**Project Description – Project Proposals**

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**Title: Geometric Combinatorics of Coxeter Groups: structures and applications**

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## Project Description

### 1. State of the art and preliminary work

Discrete geometry stands at the crossroads of rapidly developing areas of research in mathematics. Many exciting problems in algebraic geometry, combinatorics, theoretical physics and representation theory share deep, and sometimes stunning, relationships with geometric objects and their combinatorial properties. These relationships, along with the problems at the core of discrete geometry, foster the emergence of a plethora of important mathematical challenges. Coxeter groups form a flagship example of such an object. *Coxeter groups* are abstract versions of reflection groups of geometric or topological objects. By their simple definition based on the concept of symmetry, they provide a flexible combinatorial framework to study internal properties of geometric and algebraic objects. Examples of relations between Coxeter groups and other geometric and combinatorial objects include Kazhdan–Lusztig polynomials, Stanley symmetric functions, cluster algebras of finite types, and constructions of hyperbolic manifolds.

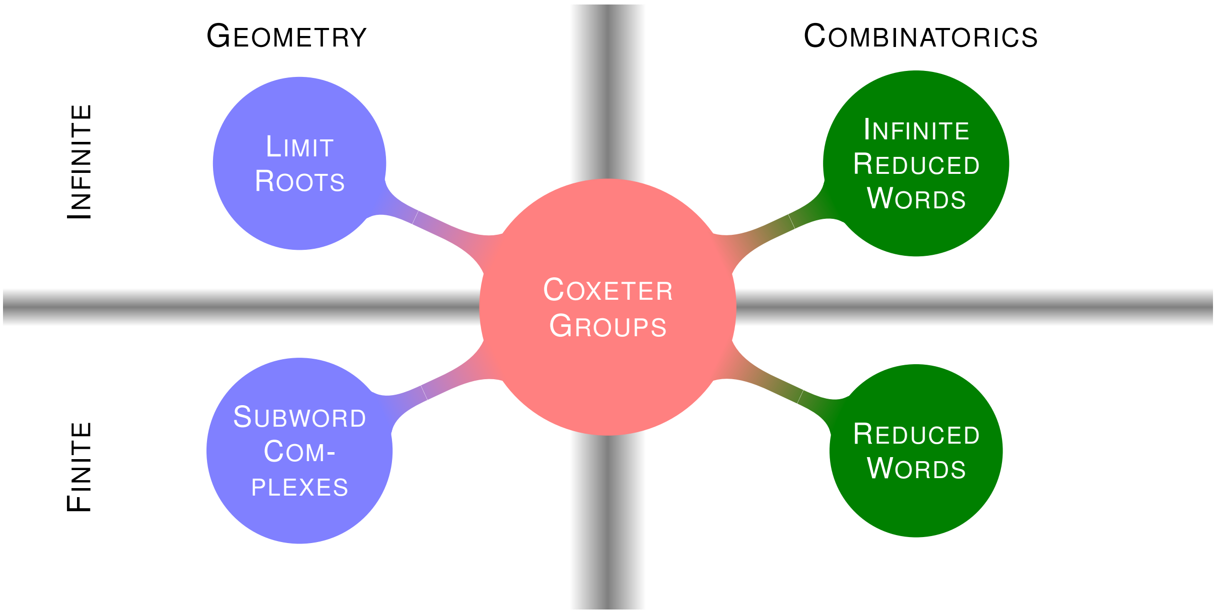
The proposed project intends to study novel fundamental structural properties of Coxeter groups and their interactions with geometric objects such as simplicial complexes, polytopes, root systems, fractals and combinatorial objects such as (infinite and finite) reduced words, graphs, posets, and their enumerative invariants.

Coxeter groups are defined using generators and relations. They form a particularly well-behaved family of finitely presented groups. For instance, they are characterized by the Exchange condition, the word-problem can be solved in polynomial time, and the Bruhat order lays at the center of the combinatorial structure. Their elements are represented by words, the shortest of which are called *reduced words*. Considered otherwise, Coxeter groups act through *root systems* on vector spaces. Root systems originated in the classification of semi-simple Lie algebras. They form special minimal sets of vectors configured precisely to describe this geometric action through reflections. Much of the combinatorial structure (e.g. Bruhat order and weak order posets) can be described using root systems, allowing the combinatorics of Coxeter groups to be used to study objects related to root systems, such as permutahedra or group representations.

Simplicial complexes and polytopes are two important objects studied in discrete geometry. *Simplicial complexes* are higher-dimensional generalizations of graphs, while *polytopes* are geometric bodies obtained as the convex hulls of finite point sets in a real vector space, so they are generalizations of polygons. They are used particularly as discretization tools to model topological spaces via triangulations or cell decompositions. Faces of simplicial complexes are the simplices forming it, while faces of polytopes are its surfaces of contact with “supporting” hyperplanes. Both objects share combinatorial invariants: the *$f$-vector* recording the number of faces in each dimension, and the *diameter* giving a significant complexity indicator for algorithms defined on these structures. The family of simplicial complexes called *subword complexes* were introduced in the context of Gröbner geometry of Schubert varieties using finite Coxeter groups [KM04, KM05]. Through their combinatorial definition using combinatorics of reduced words, they can be used as an intermediary to translate between geometric and combinatorial concepts. Namely, studying the diameter and the $f$-vector of subword complexes directly calls for a deep understanding of reduced words of Coxeter groups. On the other hand, the structure of reduced words of Coxeter groups has a dramatic effect on the geometrical realizations of subword complexes and therefore on potential properties of the associated Schubert varieties, for example.

What is more, the study of infinite Coxeter groups still offers a wide range of recent research avenues. The development of the theory of *limit roots* provides an innovative approach to the study of the weak order. A limit root is a limit direction arising from the action of the group on the root system. This approach is both geometric through its definition and combinatorial in its close connection to infinite reduced words. An *infinite reduced word* is a right-infinite word with reduced finite prefixes. For a given infinite Coxeter group, the set of limit roots is a minimal set of directions on which the group acts and encodes much of its asymptotical structure. Although very natural, the general relationship between infinite reduced words and limit roots in not fully understood.

Through the above discussion, the study of Coxeter groups can be arranged into four parts laid out on two axis depending on whether one is interested in *finite* or *infinite* groups first, and then depending on whether *geometric* or *combinatorial* aspects or tools are at play.




Coxeter groups involve a rich mixture of combinatorial and geometric structures. With regard to the geometric aspect of finite Coxeter groups, the current project plans to focus on *subword complexes*, while for infinite Coxeter groups, we center our attention on *limit roots*. On the combinatorial side, both *finite* and *infinite reduced words* constitute on of the main objects at play. Of course, as far as Coxeter groups are concerned, many other objects intervene. Among them, root systems, polytopes, triangulations, graphs and their related enumerative statistics play a significant role in this theory.

Let us highlight further two objects of particular importance. On the one hand, the weak order poset encodes the combinatorial structure of Coxeter groups, and is useful to study the Bruhat order, Hecke algebras, and Kazhdan–Lusztig polynomials. On the other hand, root systems are a prime tool to represent Coxeter groups as reflection groups and many combinatorial aspects of the Coxeter groups translate into properties of polytopes associated to root systems. The interplay between root systems and the weak order lead to fascinating relations, for example, between combinatorial objects (for example, automata and reduced words), algebraic objects (for example semi-simple Lie algebras, and hyperbolic groups), and geometric objects (for example, billiards trajectories, and limit sets).

## On Finite Coxeter groups

Geometry of finite Coxeter groups

*Subword complex approach to cluster complexes.* In [CLS14], Ceballos, Stump and I expose a deep relationship between subword complexes and cluster complexes in the theory of cluster algebras, which led to a fruitful research avenue. In particular, subword complexes have shown further connections to cluster algebras [CP15], toric geometry [Esc16], root polytopes [EM18], Hopf algebras [BC17], among many others. This approach also led to a simple and unified approach to different topological objects such as the simplicial complex of multi-triangulations of a convex polygon [Jon05], the simplicial complex of centrally symmetric multi-triangulations of a convex regular polygon [SW09].

*Modelling of Tropical planes.* Tropical geometry studies algebraic geometry questions after a so-called *tropicalization*. This operation replaces usual ring operations with semi-ring operations: additions becomes minima and products becomes additions. Usual varieties–like hyperplanes–become “tropical” under this mapping. The combinatorial types of tropical two-dimensional planes were classified geometrically using matroid subdivisions of hypersimplices for small dimensions [Her+09]. A more combinatorial approach uses the cluster complex of type $D_4$ with the pseudo-triangulation model, developed by C. Ceballos and V. Pilaud, and the combinatorics of subword complexes to describe the combinatorial types of tropical planes of dimension $5$ [BCL17]. Inspired by this model, it seems plausible to extend the results on tropical planes in dimension $6$ relying *solely* on the subword complex of type $E_6$.

Combinatorics of finite Coxeter groups

*Enumeration of singletons.* The associahedra can be realized using the permutahedra by pulling certain facets away to infinity, see [HL07, HLT11]. Subword complexes and the combinatorics of Coxeter groups lead to formulas counting the number of common vertices of permutahedra and generalized associahedra for arbitrary finite Coxeter groups and Coxeter elements [LL19]. This enumeration has deep connections with the notion of acyclic sets in social choice theory as shown by Á. Galambos and V. Reiner [GR08].

*Graph on reduced expressions.* It is a well-known property of Coxeter groups that reduced expressions of elements are connected via finite sequences of braid moves, in particular that no reductions $s_i^2 = e$ are necessary [Mat64, Tit69]. The graph $\mathcal{G}(w)$ whose vertices are reduced expressions of $w$ and edges represent braid moves between expressions is hence connected. The diameter of $\mathcal{G}(w)$ has been studied in [AD10, RR13] and other closely related enumerative properties in [Ten17]. Of particular importance is the fact that the graph $\mathcal{G}(w_\circ)$ and certain minors are bipartite graphs. This was proved for finite Coxeter group by Bergeron, Ceballos, and I using a geometric argument in [BCL15] and generalized to infinite Coxeter groups and extended to a finer description by Grinberg and Postnikov in [GP17] using only conjugations instead of automorphisms. An asymptotic study of expected number of commutations in reduced words was done in type $A$ [Rei05] and type $B$ [Ten15] and it is possible to determine the number of elements that have a unique reduced word [Har17]. Further, it is possible to define a metric on this graph which relates naturally to balanced tableaux [Ass19]. These results establish a relatively good knowledge of reduced expressions. However, it seems that many important properties of reduced words stemming from the geometry of subword complexes still have to be studied in detail.

Geometric Combinatorics of finite Coxeter groups

At the intersection of algebraic combinatorics and discrete geometry, the following open question resisted for the last 16 years [KM04, Question 6.4]:

*Are the spherical subword complexes realizable as the boundary of simplicial polytopes?*

Checking the polytopal realizability of a spherical simplicial complex is NP-hard. An answer to the above question carries striking consequences: The existence of such polytopes would provide a distinguished family of polytopes with exceptional combinatorial properties and connections to many areas of mathematics, thus opening the door to the use of new discrete geometric tools and the study of their associated toric varieties, for example. On the other hand, if subword complexes are not all polytopal, they would constitute a large and combinatorially simple family of vertex-decomposable simplicial spheres arising *naturally* that *are not* polytopal. But how large is this family exactly? This would be of extraordinary interest, as presently such obstructions are usually obtained using brute force enumeration or *ad hoc* constructions.

Multi-triangulations offer a wide generalization of the simplicial sphere dual to the associahedron [PP12]. The simplicial complex whose faces are multi-triangulations is conjectured to be a polytopal sphere. This conjecture first appeared in writing in the Oberwolfach Book of Abstract of Jonsson in 2003 [Jon03]. Currently, the only known polytopal construction is for the $2$-triangulations of the $8$-gon [BP09, Ceb12, BCL15], and certain cases are known to be realizable as geodesic spheres [BCL15, Man18]. The complex of multi-triangulations turns out to be an example of subword complexes. Optimists may wish that a notion from related areas is key to determine the polytopality of subword complexes and therefore determine if multi-associahedra exist. Answering this question would settle three conjectures simultaneously [KM04, Jon05, SW09]. Two of these conjectures have been open since the beginning of the 2000's and yet little progress has been made.

In the article [BCL15], Bergeon, Ceballos and I lay down necessary conditions for the polytopality of subword complexes. The first step consisted in showing the existence of a certain sign function, which is then used to formulate sign conditions on minors of matrices to obtain *signature matrices*. This sign function is a natural generalization of the notion of odd and even permutations in the symmetric group and relate to work on scattering amplitudes in quantum physics [Ark+16] and permutation patterns in algebraic combinatorics [Ten17]. Then, a combinatorial construction is given that provides signature matrices and it was possible to prove that they lead to complete simplicial fans for subword complexes of type $A_3$ and for certain cases in type $A_4$. In spite of these positive results, the reason *why* the construction works is still mysterious. Additionally, the general knowledge on subword complexes is still scarce. Namely, certain combinatorial aspects of reduced words that lay at the center of the problem are still not explored in details. The geometric interpretation of these aspects is hence inexistant. In [Man18], Manneville gives a complete fan construction of some subword complexes and conjecture that the construction extends to further cases. To prove the polytopality, one can first construct complete simplicial fans with the corresponding combinatorial structure and then show that these fans are the normal fan of convex polytopes.

### On Infinite Coxeter groups

In contrast with the finite case, the infinite weak order is delicate: it is only a meet-semilattice. However, it is possible to embed it into a bigger lattice for Coxeter groups of rank $\leqslant 3$ [Lab13, Theorem 2.35] and this allows to describe the join in the weak order geometrically [HL16]. Nevertheless, a better geometric understanding of the weak order is desirable. When studying and visualizing root systems of infinite Coxeter groups, we obtained intriguing fractal-like pictures using `Sagemath`. These pictures show that roots tend to the isotropic cone of the vector space.

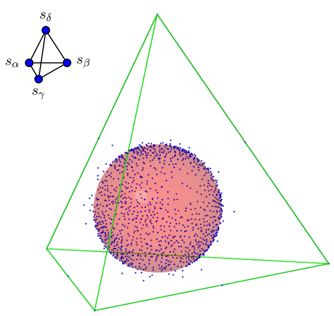



FIGURE 1. The normalized isotropic cone (sphere) and the first thousand normalized roots (dots) for the root system with diagram the complete graph with labels $3$.

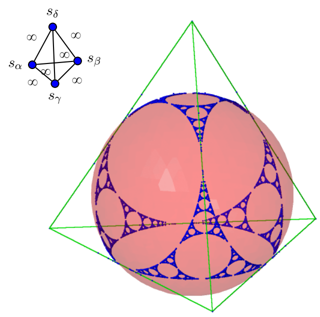



FIGURE 2. The normalized isotropic cone (sphere) and the first thousand normalized roots (dots) for the based root system with diagram the complete graph with labels $\infty$.

The investigations of these pictures led to many developments surrounding infinite Coxeter groups of which the following problems are still open.

(1) How to understand the weak order of Coxeter groups geometrically in general?
(2) How to unite the combinatorial and geometric weak order?
(3) How to characterize Lorentzian Coxeter groups by their graphs?

<u>Geometry of infinite Coxeter groups</u>

*Limit roots approach to the weak order.* *Limit roots* were introduced to obtain a geometric understanding of the asymptotic behavior of roots shown in the pictures above [HLR14], and computational and visualization tools were implemented in `Sagemath` [Lab17]. Thus far, limit roots have shown connections in particular with sphere packings [CL15], the imaginary cone [DHR16], and Garside theory [DH16].

*Relation to sphere packings.* The relationship between the Coxeter groups with sphere packings can be phrased as follows: The set of limit roots forms a ball packing if and only if the Coxeter graph of the group is so-called *of level 2* [CL15]. This is a restrictive property introduced by Maxwell [Max82] to give a dual version of this result, which leads to a finite set of graphs, which can be classified. Chen and I also corrected the previous enumeration of level $2$ graphs done by Maxwell.

*A Perron theorem and applications to Coxeter groups.* To comprehend the dynamics of Coxeter groups, it is essential to have a precise description of the spectrum of infinite order elements. So far, only Coxeter elements have a complete description with striking properties [ACa76] and some other elements were studied in [How82, McM02]. Using tools from Perron–Frobenius theory on primitive matrices, it is possible to get a better view of the spectrum of infinite-order elements [LL15].

*Lorentzian Coxeter groups.* When a Coxeter group acts on a Lorentz space, it is called *Lorentzian*. Which Coxeter groups are Lorentzian? Vinberg discussed this long standing problem in geometry at the ICM in 1983 in Warsaw [Vin84]. Although a lot of progress has been made since then [PV05, Duf10, FT14], the general problem still eludes our comprehension.

Combinatorics of infinite Coxeter groups

*Limit directions and infinite reduced words.* Given a random vector in space, what is the limit set of its orbit under the action of a Coxeter group? These limit sets can be characterized using eigenspaces of infinite order element in the case of Lorentz spaces [CL17]. This approach exhibits peculiar properties of the infinite arrangements of hyperplanes. Limit roots are related to infinite reduced words, introduced by T. Lam and P. Pylyavskyy [LP13, LT15]: To an infinite reduced word corresponds a unique limit root [CL17]. This exhibits yet another a deep relationship between the geometry of the limit set with the combinatorics of the weak order of the group that deserves to be explored in more detail. This builds on the work of M. Dyer, C. Hohlweg, J.-P. Préaux and V. Ripoll [DHR16, HPR13].

*On inversion sets and the weak order in Coxeter groups.* It turns out that the join in the weak order can be described geometrically in terms of inversion set and their relative position with respect to the imaginary cone [HL16]. Therein, Hohlweg and I also presented many conjectures related to the weak order, biclosed sets, and infinite reduced words.

Geometric Combinatorics of Infinite Coxeter groups.

The peculiar configurations of the infinite hyperplane arrangements observed in [CL17] suggest a deeper study of their geometric and combinatorial properties. There are currently very little knowledge on infinite hyperplane arrangements in spaces other than Euclidean spaces. What can be said about infinite and not discrete hyperplane arrangements in Lorentz space? This suggests the introduction of an infinite poset of flats of finite rank that would describe infinite Coxeter arrangements.

## On the Computational Framework

*`Sagemath` and combinatorics on words.* Subword complexes involve a great deal of combinatorial structures. Their study involves challenging computational problems in order to obtain even relatively small examples. To face the exponential complexity of the generation of these objects, we implemented a large-scale database storing sign functions and subwords. This is a powerful tool that made it possible to look at examples that could not previously be computed. Using tools from multilinear algebra, Schur functions and combinatorics of Coxeter groups, we layed down the fundation of the convex geometry of subword complexes and present a universal oriented matroid that *realizes them all*, i.e., whose realizability is equivalent to the realizability of “essentially all” subword complexes [Lab19].

*`Sagemath` implementation of limit roots.* Besides the theoretical setting, the implementation of experimental tools in `Sagemath` [Sage] is well underway [Lab17]. This infrastructure was very successful in exhibiting behaviors before they were proved. As a consequence, researchers around the world benefit from the latest developments in the current theory.

*Polyhedral Geometry in `Sagemath`.* Since February 2017, a lot of efforts and energy was put into the polyhedral geometry libraries available in `Sagemath`. There is a concrete need from various researchers to be able to use the cutting-edge algorithms offered through different softwares “under the same hood”. `Sagemath` offers a well-established user-friendly interface to several open-source softwares. The following meetings were organized to bring together the different communities and initiate closer cooperations.

- SageDays 84, February–March 2017, 2 weeks, (Olot, Spain) (`latte`, `polymake`, and `Sagemath` communities)
- IMA Coding Sprint on Polyhedral Geometry, April 2018, 2 weeks, (Minneapolis, USA) (`latte`, `Normaliz`, `PARI/GP`, `polymake`, and `Sagemath` communities)
- Research in Pairs MFO, April–May 2019, 2 weeks, (`latte`, `Normaliz`, `polymake`, and `Sagemath` communities)
- Research in Pairs CIRM, June 2020, 1 week, (`e-antic`, `latte`, `Normaliz`, and `Sagemath` communities)

These meetings included around 25 developers from a broad spectrum of research areas whose expertise and diversity created a vivid workflow. The exchanges were very fruitful, brought the different communities closer, and made a significant difference in the progress of the implementation framework. Through this series of workshops we reached several important milestones. The first major one was the successful implementation and development of interfaces to `Normaliz` and `polymake` in `Sagemath`. Furthermore, the documentation and stability of the libraries have been improved greatly for a greater accessibility for new users. These developments were paramount in the study of obstructions to inscribability of polytopes [Doo+19]. Currently, `Sagemath` offers the leading algorithms of `polymake`, `Normaliz`, and `latte` and further its own cutting-edge implementation of combinatorial invariants of polyhedral objects. A particularly innovative outcome of this initiative is the possibility to execute *fast* exact convex hull calculations on arbitrary number fields using a combination of number theory libraries and `Normaliz` using `python` and is now fully accessible in `Sagemath`.

### 1.1. **Project-related publications.**

#### 1.1.1. *Articles published by outlets with scientific quality assurance, book publications, and works accepted for publication but not yet published.*


[1] Nantel Bergeron, Cesar Ceballos, and Jean-Philippe Labbé, *Fan realizations of type $A$ subword complexes and multi-associahedra of rank 3*, Discrete Comput. Geom. **54** (2015) no. 1, 195–231.

[2] Sarah B. Brodsky, Cesar Ceballos, and Jean-Philippe Labbé, *Cluster algebras of type $D_4$, tropical planes, and the positive tropical Grassmannian*, Beitr. Algebra Geom. **58** (2017) no. 1, 25–46.

[3] Cesar Ceballos, Jean-Philippe Labbé, and Christian Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, J. Algebraic Combin. **39** (2014) no. 1, 17–51.

[4] Hao Chen and Jean-Philippe Labbé, *Lorentzian Coxeter systems and Boyd-Maxwell ball packings*, Geom. Dedicata **174** (2015) 43–73.

[5] Hao Chen and Jean-Philippe Labbé, *Limit directions for Lorentzian Coxeter systems*, Groups Geom. Dyn. **11** (2017) no. 2, 469–498.

[6] Christophe Hohlweg and Jean-Philippe Labbé, *On inversion sets and the weak order in Coxeter groups*, European J. Combin. **55** (2016) 1–19.

[7] Christophe Hohlweg, Jean-Philippe Labbé, and Vivien Ripoll, *Asymptotical behaviour of roots of infinite Coxeter groups*, Canad. J. Math. **66** (2014) no. 2, 323–353.

[8] Jean-Philippe Labbé and Carsten Lange, *Cambrian acyclic domains: counting $c$-singletons*, Order (to appear) (2019) 24 pp.

[9] Jean-Philippe Labbé, Thibault Manneville, and Francisco Santos, *Hirsch polytopes with exponentially long combinatorial segments*, Math. Program. **165** (2017) no. 2, Ser. A, 663–688.

[10] Jean-Philippe Labbé and Eran Nevo, *Bounds for entries of $\gamma$-vectors of flag homology spheres*, SIAM J. Discrete Math. **31** (2017) no. 3, 2064–2078.


#### 1.1.2. *Other publications.*


[11] Winfried Bruns, Vincent Delecroix, Matthias Köppe, and Jean-Philippe Labbé, *Algebraic polyhedra in Sagemath with Normaliz*, in preparation (2019) 21 pp.

[12] Sophia Elia and Jean-Philippe Labbé, *Congruence normality and oriented matroids*, in preparation (2019) 17 pp.

[13] Jean-Philippe Labbé, *Convex geometry of subword complexes of Coxeter groups*, in preparation (2019) 30 pp.


#### 1.1.3. *Patents.*

1.1.3.1. Pending.

not applicable

1.1.3.2. Issued.

not applicable

## 2. **Objectives and work programme**

### 2.1. **Anticipated total duration of the project.**

The duration of the project is intended to be **3 years** (**36 months**) and the DFG funds will be necessary for the entire duration of the project.

### 2.2. **Objectives.**

The overarching goal of this project is two-fold. First, this project will reveal new knowledge on fundamental structural properties of Coxeter groups. Second, this acquired knowledge will be exploited to expose and detail new connections with a variety of closely related geometric objects. This project serves as the first step towards accomplishing the following long term goals: 1) establish an explicit pathway to determine whether subword complexes are boundaries of polytopes 2) clarify how Coxeter groups generalize cyclic polytopes 3) lay out the foundations of the study of equivariant infinite hyperplane arrangements that do not yield CAT-$0$ spaces, and 4) classify Lorentzian Coxeter groups of bounded dimensions. More specifically, the project will have a constructive impact on the understanding of hyperbolic manifolds, subword complexes, the associahedron and its generalizations, and Weyl groupoids.

We will start by studying properties of Coxeter Groups through their graphs and reduced words, in order to quantify their similarities or differences. Then, we plan to study a partial order on Coxeter graphs introduced by McMullen [McM02] that respects properties related to the Poincaré series and limit roots. We want to have a deep understanding of this poset. We anticipate many potential applications of this poset and will use it to obtain necessary and sufficient conditions to obtain Lorentzian Coxeter groups. This knowledge will later be helpful to construct more hyperbolic manifolds arising from infinite Coxeter groups. Also at the beginning of the project, we will study several properties of reduced words: determine which braid equivalence relations preserve geometric properties of limit roots, give a combinatorial characterization of infinite order elements, and investigate word statistics of reduced words. As for subword complexes, we want to describe explicitly the realization space of rank-$3$ subword complexes first, then get some numerical evidence to approach the upper bound conjecture for subword complexes and to enumerate their number of combinatorial types.

Finally, the project aspires to uncover exciting, yet unexplored, relationships between Coxeter groups and the following objects: Bottman's 2-associahedron in symplectic geometry, exceptional type sextonions in Lie algebras, and poset of regions of simplicial hyperplane arrangements arising from Weyl groupoids.

### 2.3. **Work programme incl. proposed research methods.**

The project will pursue research simultaneously on several fronts. As the knowledge on structural properties of Coxeter groups is capital for the good progression of the project, it should be addressed in priority at the beginning of the funding period. Below we detail the subprojects organized according to the Finite/Infinite and Geometry/Combinatorics divisions previously established and their interdependence with respect to the research agenda. Further, where applicable, we detail how the travel funding, research visits, and workshop will support the progression of the subproject.

#### 2.3.1. *On Finite Coxeter Groups.*

##### 2.3.1.1. Combinatorics of Finite Coxeter Groups.

▪ *Reduced expressions in exceptional cases* 09/2020 to 04/2021

A formula for the number of reduced expressions of the longest element in the symmetric group using Young tableaux was given by Stanley [Sta84], and further instructive descriptions were obtained by Edelman and Greene [EG87] and Lascoux and Schützenberger [LS82]. We aim at obtaining all reduced expressions of the longest element in the exceptional cases with rank $4$: so far the types $H_4$ and $F_4$ have resisted many attempts. There is a straightforward algorithm to produce them, but it has the pitfall of using exponential diskspace. We will develop intuition coming from combinatorics on words in order to approach the problem from a different angle than using braid moves to generate all reduced words.

▪ *Structural properties of Subword Complexes* 09/2020 to 04/2021

Multi-cluster complexes form a very structured subfamily of subword complexes. They unify cluster complexes and the simplicial complex of multi-triangulations. Narayana numbers are a specialization of Catalan numbers that use one more parameter which depends on the Catalan object. Narayana numbers form what is called the $h$-vector of the associahedra and have an important connection to commutative algebra via the face ring of the associahedra. Multi-cluster complexes of type $A$, or multi-associahedra, provide a natural generalization of Narayana numbers via their $h$-vectors. These generalized Narayana numbers count a similar generalized structure; nevertheless no general formula is known, apart for the number of facets. In contrast, for general types, there is still no known uniform formula for the number of facets. Multi-cluster complexes are conjectured to have extremal $f$-vectors among subword complexes on prescribed number of vertices. That is, they would give an upper bound on the number of faces that a subword complex can have. Using brute force enumeration and polynomial interpolation manipulations, it is possible to get a formula for the $h$-vector for the family of multi-associahedra $A_3$. Inspired by the work of R. Simion and D. Ullman [SU91] on symmetric chain decompositions of the lattice of noncrossing partitions in relation with the $\gamma$-vector of the associahedra [PRW08], we seek to obtain a general formula for the $h$-vector of the multi-associahedra.

Another potential approach is to study the shifted multi-cluster complexes. This question will necessitate to use a computational framework to study the result of the shifting operation on multi-cluster complexes. Indeed, shifting involves very technical algebraic manipulations and perhaps the only practical way to study them is by using a non-deterministic algorithm. We seek a description of their the structure once shifted.

In addition, in order to place the subword complexes' upper bound result in context with the set of simplicial spheres, it would be important to obtain an asymptotic formula as to quantify their size compared to all simplicial vertex-decomposable spheres.

Also, we expect to extend the work presented in [Ceb+, CP16, Pou14, Pou17] to study the diameter of multi-cluster complexes. For this, the computations of reduced words of exceptional finite types will play a prominent role.

- *Word statistics of reduced words* 01/2021 to 12/2021

Let $(W, S)$ be a Coxeter group and $w \in W$. For each letter $s \in S$, what is the minimum and maximum number of times that $s$ can appear in a reduced word for $w$? In type $A$, this simple question is related to the halving line problem in discrete geometry: Given $n$ points in generic position on the plane, a halving line is a line that partitions the points into two sets of $n/2$ points. What is the largest possible number of halving lines for a set of $n$ points in the plane? Initially studied by L. Lovász and P. Erdős in the 1970s, the current best upper bound is $O(n^{4/3})$, obtained by T. Dey [Dey98]. The current lower bound $\Omega(ne^{c(\log n/2)^{1/2}})$, for some constant $c$, was obtained by G. Tóth [Tót01].

Halving lines are related to reduced words of the reverse permutation in the symmetric group using planar point-line duality. It turns out that the computational complexity of subword complexes is deeply related to this question. We seek to describe bounds for the possible number of occurrences of letters in reduced words. Again, the first bounds will build upon evidence brought by the computations of reduced words in the exceptional cases.

- *Study shortest superstring problems of Coxeter groups* 01/2022 to 12/2022

Problem 6.1 of [KM04] asks: What is the smallest size of a word in $S$ containing every reduced expression for $w$ as a subword? This turns out to be a particular instance of the shortest superstring problem. One may also ask relevant variations, where one asks to contain every reduced words for $w$, up to commutations. The shortest superstring problem is NP-complete already on alphabets of size $2$ [RU81]. By improving knowledge on reduced words, we pursue an answer to this problem for reduceds words of finite Coxeter groups. Further, the study of word statistics of reduced words will help in getting a hand on such shortest superstrings. As it turns out, the length of such a superstring is a direct indicator of how difficult it is to realize subword complexes geometrically [Lab19].

2.3.1.2. Geometry of Finite Coxeter Groups.

- *Geometric Realizations of Subword Complexes* 09/2020 to 08/2022

This project's long term goal is to determine whether subword complexes are boundaries of convex polytopes. The recent developments lay out in the article [Lab19] suggest to a part of the realization space of small rank subword complexes as a semi-algebraic set using Gale duality and Schur functions. We plan to examine the recently developed theory of slack realization spaces in the case of subword complexes with the help of Amy Wiebe and Rainer Sinn. The connection between realization spaces of subword complexes and Schur functions deserves to be examined in more detail.

▪ *Bottman's 2-associahedron* 09/2020 to 08/2021
Nathaniel Bottman introduced a poset that widely generalizes the face lattice of the associahedron [Bot19]. This generalization stems from symplectic geometry in relation with Fukaya categories of different symplectic manifolds. Further, the structure of these posets are combinatorially very close to face lattices of polytopes, i.e. they were also proved to be Eulerian [BM19]. It is thus very natural to ask whether they can be realized at the boundary of convex polytopes. In a cooperation with Nathaniel Bottman, we plan to establish a translation of the 2-associahedron into purely combinatorial terms, study its enumerative properties and finally obtain geometric realization. For this, we will compare it to the other myriad of generalizations of the associahedron available [MPS12]. This subproject does not require new methods *a priori* and should therefore start at the beginning of the project by a visit of Nathaniel Bottman in Berlin.

▪ *Combinatorics of Weyl Groupoids* 09/2020 to 09/2021
When Weyl groupoids admit a finite root system, one can generalize the usual weak order using the poset of regions of the associated simplicial hyperplane arrangement [HW11, CH15]. Preliminary direct evidence suggests that finite Weyl groupoids are congruence uniform. Sophia Elia and I will examine how the property of congruence uniformity behaves for finite Weyl groupoids exactly. Are they all congruence uniform? We will use oriented matroids as a main tool to study the combinatorics of finite Weyl groupoids and extract the necessary notions that lead to congruence uniformity.

▪ *Relate subword complexes to Lie algebras* 01/2021 to 12/2021
In [LM06], Joseph Landsberg and Laurent Manivel describe an exceptional Lie algebra called $E_{7\frac{1}{2}}$ with similar properties as the Lie algebras $E_7$ and $E_8$. They also study some other subexceptional Lie algebras whose dimension, in a certain case, correspond to the number of facets of a multi-cluster complex of type $H_3$. In collaboration with Laurent Manivel throughout our planned research visits in 2021, we will study the internal symmetries of the related Lie algebras and determine how they relate to the symmetries of the relevant subword complexes.

▪ *Generalization of Cyclic Polytopes* 01/2022 to 08/2023
The study of realizations of subword complexes presented in [Lab19] motivates the following question in discrete geometry. Consider $n$ generic skew curves of degrees $k_1, k_2, \dots, k_n$ and $m$ points distributed on them. What kind of polytopes arise as the convex hull of these $m$ points? When the number of curves is $1$, this correspond to the construction of the cyclic polytope. It turns out that subword complexes are related to such polytopes. We expect to be able to describe polytopes that arises using such a construction using Gale duality and oriented matroids.

### 2.3.2. *On Infinite Coxeter Groups.*

#### 2.3.2.1. Geometry of Infinite Coxeter Groups.

We will expand the theory of geometric representations of Coxeter groups by clarifying the relation between limit roots and dynamical systems, and properly understand the dynamics and combinatorics of infinite order elements. First, an effective geometric characterizations of limit roots, limit weights and infinite order elements of Coxeter groups is seeked. These characterizations will help to reveal further the structure of Tits cones. Another step is to overcome the difficulties when passing from Lorentz space to the general case. Indeed, many of the current results rely deeply on the geometry of Lorentz space. Therefore calling for a better understanding of the dynamics of infinite Coxeter groups in general. The theory of primitive matrices seems to fit the problem well and further research in this direction should lead to an explicit description of their spectra.

▪ *Construct more hyperbolic manifolds* 01/2021 to 12/2021
Reflection groups in hyperbolic spaces are important in relation to Poincaré's theorem on fundamental polyhedra of group acting on hyperbolic spaces, see e.g. [Kap09, Section 4.11] and [Mar16, Section 3.5]. The notion of face-pairing transformation is used to give the characterization. It turns out that obtaining face-pairings in higher dimension is a challenge. Indeed, much less is known on hyperbolic polyhedra in dimension higher than $3$: their combinatorics and geometric relations can be wild [APT15]. This subproject envisage to study Lorentzian Coxeter groups with a perspective on getting face-pairings for the level $2$ Coxeter graphs. This will help in getting many test cases for a generalization of an algorithm that determines the fundamental polyhedra

of arithmetic Kleinian groups [Pag15]. This project will benefit from the collaboration with Vincent Delecroix, who has a specialized expertize on the dynamics of Kleinian and Bianchi groups. In relation with this investigation, Vincent Delecroix posed the following quite intriguing question: Is there a pair of combinatorially distinct polytopes $P$ and $Q$ such that their facet–ridge graphs—enriched with their ridges angles—are isomorphic? To start the examination, we will look at the case of polyhedra coming from representations of Coxeter groups. This subproject is expected to start at the beginning of 2021 by a visit to the University of Bordeaux and continue during visits of Vincent Delecroix in Berlin.

▪ *Connection between Limit Roots and irreducible reduced words* 01/2022 to 12/2022
In [CL17], Chen and I show that the limit roots of Lorentzian Coxeter groups are the only isotropic limit directions and that to every infinite reduced word corresponds a unique limit root. The proof uses the convexity of the isotropic cone and its interior. To extend the result to the general case, we have to reobtain the same result without making use of convexity. For this, we need to study eigenvalues of the associated matrices and bring further the results presented in [LL15].

▪ *Study infinite hyperplane arrangement* 01/2022 to 08/2023
To study infinite hyperplane arrangements, we will first need to extend the notion of lattice of flats to the case when hyperplane arrangements are not locally finite and see how the known properties behave in this extension. The theory available in [CL17] and the computational framework [Lab17] will make it possible to test the developed theory of locally infinite lattice of flats on several examples. This will also necessitate an adapted notion of oriented matroid to the infinite case.

▪ *Classify Lorentzian Coxeter Groups* 01/2022 to 08/2023
It is possible to encode Coxeter graphs as certain set of points in a vector space. We propose to classify Lorentzian Coxeter groups using strategies from lattice polytope theory and Robertson-Seymour's theorem about forbidden minors [RS04]. Indeed, the property of being Lorentzian can be phrased in the language of forbidden minors. Thus there exists a finite set of obstruction which describes the non-Lorentzian Coxeter groups. Obtaining a list of these obstructions seems to be within reach. Further, using the computational framework, we expect to obtain new sufficient conditions for a Coxeter graph to yield a Lorentzian Coxeter group.

2.3.2.2. Combinatorics of Infinite Coxeter Groups.

▪ *Minor partial order on Coxeter graphs* 01/2021 to 12/2021
In order to study Lorentzian Coxeter groups and more generally to describe the types of spaces on which Coxeter groups act, it seems that the partial order defined by McMullen [McM02] might be useful. McMullen already described certain minimal elements, and we plan to extend his poset to the general Coxeter graphs involved in the theory of limit roots and determine how the poset interacts with the notion of level.

▪ *Combinatorial characterization of LR, LW, infinite order element* 01/2022 to 12/2022
Provided a reduced word of an infinite reduced word, is it possible to determine if this word has infinite order solely using combinatorics on words? We will investigate the various automata related to reduced words to know whether it is possible to modify them to provide automata that recognize infinite order elements. In view of the fact that infinite order elements lead to a very precise description of the Tits cone and limit roots, having an automaton to generate them would be very useful. Finally, we will look into the spectrum of more elements using ideas coming from dynamical systems such as the subshift of finite type model and iterated function systems to mimic the group action.

2.3.3. *Computational Framework.*
*Develop a database of reduced words.* 08/2020 to 04/2021
As the information on reduced words is paramount for this project, we aim to build a database of reduced words available online and through `Sagemath`.

*Implementations in `Sagemath`.* 08/2020 12/2021
Many computations will be necessary to support this project. Motivated by this necessity, we will make these computations available to the community through the peer-reviewed implementation of `Sagemath`. We aim simultaneously to make `Sagemath` *the integrated user-friendly interface of choice for fast and reliable computations* that support the usage of many available open source geometric software. These computations include: the shifting operation (that will interest researchers in commutative algebra), algebraic polytopes, Coxeter graphs. Further we plan to

provide interfaces to the functionalities of `Bertini` and `JuliaHomotopyContinuation` in `Sagemath`. The team of developers Sophia Elia, Jonathan Kliem, and Laith Rastanawi that I formed in the Discrete Geometry workgroup is expected to expand its computational capacities all along the duration of this project.

2.3.4. *Choice of host institution.* Along with my doctoral student Sophia Elia (supervised jointly with Pr. Christian Haase through the Research Training Group "Facets of Complexity"), we will continue to study the geometric combinatorics of Weyl groupoids and further investigate equivariant Ehrhart theory which joins the representation theory of finite Coxeter groups and Ehrhart theory. Further, with Dr Amy Wiebe (Dirichlet Postdoctoral Fellow, Discrete Geometry Group 2019-2021) and Prof. Rainer Sinn, we will study the slack realization spaces of subword complexes. The computational framework layed out in `Sagemath` underlying the project will benefit from the cooperation with the doctoral students Laith Rastanawi (RTG "FoC", Discrete Geometry Group, time period 2018-2021) and Jonathan Kliem (Ph.D. Student, Discrete Geometry Group, until March 2023).

The MATH+ Research Center and the Research Training Group "Facets of Complexity" will act as a catalyst for the present project. Further the deep integration of the Berlin research groups related to discrete and geometric structures and their computational aspects makes it a top-level source of feedback and provides access to high-quality infrastructures such as the computing facilities at the Freie Universität Berlin.

### 2.4. Data handling.

Research data steming from the results obtained during this research project will be made available via the `Sagemath` open source software through its code reviewing platform.

### 2.5. Other information.

not applicable

### 2.6. Descriptions of proposed investigations involving experiments on humans, human materials or animals as well as dual use research of concern.

This project does not involve any experiments on humans, human materials or animals as well as dual use research of concern as referenced by the DFG guidelines.

### 2.7. Information on scientific and financial involvement of international cooperation partners.

not applicable

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## 4. **Requested modules/funds**

The present project request a total funding of 248,050 EUR in order to successfully achieve the inherent objectives. The funding covers a postdoctoral research position 222,300 EUR, travel expenses 11,000 EUR, visiting researchers 5,000 EUR, publication expenses 2,250 EUR, and workshop funding 7,500 EUR.

### 4.1. **Basic Module.**

Within the framework of the project, the funding request for the Basic Module amounts to 18,250 EUR. The supporting justification for the use and necessity of these funds for the research project are detailed in the next sections.

#### 4.1.1. *Funding for Staff.*

not applicable

#### 4.1.2. *Direct Project Costs.*

Some parts of this project will be done in cooperation with collaborators abroad. In order to make the collaborations effective, we are asking for financial support at the level of 11,000 EUR for travel expenses, 5,000 EUR to host the collaborators at the host institution, and finally 2,250 EUR to fund the OpenAccess publications of the results of the project.

4.1.2.1. Equipment up to Euro 10,000, Software and Consumables.

not applicable

4.1.2.2. Travel Expenses.

To strengthen research production and to favor the successful completion of the project, we established collaborations with experts in areas surrounding the project. These experts were invited to collaborate in the relevant subprojects. These collaborations are based on regular scientific visits to first establish the collaborations and afterwards work on the respective project strands. We plan to visit the University of Bordeaux–LaBRI and the University of Toulouse. The research visits in 2021 will start the collaborations and lay down the research tasks. The research visits in 2022 are intended to ensure the continuity of research and the dissemination of the knowledge gained.

Furthermore, during the course of the research project, we intend to submit the latest results to international conferences and symposiums (e.g. FPSAC, SIAM Conferences, CanaDAM, and the European Congress of Mathematics) to bring exposure to the research developments of the project.

| Funding for | 2020/2 | | 2021 | | 2022 | | 2023/1 | |
|---|---|---|---|---|---|---|---|---|
| | Quant. | Sum | Quant. | Sum | Quant. | Sum | Quant. | Sum |
| *Conferences* | 0 | 0 | 1 | 2,000 | 1 | 2,000 | 1 | 2,000 |
| *Research visits* | 1 | 1,000 | 2 | 1,500 | 2 | 1,500 | 1 | 1,000 |
| *Total* | | 1,000 | | 3,500 | | 3,500 | | 3,000 |

(All figures in Euro)

4.1.2.3. Visiting Researchers (excluding Mercator Fellows).

The research agenda relies on the visit of researchers to cooperate in certain parts of the project. The research agenda plans a visit of Nathaniel Bottman (University of Southern California) in Berlin in the Fall of 2020. Further, we plan to invite Vincent Delecroix (University of Bordeaux) and Laurent Manivel (University of Toulouse) for annual research stays in Berlin.

| Funding for | 2020/2 | | 2021 | | 2022 | | 2023/1 | |
|---|---|---|---|---|---|---|---|---|
| | Quant. | Sum | Quant. | Sum | Quant. | Sum | Quant. | Sum |
| *Research visits* | 1 | 2,000 | 2 | 1,000 | 2 | 1,000 | 2 | 1,000 |
| *Total* | | 2,000 | | 1,000 | | 1,000 | | 1,000 |

(All figures in Euro)

4.1.2.4. Expenses for Laboratory Animals.

not applicable

4.1.2.5. Other Costs.

not applicable

4.1.2.6. Project-related publication expenses.

During this project, we aim to publish in excellent OpenAccess journals. In conjunction with the OpenAccess policy of the Freie Universität Berlin we request further funding for the purpose of publishing in OpenAccess journal satisfying the DFG criteria. We request the following funding for the project.

| Funding for | 2020/2 | | 2021 | | 2022 | | 2023/1 | |
|---|---|---|---|---|---|---|---|---|
| | Quant. | Sum | Quant. | Sum | Quant. | Sum | Quant. | Sum |
| *Publication Expenses* | 0 | 0 | 1 | 750 | 1 | 750 | 1 | 750 |
| *Total* | | 0 | | 750 | | 750 | | 750 |

(All figures in Euro)

4.1.3. *Instrumentation.*

4.1.3.1. Equipment exceeding Euro 10,000.

not applicable

4.1.3.2. Major Instrumentation exceeding Euro 50,000.

not applicable

4.2. **Module Temporary Position for Principal Investigator.**

This Module is requested for Dr. Jean-Philippe Labbé to be employed full-time to work on this project. At the time of employment, the cumulated number of years of postdoctoral research experience will be of seven (7) years. The incumbent employment contract expires at the end of June 2020.

| Funding for | 2020/2 | | 2021 | | 2022 | | 2023/1 | |
|---|---|---|---|---|---|---|---|---|
| | Quant. | Sum | Quant. | Sum | Quant. | Sum | Quant. | Sum |
| *Staff* | | | | | | | | |
| Postdoctoral researcher | 1 | 24,700 | 1 | 74,100 | 1 | 74,100 | 1 | 49,400 |
| *Total* | | 24,700 | | 74,100 | | 74,100 | | 49,400 |

(All figures in Euro)

**Job description of staff (requested):**

1. J.-P. Labbé is expected to work full-time on the research agenda and advance the project in all its components.

**Job description of staff (supported through available funds):**

2. S. Elia joined FU Berlin in October 2016 as a BMS Phase I student. She started her PhD studies in October 2018 and is supported by the "Facets of Complexity" Research Training Group funded by the DFG. Her work centers on combinatorics of hyperplane arrangements, tropical convexity and equivariant Ehrhart theory.

4.3. **Module Workshop Funding.**

4.3.1. *Description.*

`Sagemath` is a free, open-source computer algebra system that brings together cutting-edge packages in various areas of mathematics. We plan to organize a workshop under the title "Sage Days on Geometric Combinatorics". There were more than 140 meetings in the last 15 years, of which I personally attended seven and organized two (in Jerusalem in November 2016, and in Olot (Spain/Catalunya) in March 2017). The Sage Days are now an established meeting where `Sagemath` power users and development experts gather with several goals: expand the community by giving tutorials to unfamiliar mathematicians from Master level to Professors, bring further the capacities of `Sagemath`, and obtain novel theoretical results obtained through experimental results.

Until now, only one Sage Days aimed broadly at new users occured in Germany. It was held in Bonn in June 2019, and organized by Vincent Delecroix. Berlin is well-known for its application-driven mathematics and this workshop aims at bringing together the `Sagemath` community and researchers from new research projects from the MATH+ initiative. Indeed, many new projects involve computational components, for example the *Research Area 8: Mathematics of data science*. We aim to make Machine-Learning-Frameworks like TensorFlow and PyTorch accessible to theoretical mathematicians for the benefit of their research.

4.3.2. *Objectives.*

Sage Days are meant to be flexible and fit the needs required to achieve the organizers' goals. The goals of the Sage Days in Berlin are:

- provide a proper introduction to `Sagemath` and `python` for beginners and intermediate users,
- provide support to participants familiar with `Sagemath` from expert `Sagemath` developers,
- showcase cutting-edge results and techniques related to `Sagemath` and computational mathematics in general,
- familiarize the participants to open-source softwares,
- enlarge the `Sagemath` community in Germany.

Finally, the last goal is to contribute explicitly to the progress of `Sagemath` by providing new code to `Sagemath` from the participants. Experience has shown that the following format brings a maximum of impact on scientific innovation.

4.3.3. *Format.*

The workshop will be two weeks long. The first will take the form of a school, where experts welcome beginners in the community and teach them how to use the software. It will be hosted on a Berlin University campus.

*Week 1. School:* Intended to a larger public that includes Master, PhD students as well as Post-Doc and Professors that are interested in learning `Sagemath`. It will consists in building-up talks and tutorials on specific topics, completed by a scientific component involving plenary talks by theoretical mathematicians where potential for computational applications has been suspected.

The second week will take the form of an intensive workshop focused on implementing new features. The second week's format will greatly benefit from taking place in a small conference center to maximize working time.

*Week 2: Workshop:* Intended to a restricted group of intermediate to advanced users, `Sagemath` developers, and researchers in Geometric Combinatorics. The week will consist in intense coding sessions and scientific presentations on advanced topics related to `Sagemath` and Geometric Combinatorics. Further, much of the time of the second week will be dedicated to scientific exchange between the participants. At the end of each day, the participants will expose their progress in a common status report, where feedback from different groups will be given.

4.3.4. *Selection procedure.*

In order to get the maximum out of the first week, the number of participants will be restricted to at most 40 to 45 participants. The Master and PhD Student should provide a reference person and explain the relevance of the workshop for their research project. This selection process done in conjunction with scientific members of the MATH+ and the Research Training Group "Facets of Complexity" will help in planning the tutorials. For the second week, the number of participants should not exceed 20 to maximize the interaction between participants.

4.3.5. *Projected Budget.*

| | | |
|---|---|---|
| *Funding* | | |
| DFG Module – Workshop Funding | 7,500 | |
| MATH+ | 5,000 | |
| GRK-2434 | 2,500 | |
| *Expenses* | | |
| Travel support | | 7,500 |
| Accommodation support | | 6,000 |
| Catering | | 1,500 |
| Total | 15,000 | 15,000 |

(All figures in Euro)

For the successful completion of the workshop's goals, we request funding through the DFG Module Workshop at the level of 6,000 EUR. We plan to submit funding requests to the Berlin Mathematics Research Center MATH+ and the Research Training Group "Facets of Complexity" through their dedicated scientific activities organized for local and international participants. The infrastructure and experience of MATH+ will guarantee a maximal impact of the event on the community and the herein proposed project.

## 5. **Project requirements**

### 5.1. **Employment status information.**

The sole applicant Jean-Philippe Labbé has occupied the following positions after obtaining his doctorate up to this day.

| Employment status | Period | Funding body |
|---|---|---|
| Postdoctoral researcher | 10/2016 to 06/2020 | DFG, SFB-109 Project A03 |
| Postdoctoral researcher | 10/2014 to 09/2016 | Israel Science Foundation (805/11, PI: Nevo) |
| | | Hebrew University Jerusalem |
| Postdoctoral fellow | 01/2015 to 12/2016 | FRQNT (Québec Science Foundation) |
| | | Hebrew University Jerusalem, and Universidad de Cantabria (Spain) |
| Postdoctoral researcher | 07/2013 to 06/2014 | DFG, SFB-109 Project A03 |

### 5.2. **First-time proposal data.**

This is my first proposal for a DFG "Einzelprojekt – Sachbeihilfe" with the Modules "Temporary Position for Principal Investigator" and "Workshop Funding".

### 5.3. **Composition of the project group.**

My doctoral student will be involved in one branch of the project.

| Name | Academic title | Employment status | Funding Type |
|---|---|---|---|
| Sophia Elia | M. Sc. | Doctoral researcher (10/2018 to 09/2021) | GRK-2434 (DFG) |

### 5.4. **Cooperation with other researchers.**

#### 5.4.1. *Researchers with whom you have agreed to cooperate on this project.*

The project will benefit from cooperations with members of the Discrete Geometry Group at the Freie Universität Berlin and the Research Training Group "Facets of Complexity".

| | |
|---|---|
| Sophia Elia, M. Sc. | is a BMS doctoral student in the Discrete Geometry group at the Freie Universität Berlin. |
| Amy Wiebe, Ph.D. | is a Math+ Dirichlet postdoctoral fellow in the Discrete Geometry group at the Freie Universität Berlin. |
| Rainer Sinn, Prof. Dr. | is a junior professor in the Discrete Geometry group at the Freie Universität Berlin. |
| Laith Rastanawi, M. Sc. | is a BMS doctoral student in the Discrete Geometry group at the Freie Universität Berlin. |
| Jonathan Kliem, M. Sc. | is a BMS doctoral student in the Discrete Geometry group at the Freie Universität Berlin. |

Further, the project involves cooperations with the following partners located abroad.

| | |
|---|---|
| Nathaniel Bottman, Ph.D. | is an assistant professor at the University of Southern California (Los Angeles). He is working in symplectic geometry, algebraic geometry and combinatorics. |
| Laurent Manivel, Dr. | is directeur de recherche at the Toulouse Mathematics Institute. He is working in particular in complex algebraic geometry and studies homogeneous spaces and representations of algebraic groups. |
| Vincent Delecroix, Dr. | is chargé de recherche at the University of Bordeaux (LaBRI). He works in combinatorics, number theory, geometry and dynamical systems. |

5.4.2. *Researchers with whom you have collaborated scientifically within the past three years.*

| | |
|---|---|
| Winfried Bruns | is professor at the Universität Osnabrück |
| Matthias Köppe | is professor at the University of California, Davis |
| Vincent Delecroix | is chargé de recherche at the University of Bordeaux |
| Ana Maria Botero | is postdoctoral researcher at the University of Regensburg |
| Lauren Williams | is Dwight Parker Robinson professor of Mathematics at Harvard |
| Joseph Doolittle | is postdoctoral researcher at the Freie Universität Berlin |
| Carsten Lange | is Akademischer Rat at the Technische Universität München |
| Rainer Sinn | is junior professor at the Freie Universität Berlin |
| Jonathan Spreer | is lecturer at the University of Sydney |
| Günter M. Ziegler | is president of the Freie Universität Berlin |
| Raman Sanyal | is professor at the Goethe-Universität Frankfurt |
| Günter Rote | is professor at the Freie Universität Berlin |
| Thibault Manneville | is responsable du service économique de l'État en région à Paris |
| Francisco Santos | is professor at the Universidad de Cantabria |
| Eran Nevo | is professor at the Hebrew University of Jerusalem |

5.5. **Scientific equipment.** The Department of Mathematics and Computer Science at the Freie Universität Berlin has a computer cluster made available to its researchers. The project necessitate these ressources, especially with regard to the calculation of reduced expressions that require parallelization and large memory capacities.

| Processors | Cores | Memory | Storage |
|---|---|---|---|
| x86 CPUs and GPUs | $\sim$1,000 | 3.5 TB RAM | 20 TB |

5.6. **Project-relevant cooperation with commercial enterprises.**

not applicable

5.7. **Project-relevant participation in commercial enterprises.**

not applicable

6. **Additional information**

This application has not been submitted to another Module from the DFG.