Report on Research programme – Section 06/03

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Contents

1. Objectives

The overarching goal of my research endeavour is two-fold. First, my research aims to reveal unseen knowledge on fundamental structural properties of Coxeter groups. Second, I will exploit this acquired knowledge to expose and detail new connections with a variety of closely related geometric objects such as simplicial complexes, polytopes, root systems, fractals and combinatorial objects such as (infinite and finite) reduced words, graphs, posets, and their enumerative invariants.

The subprojects described hereafter serve as the initial and intermediate steps towards accomplishing the following long term goals:

- Establish an explicit pathway to determine whether subword complexes are boundaries of polytopes,
- Clarify how Coxeter groups generalize cyclic polytopes through subword complexes,
- Lay out the foundations of the study of equivariant infinite hyperplane arrangements that do not yield CAT-0 spaces, and
- Classify Lorentzian Coxeter groups of bounded dimensions.

More specifically, my project will impact constructively on the understanding of hyperbolic manifolds, subword complexes, the associahedron and its generalizations, and Weyl groupoids.

The study of geometric and combinatorial properties of Coxeter Groups is a cornerstone of this journey. For instance, the project proposes to study their graphs and reduced words, in order to quantify their similarities or differences. Particularly, I plan to study a partial order on Coxeter graphs introduced by McMullen [McM02] that respects properties related to the Poincaré series and limit roots. I want to forge a deep understanding of this poset. I anticipate many potential applications of this poset and will use it to obtain necessary and sufficient conditions to obtain Lorentzian Coxeter groups. This knowledge will later be helpful to construct and study more hyperbolic manifolds arising from infinite Coxeter groups.

I will look into several new properties of reduced words: determine which braid equivalence relations preserve geometric properties of limit roots, give a combinatorial characterization of infinite order elements, and investigate word statistics of reduced words.

As for subword complexes, I want to describe explicitly the realization space of rank-3 subword complexes. Followingly, I will find some numerical evidence to approach the upper bound conjecture for subword complexes and also to enumerate their number of combinatorial types.

Finally, I aspire to uncover exciting, yet unexplored, relationships between Coxeter groups and the following objects: Bottman's 2-associahedron in symplectic geometry, exceptional type sextonions in Lie algebras, and poset of regions of simplicial hyperplane arrangements arising from Weyl groupoids.

I will pursue research simultaneously on several fronts. As the knowledge on structural properties of Coxeter groups is capital for the good progression of my investigation, it should be addressed in priority. Below I detail subprojects organized according to the Finite/Infinite and Geometry/Combinatorics divisions and their interdependence with respect to the research agenda.

2. Expand the geometric combinatorics of finite Coxeter groups

2.1. Novel perspectives on the combinatorics of finite Coxeter groups.

- **Project 1:** *Characterize reduced expressions in exceptional types*

A formula for the number of reduced expressions of the longest element in the symmetric group using Young tableaux was given by Stanley [Sta84], and further instructive descriptions were obtained by Edelman and Greene [EG87] and Lascoux and Schützenberger [LS82]. I will obtain all reduced expressions of the longest element in the exceptional cases with rank 4: so far the types F_4 and H_4 have resisted many attempts. There is a straightforward algorithm to produce them, but this method has computational pitfalls. I will develop a new intuition coming from combinatorics on words in order to approach the problem from a more theoretical angle than using braid moves to generate all reduced words.

- **Project 2:** *Establish structural properties of Subword Complexes*

Multi-cluster complexes form a very structured subfamily of subword complexes. They unify cluster complexes and the simplicial complex of multi-triangulations. Narayana numbers are a specialization of Catalan numbers that use one more parameter which depends on the Catalan object. Narayana numbers form what is called the h-vector of the associahedra and have an important connection to commutative algebra via the face ring of the associahedra. Multi-cluster complexes of type A, or multi-associahedra, provide a natural generalization of Narayana numbers via their h-vectors. These generalized Narayana numbers count a similar generalized structure; nevertheless no general formula is known, apart for the number of facets. In contrast, for general types, there is still no known uniform formula for the number of facets. Multi-cluster complexes are conjectured to have extremal f-vectors among subword complexes on prescribed number of vertices. That is, they would give an upper bound on the number of faces that a subword complex can have. Using brute force enumeration and polynomial interpolation manipulations, it is possible to get a formula for the h-vector for the family of multi-associahedra A_3. Inspired by the work of Simion and Ullman [SU91] on symmetric chain decompositions of the lattice of noncrossing partitions in relation with the γ-vector of the associahedra [PRW08], I aim at a general formula for the h-vector of the multi-associahedra.

Another approach with potential to bring new light on subword complexes is to study the shiftings of multi-cluster complexes. The shifting operation allows to reveal hidden structures of incidence structures while preserving certain invariants like the f-vector and the Betti numbers. This question will necessitate to use a computational framework to study the result of the shifting operation on multi-cluster complexes. Indeed, shifting involves very technical algebraic manipulations and perhaps the only practical way to study them is by using a non-deterministic algorithm. I will give a description of their structure once shifted.

In addition, in order to place the subword complexes' upper bound result in context with the set of simplicial spheres, it is important to obtain an asymptotic formula as to quantify their size compared to all simplicial vertex-decomposable spheres.

Also, I will extend the work presented in [Ceb+15, CP16, Pou14, Pou17] to study the diameter of multi-cluster complexes. For this, the computations of reduced words of exceptional finite types from Project 1 will play a prominent role.

- **Project 3:** *Reveal further word statistics of reduced words*

Let (W, S) be a Coxeter group and $w \in W$. For each letter $s \in S$, what is the minimum and maximum number of times that s can appear in a reduced word for w? In type A, this simple question is related to the famous halving line problem in discrete geometry: Given n points in general position on the plane, a halving line is a line that partitions the points into two sets of $n/2$ points. What is the largest possible number of halving lines for a set of n points in the plane? Initially studied by Lovász and Erdős in the 1970s, the current best upper bound is $O(n^{4/3})$, obtained by Dey [Dey98]. The current lower bound $\Omega(ne^{c(\log n/2)^{1/2}})$, for some constant c, was obtained by Tóth [Tót01].

Halving lines are related to reduced words of the reverse permutation in the symmetric group using planar point/pseudoline duality. It turns out that the space of geometric realizations of subword complexes is deeply related to this question [⋆5]. I want to describe bounds for the possible number of occurrences of letters in reduced words. This has not been addressed for general Coxeter groups. Again, the first bounds will build upon evidence brought by the computations of reduced words in Project 1.

- **Project 4:** *Study shortest superstring problems of Coxeter groups*

Problem 6.1 of [KM04] asks: What is the smallest size of a word in S containing every reduced expression for w as a subword? This turns out to be a particular instance of the *shortest superstring problem*. One may also ask relevant variations, where one asks to contain every reduced words for w, up to commutations. The shortest superstring problem is NP-complete already on alphabets of size 2 [RU81]. By improving knowledge on reduced words, I pursue an answer to this challenging problem for reduceds words of finite Coxeter groups. Further, the study of word statistics of reduced words will help in getting a hand on such shortest superstrings. As it turns out, the length of such a superstring is a direct indicator of how difficult it is to realize subword complexes geometrically [⋆5], therefore making such study of prime importance.

2.2. **Expand the geometric applications of finite Coxeter groups.**

- **Project 5:** *Describe explicitly the geometric realizations of subword complexes*

This project's long term goal is to determine whether subword complexes are boundaries of convex polytopes. I plan to examine the recently developed theory of *slack realization spaces* in the case of subword complexes to make progress in this direction [Gou+19].

- **Project 6:** *Clarify the role of Schur functions in geometric realizations of subword complexes*

The recent developments layed out in the article [⋆5] suggest to describe a part of the realization space of small rank subword complexes as a semi-algebraic set using Gale duality and Schur functions. The role of Schur functions in the description of realization spaces of subword complexes should be thoroughly clarified.

- **Project 7:** *Bring forward a combinatorial point of view on Bottman's 2-associahedron*

Bottman recently introduced a poset that widely generalizes the face lattice of the associahedron [Bot19]. This generalization stems from symplectic geometry in relation with Fukaya categories of different symplectic manifolds. Further, the structure of these posets are combinatorially very close to face lattices of polytopes, i.e. they were also proved to be Eulerian [BM19]. It is thus very natural to ask whether they can be realized as the boundary of convex polytopes. I will translate the 2-associahedron into purely combinatorial terms, study its enumerative properties, and finally obtain geometric realization. For this, I will compare it to the other myriad of generalizations of the associahedron available [MPS12].

- **Project 8:** *Study the lattice theory of Weyl groupoids*

When Weyl groupoids admit a finite root system, one can generalize the usual weak order using the poset of regions of the associated simplicial hyperplane arrangement [HW11, CH15]. Preliminary direct evidence suggests that finite Weyl groupoids are congruence uniform [⋆3]. My PhD Student Sophia Elia and I will examine how the property of congruence uniformity behaves for finite Weyl groupoids exactly. Are they all congruence uniform? We will use oriented matroids as a main tool to study the combinatorics of finite Weyl groupoids and extract the necessary notions that would lead to congruence uniformity.

- **Project 9:** *Relate subword complexes to Lie algebras*

In [LM06], Joseph Landsberg and Laurent Manivel describe an exceptional Lie algebra called $E_{7\frac{1}{2}}$ with similar properties as the Lie algebras E_7 and E_8. They also study some other subexceptional Lie algebras whose dimension, in a certain case, correspond to the number of facets of multi-cluster complexes of type H_3. I will study the internal symmetries of the related Lie algebras and determine how they relate to the symmetries of the relevant subword complexes to uncover the relation between them.

- **Project 10:** *Generalize cyclic polytopes through subword complexes*

The study of realizations of subword complexes presented in [⋆5] motivates the following question in discrete geometry. Consider n generic skew curves of degrees $k_1, k_2, \dots, k_n$ and m points distributed on them. What kind of polytopes arise as the convex hull of these m points? When the number of curves is 1, this correspond to the construction of the cyclic polytope. Is it possible to describe these polytopes using known constructions? It turns out that subword complexes are related to such polytopes. I expect to be able to describe polytopes that arises using such a construction using Gale duality and oriented matroids.

3. Unify the geometric and combinatorial theories of infinite Coxeter groups

3.1. **Describe the geometry and dynamics of infinite Coxeter groups.**

I will expand the theory of geometric representations of Coxeter groups by clarifying the relation between limit roots and dynamical systems, and properly understand the dynamics and combinatorics of infinite order elements. First, an effective geometric characterizations of limit roots, limit weights and infinite order elements of Coxeter groups is seeked. These characterizations will help to reveal further the structure of Tits cones. Another step is to overcome the difficulties when passing from Lorentz space to the general case. Indeed, many of the current results rely deeply on the geometry of Lorentz space, see e.g. [⋆1]. Therefore calling for a better understanding of the dynamics of infinite Coxeter groups in general. The theory of primitive matrices seems to fit the problem well [⋆6], and further research in this direction should lead to an explicit description of their spectra.

- **Project 11:** *Construct more hyperbolic manifolds*

Reflection groups in hyperbolic spaces are important in relation to Poincaré's theorem on fundamental polyhedra of group acting on hyperbolic spaces, see e.g. [Kap09, Section 4.11] and [Mar16, Section 3.5]. The notion of face-pairing transformation is used to give the characterization. It turns out that obtaining face-pairings in higher dimension is a challenge. Indeed, much less is known on hyperbolic polyhedra in dimension higher than 3: their combinatorics and geometric relations can be wild [APT15]. For example, ideal hyperbolic polyhedra are related to the inscribability of polytopes in dimension larger than 3; promising novel techniques were recently developed and could be applied [⋆2]. This project envisage to study Lorentzian Coxeter groups with a perspective on getting face-pairings for the level-2 Coxeter graphs. This will help in getting many test cases for a generalization of an algorithm that determines the fundamental polyhedra of arithmetic Kleinian groups [Pag15]. This project will benefit from the collaboration with Vincent Delecroix, who has a specialized expertize on the dynamics of Kleinian and Bianchi groups. In relation with this investigation, Vincent Delecroix posed the following quite intriguing question: Is there a pair of combinatorially distinct polytopes P and Q such that their facet–ridge graphs—enriched with their ridges angles—are isomorphic? To start the examination, I will look at the case of polyhedra coming from representations of Coxeter groups.

- **Project 12:** *Connect limit roots and infinite reduced words*

In [⋆1], Chen and I show that the limit roots of Lorentzian Coxeter groups are the only isotropic limit directions and that to every infinite reduced word corresponds a unique limit root. The proof uses the convexity of the isotropic cone and its interior. To extend the result to the general case, it is necessary to reobtain the same result without making use of convexity. For this, a new study of eigenvalues of the associated matrices is required and I should bring further the results presented in [⋆6].

- **Project 13:** *Establish a theory of infinite hyperplane arrangements*

To study infinite hyperplane arrangements, I will extend the notion of lattice of flats to the case when hyperplane arrangements are not locally finite and see how the known properties behave in this extension. The theory available in [⋆1] and the computational framework [⋆4] will make it possible to test the developed theory of locally infinite lattice of flats on several examples. This will also necessitate an adapted notion of oriented matroid to the infinite case.

- **Project 14:** *Classify Lorentzian Coxeter Groups*

When a Coxeter group acts on a Lorentz space, it is called *Lorentzian*. Which Coxeter groups are Lorentzian? Vinberg discussed this long standing problem in geometry at the ICM in 1983 in Warsaw [Vin84]. Although a lot of progress has been made since then [PV05, Duf10, FT14], the general problem still eludes our comprehension.

It is possible to encode Coxeter graphs as certain set of points in a vector space. I propose to classify Lorentzian Coxeter groups using strategies from lattice polytope theory and Robertson-Seymour's theorem about forbidden minors [RS04]. Indeed, the property of being Lorentzian can be phrased in the language of forbidden minors. Thus there exists a finite set of obstruction which describes the non-Lorentzian Coxeter groups. Obtaining a list of these obstructions seems to be within reach. Further, using an experimental framework, I expect to obtain sufficient conditions for a Coxeter graph to yield a Lorentzian Coxeter group.

3.2. Develop the algebraic combinatorics approach to infinite Coxeter groups.

- **Project 15:** *Study McMullen's minor partial order on Coxeter graphs*

In order to study Lorentzian Coxeter groups and more generally to describe the types of spaces on which Coxeter groups act, it seems that the partial order defined by McMullen [McM02] might be useful. McMullen already described certain minimal elements, and I plan to extend his poset to the general Coxeter graphs involved in the theory of limit roots and determine how the poset interacts with the notion of level.

▪ **Project 16:** *Obtain combinatorial characterizations of limit roots, limit weights, and infinite order elements*
Provided a reduced word of an infinite reduced word, is it possible to determine if this word has infinite order solely using the word itself? I will investigate the various automata related to reduced words to know whether it is possible to modify them to provide automata that recognize infinite order elements. In view of the fact that infinite order elements lead to a very precise description of the Tits cone and limit roots, having an automaton to generate them would be very useful. Finally, I will look into the spectrum of more elements using ideas coming from dynamical systems such as the subshift of finite type model and iterated function systems to mimic the group action.

4. Computational Framework

▪ **Project 17:** *Expand the Brocoli package.*
The package `Brocoli` [⋆4] should be maintained and expanded by the developments in my research to include more dynamical aspects and combinatorial tools.
▪ **Project 18:** *Develop a database of reduced words.*
As the information on reduced words is paramount for this project, I will build a database of reduced words available online and through `Sagemath`.

▪ **Project 19:** *Implementations in `Sagemath`.*
Many computations will be necessary to support this project. Motivated by this necessity, I will make these computations available to the community through the peer-reviewed implementation of `Sagemath`. Since 2017, I have been involved in the development of the geometry component of the open source software `Sagemath`. We plan to have a Research in Pairs at CIRM in 2020 to finish our novel implementation framework for fast exact computations on algebraic polytopes. We aim simultaneously to make `Sagemath` *the integrated user-friendly interface of choice for fast and reliable geometric computations* that support the usage of many available open source softwares. Some desired capacities include: the shifting operation (that will interest researchers in commutative algebra), algebraic polytopes, and Coxeter graphs. Further we plan to provide interfaces to the functionalities of `Bertini` and `JuliaHomotopyContinuation` in `Sagemath`.

5. Integration with the host institution

Because of its multiple facets, this research project will be able to place itself in an interesting strategic position within several research groups. Among these, three laboratories are studying problems directly related to or at the borders of the proposed project: LRI-GALAC (Paris Saclay), LIPN-CALIN (Paris Nord) and LIP-MC2 (UMR 5668).

LRI-GALAC (UMR 8623) – The *Laboratoire de Recherche en Informatique* (LRI) is an institute that stands out as a perfect host for my research project. My research project involves a large combinatorial program where graphs, posets and words play a key role. These research topics are studied by the *Graphes, Algorithmique et Combinatoire* (GALaC) group at the LRI. Moreover, their research on algebraic structures such as Hopf algebras and monoids will be added to my expertise on Coxeter groups which will facilitate the progress of my research. Within the GALaC group, my research interests fit well with the research interests of Nicolas Thiery, Florent Hivert, Viviane Pons, and Joël Gay. Their research focuses in particular on representation theory, monoids, Tamari lattice, Coxeter groups, and more generally on algebra. In addition, we have been collaborating since 2010 through the free software `Sagemath`. We have organized several meetings where we have been able to work together on the objectives of the OpenDreamKit project. Finally, the geographical location of the GALaC group also allows the close collaboration of the *Modèles combinatoire* group and the *Algèbre et arithmétique* of the *Centre de Mathématiques Laurent Schwartz* present at LIX, where Marie Albenque, Éric Fusy, Vincent Pilaud and Bertrand Rémy carry out their research. I visited Vincent Pilaud several times to present my research on the geometry of finite Coxeter groups and the associahedron and my research on infinite groups may be useful to him. I met Bertrand

Rémy during a workshop in Olot, Spain while working on computational tools for polyhedral computations and visualizing tools for geometric group theory. Many questions related to limit roots and weights relate to the theory of infinite dimensional Lie algebras and our discussions opened several interesting research possibilities.

LIPN-CALIN (UMR 7030) – The *Laboratoire d'Informatique de Paris Nord* (LIPN) constitutes an ideal place to pursue this research project as well. Indeed, my research involves combinatorics, optimization and language of reduced words which lays at the center of the research focus of this laboratory. My research fits particularly within the *Combinatoire, algorithmique et Intéractions* (CALIN) group. In addition, I will be happy to work with members of the *Algorithmes et Optimisation Combinatoire* (AOC) group, since some of my problems will benefit from an optimization approach. Among the members of the CALIN team, Thomas Fernique, Thierry Monteil and Lionel Pournin do their research in particular on geometric combinatorics, and discrete geometry. In addition, Olivier Bodini and Frédérique Bassino are interested in combinatorics on words and rhombus tilings, concepts that are important when considering reduced words of Coxeter groups. I met Thierry Monteil in 2010 during a conference in Montréal and we kept in touch through the development of the open source software `Sagemath` since then. The knowledge of Thierry Monteil on minimal subshifts will certainly act as a catalyst for Project 16. It will be interesting to exchange on my research on sphere packings in terms of Coxeter groups with Thomas Fernique who did research on tilings and packings. At FPSAC 2013, I met Lionel Pournin when he gave his talk on the diameter of the associahedron, a problem that I have considered during my doctoral studies. Since then, we have been studying the concept of diameter for various generalizations of the associahedron. I have invited Lionel in Berlin in February 2020 to present his recent research on triangulations, an important topic related to the geometric realizations of subword complexes. My research on subword complexes and their diameters will surely interest Lionel, greatly benefit from his feedback, and lead to further collaborations.

LIP-MC2 (UMR 5668) – This laboratory focuses on fundamental aspects of computer and information science, which fits my theoretical background. The *Models of computation, Complexity, Combinatorics* group of the *Laboratoire d'Informatique du Parallèlisme* (LIP) uses tools from combinatorics and algebra to study problems involving large scale computational framework. The work of Michael Rao on the final step to classify the pentagonal tilings of the plane attracted the attention of the discrete geometry community, since he used exact arithmetic for polyhedral computations. I have met Michael Rao during conferences and seminars in Montréal and Bordeaux, and subsequently invited him to describe his latest development in polyhedral combinatorics in February 2018. We have discussed shortly about Dürer's conjecture in differential geometry that could perhaps be attacked through similar methods.

Further, the *Arithmetic and Computing* (AriC) group in the *Laboratoire d'Informatique du Parallèlisme* (LIP) offers a great opportunity to connect my research done in solving polynomial system of equations and optimization and relate it to cryptography through the solving of Groebner basis and the study of properties of infinite lattices. The group AriC works in particular on arithmetic algorithms, approximation methods, Euclidean lattices and cryptology, certified computing and computer algebra. My work on exact algebraic computations in polyhedral geometry fits this group particularly well.

6. Personal publications

6.1. Refereed publications.

[⋆1] Hao Chen and **Jean-Philippe Labbé**, *Limit directions for Lorentzian Coxeter systems*, Groups Geom. Dyn. **11** (2017) no. 2, 469–498.

6.2. Other publications.

[⋆2] Joseph Doolittle, **Jean-Philippe Labbé**, Carsten Lange, Rainer Sinn, Jonathan Spreer, and Günter M. Ziegler, *Combinatorial inscribability obstructions for higher-dimensional polytopes*, arXiv:1910.05241 (2019) 27 pp.

[⋆3] Sophia Elia and **Jean-Philippe Labbé**, *Congruence normality and oriented matroids*, in preparation (2019) 17 pp.

[⋆4] **Jean-Philippe Labbé**, *Brocoli: Sagemath package dealing with LImit ROots of COxeter groups*, `https://github.com/jplab/brocoli` (2017) version 1.0.0 3500 lines.

[⋆5] **Jean-Philippe Labbé**, *Convex geometry of subword complexes of Coxeter groups*, in preparation (2019) 30 pp.

[⋆6] **Jean-Philippe Labbé** and Sébastien Labbé, *A Perron theorem for matrices with negative entries and applications to Coxeter groups*, `arXiv:1511.04975` (November 2015) 14 pp.

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