Report on conducted research work

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The publications resulting from the work described below are listed in the relevant document attached to this application and are available on the webpage:

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1. Introduction

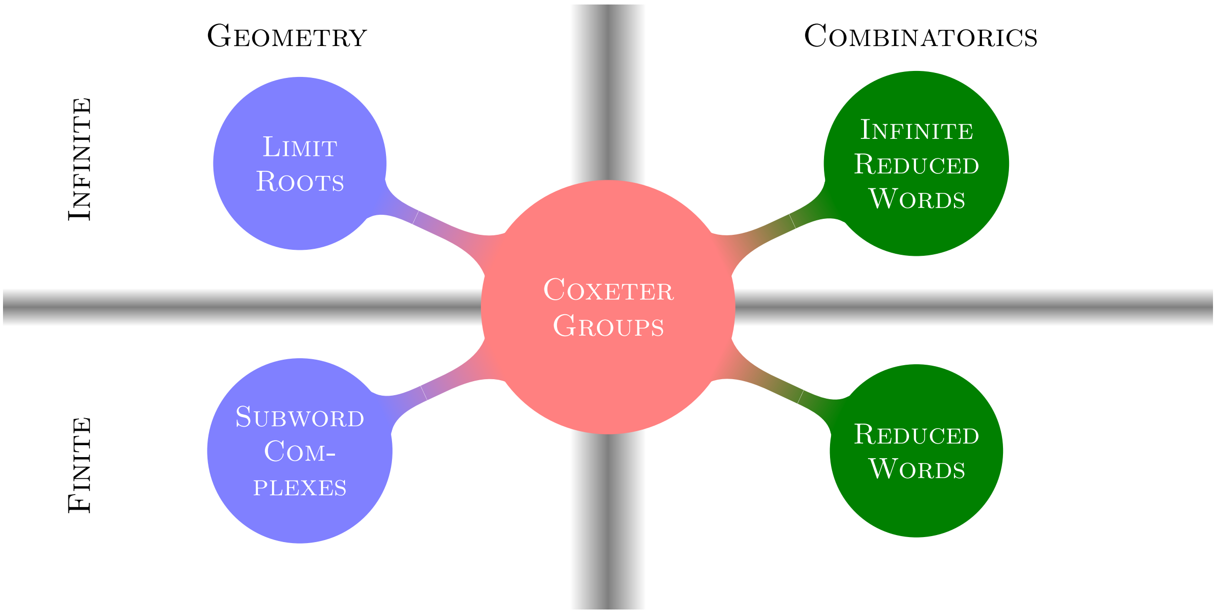
Discrete geometry stands at the crossroads of rapidly developing areas of research in mathematics. Many exciting problems in algebraic geometry, combinatorics, theoretical physics and representation theory share deep, and sometimes stunning, relationships with geometric objects and their combinatorial properties. These relationships, along with the problems at the core of discrete geometry, foster the emergence of a plethora of important mathematical challenges. Coxeter groups form a flagship example of such an object. *Coxeter groups* are abstract versions of symmetry groups of geometric or topological objects. By their simple definition based on the concept of symmetry, they provide a flexible combinatorial framework to study internal properties of these objects. Examples of geometric and combinatorial objects with connections to Coxeter groups include Kazhdan–Lusztig polynomials, Stanley symmetric functions, cluster algebras of finite types, and hyperbolic manifolds.

Coxeter groups are defined using generators and relations. They form a particularly well-behaved family of finitely presented groups. For instance, they are characterized by the "Exchange condition", which gives a simple criterion to determine whether two words represent the same element. Hence the word-problem can be solved in polynomial time, and further, the Bruhat order lays at the center of their combinatorial structure. Their elements are represented by words, the shortest of which are called *reduced words*. Considered otherwise, Coxeter groups act through *root systems* on vector spaces. Root systems originated in the classification of semi-simple Lie algebras. They form special minimal sets of vectors configured precisely to describe this geometric action through reflections. Much of the combinatorial structure (e.g. Bruhat order and weak order posets) can be described using root systems, allowing the combinatorics of Coxeter groups to be used to study objects related to root systems, such as permutahedra or group representations.

Besides, simplicial complexes and polytopes are two important objects studied in discrete geometry. *Simplicial complexes* are higher-dimensional generalizations of graphs, while *polytopes* are geometric bodies obtained as the convex hulls of finite point sets in a real vector space, so they are generalizations of convex polygons. They are used particularly as discretization tools to model topological spaces via triangulations or cell decompositions. Faces of simplicial complexes are the simplices forming it, while faces of polytopes are its surfaces of contact with "supporting" hyperplanes. Both objects share combinatorial invariants: the *f-vector* recording the number of faces in each dimension, and the *diameter* giving a significant complexity indicator for algorithms defined on these structures. The family of simplicial complexes called *subword complexes* were introduced in the context of Gröbner geometry of Schubert varieties using finite Coxeter groups [KM04, KM05]. Through their combinatorial definition using combinatorics of reduced words, they can be used as an intermediary to translate between geometric and combinatorial concepts. Namely, studying the diameter and the f-vector of subword complexes directly calls for a deep understanding of reduced words of Coxeter groups. On the other hand, the structure of reduced words of Coxeter groups has a dramatic effect on the geometrical realizations of subword complexes as polytopes and therefore on potential properties of the associated Schubert varieties, for example.

What is more, the study of infinite Coxeter groups still offers a wide range of recent research avenues. The development of the theory of *limit roots* provides an innovative approach to the study of the weak order. A limit root is a limit direction arising from the action of the group on the root system. This approach is both geometric through its definition and combinatorial in its close connection to infinite reduced words. An *infinite reduced word* is a right-infinite word with reduced finite prefixes. For a given infinite Coxeter group, the set of limit roots is a minimal set of directions on which the group acts and encodes much of its asymptotical structure. Although very natural, the general relationship between infinite reduced words and limit roots is not yet fully understood.

Through the above discussion, the study of Coxeter groups can be arranged into four parts laid out on two axis depending on whether one is interested in *finite* or *infinite* groups first, and then depending on whether *geometric* or *combinatorial* aspects or tools are at play.



Coxeter groups involve a rich mixture of combinatorial and geometric structures. With regard to the geometric aspect of finite Coxeter groups, my research focuses on *subword complexes*, while for infinite Coxeter groups, my attention is centered on *limit roots*. On the combinatorial side, both *finite* and *infinite reduced words* constitute one of the main objects at play. Of course, as far as Coxeter groups are concerned, many other objects intervene. Among them, root systems, polytopes, triangulations, graphs and their related enumerative statistics play a significant role in this theory.

Let us highlight further two objects of particular importance. On the one hand, the weak order poset encodes the combinatorial structure of Coxeter groups, and is useful to study the Bruhat order, Hecke algebras, and Kazhdan–Lusztig polynomials. On the other hand, root systems are a prime tool to represent Coxeter groups as reflection groups and many combinatorial aspects of the Coxeter groups translate into properties of polytopes associated to root systems. The interplay between root systems and the weak order lead to fascinating relations, for example, between combinatorial objects (for example, automata and reduced words), algebraic objects (for example semi-simple Lie algebras, and hyperbolic groups), and geometric objects (for example, billiards trajectories, and limit sets).

In Section 2, I describe my work related to the geometric combinatorics of finite Coxeter groups, while Section 3 is dedicated to the infinite case. In Section 4, I describe my work related to the study of extremal combinatorial and geometric properties of simplicial complexes and polytopes. In Section 5, I describe my work on the study of the discrepancy of dissections of the square with triangles of the same area. Finally, in Section 6, I describe my work in computational and experimental frameworks that support my theoretical research.

Convention. In the sections below, I review my research work conducted during and since obtaining my PhD degree according to the aforementionned topical division. Research work done during the doctoral phase are underlined. Citations to my own work and work with collaborators are indicated by numbers preceeded by a "⋆", while other citations use the lastname initials-year convention.

2. Research on finite Coxeter groups

2.1. **Geometry of finite Coxeter groups.**

2.1.1. *Subword complex approach to cluster complexes*. In [⋆10], Ceballos, Stump and I expose a strong relationship between subword complexes and cluster complexes in the theory of cluster algebras, which led to a fruitful research avenue. In particular, subword complexes have shown connections to cluster algebras [CP15], toric geometry [Esc16], root polytopes [EM18], Hopf algebras [BC17], among many others. This approach also led to a simple and unified approach to different topological objects such as the simplicial complex of multi-triangulations of a convex polygon [Jon05], and the simplicial complex of centrally symmetric multi-triangulations of a convex regular polygon [SW09].

2.1.2. *Modelling of Tropical planes.* Tropical geometry studies algebraic geometry questions after a so-called *tropicalization*. This operation replaces usual ring operations with semi-ring operations: additions becomes minima and products becomes additions. Usual varieties–like hyperplanes–become "tropical" under this mapping. The combinatorial types of tropical two-dimensional planes were classified geometrically using matroid subdivisions of hypersimplices for small dimensions [Her+09]. A more combinatorial approach uses the cluster complex of type D_4 with the pseudotriangulation model, developed by Ceballos and Pilaud. Combining this approach with the combinatorics of subword complexes, Brodsky, Ceballos and I described the combinatorial types of tropical planes of dimension 5 [$\star$3]. Inspired by this model, it seems plausible to extend the results on tropical planes in dimension 6 relying *solely* on the subword complex of type E_6.

2.2. Combinatorics of finite Coxeter groups.

2.2.1. *Enumeration of singletons.* The associahedra can be realized using the permutahedra by pulling certain facets away to infinity, see [HL07, HLT11]. Subword complexes and the combinatorics of Coxeter groups lead to formulas counting the number of common vertices of permutahedra and generalized associahedra for arbitrary finite Coxeter groups and Coxeter elements [$\star$12, $\star$1]. This enumeration has deep connections with the notion of acyclic sets in social choice theory as shown by Galambos and Reiner [GR08].

2.2.2. *Graph on reduced expressions.* It is a well-known property of Coxeter groups that reduced expressions of elements are connected via finite sequences of braid moves, in particular that no reductions $s_i^2 = e$ are necessary [Mat64, Tit69]. The graph $\mathcal{G}(w)$ whose vertices are reduced expressions of w and edges represent braid moves between expressions is hence connected. The diameter of $\mathcal{G}(w)$ has been studied in [AD10, RR13] and other closely related enumerative properties in [Ten17]. Of great importance is the fact that the graph $\mathcal{G}(w_\circ)$ and certain minors are bipartite graphs. This was first proved for finite Coxeter group by Bergeron, Ceballos, and I using a geometric argument in [$\star$8] and generalized to infinite Coxeter groups and extended to a finer description by Grinberg and Postnikov in [GP17] using only conjugations instead of automorphisms. An asymptotic study of expected number of commutations in reduced words was done in type A [Rei05] and type B [Ten15] and it is possible to determine the number of elements that have a unique reduced word [Har17]. Further, it is possible to define a metric on this graph which relates naturally to balanced tableaux [Ass19]. These results establish a relatively good knowledge of reduced expressions. However, it seems that many important properties of reduced words stemming from the geometry of subword complexes still have to be studied in detail.

2.3. Geometric Combinatorics of finite Coxeter groups.

At the intersection of algebraic combinatorics and discrete geometry, the following open question resisted for the last 16 years [KM04, Question 6.4]:

Are the spherical subword complexes realizable as the boundary of simplicial polytopes?

Checking the polytopal realizability of a spherical simplicial complex is an NP-hard problem. An answer to the above question carries striking consequences: The existence of such polytopes would provide a distinguished family of polytopes with exceptional combinatorial properties and connections to many areas of mathematics, thus opening the door to the use of new discrete geometric tools and the study of their associated toric varieties, for example. On the other hand, if subword complexes are not all polytopal, they would constitute a large and combinatorially simple family of vertex-decomposable simplicial spheres arising *naturally* that *are not* polytopal. This would be of extraordinary interest, as presently such obstructions are usually obtained using brute force enumeration or *ad hoc* constructions.

Among the several possible extensions of the associahedron, multi-triangulations of a convex n-gon offer a particularly broad generalization of the underlying boundary complex of its polar [PP12]. Indeed, the simplicial complex whose facets are maximal k-crossing-free sets of diagonals of a convex polygon (i.e. multi-triangulations) is conjectured to be the boundary of a convex polytope whose polar would generalize the associahedron. This conjecture first appeared in the Oberwolfach Book of Abstract, handwritten by Jonsson in 2003, which did not subsequently appear in the official MFO Report [Jon03]. Currently, the only known non-classical polytopal construction is for the 2-triangulations of the 8-gon [BP09, Ceb12, ⋆8]. Further, certain cases are known to be realizable as geodesic spheres [⋆8, Man18]. The complex of multi-triangulations turns out to be an example of subword complexes. Optimists may wish that a notion from related areas is key to determine the polytopality of subword complexes and therefore determine if multi-associahedra exist. Answering this question would settle three conjectures simultaneously [KM04, Jon05, SW09]. Two of these conjectures have been open since the beginning of the 2000's and yet little progress has been made.

In the article [⋆8], Bergeon, Ceballos and I lay down necessary conditions for the polytopality of subword complexes. The first step consisted in showing the existence of a certain sign function, which is then used to formulate sign conditions on minors of matrices to obtain *signature matrices*. This sign function is a natural generalization of the notion of odd and even permutations in the symmetric group and relate to work on scattering amplitudes in quantum physics [Ark+16] and permutation patterns in algebraic combinatorics [Ten17]. Then, a combinatorial construction is given that provides signature matrices and it was possible to prove that they lead to complete simplicial fans for subword complexes of type A_3 and for certain cases in type A_4. In spite of these positive results, the reason *why* the construction works is still mysterious. Additionally, the general knowledge on subword complexes is still scarce. Namely, certain combinatorial aspects of reduced words that lay at the center of the problem are still not explored in details. The geometric interpretation of these aspects is hence inexistant.

3. Research on infinite Coxeter groups

In contrast with the finite case, the infinite weak order is delicate: it is only a meet-semilattice. However, in my doctoral thesis I showed that it is possible to embed it into a bigger lattice for Coxeter groups of rank $\leqslant 3$ [⋆12, Theorem 2.35] and this allows to describe the join operation in the weak order geometrically [⋆7]. Nevertheless, a better geometric understanding of the weak order is desirable. When studying and visualizing root systems of infinite Coxeter groups, we obtained intriguing fractal-like pictures using `Sagemath`. These pictures show that roots tend to the isotropic cone of the vector space.

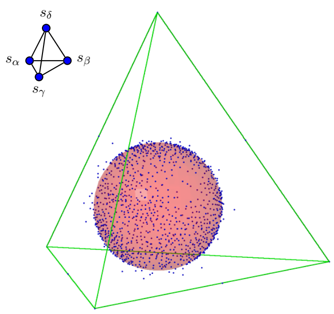


Figure 1. The normalized isotropic cone (sphere) and the first thousand normalized roots (dots) for the root system with diagram the complete graph with labels 3.

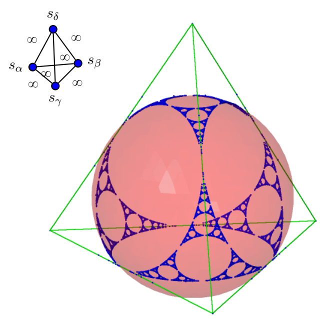


Figure 2. The normalized isotropic cone (sphere) and the first thousand normalized roots (dots) for the based root system with diagram the complete graph with labels ∞.

The investigations of these pictures led to many developments surrounding infinite Coxeter groups of which the following problems are still open.

(1) How to understand the weak order of Coxeter groups geometrically in general?
(2) How to unite the combinatorial and geometric weak order?
(3) How to characterize Lorentzian Coxeter groups by their graphs?

3.1. **Geometry of infinite Coxeter groups.**

3.1.1. *Limit roots approach to the weak order.* *Limit roots* were introduced to obtain a geometric understanding of the asymptotic behavior of roots shown in the pictures above [⋆11], and computational and visualization tools were implemented in `Sagemath` [⋆15]. Thus far, limit roots have shown connections in particular with sphere packings [⋆9], the imaginary cone [DHR16], and Garside theory [DH16].

3.1.2. *Relation to sphere packings.* The relationship between the Coxeter groups with sphere packings can be phrased as follows: The set of limit roots forms a ball packing if and only if the Coxeter graph of the group is so-called *of level 2* [⋆9]. This is a restrictive property introduced by Maxwell [Max82] to give a dual version of this result, which leads to a finite set of graphs, and this set can be completely described. In [⋆9], Chen and I also corrected the previous enumeration of level 2 graphs done by Maxwell.

3.1.3. *A Perron theorem and applications to Coxeter groups.* To comprehend the dynamics of Coxeter groups, it is essential to have a precise description of the spectrum of infinite order elements. So far, only Coxeter elements have a complete description with striking properties [ACa76] and some other elements were studied in [How82, McM02]. Using tools from Perron–Frobenius theory on primitive matrices, Sébastien Labbé and I proposed a novel point of view on the spectrum of infinite-order elements [⋆16].

3.2. **Combinatorics of infinite Coxeter groups.**

3.2.1. *Limit directions and infinite reduced words.* Given a random vector in space, what is the limit set of its orbit under the action of a Coxeter group? In a joint work with Chen, we characterized these limit sets using eigenspaces of infinite order element in the case of Lorentz spaces [⋆4]. This approach exhibits peculiar properties of the infinite arrangements of hyperplanes. Limit roots are related to infinite reduced words, introduced by Lam and Pylyavskyy [LP13, LT15]: To an infinite reduced word corresponds a unique limit root [⋆4]. This exhibits yet another a deep relationship between the geometry of the limit set with the combinatorics of the weak order of the group that deserves to be explored in more detail.

3.2.2. *On inversion sets and the weak order in Coxeter groups.* It turns out that the join operation in the weak order can be described geometrically using the corresponding inversion set and their relative position with respect to the imaginary cone [⋆12, ⋆7]. In our article, Hohlweg and I also presented many conjectures related to the weak order, biclosed sets, and infinite reduced words obtained from computational evidence.

3.3. **Geometric Combinatorics of Infinite Coxeter groups.**

The peculiar configurations of the infinite hyperplane arrangements observed in [⋆4] suggest a deeper study of their geometric and combinatorial properties. There are currently very little knowledge on infinite hyperplane arrangements in spaces other than Euclidean spaces. What can be said about infinite and not discrete hyperplane arrangements in Lorentz space? This suggests the introduction of an infinite poset of flats of finite rank that would describe infinite Coxeter arrangements.

4. Research on extremal combinatorics of simplicial complexes and polytopes

4.1. **γ-vectors of flag simplicial spheres.** Euler's formula for connected planar graphs is a classic example of restriction on the face vectors of graphs given by a topological property. In [NP11, KN13], the authors consider flag spheres, i.e., simplicial complexes with the homology of the sphere where the size of minimal nonfaces is two. Gal conjectured that the γ-vectors (obtained from the face vector via a change of basis) of flag spheres are nonnegative [Gal05]. This conjecture is still open as of today. If proven, this would give stronger inequalities on the face vector of flag spheres than the recently proven g-theorem for spheres (stated by McMullen [McM71] and a proof is presented by Adiprasito in [Adi19]) and prove the Charney–Davis conjecture in differential geometry [CD95]. My coauthor Eran Nevo and I proved Gal's conjecture in specific cases using combinatorial techniques that allowed the characterization of certain entries of the γ-vector [⋆6].

4.2. **Extremal combinatorial segments on simplicial polytopes.** The polynomial Hirsch conjecture postulates the existence of a polynomial bound on the diameter of simplicial polytopes. The diameter is a significant indicator of complexity for polytopes and is also an indication of how fast linear programming can be done over that polytope. Manneville, Santos and I showed that the techniques developed by Adiprasito and Benedetti [AB14] to prove the Hirsch bound for flag manifolds cannot be extended to give a polynomial bound on the diameter of simplicial polytopes by constructing an exotic family of polytopes [⋆5].

4.3. **Combinatorial inscribability obstructions for polytopes.** With my colleagues Doolittle, Lange, Sinn, Spreer, and Ziegler, we found the first face vector that cannot be the face vector of a polytope with vertices on a sphere [⋆13]. This vector was obtained via a combinatorial obstruction present in the face lattice of polytopes and constitutes the first example of obstructions to inscribability in dimension larger than 3 that comes solely from combinatorial data. Previously, obstructions made use of the graph of the polytope only and gave rather weak conditions for inscribability in higher dimensions.

5. Research on the discrepancy of equal-area triangulations

This project studied triangulations of polygonal domains with prescribed combinatorics and triangle areas. This is motivated by the following celebrated result by Monsky [Mon70]:

Theorem. *It is not possible to triangulate a square into an odd number of equal-area triangles.*

However, this result does not give a lower bound for the area differences that must occur. In [⋆2] Rote, Ziegler and I applied a gap theorem from semi-algebraic geometry to a polynomial area difference measure and thus got a lower bound for the area differences that decreases doubly-exponentially with the number of triangles. Further, we obtained the first superpolynomial upper bounds for this problem, derived from an explicit construction that uses the Thue–Morse sequence.

6. Development of experimental frameworks

6.1. **Combinatorics of reduced words.** Subword complexes involve a great deal of combinatorial structures, most notably reduced words. Their study involves challenging computational problems in order to obtain even relatively small examples. To face the exponential complexity of the generation of these objects, I implemented a large-scale database storing sign functions and subwords. This is a powerful tool that made it possible to look at examples that could not previously be computed. Using tools from multilinear algebra, Schur functions and combinatorics of Coxeter groups, I layed down the foundation of the convex geometry of subword complexes and present a family of oriented matroids that *realizes them all*, i.e., whose realizability is equivalent to the realizability of “essentially all” subword complexes [⋆14].

6.2. **Visualization of limit roots.** Besides the theoretical setting, the implementation of experimental tools on limit roots in `Sagemath` [Sage] is well underway [⋆15]. This infrastructure was very successful in exhibiting behaviors before they were proven. As a consequence, researchers around the world benefit from the latest developments in the current theory. They may use it to gain knowledge on limit roots and verify conjectures.

6.3. **Polyhedral Geometry.** Since February 2017, a lot of efforts and energy was put into the polyhedral geometry libraries available in `Sagemath`. There is a concrete need from researchers in various areas of research to be able to use the cutting-edge algorithms offered through different softwares “under the same hood”. `Sagemath` is a strategic choice, since it offers a well-established user-friendly interface to several open-source softwares. The following meetings were organized to bring together the different communities and initiate closer cooperations.

- SageDays 84, February–March 2017, 2 weeks, (Olot, Spain) (`latte`, `polymake`, and `Sagemath` communities)
- IMA Coding Sprint on Polyhedral Geometry, April 2018, 2 weeks, (Minneapolis, USA) (`latte`, `Normaliz`, `PARI/GP`, `polymake`, and `Sagemath` communities)
- Research in Pairs MFO, April–May 2019, 2 weeks, (`latte`, `Normaliz`, `polymake`, and `Sagemath`

communities)

- (tentative) Research in Pairs CIRM, June 2020, 1 week, (`e-antic`, `latte`, `Normaliz`, and `Sagemath` communities)

These meetings included around 25 developers from commutative algebra, category theory and homological algebra, polytope theory, and optimization, whose expertise and diversity created a vivid workflow. The exchanges were very fruitful, brought the different communities closer, and made a significant difference in the progress of the implementation framework. Through this series of workshops we reached several important milestones. The first major one was the successful implementation and development of interfaces to `Normaliz` and `polymake` in `Sagemath`. Currently, `Sagemath` offers the leading algorithms of `polymake`, `Normaliz`, and `latte` and further its own cutting-edge implementation of combinatorial invariants of polyhedral objects. A particularly innovative outcome of this initiative is the possibility to execute *fast* exact convex hull calculations on arbitrary number fields using a combination of number theory libraries and `Normaliz` using `python` and is now fully accessible in `Sagemath`. These developments were paramount in the study of obstructions to inscribability of polytopes using algebraic coordinates [⋆13]. Furthermore, the documentation and stability of the libraries have been improved greatly for a greater accessibility for new users.

7. Personal publications

7.1. Refereed publications.

[⋆1] **Jean-Philippe Labbé** and Carsten Lange, *Cambrian acyclic domains: counting c-singletons*, Order (to appear) (2019) 24 pp.

[⋆2] **Jean-Philippe Labbé**, Günter Rote, and Günter M. Ziegler, *Area difference bounds for dissections of a square into an odd number of triangles*, Exp. Math. (2018) published electronically.

[⋆3] Sarah B. Brodsky, Cesar Ceballos, and **Jean-Philippe Labbé**, *Cluster algebras of type D_4, tropical planes, and the positive tropical Grassmannian*, Beitr. Algebra Geom. **58** (2017) no. 1, 25–46.

[⋆4] Hao Chen and **Jean-Philippe Labbé**, *Limit directions for Lorentzian Coxeter systems*, Groups Geom. Dyn. **11** (2017) no. 2, 469–498.

[⋆5] **Jean-Philippe Labbé**, Thibault Manneville, and Francisco Santos, *Hirsch polytopes with exponentially long combinatorial segments*, Math. Program. **165** (2017) no. 2, Ser. A, 663–688.

[⋆6] **Jean-Philippe Labbé** and Eran Nevo, *Bounds for entries of γ-vectors of flag homology spheres*, SIAM J. Discrete Math. **31** (2017) no. 3, 2064–2078.

[⋆7] Christophe Hohlweg and **Jean-Philippe Labbé**, *On inversion sets and the weak order in Coxeter groups*, European J. Combin. **55** (2016) 1–19.

[⋆8] Nantel Bergeron, Cesar Ceballos, and **Jean-Philippe Labbé**, *Fan realizations of type A subword complexes and multi-associahedra of rank 3*, Discrete Comput. Geom. **54** (2015) no. 1, 195–231.

[⋆9] Hao Chen and **Jean-Philippe Labbé**, *Lorentzian Coxeter systems and Boyd-Maxwell ball packings*, Geom. Dedicata **174** (2015) 43–73.

[⋆10] Cesar Ceballos, **Jean-Philippe Labbé**, and Christian Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, J. Algebraic Combin. **39** (2014) no. 1, 17–51.

[⋆11] Christophe Hohlweg, **Jean-Philippe Labbé**, and Vivien Ripoll, *Asymptotical behaviour of roots of infinite Coxeter groups*, Canad. J. Math. **66** (2014) no. 2, 323–353.

[⋆12] **Jean-Philippe Labbé**. *Polyhedral Combinatorics of Coxeter Groups.* PhD thesis. Freie Universität Berlin, July 2013, xvi+103.

7.2. Other publications.

[⋆13] Joseph Doolittle, **Jean-Philippe Labbé**, Carsten Lange, Rainer Sinn, Jonathan Spreer, and Günter M. Ziegler, *Combinatorial inscribability obstructions for higher-dimensional polytopes*, arXiv:1910.05241 (2019) 27 pp.

[⋆14] **Jean-Philippe Labbé**, *Convex geometry of subword complexes of Coxeter groups*, in preparation (2019) 30 pp.

[⋆15] **Jean-Philippe Labbé**, *Brocoli: Sagemath package dealing with LImit ROots of COxeter groups*, `https://github.com/jplab/brocoli` (2017) version 1.0.0 3500 lines.

[⋆16] **Jean-Philippe Labbé** and Sébastien Labbé, *A Perron theorem for matrices with negative entries and applications to Coxeter groups*, `arXiv:1511.04975` (November 2015) 14 pp.

8. External Bibliography

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