set as described in the Ellsberg case.

Observe, then that $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\mathsf{s}_{1A,2B})(\omega) = 5$ for each ω as Red, Black or Yellow, but $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\mathsf{s}_{1B,2A})(\omega) = 6$ for each ω as Red, Black or Yellow. So for any p with $p \times \mu^* \in \mathbb{B}$, $\mathrm{EU}_{p\times\mu^*}\mathfrak{U}(\mathsf{s}_{1A,2B}) = 5$ and $\mathrm{EU}_{p\times\mu^*}\mathfrak{U}(\mathsf{s}_{1B,2A}) = 6$; so $\mathsf{s}_E = \mathsf{s}_{1A,2B} \notin \Gamma_B(\mathcal{S})$.

This example is closely related to another phenomenon: Dutch books, or paradoxes of sequential choice can be constructed against agents on the basis of

[7] In fact, we have added a small constant to the usual versions of 1B and 2A, reflecting the fact that people strictly prefer the usual version of 1A over the usual version of 1B, and so are willing to pay a penalty for making that choice; we've taken that penalty to be £1, but our point remains however small you make it.

[8] This is because, to have $\mathsf{Exp}_p[\mathfrak{U}(1A)] \geqslant \mathsf{Exp}_p[\mathfrak{U}(1B)]$ it must be that $x \leqslant 7/30$, and to have $\mathsf{Exp}_p[\mathfrak{U}(2B)] \geqslant \mathsf{Exp}_p[\mathfrak{U}(2A)]$ it must be that $x \geqslant 13/30$; and these are jointly incompatible.

such examples. In such examples we consider how she will choose in D_1 and D_2, individually, and then combine these choices and observing that the result is dominated. One response is simply to reject the package principle. Another version of the examples is presented diachronically: first evaluate D_1 then D_2; but the evaluation of D_2 surely depends on what has been selected in D_1. Such examples can be used to construct instances of undermining evaluations of strategies when we assume that your confidence over the various decisions you're faced with is uniform, so that the result is equivalent (probabilistically) to just adding up the results of each of the decisions.

3.1.1 Uncertainty about decisions

We have only shown that in this particular way of being uncertain about what decisions you'll face, Γ-Maximin is self-undermining, ruling its own strategy as impermissible. Like in our discussion in Section 1.2.3, we would like to be able to improve on the results by showing a more general result that a wide class of ways of being uncertain results in analogous undermining.

In fact, we do have a general result if we make a particular assumption: that one's uncertainty over which decision problem she'll be faced with is given by a precise probability which is broad over many decisions. We will discuss the details of this in Section 4.1.3.

check the link to the general stuff!!

3.2 E-Admissibility and Maximality

Two alternative decision theories are E-Admissibility and Maximality. When coupled with a credal set $\mathbb{P}$, E-Admissibility rejects an act a from a decision problem D when, for any p in $\mathbb{P}$, there is some a' in D that p expects to do better than a. In that case, every p in $\mathbb{P}$ expects *some* other option to be better than a, but there may be no *single* option that they all agree to better. Maximality rejects an act a from D when there is some a' in D that every p in $\mathbb{P}$ expects to do better than a, *i.e.*, when every p in $\mathbb{P}$ agree on a *single* option which they expect to be better than a. If an act is rejected according to Maximality, then it is also rejected according to E-Admissibility, but not vice versa.

Definition 6.

$$\mathrm{EAd}_{\mathbb{P}}(D) = \{a \in D \mid (\exists p \in \mathbb{P})(\forall a' \in D)(\mathsf{Exp}_p[\mathfrak{U}(a)] \geqslant \mathsf{Exp}_p[\mathfrak{U}(a')])\}$$

s picks for $\mathrm{EAd}_{\mathbb{P}}$ iff $\mathsf{s}(D) \in \mathrm{EAd}_{\mathbb{P}}(D)$ for every $D \in \mathcal{D}$.

Definition 7.

$$\mathrm{Max}_{\mathbb{P}}(D) = \{a \in D \mid (\forall a' \in D)(\exists p \in \mathbb{P})(\mathsf{Exp}_p[\mathfrak{U}(a)] \geqslant \mathsf{Exp}_p[\mathfrak{U}(a')])\}$$

s picks for $\mathrm{Max}_{\mathbb{P}}$ iff $\mathsf{s}(D) \in \mathrm{Max}_{\mathbb{P}}(D)$ for every $D \in \mathcal{D}$.

Let's treat E-Admissibility first. Suppose you are uncertain which decision problem you'll face. And suppose, as above, we represent your uncertainty over $\Omega \times \mathcal{D}$ with a credal set $\mathbb{B}$. Then we can ask, from the point of view of $\mathbb{B}$, and using E-Admissibility as our decision theory, whether a picking strategy that picks for $\mathrm{EAd}_{\mathbb{P}}$ is permissible. And it turns out that, unlike for Γ-Maximin, there always is such a strategy. Indeed, if you simply take p from $\mathbb{P}$, and take a picking strategy s that picks for EU_p, then s also picks for $\mathrm{EAd}_{\mathbb{P}}$, and s is E-Admissible, as evaluated by $\mathbb{B}$—that is, s is in $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$; at least if there is some $b = p \times \mu \in \mathbb{B}$. The following is a corollary of Propositions 1 and 2:

Proposition 4. *If s μ-surely picks for EU_p for some $p \times \mu \in \mathbb{B}$ then $\mathsf{s} \in \mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$.*

Thus, if there is some $p \in \mathbb{P}$ with some $p \times \mu \in \mathbb{B}$, then there is some s, namely any s which picks for EU_p, which picks for $\mathrm{EAd}_{\mathbb{P}}$ and which is in $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$.

There are a number of conditions which guarantee the existence of some $p \times \mu \in \mathbb{B}$ for any $p \in \mathbb{P}$, and thus ensure that every E-Admissible action in a decision problem is part of a picking strategy which is E-Admissible. For example, suppose that you have no views whatsoever about which decisions you will face, or the evidential value of information about which decisions you will face. In that case, your credal set $\mathbb{B}$ over $\Omega \times \mathcal{D}$ is given by the *natural extension* of $\mathbb{P}$ to this space, which is the largest (least informative) set of probabilities that extend the probabilities in $\mathbb{P}$ to $\Omega \times \mathcal{D}$. This is sufficient to guarantee that for every $p \in \mathbb{P}$ there is some μ on $\mathcal{D}$ such that $p \times \mu \in \mathbb{B}$.

Alternatively, suppose that you have a bit of information both about the world and which decision problem you will face. Your uncertainty about the world is given by the credal set $\mathbb{P}$ over Ω. Your uncertainty about the which decision you will face is given by the credal set $\mathbb{M}$ over $\mathcal{D}$. Suppose also that you treat information about which decisions you will face as *irrelevant* to which state of the world you are in.

In the precise setting, irrelevance is a univocal, symmetric notion: for any joint distribution b over $\Omega \times \mathcal{D}$, $\mathcal{D}$ is *stochastically independent* of (and hence irrelevant to) Ω according to b just in case $b(\omega \in A \mid D \in B) = b(\omega \in A)$ whenever $b(D \in B) > 0$.[9]

ftnte defining marginal again. Check consistency throughtou paper.

But in the imprecise setting, irrelevance fractures into a variety of distinct, not necessarily symmetric notions.[10]

Consider a case where $\mathbb{P}$ and $\mathbb{M}$ are closed and convex and you treat $\mathcal{D}$ as *epistemically irrelevant* to Ω, in the sense of **?**. This means roughly that learning information about which decision problem you face does not change your maximum buy price for any "worldly" gamble, *i.e.*, any gamble whose payout depends only on Ω. Suppose that $\mathbb{P}$, $\mathbb{M}$ and this judgment of epistemic irrelevance jointly capture the totality of your views. In that case, your credal set $\mathbb{B}$ over $\Omega \times \mathcal{D}$ is given by the *irrelevant natural extension* (*cf.* (**?**, Thm 13)). This is the largest (least informative) set $\mathbb{B}$ of probabilities b over $\Omega \times \mathcal{D}$ that

[9] For any $A \subseteq \Omega$, $\omega \in A := \{\langle \omega, D\rangle \in \Omega \times \mathcal{D} \mid \omega \in A\}$. Likewise, for any $B \subseteq \mathcal{D}$, $D \in B := \{\langle \omega, D\rangle \in \Omega \times \mathcal{D} \mid D \in B\}$.

[10] A short survey of independence notions for imprecise probability: complete independence for sets of probabilities (**??**); independence in selection for lower previsions (**?**); strong independence for lower previsions and sets of desirable gambles (**?**); epistemic independence (value and subset) for sets of desirable gambles (**?**); epistemic h-independence for lower previsions and credal sets (**?**); S-independence for choice functions (**?**).

marginalize to $\mathbb{P}$ and $\mathbb{M}$ and satisfy the following inequality constraints: for any gamble $g : \Omega \to \mathbb{R}$ and any $B \subseteq \mathcal{D}$ with $b(D \in B) > 0$

$$\inf\left\{\mathsf{Exp}_p[g] \mid p \in \mathbb{P}\right\} \leqslant \mathsf{Exp}_b[g^+] \leqslant \sup\left\{\mathsf{Exp}_p[g] \mid p \in \mathbb{P}\right\}$$

and

$$\inf\left\{\mathsf{Exp}_p[g] \mid p \in \mathbb{P}\right\} \leqslant \mathsf{Exp}_b[g^+ \mid D \in B] \leqslant \sup\left\{\mathsf{Exp}_p[g] \mid p \in \mathbb{P}\right\}$$

where $g^+ : \Omega \times \mathcal{D} \to \mathbb{R}$ is the "cylindrical extension" of g defined by $g^+(\omega, D) = g(\omega)$ for all $\omega \in \Omega$ and $D \in \mathcal{D}$. As many authors have noted, individual probabilities b in the irrelevant natural extension $\mathbb{B}$ will not in general treat $\mathcal{D}$ as irrelevant to Ω (*cf.* (**?**, pp. 96-7)). Nonetheless, $\mathbb{B}$ itself will do so, in the sense described above. Moreover, $\mathbb{B}$ will contain any b that treats $\mathcal{D}$ as stochastically independent of Ω. This is sufficient to guarantee that the condition of Proposition 4 holds.

Rather than treating $\mathcal{D}$ as epistemically irrelevant to Ω, you might treat $\mathcal{D}$ and Ω as *completely independent*, in the sense of **??**, *i.e.*, $\mathcal{D}$ and Ω are stochastically independent according to every $b \in \mathbb{B}$. This is a more stringent notion of irrelevance than epistemic irrelevance (and is also symmetric). If $\mathbb{P}$ and $\mathbb{M}$ capture your opinions about Ω and $\mathcal{D}$, respectively, you judge $\mathcal{D}$ and Ω as completely independent, and nothing more (this captures the totality of your views), then your credal set $\mathbb{B}$ over $\Omega \times \mathcal{D}$ is the largest (least informative) set $\mathbb{B}$ of probabilities b over $\Omega \times \mathcal{D}$ that marginalize to $\mathbb{P}$ and $\mathbb{M}$ and satisfies completely independent, *i.e.*, $\mathbb{B} = \{p \times \mu \mid p \in \mathbb{P}, \mu \in \mathbb{M}\}$. This is also sufficient to guarantee that the condition of Proposition 4 holds.

The upshot is that E-Admissibility is not self-undermining in the same way that Γ-Maximin is self-undermining: under a broad range of conditions, there are strategies that pick for it that it does not deem impermissible.

I rewrote the start of this. JK to check.

Do we obtain a converse to Proposition 4? Are these the only E-Admissible strategies? A strategy is E-Admissible iff there is some $b \in \mathbb{B}$ which expects it to be optimal. We might hope to be able to apply Proposition 2 to get that it is only these strategies which are E-Admissible. For this, we need to assume that every b in $\mathbb{B}$ has the form $p \times \mu$, i.e., that you treat $\mathcal{D}$ as completely irrelevant to Ω:

Proposition 5. *Suppose $\mathcal{D}$ and Ω are completely independent in $\mathbb{B}$. Then every $b \in \mathbb{B}$ has the form $p \times \mu$. In that case,* $\mathsf{s} \in \mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$ *iff* s *μ-surely picks for EU_p for some $p \times \mu \in \mathbb{B}$.*

So the rather strong judgment of complete independence has rather strong implications for your views about picking strategies. The only strategies that are permissible by the lights of E-Admissibility in this case are ones that pick for expected utility theory, *i.e.*, always pick options that maximize p-expected utility, for some $p \times \mu \in \mathbb{B}$.

For example, in the Ellsberg case (Section 3.1), $\mathrm{EAd}_{\mathbb{P}}(D_1) = \{1A, 1B\}$ and $\mathrm{EAd}_{\mathbb{P}}(D_2) = \{2A, 2B\}$; so every strategy picks for $\mathrm{EAd}_{\mathbb{P}}$. However, the $\mathsf{s}_{1A,2B}$

strategy, which is the empirically observed strategy, is not an EU_p strategy for any p: it is not rationalisable by expected utility theory.[11] I.e., it is not in $\mathrm{EU}_p(\mathcal{S})$ for any p. If every $b \in \mathbb{B}$ has the form $p \times \mu$ with each μ giving positive probability to facing both of the decisions in the Ellsberg case, then it also does not μ-surely pick for EU_p for any $p \times \mu \in \mathbb{B}$; and thus, is not in $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$, despite picking for $\mathrm{EAd}_{\mathbb{P}}$. However, the other strategies also pick for $\mathrm{EAd}_{\mathbb{P}}$ and they are EU_p and thus in $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$, for example $\mathsf{s}_{1A,2A}$.

This is a more general feature: there are always some strategies which picks for $\mathrm{EAd}_{\mathbb{P}}$ but which are rejected by $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$, at least if $\mathrm{EAd}_{\mathbb{P}}$ is not sufficiently precise, i.e., that it does not μ-surely look like EU_p, or a restriction thereof, for any $p \times \mu \in \mathbb{B}$.

Proposition 6. *Suppose $\mathcal{D}$ and Ω are completely independent in $\mathbb{B}$. Then every $b \in \mathbb{B}$ has the form $p \times \mu$. Suppose further that for every $p \times \mu \in \mathbb{B}$, $\mu\{D \mid \mathrm{EAd}_{\mathbb{P}}(D) \subseteq \mathrm{EU}_p(D)\} \neq 1$. That is, for all $p \times \mu \in \mathbb{B}$, $\mu\{D \mid$ there is $p' \in \mathbb{P}$ with $\mathrm{EU}_{p'}(D) \not\subseteq \mathrm{EU}_p(D)\} > 0$.*

Then there is some s which picks for $\mathrm{EAd}_{\mathbb{P}}$ but which is not in $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$.

E-Admissibility rejects some picking strategies which are compatible with its own choice set. But in every decision problem you're faced with, those acts which are E-Admissible are exactly those that arise from E-Admissible picking strategies, unlike in Section 1.1, although E-Admissible picking strategies require coordinating across the decision problems so that they could arise as expected utility strategies for some probability in one's credal set.

Does this make E-Admissibility self-undermining? This feels a bit like Consumer Bulletin advising you that it is fine to hire a personal shopper who invariably comes back with Consumer Bulletin best buys, but only if they are *guaranteed* to pick Consumer Bulletin best buys that are also Consumer Reports best buys. This is not quite self-undermining, but close, you might think.

One might see this result as a problem for E-Admissibility. If every b in $\mathbb{B}$ agrees that a given picking strategy could *possibly* saddle you with an E-Admissibile but non-EU option in *some* decision problem or other, then E-Admissibility will reject that strategy. The only picking strategies it does not reject are ones that are certain, according to some b in $\mathbb{B}$, to yield an EU option in every decision problem. This feels a bit like Consumer Bulletin advising you that it is fine to hire a personal shopper who invariably comes back with Consumer Bulletin best buys, but only if they are *guaranteed* to pick Consumer Bulletin best buys that are also Consumer Reports best buys. This is not quite self-undermining, but close, you might think.

On the other hand, one might not see this as a concern for E-Admissibility. Firstly, as Proposition 8 makes clear, there are many picking strategies that E-Admissibility does not reject. In the Consumer Bulletin analogy, it is as if there are many other magazines that your personal shopper could cross-reference, not just Consumer Reports. Secondly, and more to the point, the phenomenon here just reflects the fact that E-Admissibility sees value in coordinating how you resolve incomparability. Take a simple example.

[11] See Footnote 8.

D_1	X	$\neg X$
p	x	$1-x$
a	£10	£10
b_1	£0	£20

D_2	X	$\neg X$
p	x	$1-x$
a	£10	£10
b_2	£20	£0

Any EU-maximizer will coordinate their choices in D_1 and D_2. They will (assuming utility linear in GBP) choose a (reject b_1) in D_1 just in case they choose b_2 (reject a) in D_2. Likewise, they will choose b_1 (reject a) in D_1 just in case they choose a (reject b_2) in D_2.

unless $p =$ 0.5!!!

Suppose $\mathbb{P} = \{p_1, p_2\}$, where p_1 expects a to be strictly better than b_1, while p_2 expects b_1 to be strictly better than a. In that case, $\text{EAd}_{\mathbb{P}}(D_1) = \{a, b_1\}$ and $\text{EAd}_{\mathbb{P}}(D_2) = \{a, b_2\}$. You find both options in both options incomparable, *i.e.*, not rejected, but also not equally good. Just as each of p_1 and p_2 coordinates their choices in D_1 and D_2, so too does E-Admissibility advise you to coordinate how you resolve incomparability by picking. You ought to pick a in D_1 just in case you pick b_2 in D_2. Likewise, you ought to pick b_1 in D_1 just in case you pick a in D_2.

You might doubt that there is *really* any value in this sort of "modal coordination." (Recall, you will actually only face one of D_1 or D_2. You are not coordinating across time.) But the fact that E-Admissibility *sees* value in coordinating how you resolve incomparability does not render it self-undermining.

I like this paragraph a lot!!!!

You might also be perplexed by the sheer quantity of strategies that pick for E-Admissibility but are nonetheless rejected by E-Admissibility. Here it is important to remember that picking for E-Admissibility is simply a matter of being guaranteed to select options that are *not rejected* by E-Admissibility. The utility of an option, $\mathfrak{U}(a)$, is a gamble on Ω. The utility of picking strategy, in contrast, $\mathfrak{U}(\mathsf{s})$, is a gamble on $\Omega \times \mathcal{D}$—a larger, refined sample space. Reasons for rejection that are not visible at one scale, or level of resolution, might nonetheless become apparent at others. The fact that E-Admissibility identifies some reasons for rejection at the scale of picking strategies—reasons grounded in the (putative) value of coordination—does not conflict in any way with more pervasive non-rejection at the scale of actions or options, nor is it altogether surprising.

sort out here!!!! I

3.2.1 Maximality

Since Maximality is a more permissive decision theory than E-Admissibility, Proposition 4 entails:

Proposition 7. *If* s *μ-surely picks for* EU_p *for some* $p \times \mu \in \mathbb{B}$*, then* s *picks for* $\text{Max}_{\mathbb{P}}$ *and* s *is in* $\text{Max}_{\mathbb{B}}(\mathcal{S})$.

And so, like E-Admissibility, Maximality is not self-undermining in the same way that Γ-Maximin is self-undermining: there are always strategies that pick for it that it does not deem impermissible.

Unlike for E-Admissibility, we do not get the converse result (even under the assumption of complete independence). There can sometimes be some strategies which are not ruled out by Maximality but which nonetheless are not EU_p strategies. This is because a strategy is only ruled out as impermissible if there's a single alternative which is preferable according to every $b \in \mathbb{B}$. This happens, for example, in the Ellsberg case if one's probability over which decision problem you think you'll face is sufficiently imprecise.

If, however, you think it's precise, and equally likely, that you'll face each of D_1 and D_2, so your credal set is given by $\mathbb{B} = \{p \times \mu^* \mid p \in \mathbb{P}\}$, where $\mu^*(D_1) = \mu^*(D_2) = 0.5$, then as we observed in Section 3.1, $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\mathsf{s}_{1A,2B})(\omega) = 5$ for each ω and $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\mathsf{s}_{2A,1B})(\omega) = 6$; so then for every probability p, $\mathsf{Exp}_{p\times\mu^*}\mathfrak{U}(\mathsf{s}_{1A,2B}) = 5$ and $\mathsf{Exp}_{p\times\mu^*}\mathfrak{U}(\mathsf{s}_{2A,1B}) = 6$, so $\mathsf{s}_{1A,2B} \notin \mathrm{Max}_{\mathbb{B}}(\mathcal{S})$.

We will be able to show that if your credence over which decision problem you'll be faced with is precise, and also has a further property, that it requires almost everywhere decisiveness,[12] then we will be able to show that the only strategies that Maximality does not rule as impermissible are the expected utility strategies. We will discuss this and give the details in Section 4.1.3.

3.2.2 Decision-State Dependence

To avoid the various independence assumptions that we employed in Section 3.2, we will now generalize some of our earlier results. To do this, in any decision problem, we must bring $\mathbb{B}$ up to speed on the problem that you face (by conditionalizing on that information), and then using the updated credal set to determine which options to reject. todoWe already talked about this earlier!

Definition 8.

$$\mathrm{EAd}_{\mathbb{B}(\cdot|-)}(D) = \{a \in D \mid (\exists b \in \mathbb{B})(\forall a' \in D)(\mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a)] \geqslant \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a')])\}$$

That is, $\mathrm{EAd}_{\mathbb{B}(\cdot|-)}(D) = \bigcup_{b\in\mathbb{B}} \mathrm{EU}_{b(\cdot|-)}(D)$. We thus have, as a consequence of Proposition 3:

Proposition 8. *$\mathsf{s} \in \mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$ iff s b-surely picks for $\mathrm{EU}_{b(\cdot|-)}$ for some $b \in \mathbb{B}$.*

Thus there exists some s which picks for $\mathrm{EAd}_{\mathbb{B}(\cdot|-)}$ which is not rejected by $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$; but there will also be some which are not, at least if $\mathrm{EAd}_{\mathbb{B}(\cdot|-)}$ is not a restriction of some precise choice function.

Proposition 9. *Suppose that for every $b \in \mathbb{B}$, $b_{\mathcal{D}}\{D \mid \mathrm{EAd}_{\mathbb{B}(\cdot|-)}(D) \subseteq \mathrm{EU}_{b(\cdot|-)}\} < 1$. That is, for all $b \in \mathbb{B}$, $b_{\mathcal{D}}\{D \mid$ there is $b' \in \mathbb{B}$ with $\mathrm{EU}_{b'(\cdot|-)}(D) \nsubseteq \mathrm{EU}_{b(\cdot|-)}(D)\} > 0$.*

Then there is some s which picks for $\mathrm{EAd}_{\mathbb{B}(\cdot|-)}$ that is rejected by $\mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$.

Similarly, for Maximality, we can define:

Definition 9.

$$\mathrm{Max}_{\mathbb{B}(\cdot|-)}(D) = \{a \in D \mid (\forall a' \in D)(\exists p \in \mathbb{P})(\mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a)] \geqslant \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a')])\}$$

[12] The example using the Ellsberg case does not have this property, which requires many decision problems to be possible. If, for example, we had selected $\mu^*(D_1) = 0.1$, then one can check that no strategies are ruled out by Maximality.

Since $\mathsf{s} \in \mathrm{EAd}_{\mathbb{B}}(\mathcal{S})$ implies $\mathsf{s} \in \mathrm{Max}_{\mathbb{B}}(\mathcal{S})$, we obtain, as a consequence of Proposition 8:

Proposition 10. *If* s *b-surely picks for* $\mathrm{EU}_{b(\cdot|-)}$ *for some* $b \in \mathbb{B}$*, then* $\mathsf{s} \in \mathrm{Max}_{\mathbb{B}}(\mathcal{S})$.

And thus, there is always some s which picks for $\mathrm{Max}_{\mathbb{B}(\cdot|-)}$ which is itself not rejected according to $\mathrm{Max}_{\mathbb{B}}(\mathcal{S})$. However, again, we do not have an analogue of Proposition 9 unless we impose additional particular restrictions on $\mathbb{B}$ (Section 4.1.3).

4 The utility of using a decision theory

Up to this point, we have asked how a decision theory evaluates the picking strategies that pick for the choice function to which that decision theory gives rise. This is one way to answer the question of whether the decision theory undermines its own recommendations, and we've seen that Allais-permitting decision theories fare poorly, as does Γ-Maximin both of which rule out as impermissible the strategy which they require; while E-Admissibility and Maximality fare better, as some compatible strategies are evaluated as acceptable (although not all).

check that

However, other approaches are available too. We are interested in judging a decision theory as a means to your ends, and we have been using the proposed decision theory itself to do the judging, for it is, after all, a theory of which means to your ends are rational. Judging picking strategies that pick for the choice function that a decision theory produces furnish us with an straightforward approach to this question because they determine what the outcomes are: given a decision problem and a state of the world, the utility of a picking strategy is the utility, at that state of the world, of the act it picks from the decision problem. Since decisions theories don't always have definitive guidance on what to pick when faced with a decision, we considered various strategies compatible with its recommendations, which we called, the strategies which pick for it. How else might we evaluate what a decision theory will lead you to do when there are various acts it leaves open?

We propose that you might have a precise probability over those permissible acts—what we'll call a probabilistic picking strategy—and you might take the utility of this probabilistic picking strategy at a state of the world to be its expected utility at that world. There are two reasons you might think this is the right way to evaluate a decision theory:

Firstly, you might think that, once your decision theory gives you its choice set, you will pick by applying some randomisation method, such as tossing a coin or rolling a die. Perhaps you think that we are freely selecting amongst various randomisation methods as well as the choice functions to which your decision theory gives rise, or perhaps you think that when selecting a choice function, it simply comes with a specified randomisation method.

Secondly, you might think that, once your decision theory gives its choice set, you don't know what happens next except that, in the end you do in fact pick a particular act from that set. We then want to represent your uncertainty about how you'll end up picking when you've adopted a particular decision theory whose choice set is not a singleton. And it might just be that your uncertainty over how you'll pick is best represented by a precise probability. (In Section 4.2.1, we will extend this to the case where your uncertainty over how you'll pick is imprecise.)

Before discussing some alternatives, we will now show that under either of these ways of thinking about judging the outcomes of adopting a decision theory, all our previous claims carry over, and in fact in some cases even get worse since the strategies that align with expected utility theory arise from extremal picking strategies which we might want to rule out under this way of thinking.

4.1 Probabilistic picking strategies

We begin by extending our definitions:

Definition 10.

- *A* probabilistic picking strategy ν, *specifies, for each decision problem,* $D \in \mathcal{D}$, *a probability function* ν_D *over* D, *i.e., over the acts available in the decision problem* D.
- *For a choice function* c, ν picks for c *iff* $\nu_D(\mathsf{c}(D)) = 1$ *for each* $D \in \mathcal{D}$, *i.e., it is certain that what it picks will be compatible with C's recommendations.*
- *For a choice function* c, ν μ-surely picks for c, *if* $\mu\{D \mid \nu_D(\mathsf{c}(D)) = 1\} = 1$.

And we add a new one:

Definition 11. *For a choice function* c, ν is regular for c, *if, for each* D *in* $\mathcal{D}$, $\nu_D(a) > 0$ *iff* $a \in \mathsf{c}(D)$.

Observe that in the special case where ν is extremal—that is, when for every D it assigns all its probabilistic weight to an individual member of D—then we recover our original notion of a picking strategy.

For a non-probabilistic picking strategy, s, we simply took its utility to be the utility of the act it requires you to pick: given a state of the world ω and a decision problem D, $\mathfrak{U}(\mathsf{s})(\omega, D) := \mathfrak{U}(\mathsf{s}(D))(\omega)$, the utility of the act $\mathsf{s}(D)$ at ω. For ν, we take its utility to be the *expected* utility of the act it lead you to pick: given a state of the world ω and a decision problem D,

$$\mathfrak{U}(\nu)(\omega, D) := \sum_{a \in D} \nu_D(a)\mathfrak{U}(a)(\omega).$$

We have thus far been assuming that decision problems D are non-empty finite sets of acts. If we were to allow D to be infinite (although compact), then we should have $\mathfrak{U}(\nu)(\omega, D) := \int_D \mathfrak{U}(a)(\omega)\nu_D(\mathrm{d}a)$.

In the next few sections, we note how our earlier results concerning non-probabilistic picking strategies generalize to probabilistic picking strategies.

4.1.1 Expected Utility Theory

We will judge whether the decision theory evaluates a probabilistic picking strategy ν to be impermissible. This will depend on the range of alternatives available. That is, we will be judging whether ν is an impermissible picking strategy from a set of picking strategies, $\mathbf{N}$. There are various natural proposals for what is considered in $\mathbf{N}$ depending on one's interpretation and applications of our results. When $\mathbf{N}$ consists just of extremal picking strategies, it is equivalent to our set of (deterministic) picking strategies, $\mathcal{S}$, considered in the first half of the paper. If you think you get to pick by randomisation, and can select any randomisation process, then $\mathbf{N}$ will be the collection of all probabilistic picking strategies. If you think we are just evaluating some choice functions, and each one just comes along with a single randomisation process, then $\mathbf{N}$ will have a particular ν^{c} for each c, where ν picks for c. If you instead are just uncertain over how you'll pick when using a choice function c, and assume that this is a matter governed by a precise probability, then again we'll have a ν^{c} representing your probabilistic uncertainty over how you'll pick once you've selected a choice function c and are faced with a decision D.

Our results will all hold under various choices of $\mathbf{N}$, so long as it has a particular feature:

Definition 12. *A set of probabilistic picking strategies,* $\mathbf{N}$*, is* EU-complete *if, for every probability p over Ω, there is some ν in $\mathbf{N}$ such that ν picks for* EU_p*.*

In fact, for all the results, one only needs that $\mathbf{N}$ is sufficiently EU-complete in that it contains a strategy which is μ-surely an EU_p strategy for relevant p and μ. We don't think that such weakenings are significantly interesting, so we simply impose EU-completeness for ease, noticing that it applies both when $\mathbf{N}$ is the collection of all deterministic strategies and when $\mathbf{N}$ is the collection of all probabilistic picking strategies.

Propositions 1 and 2 extend to this setting. Suppose p is a probability over Ω and μ is a probability measure over $\mathcal{D}$.

Proposition 11. *If $\mathbf{N}$ is* EU*-complete, we have:*

$$\nu \in \mathrm{EU}_{p\times\mu}(\mathbf{N}) \Leftrightarrow \nu \ \mu\text{-surely picks for } \mathrm{EU}_p.$$

and, more generally,

$$\nu \in \mathrm{EU}_b(\mathbf{N}) \Leftrightarrow \nu \ b\text{-surely picks for } \mathrm{EU}_{b(\cdot|-)}.$$

4.1.2 E-Admissibility

Propositions 4 to 6, 8 and 9 also generalise to the probabilistic picking strategy setting.

Since $\mathrm{EAd}_{\mathbb{B}} = \bigcup_{b\in\mathbb{B}} \mathrm{EU}_b$, we get as an immediate consequence of Proposition 11:

Proposition 12. *If ν μ-surely picks for EU_p for some $p \times \mu \in \mathbb{B}$, then $\nu \in \mathrm{EAd}_{\mathbb{B}}(\mathbf{N})$. More generally, if ν b-surely picks for $\mathrm{EU}_{b(\cdot|-)}$ for some $b \in \mathbb{B}$ then $\nu \in \mathrm{EAd}_{\mathbb{B}}(\mathbf{N})$.*

Moreover, these are the only members of $\mathrm{EAd}_{\mathbb{B}}$*, at least assuming that $\mathbf{N}$ is* EU*-complete:*

$$\nu \in \mathrm{EAd}_{\mathbb{B}}(\mathbf{N}) \Leftrightarrow \nu \ b\text{-surely picks for } \mathrm{EU}_{b(\cdot|-)}, \text{ for some } b \text{ in } \mathbb{B}.$$

Proposition 13. *Suppose that $\mathbf{N}$ is* EU*-complete.*

Suppose that for every $b \in \mathbb{B}$, $b_{\mathcal{D}}\{D \mid \mathrm{EAd}_{\mathbb{B}(\cdot|-)}(D) \subseteq \mathrm{EU}_{b(\cdot|-)}\} < 1$. That is, for all $b \in \mathbb{B}$, $b_{\mathcal{D}}\{D \mid$ there is $b' \in \mathbb{B}$ with $\mathrm{EU}_{b'(\cdot|-)}(D) \not\subseteq \mathrm{EU}_{b(\cdot|-)}(D)\} > 0$.

Then, if ν is a regular picking strategy for $\mathrm{EAd}_{\mathbb{B}(\cdot|-)}$, then $\nu \notin \mathrm{EAd}_{\mathbb{B}}(\mathbf{N})$.

Let's see this at work. Consider again the Ellsberg setup, Section 3.1. Recall that $\mathrm{EAd}_{\mathbb{P}}(D_1) = \{1A,1B\}$ and $\mathrm{EAd}_{\mathbb{P}}(D_2) = \{2A,2B\}$. Every probability in $\mathbb{P}$ rules out at least one of the E-Admissible options as impermissible. For example, any $p(Black) > 7/30$ rules 1A as excluded, i.e., not in EU_p, but there's some positive chance that ν_{D_1} picks 1A, by the assumption that it is regular for $\mathrm{EAd}_{\mathbb{P}}$. Thus, ν does not pick for EU_p. It also does not even μ-surely pick for EU_p if we assume that each μ assigns positive probability to both D_1 and D_2. It is thus not E-Admissible.

In fact, if $\mathbb{P}$ is allowed to be non-convex, we get cases where ν will be judged as impermissible even when you know what decision problem you'll be faced with.[13] Suppose you're certain you'll face a decision problem $D = \{a_1, a_2\}$. So $\mu(D) = 1$. Suppose $\mathbb{P} = \{p_1, p_2\}$, where p_1 expects a_1 to be strictly better than a_2, while p_2 expects a_2 to be strictly better than a_1. So E-Admissibility with $\mathbb{P}$ says that neither a_1 or a_2 are rejected. Then for any regular picking strategy for $\mathrm{EAd}_{\mathbb{P}}$, ν_D will give positive probability to both a_1 and a_2. p_1 doesn't expect it to be best, and nor does p_2. So, by E-Admissibility, ν is not rationally permissible.

double check wording of above. And make it match JKs eg

One motivation for introducing probabilistic picking strategies, and in particular regular probabilistic picking strategies was to judge one's decision theory as a means to her ends, and trying to give a particular judgement of how good it would be to adopt a given decision theory is, rather than simply leaving open a whole range of strategies open. If this is how we are trying to judge E-Admissibility, then E-Admissibility is self-undermining; although the defender of E-Admissibility will argue against this way of understanding the value of E-Admissibility, instead highlighting the importance of coordination encoded in E-Admissibility.

yes, this was JKs stuff. I am still spiritially trying to say what I had said.

The point is also a more general one. If one tries to give any way of scoring, or measuring the utility of choice functions or decision rules at each world, and at each decision problem, then avoid being ruled out as impermissible by the lights of E-Admissibility, it must be evaluated as equivalent to expected utility theory.

this new

4.1.3 Maximality

As in Section 3.2, since Maximality is more permissive than E-Admissibility, all EU_p strategies are evaluated as acceptable according to Maximality. We thus have, as a corollary to Proposition 11, and extending Proposition 7:

Proposition 14. *If ν μ-surely picks for EU_p for some $p \times \mu \in \mathbb{B}$, then $\nu \in \mathrm{Max}_{\mathbb{B}}(\mathbf{N})$.*

[13] These examples are avoided when $\mathbb{P}$ is convex as then there will be a probability which is indifferent between the two actions, and thus, ν is Bayes for this probability.

More generally, if ν μ-surely picks for $\mathrm{EU}_{b(\cdot|-)}$ *for some $b \in \mathbb{B}$, then* $\nu \in \mathrm{Max}_{\mathbb{B}}(\mathbf{N})$.

For E-Admissibility, we were able to show that it was only these strategies which were judged by the decision theory as acceptable. This result does not immediately apply to Maximality in a similar way because for a strategy to be deemed impermissible, the various probabilities $b \in \mathbb{B}$ have to agree on a particular alternative as better.

However, if we impose an additional restriction on $\mathbb{B}$ we can obtain an analogous result: suppose your credence over which decision you'll be faced with is given by a single, *precise* probability, μ^*, and that $\mathbb{B}$ has the form $\{p \times \mu^* \mid p \in \mathbb{P}\}$. All our results equally apply when $\mathbb{B}$ is the convex hull of this, as taking a convex hull doesn't affect which acts are rejected by Maximality. But since $\mathbb{P}$ being convex implies that this $\{p \times \mu^* \mid p \in \mathbb{P}\}$ is convex (since μ^* is fixed), we do not bother with presenting this strengthening.

check!!!

We will also place a further condition on μ^*:

Definition 13. μ^* requires almost everywhere decisiveness *iff for all probabilities p,*

$$\mu^*\{D \mid \mathrm{EU}_p(D) \textit{ is a singleton}\} = 1.$$

That is, for each probability function p, μ^* is certain that you'll face a decision problem in which only one act maximizes expected utility. That is, the set of decision problems in which there are ties for expected utility has measure 0. Just considering a single proposition, this will hold, for example, if you might be faced with various decision problems indexed by $t \in [0,1]$: are you willing to pay \$$t$ for a bet paying out \$1 if p and \$0 if $\neg p$, where μ^* is a measure assigning strictly positive weight to every non-degenerate interval $[x, y]$.

Then we have the following result.[14]

Proposition 15. *Suppose* $\mathbf{N}$ *is* EU*-complete. Suppose that $\mathbb{B}$ has the form $\{p \times \mu^* \mid p \in \mathbb{P}\}$ and μ^* requires almost everywhere decisiveness and is countably additive. Then if $\nu \in \mathrm{Max}_{\mathbb{B}}(\mathbf{N})$, then ν μ^*-surely picks for EU_p, for some probability p.*

I removed the converse because I don't know if the probability has to be in the original set, not sure if that follows from the Wald..???

This follows from a version of Wald's Complete Class Theorem. We will be able to show that if is not in $\mathrm{EU}_{p\times\mu^*}$ for any p, then there is some alternative ν', in fact, an ν' which picks for some EU_p, where $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu')(\omega) > \mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu)(\omega)$ for all ω, and thus for all p, $\mathsf{Exp}_p[\mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu')] > \mathsf{Exp}_p[\mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu)]$, i.e., $\mathsf{Exp}_{p\times\mu^*}\mathfrak{U}(\nu') > \mathsf{Exp}_{p\times\mu^*}\mathfrak{U}(\nu)$, so $\nu \notin \mathrm{Max}_{\mathbb{B}}(\mathbf{N})$.

The deterministic picking strategies discussed in the earlier part of the paper, $\mathcal{S}$, form a class which is EU-complete, since we assumed that all strategies were

[14]The assumptions on μ^* are not required if one instead assumes that $\mathbf{N}$ is convex, which is motivated when one considers randomisations as available options. However, one should usually only apply these decision theories to compact sets, otherwise all options can be rejected, so we should ensure that $\mathbf{N}$ is also closed, for which one needs to allow merely finitely additive randomisations (**?**).

in it; and thus we obtain the result that we hinted at in Section 3.2 that if we have such a $\mathbb{B}$, then it is only EU_p strategies which are in $\mathrm{Max}_{\mathbb{B}}(\mathcal{S})$.

Just as we got a more challenging result for E-Admissibility when we restricted attention to *regular* picking strategies, as these won't look like EU_p strategies, similarly we get a more challenging result for Maximality when we restrict to regular picking strategies because such regular picking strategies will not μ^* surely pick for any EU_p, unless Maximality just collapses to being EU_p for some p, or at least μ^*-surely does so.

Proposition 16. *Suppose* $\mathbf{N}$ *is* EU*-complete. Suppose that* μ^* *requires almost everywhere decisiveness and is countably additive.*

Suppose that for every $p \in \mathbb{P}$, $\mu^*\{D \mid \mathrm{Max}_{\mathbb{P}}(D) \subseteq \mathrm{EU}_p(D)\} < 1$. *That is, for all* $p \in \mathbb{P}$, $\mu^*\{D \mid \exists a \in D \left[(\exists b{\in}D\, \mathsf{Exp}_p(a) < \mathsf{Exp}_p(b)) \text{ and } \forall b{\in}D\, \exists p'{\in}\mathbb{P}\, \mathsf{Exp}_{p'}(b) \leqslant \mathsf{Exp}_{p'}(a)\right]\}$

Then, if ν *is regular picking strategy for* Max *then* $\nu \notin \mathrm{Max}_{\mathbb{B}}(\mathbf{N})$, *where* $\mathbb{B}$ *has the form* $\{p \times \mu^* \mid p \in \mathbb{P}\}$.

The same sorts of considerations that potentially diffuse this challenge for E-Admissibility also diffuse it for Maximality.

This needs double checking

Since Γ-Maximin is a more restrictive theory than Maximality, we thus also have that the only strategies which are Γ-Maximin are those which μ^* surely pick for EU_p for some $p \in \mathbb{P}$.

Proposition 17. *Suppose* $\mathbf{N}$ *is* EU*-complete. Suppose that* $\mathbb{B}$ *has the form* $\{p \times \mu^* \mid p \in \mathbb{P}\}$ *and* μ^* *requires almost everywhere decisiveness and is countably additive. Then if* $\nu \in \Gamma_{\mathbb{B}}(\mathbf{N})$, ν μ^**-surely picks for* EU_p*, for some probability* p.

If also for every $p \in \mathbb{P}$, $\mu^*\{D \mid \Gamma_{\mathbb{P}}(D) \subseteq \mathrm{EU}_p(D)\} < 1$, *then there is some* ν *which picks for* $\Gamma_{\mathbb{P}}$ *and which is not in* $\Gamma_{\mathbb{B}}(\mathbf{N})$ *(in fact, any* ν *which picks for* EU_p *with* $p \in \mathbb{P}$ *will do). And if* ν *is regular for* $\Gamma_{\mathbb{P}}$ *then* $\nu \notin \Gamma_{\mathbb{B}}(\mathbf{N})$.

The idea of N being closed and convex has now totally been deleted.

Uncertainty about decisions Like in Section 1.2.3, these results hold when we assume something about your uncertainty concerning the decision you'll face: we assume you have precise probabilities over the possible decision problems, and those probabilities are broad enough to ensure that they require almost everywhere decisiveness. It seems troubling enough that, should you acquire sufficient evidence to become uncertain about the decision problems you'll face in a way that is represented by precise probabilities, you would have to abandon the decision theory or the picking strategy you're using.

4.2 Alternatives

4.2.1 Imprecise picking strategies

So far, we've used decision theories to judge deterministic picking strategies, s, and probabilistic picking strategies, ν. But perhaps the proponent of imprecise probabilities thinks the way you pick is better represented by imprecise probabilities, indeed, a set of probabilistic picking strategies. Perhaps, it is the set of all strategies that pick for the choice function, c.

These are imprecise acts, and it is imprecise what utility they lead to. How might we apply our theories of imprecise probability to judge such acts? We understand an imprecise act as a set of acts. So the imprecise acts available in decision problem D is any $\mathbb{A} \subseteq D$. For precise acts, E-Admissibility rejects an act when, for every p in $\mathbb{P}$, there is some alternative act a' that p expects to do better. When extending E-Admissibility to judge imprecise acts, we have to ask what it means for p to expect an imprecise act $\mathbb{A}'$ to do better than $\mathbb{A}$.

A first suggestion is to say that p expects $\mathbb{A}'$ to do better than $\mathbb{A}$ when, for every $a \in \mathbb{A}$ and $a' \in \mathbb{A}'$, $\mathsf{Exp}_p[\mathfrak{U}(a')] > \mathsf{Exp}_p[\mathfrak{U}(a)]$. This is a very hard condition to meet, so very few imprecise acts will be ruled out as impermissible on this basis. This can already rule as impermissible any imprecise act each of whose members is a regular picking strategy for $\mathrm{EAd}_{\mathbb{P}}$, but it does not deem impermissible the imprecise act consisting of all picking strategies, or all deterministic picking strategies; a natural application of the idea of imprecise picking strategies.

There is a slight weakening: p expects $\mathbb{A}'$ to do better than $\mathbb{A}$ when, for every $a \in \mathbb{A}$ and $a' \in \mathbb{A}'$, $\mathsf{Exp}_p[\mathfrak{U}(a')] \geqslant \mathsf{Exp}_p[\mathfrak{U}(a)]$, and there is some $a \in \mathbb{A}$ such that for all $a' \in \mathbb{A}'$, $\mathsf{Exp}_p[\mathfrak{U}(a')] > \mathsf{Exp}_p[\mathfrak{U}(a)]$. This condition generates a version of E-Admissibility for imprecise acts which rules as impermissible the set of all picking strategies for $\mathrm{EAd}_{\mathbb{P}}$.

We get a result paralleling that for precise picking strategies: to avoid being ruled out as impermissible, one's strategy needs to μ-surely be an EU_p strategy for some $p \times \mu \in \mathbb{B}$. (Or equivalently for the conditional version.) If $\mathbb{N}$ is a set of picking strategies, and there is no $p \times \mu$ in $\mathbb{B}$ such that all strategies in $\mathbb{N}$ μ-surely pick for EU_p; then each probability $b = p \times \mu$ evaluates the precise picking strategy $\{\nu^p\}$, where ν^p picks for EU_p, be better, in this sense: every $\nu \in \mathbb{N}$, $\mathsf{Exp}_p[\mathfrak{U}(\nu)] \leqslant \mathsf{Exp}_p[\mathfrak{U}(\nu^p)]$, and there is some $\nu \in \mathbb{N}$ with $\mathsf{Exp}_p[\mathfrak{U}(\nu)] < \mathsf{Exp}_p[\mathfrak{U}(\nu^p)]$.

check that again

We should not expect to obtain similar results for Maximality. When we assume that one's probability over which decision she'll be faced with is precise, we showed that for every ν which is not μ^*-surely an EU_p strategy we get some ν' which every probability p agrees is preferable. But we can't guarantee that given a set of νs, $\mathbb{N}$, we get any agreement on which ν' are preferable.

So, if picking strategies may be imprecise acts, represented by sets of precise probabilistic picking strategies, then Maximality sometimes evades the charge of being self-undermining. However, it is hard to see how to implement an imprecise picking strategy. Perhaps one could choose by tossing a coin about whose bias you have imprecise probabilities, or perhaps one should just have imprecise probabilistic uncertainty about how one will pick. But to maintain this response, one had better not gain additional information sufficient to make one precise.

4.2.2 Utility of a choice function not given by a picking strategy

Perhaps one should specify the utility of a choice function in an alternative way, specifying $\mathfrak{U}(\mathsf{c}, D, \omega)$ for the various D and ω. Even if we impose that $\mathfrak{U}(\mathsf{c}, D, \omega)$ should be a mixture of $\mathfrak{U}(a, \omega)$ for $a \in \mathsf{c}(D)$ our results do not follow. For example, if we specify that $\mathfrak{U}(\mathsf{c}, D, \omega) = \sup\{\mathfrak{U}(a, \omega) \mid a \in \mathsf{c}(D)\}$, then our results clearly do not hold, in fact, then one should be maximally imprecise in every decision problem, setting $\mathsf{c}(D) = D$. This, however, needs more justification. Why should it be evaluated this way? Perhaps an approach which says that when there's an imprecise decision set one should do something else to make the decision, for example to consult an expert (**?**); although we wonder then why the option to consult an expert or gather more evidence wasn't available in the setting up of the decision problem.

5 Conclusion

We proposed to use a decision theory to judge itself, or to judge strategies compatible with its recommendations, and found significant challenges for a host of theories that diverge from expected utility theory. Our formal analysis often parallels that of existing known challenges for these theories such as diachronic inconsistencies exhibited by Dutch book arguments or paradoxes of sequential choice, but our interpretation of the results is different, understanding it as the decision theory undermining its own recommendations.

In Section 1 we showed that risk-weighted expected utility theory, and other theories that accommodate the Allais preferences are self-undermining in a particular way: for any such theory, there are particular ways of being uncertain about which decisions you'll face, where there is a single picking strategy that chooses in line with the recommendations the decision theory would make were you to face each possible decision you might face, and that strategy is not itself acceptable according to the decision theory. These decision theories undermine their own recommendations, recommending instead choosing in a way that the theory itself rejects. We generated these examples on the basis of the Allais preferences—and indeed any failure of the Independence Axiom would do. We then noted that we see the same phenomenon if we know we'll face a binary decision defined over two possible states of the world, and we place a uniform distribution over these different possible decisions; and similarly for a number of beta distributions we might place over them. And so the extent of the self-undermining is reasonably broad, but we don't have a precise general result that shows how broad it is.

are we using independence still?

In Section 2 we showed that traditional Savage-style expected utility theory does not have the same flaw: that it always recommends its own picking strategies.

We then turned to decision theories that accommodate ambiguity and imprecision. In Section 3.1, we say that Γ-maximin is self-undermining in the way the Allais-permitting theories were, and we generated the witness to this using the Ellsberg preferences. That is, they can rule out their (only) picking strategy as impermissible.

are we going to say something about

In Section 3.2 we observed that E-Admissibility and Maximility aren't vulnerable to the same challenge as they judge some of their picking strategies as permissible. However, in the case of E-Admissibility, we noted that it was only the EU_p strategies that they judge as acceptable, thus requiring their picking strategy to coordinate across decisions.

In Section 4 we reevaluated what we are judging and instead proposed evaluating not the deterministic picking strategies compatible with the recommendations of the decision theory but instead using probabilistic picking strategies. This might be because you are using a randomisation process, or just that you have uncertainty about how you'll pick which is governed by a precise probability, or simply that how we judge the utility of a choice rule should be given by a mixture of the utilities of the actions that are not ruled out. We noted that all our previous results generalised to this setting and moreover, that the situation became worse for the imprecise decision theories. E-Admissibility now judges as impermissible any regular ν which picks for it, as it no longer looks like EU_p for any p, at least given some mild assumptions on the credal set. For Maximality, we were able to show something similar, that it deems impermissible any regular ν that picks for it, although in this case we needed a much stronger assumption: that one's uncertainty over which decision she'll be faced with is governed by a precise probability, along with a broadness assumption that this requires almost everywhere decisiveness and is countably additive. We saw that these considerations while *prima facie* challenging, are perhaps less worrying or surprising than one might expect.

We also considered imprecise picking strategies. This is still susceptible to the analogue result in the E-Admissibility case: even if one's picking strategy is imprecise, to be E-Admissible it must look like an EU_p strategy, thus requiring coordination. We don't have results for Maximality for the imprecise picking strategies, so this is a line of defence for these theories. Another line of defence that any theory could take is to argue that the utility of an imprecise choice set should not amount to being a mixture, a defence which we acknowledge is open. It is, for example, very easy if one sets $\mathfrak{U}(\mathsf{c}, D, \omega) = \sup\{\mathfrak{U}(a)(\omega) \mid a \in \mathsf{c}(D)\}$ then of course one should have maximally imprecise choice function. But motivating and justifying any such analysis remains an important task.

To summarise: we have found challenges for any of the decision theories we've considered that depart from expected utility theory. When we ask a whole range of decision theories how they think one should pick, they pretty systematically recommend picking in accordance with expected utility theory. For some of the theories we considered (REU, Γ-maximin), this undermines their own recommendations whenever they don't collapse into the recommendations of expected utility theory. For others (E-Admissibility, Maximality), it rather reflects a certain value, *viz.*, the value of coordinating how you resolve incomparability across decision problems.

Recheck the conclusion??

A EU and E-admissibility

Suppose b is probability measure over $\Omega \times \mathcal{D}$ which is absolutely continuous with respect to $u \times \lambda$, where u is the uniform distribution on Ω and λ is the restriction of the Lebesgue measure to $\mathcal{D}$. Let $f : \Omega \times \mathcal{D} \to \mathbb{R}_{\geqslant 0}$ be a Radon-Nikodym derivative (or density) of b, so that for any measurable $A \subseteq \Omega \times \mathcal{D}$

$$b(A) = \sum_{\omega \in \Omega} \int_{A_\omega} f(\omega, D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D)$$

where, for any $\omega \in \Omega$, $A_\omega := \{\langle \omega^*, D^* \rangle \in A \mid \omega = \omega^*\}$.

The marginal density, $f_{\mathcal{D}}$, is given by $f_{\mathcal{D}}(D) = \sum_{\omega \in \Omega} f(\omega, D)$.

Let $\mathcal{D}' := \{D \in \mathcal{D} \mid f_{\mathcal{D}}(D) > 0\}$.

Definition 14. *For any $D \in \mathcal{D}'$ and any $a \in D$,*

$$\mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a)] := \sum_{\omega \in \Omega} \frac{f(\omega, D)}{f_{\mathcal{D}}(D)} \mathfrak{U}(a)(\omega)$$

Definition 15. *For any $D \in \mathcal{D}'$,*

$$\mathrm{EU}_{b(\cdot|D)}(D) := \left\{ a \in D \mid \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a)] \geqslant \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a')] \textit{ for all } a' \in D \right\}$$

Definition 16. $\mathrm{EU}_{b(\cdot|-)} : \mathcal{D}' \to \mathcal{D}$ *is the choice function defined by*

$$\mathrm{EU}_{b(\cdot|-)}(D) := \mathrm{EU}_{b(\cdot|D)}(D).$$

Definition 17. *A picking strategy* s b-surely picks for $\mathrm{EU}_{b(\cdot|-)}$ *iff*

$$b\{\langle \omega, D \rangle \mid \omega \in \Omega, \mathsf{s}(D) \in EU_{b(\cdot|-)}(D)\} = 1$$

Theorem 18.

(i) *If* s *b-surely picks for* $\mathrm{EU}_{b(\cdot|-)}$ *then, for any* s', $\mathsf{Exp}_b[\mathfrak{U}(\mathsf{s})] \geqslant \mathsf{Exp}_b[\mathfrak{U}(\mathsf{s}')]$

(ii) *If* s *b-surely picks for* $\mathrm{EU}_{b(\cdot|-)}$ *and* s' *does not, then* $\mathsf{Exp}_b[\mathfrak{U}(\mathsf{s})] > \mathsf{Exp}_b[\mathfrak{U}(\mathsf{s}')]$.

Proof. For any $\omega \in \Omega$ and $D \in \mathcal{D}'$, let $t_b(\omega, D) := \sup \left\{ \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a)] \mid a \in D \right\}$. For any $\omega \in \Omega$ and $D \in \mathcal{D} \setminus \mathcal{D}'$, let $t_b(\omega, D) := 0$. Note that $t_b(\omega, D)$ only depends on D. Nevertheless, we define it as a function on $\Omega \times \mathcal{D}$ in order to take expectations relative to b.

So $\mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(a)] \leqslant t_b(\omega, D)$ for all $a \in D$, with equality iff $a \in \mathrm{EU}_{b(\cdot|-)}(D)$.

$$\begin{aligned}
\mathsf{Exp}_b[\mathfrak{U}(\mathsf{s})] &= \sum_{\omega \in \Omega} \int_{\mathcal{D}} \mathfrak{U}(\mathsf{s}(D))(\omega) f(\omega, D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D) \\
&= \sum_{\omega \in \Omega} \int_{\mathcal{D}'} \mathfrak{U}(\mathsf{s}(D))(\omega) f(\omega, D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{D}'} \left[\sum_{\omega \in \Omega} \frac{f(\omega, D)}{f_{\mathcal{D}}(D)} \mathfrak{U}(\mathsf{s}(D))(\omega) \right] f_{\mathcal{D}}(D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D) \\
&= \int_{\mathcal{D}'} \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(\mathsf{s}(D))] f_{\mathcal{D}}(D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D) \\
&= \sum_{\omega \in \Omega} \int_{\mathcal{D}'} \mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(\mathsf{s}(D))] f(\omega, D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D) \\
&\leqslant \sum_{\omega \in \Omega} \int_{\mathcal{D}'} t_b(\omega, D) f(\omega, D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D) \\
&= \sum_{\omega \in \Omega} \int_{\mathcal{D}} t_b(\omega, D) f(\omega, D) \frac{1}{|\Omega|} \, \mathrm{d}\lambda(D) \\
&= \mathsf{Exp}_b[t_b]
\end{aligned}
$$

Thus, $\mathsf{Exp}_b[\mathfrak{U}(\mathsf{s})] \leqslant \mathsf{Exp}_b[t_b]$ with equality iff $\mathsf{Exp}_{b(\cdot|D)}[\mathfrak{U}(\mathsf{s}(D))] = t_b(\omega, D)$ almost everywhere, which is true iff $b\{\langle \omega, D \rangle \in \Omega \times \mathcal{D} \mid \mathsf{s}(D) \in \mathrm{EU}_{b(\cdot|-)}(D)\} = 1$, i.e., iff s b-surely picks for $\mathrm{EU}_{b(\cdot|-)}$. Both (i) and (ii) follow immediately from this.

□

All the theorems about Expected Utility Theory and E-Admissibility, as well as the existence of Maximality strategies can be seen as quick corollaries of this: Propositions 1 to 14

B Maximality

To prove Propositions 15 and 16, we turn to two versions of Abraham Wald's (**?**) Complete Class Theorem. We'll state them in a general setting and then explain how they apply to our case. To state them, we need some definitions.

B.1 Wald theorem

- Ω is a finite set of states.
- Probability p is a normalised non-negative function. For X a random variable, $X \in \Re^\Omega$, i.e., $X : \Omega \to \Re$, will also write $p(X) = \sum_{\omega \in \Omega} p(\omega) X(\omega)$ applying the probability function to random variables.
- $\mathcal{O}$ is a set of "options". In our application they will be various picking functions, ν.
- $\mathcal{U} : \mathcal{O} \times \Omega \to [l, u]$ is a bounded "utility function". In our application, it will be $\mathcal{U}(\nu, \omega) = \mathsf{Exp}_{\mu^*} \mathfrak{U}(\nu, \omega) = \mathsf{Exp}_{\mu^*} \mathsf{Exp}_{\nu_D} \mathfrak{U}(a)(\omega)$
    - We have assumed it is bounded; however all that is actually needed for the proof is that it is bounded above, i.e., we could allow $\mathcal{U} : \mathcal{O} \times \Omega \to (-\infty, 0]$.
    - The utility profile of an option is $\mathcal{U}(o) \in \Re^\Omega$ as expected, i.e., $\mathcal{U}(o)(\omega) = \mathcal{U}(o, \omega)$, however $\mathcal{U}$ is specified.

Definition 18. *o is* strictly dominated *iff there is $o' \in \mathcal{O}$ with $\mathcal{U}(o', \omega) > \mathcal{U}(o, \omega)$ for all ω. o is called* admissible *if it is not strictly dominated.*

o is weakly dominated *iff there is $o' \in \mathcal{O}$ with $\mathcal{U}(o', \omega) \geqslant \mathcal{U}(o, \omega)$ for all ω and $\mathcal{U}(o', \omega) > \mathcal{U}(o, \omega)$ for some ω*

Lemma 19. *If o is Bayes then it is not strictly dominated.*

Proof. Suppose it is strictly dominated. Then there is o' with $\mathcal{U}(o, \omega) < \mathcal{U}(o', \omega)$ for all ω. So every probability p will have $p(\mathcal{U}(o)) < p(\mathcal{U}(o'))$. Thus it is not Bayes. □

We will start thinking directly about vectors in $\Re^\Omega$.

$$\mathcal{U}(\mathcal{O}) := \{\mathcal{U}(o) \mid o \in \mathcal{O}\} \subseteq \Re^\Omega$$
$$\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})) := \mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$$

We will also work with $\mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$, which we define as the closure in the product topology. This can also be characterised by limits of sequences, or more generally of nets: if a sequence (or net) of members of $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$ is such that for each coordinate, ω, $X_\alpha(\omega) \longrightarrow X^*(\omega)$, then $X^* \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$.

The notion of being dominated by something in a set is defined for vectors as is obvious.

Our first lemma says that if a vector is not Bayes in $\mathcal{U}(\mathcal{O})$ then it is strictly dominated in the convex hull.

Lemma 20. *If $X \in \Re^\Omega$ is such that there is no probability p with $p(X) \geqslant p(\mathcal{U}(o))$ for all $o \in \mathcal{O}$, then there is $Y \in \mathsf{ConvHull}(\mathcal{U}(O))$ which strictly dominates X, i.e., $Y(\omega) > X(\omega)$ for all ω.*

Proof. Suppose X is not strictly dominated in $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$. Let

$$D := \{Y \in \Re^\Omega \mid Y(\omega) > X(\omega) \text{ for all } \omega\},$$

i.e., the strict dominators of X; so we know that D and $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$ are disjoint by assumption. They are also both convex, so they can be separated by a continuous linear functional, by the separating hyperplane theorem (**?**, Theorem 5.61). That is, there is a non-zero linear functional, $f : \Re^\Omega \to \Re$ with $f \neq 0$ and constant c s.t., $f(Y) \geqslant c \geqslant f(Z)$ for any $Y \in D$ and $Z \in \mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$.[15]

We need to show that f is non-negative. Let $Z \geqslant 0$ and suppose $f(Z) < 0$. Then take any $Y \in D$ and consider $Y + kZ$, observing that it is still in D. And $f(Y + kZ) = f(Y) + kf(Z)$ can be arbitrarily small, and in particular less than c by making k large enough. Contradicting.

We also know that $f(\omega) \neq 0$ for some ω, otherwise it would be 0 everywhere and the theorem gives us a non-zero linear functional.

[15] In fact, to apply their result, we need that at least one of them has an interior point, i.e., a point such that any sequence converging to it must have some tail inside that set. That's fine because D is open, so long as D is non-empty, which is clear.

We can thus normalise f to obtain our probability p p with $p(Y) \geqslant c \geqslant p(Z)$ for any $Y \in D$ and $Z \in \mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$.

Since $X \in \mathrm{cl}(D)$, also $p(X) \geqslant c$. Thus, $p(X) \geqslant p(Z)$ for all $Z \in \mathcal{U}(\mathcal{O})$ (as $\mathcal{U}(\mathcal{O}) \subseteq \mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$) as required. □

This now gives us:

Corollary 21. *If $\mathcal{U}(\mathcal{O})$ is convex, then o is Bayes iff o is not strictly dominated.*

Note that this requires $\mathcal{U}(\mathcal{O})$ to be convex, not $\mathcal{O}$. This doesn't however, show that if it is not Bayes it is strictly dominated by something with is itself not dominated, and thus is Bayes. We will get that from the next result.

Lemma 22. *Suppose X is strictly dominated in* $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$. *Then we can find $Z \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$ which strictly dominates X and is itself not even weakly dominated in* $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$.

In the Wald setting, the $Z \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$ which are not weakly dominated is called the lower boundary of the set [in our case, it would be "upper boundary" because we're working with positive utility rather than risk or disutility]. This then says that being dominated entails that one is dominated by something in the lower boundary. It essentially depends on the fact that $\mathcal{U}(\mathcal{O})$ is bounded from above.

Proof. Take any $Y \in \mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$ which dominates X. Consider $A := \{Z \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))) \mid Z(\omega) \geqslant Y(\omega) \text{ for all } \omega\}$. Observe that this is closed (as it is the intersection of two closed sets) and bounded (as we assumed that utilities were bounded above), and thus compact. Let $f(Z) := \sum_\omega Z(\omega)$, observing that it is a continuous function. Thus, by the Extreme Value Theorem, it obtains its maximum somewhere in A. This maximum point will be as required. □

It can alternatively be proved with an application of Zorn's lemma, an argument which also works when Ω is infinite.[16]

This will not apply to our general case yet. For that, we need a new assumption on $\mathcal{U}$.

Definition 19.

- $\mathcal{O}, \mathcal{U}$ *is* Bayes-existing *iff for every probability p there is some $o \in \mathcal{O}$ which is Bayes for p.*
- $\mathcal{U}$ *is* Bayes-continuous *iff if o is Bayes for p and $p_1, p_2, \ldots$ are probabilities converging to p then there is some sequence of Bayes options for p_n whose utility profiles converge to those of o.*

For example, if $\mathcal{U}$ is a measure of accuracy which is continuous and strictly proper, then these will be satisfied. They are, however, *much* weaker than that.

[16]Define $\preccurlyeq$ a partial order on A as the natural coordinatewise order. For any chain, consider its pointwise supremum, which exists because utilities are bounded above, and checking that it is in the A since it is closed and this is a pointwise limit. Consequently, every chain has an upper bound, allowing the application of Zorn's lemma to guarantee the existence of a maximal element, which will be as required.

Lemma 23. *Suppose $\mathcal{U}$ is Bayes-continuous and Bayes-existing.*

Suppose $Z \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$ is not weakly dominated in $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$, then in fact $Z \in \mathcal{U}(\mathcal{O})$, moreover, it is in fact Bayes optimal in $\mathcal{U}(\mathcal{O})$.

In the Wald setting, that is, that the lower boundary of $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$ is a subset of $\mathcal{U}(\mathcal{O})$. This is then called $\mathcal{U}(\mathcal{O})$ being "closed from below" [in our case, above, as we're doing positive utility not loss]. This proof is where the assumptions on $\mathcal{U}$ are vital.

Proof. Since Z is not weakly dominated, it is also not strictly dominated. So by Lemma 20, there is some p^* with $p^*(Z) \geqslant p^*(\mathcal{U}(o))$ for all o.

By Bayes existing, there is also some o_{p^*} which is Bayes optimal for p^* in $\mathcal{U}(\mathcal{O})$. We will use Bayes continuity and existing to show that $\mathcal{U}(o_{p^*})(\omega) \geqslant Z(\omega)$ for every ω, i.e., either $\mathcal{U}(o_{p^*}) = Z$ or $\mathcal{U}(o_{p^*})$ weakly dominates Z, which would contradict our assumption that Z is not weakly dominated.

So, what we need to show is that for every $\omega, \mathcal{U}(o_{p^*})(\omega) \geqslant Z(\omega)$.

The argument we will make will work for every ω. So hold fixed a single ω, call it ω^*. Let π_{ω^*} be the projection function for the ω^* that we are considering, i.e., $\pi_{\omega^*}(Y) := Y(\omega^*)$. Then define p_n by:

$$p_n = (1 - 1/n)p^* + 1/n\pi_{\omega^*}$$

(this depends on the ω^* under consideration). Observe that p_n is probabilistic, so by Bayes existing, for each n, there is some o_{p_n} which is Bayes optimal for p_n, i.e., $p_n(\mathcal{U}(o_{p_n})) \geqslant p_n(\mathcal{U}(o))$ for all o. Since we have assumed $Z \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$, also $p_n(\mathcal{U}(o_{p_n})) \geqslant p_n(Z)$.[17]

should it be an official separate lemma?

We also know that $p^*(Z) \geqslant p^*(\mathcal{U}(o_{p_n}))$, so, since p_n is a mixture of p^* and π_{ω^*}, to get that $p_n(\mathcal{U}(o_{p_n})) \geqslant p_n(Z)$ we must in fact have that $\pi_{\omega^*}(\mathcal{U}(o_{p_n})) \geqslant \pi_{\omega^*}(Z)$. That is, we can conclude that $\mathcal{U}(o_{p_n})(\omega^*) \geqslant Z(\omega^*)$.

Observe that $p_n \longrightarrow p^*$. So, by Bayes continuity, $\mathcal{U}(o_{p_n}) \longrightarrow \mathcal{U}(o_{p^*})$, so $\mathcal{U}(o_{p_n})(\omega) \longrightarrow \mathcal{U}(o_{p^*})(\omega)$ for each ω. Thus, since $\mathcal{U}(o_{p_n})(\omega^*) \geqslant Z(\omega^*)$ for all n, also $\mathcal{U}(o_p)(\omega^*) \geqslant Z(\omega^*)$.

Since this worked for any ω, i.e., for any ω we could construct the relevant sequence and apply this argument, we have in fact obtained that $\mathcal{U}(o_{p^*})(\omega) \geqslant Z(\omega)$ for all ω, i.e., either $\mathcal{U}(o_{p^*}) = Z$ or $\mathcal{U}(o_{p^*})$ weakly dominates Z, which would contradict our assumption that Z is not weakly dominated. □

This proof in fact shows that the assumptions on $\mathcal{U}$ are very strong. It shows that for every probability there is a unique member of $\mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$ which is not weakly dominated; thus also that for every regular probability, there is a unique Bayes option.

Theorem 24. *Suppose $\mathcal{U}$ is Bayes existing and Bayes continuous. If o is not Bayes then there is o' which strictly dominates it; moreover, it is strictly dominated by an option which is itself Bayes and not even weakly dominated.*

[17] To show this, we first observe it for any $Z \in \mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$, just taking a mixture, and then taking limits can't break a $\geqslant$.

Proof. If o is not Bayes in $\mathcal{U}(\mathcal{O})$, then, by Lemma 20, $\mathcal{U}(o)$ is strictly dominated in $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$.

By Lemma 22 it is thus strictly dominated by some $Z \in \mathrm{cl}(\mathsf{ConvHull}(\mathcal{U}(\mathcal{O})))$ which is itself not weakly dominated in $\mathsf{ConvHull}(\mathcal{U}(\mathcal{O}))$. By Lemma 23, in fact $Z \in \mathcal{U}(\mathcal{O})$, and moreover, it is in fact a Bayes point; as required. □

Theorem 25 (Main theorem: a version of Wald's Complete Class Theorem)**.** *Suppose $\mathcal{U}(\mathcal{O})$ is closed and convex; or that $\mathcal{U}$ is Bayes-continuous and Bayes-existing. Then o is Bayes iff o is not strictly dominated.*

B.2 Applying Wald's Complete Class Theorem to probabilistic picking strategies for imprecise decision theories

Suppose

- $\mathcal{O}$ is a specified collection of probabilistic picking strategies, **N**.
- μ^* is a measure over $\mathcal{D}$.
- $\mathcal{U}(\nu)(\omega) := \mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu)(\omega) = \mathsf{Exp}_{\nu_D}\mathfrak{U}(a)(\omega)$

Recall Definition 12 of **N** being EU-complete, and observe that Bayes-existing immediately follows.

Lemma 26. *If* **N** *is EU-complete, then it is Bayes-existing.*

Proof. Observe that ν is Bayes for p iff it μ^*-surely picks for EU_p using Theorem 18. □

Recall the definition of requires almost everywhere decisiveness from Definition 13.

Lemma 27. *If μ^* requires almost everywhere decisiveness and countably additive, then $\mathcal{U}(\nu)(\omega) := \mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu_D)(\omega) = \mathsf{Exp}_{\mu^*}\mathsf{Exp}_{\nu_D}\mathfrak{U}(a)(\omega)$ is Bayes continuous.*

Proof. Suppose p^* is a probability function over Ω. And suppose $p_1, p_2 \ldots$ is a sequence of probability functions over Ω that converges on p^*, that is, $p_1, p_2 \ldots \longrightarrow p^*$. We will show that for any ν^{p^*} which is Bayes for p^* and ν^{p_n} which are Bayes for p_n, then $\mathcal{U}(\nu^{p_n})(\omega) \longrightarrow \mathcal{U}(\nu^{p^*})(\omega)$.

Now suppose that D is such that: (i) $\mathrm{EU}_{p^*}(D) = \{a^*\}$, (ii) $\nu_D^{p_n}(\mathrm{EU}_{p_n}(D)) = 1$, and (iii) $\nu_D^{p^*}(\mathrm{EU}_{p^*}(D)) = 1$, so that $\nu_D^{p^*}(\{a^*\}) = 1$. For every a, $\mathsf{Exp}_{p_n}(a) \longrightarrow \mathsf{Exp}_{p^*}(a)$. If D is finite, there must be some N such that for all $n > N$, also $\mathrm{EU}_{p_n}(D) = \{a^*\}$. And then $\mathfrak{U}(\nu^{p_n})(\omega, D) = \mathsf{Exp}_{\nu_D^{p_n}} a(\omega) = a^*(\omega) = \mathsf{Exp}_{\nu_D^{p^*}} a(\omega) = \mathfrak{U}(\nu^{p^*})(\omega, D)$.

If D is infinite, although compact, one can use the Berge's Maximum Theorem to observe that $\mathrm{EU}_p(D)$ is upper hemi-continuous, so that if $p_n \longrightarrow p^*$ and V is an open set with $\mathrm{EU}_{p^*}(D) \subseteq V$, then there is some N such that for all $n > N$, $\mathrm{EU}_{p_n}(D) \subseteq V$. Let $V_\epsilon = \{a \mid |a(\omega) - a^*(\omega)| < \epsilon\}$, which is open containing $\mathrm{EU}_{p^*}(D) = \{a^*\}$. So there is some N such that for all $n > N$, any $a_n \in \mathrm{EU}_{p_n}(D)$

is in V_ϵ, and so $|a(\omega) - a^*(\omega)| < \epsilon$; thus also $|\mathsf{Exp}_{\nu_D^{p_n}} a(\omega) - a^*(\omega)| < \epsilon$; so the utility profiles converge.

Next, we show that the set of D for which (i), (ii), and (iii) hold has measure 1, and so (since the utilities are bounded, by a Dominated Convergence Theorem):

$$\mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu^{p_n})(\omega) \longrightarrow \mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu^{p^*})(\omega)$$

which is what we wish to prove.

We have supposed that ν^{p^*} is Bayes for p^* and ν^{p_n} are Bayes for p_n. Thus, by Theorem 18, ν^{p^*} μ^*-surely picks for EU_{p^*} and ν^{p_n} μ^*-surely pick for EU_{p_n}. Let:

- $\mathcal{X}^* = \{D \in \mathcal{D} \mid \nu_D^{p^*}(\mathrm{EU}_{p^*}(D)) = 1\}$
- $\mathcal{X}^n = \{D \in \mathcal{D} \mid \nu_D^{p_n}(\mathrm{EU}_{p^n}(D)) = 1\}$
- $\mathcal{Y}^* = \{D \in \mathcal{D} \mid \mathrm{EU}_{p^*}(D) \textit{ is a singleton}\}$

We thus know that $\mu^*(\mathcal{X}^*) = \mu^*(\mathcal{X}^n) = 1$. We also know that $\mu^*(\mathcal{Y}^*) = 1$ by assumption of μ^* requiring almost everywhere decisiveness. And so, since μ^* is countably additive,

$$\mu^*\left(\mathcal{X}^* \cap \bigcap_{n=1}^{n} \mathcal{X}^n \cap \mathcal{Y}^*\right) = 1.$$

This completes the proof. □

Corollary 28. *Suppose that* **N** *is EU-complete, μ^* requires almost everywhere decisiveness and is countably additive.*

If there is no probability p over Ω for which ν maximises $\mathsf{Exp}_p[\mathsf{Exp}_{\mu^*}[\mathfrak{U}(\nu)]]$; then there is some ν' such that $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu')(\omega) > \mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu)(\omega)$ for all $\omega \in \Omega$.

Proof. This is immediate from Theorem 24 and Lemmas 26 and 27. □

Corollary 29. *Suppose that* **N** *is EU-complete, μ^* requires almost everywhere decisiveness and is countably additive.*

If ν does not μ^*-surely pick for any probability p, then there is some ν' such that for all probabilities p, $\mathsf{Exp}_{p\times\mu^*}[\mathfrak{U}(\nu')] > \mathsf{Exp}_{p\times\mu^*}[\mathfrak{U}(\nu)]$.

Proof. From Theorem 18, such ν thus does not maximise $\mathsf{Exp}_{p\times\mu^*}[\mathfrak{U}(\nu)]$ for any p. Observe, also that $\mathsf{Exp}_p[\mathsf{Exp}_{\mu^*}[\mathfrak{U}(\nu)]] = \mathsf{Exp}_{p\times\mu^*}[\mathfrak{U}(\nu)]$. So by Corollary 28, there is some ν' with $\mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu')(\omega) > \mathsf{Exp}_{\mu^*}\mathfrak{U}(\nu)(\omega)$ for all ω; and thus, $\mathsf{Exp}_p[\mathsf{Exp}_{\mu^*}[\mathfrak{U}(\nu')]] > \mathsf{Exp}_p[\mathsf{Exp}_{\mu^*}[\mathfrak{U}(\nu)]]$ for all probabilities p; which gives us the claim. □

Propositions 15 to 17 follow from this.

If we're rewriting the EU result, check how this looks!