Exercise Sheet 2

Discrete Mathematics I - SoSe17

Lecturer Jean-Philippe Labbé

Tutors Johanna Steinmeyer and Patrick Morris

Due date 3 May 2017 -- 16:00

You should try to solve all of the exercises below, and select three to four solutions to be submitted and graded. We encourage you to submit in pairs, please remember to indicate the author of each individual solution.

Problem 1

Prove combinatorially the following equality

$$n\binom{n-1}{k-1} = \binom{n}{k}k.$$

Problem 2

Prove combinatorially the following equality

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Problem 3

Place n points on a circle and draw all line segments joining pairs of of these points and assume that no three line segments intersect in one point inside the circle. The line segments determine bounded regions inside the circle.

a. How many regions are determined by $n = 1, 2, 3, 4, 5, 6$ points?

b. Prove combinatorially (using a bijective proof), that the number of regions is

$$\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}.$$

Hint: Label the vertices clockwise and label the regions using 0, 1, 2, 3, or 4 vertex labels from $n-1$ labels.

Problem 4

Prove that the number of regions inside the circle in Problem 3 is

$$1+\binom{n}{2}+\binom{n}{4}$$

using an inductive argument (and not the arguments or proof from the previous problem).

Problem 5

Let $n \geq 2$ and $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ its prime factorization. Show that its number of divisors is

$$\prod_{i=1}^{k}(a_i+1).$$

Problem 6

Prove that there exists a power of 3 that ends with "001".

Problem 7

Show that at a party with at least 2 people, there are 2 people that know exactly the same number of people at the party. (We assume that knowing is reflexive and symmetric: A knows A and if A knows B, then B knows A)

Problem 8

Prove the Vandermonde identity using an algebraic argument

$$\binom{m+n}{r}=\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}.$$

Problem 9

What is the coefficient of

a. x^9 in $(2-x)^{19}$?

b. x^k in $(x+1/x)^{100}$?

c. $x^{50}y^{40}$ in $(x+y-2/x)^{100}$?

Problem 10

Find the number of lattice paths from (0,0) to (10,10) (using only "East" and "North" steps, and of minimal length, i.e. 20) not passing through (2,4), nor (4,6), nor (6,9).