RESEARCH STATEMENT

Discrete geometry stands at the crossroads of rapidly developing areas of research in mathematics. Many exciting problems in algebraic geometry, combinatorics, theoretical physics and representation theory share deep, and sometimes stunning, relationships with geometric objects and their combinatorial properties. These relationships, along with the problems at the core of discrete geometry, foster the emergence of a plethora of important mathematical challenges. Among them, open problems relating Coxeter groups, polytopes, triangulations, and simplicial complexes intrigue me ceaselessly, even more so when I discover connections with other areas of mathematics and when computations and optimization are involved. This summarizes shortly why I am passionate about algebraic combinatorics and discrete geometry and do research in those areas.

Through my international research experiences that took place in leading research institutes, I bring together a unique blend of cutting-edge combinatorial, geometric, and large scale computational approaches to investigate algebraic and topological structures. My research interests are grouped into three areas. Below, their central open problems and implications are described, along with how my recent advances make new research avenues arise.

A) Extremal Properties of Simplicial Complexes and Polytopes

Summary. Simplicial complexes and polytopes are fundamental objects studied in discrete geometry. *Simplicial complexes* are higher-dimensional generalizations of graphs, while *polytopes* are geometric bodies obtained as the convex hulls of finite point sets in a real vector space, i.e. they are generalizations of convex polygons. They are used as discretization tools to model topological spaces via triangulations or cell decompositions. Faces of simplicial complexes are the simplices forming it, while faces of polytopes are its surfaces of contact with "supporting" hyperplanes. Both objects share combinatorial invariants: the *f-vector* recording the number of faces in each dimension, and the *diameter* giving a significant complexity indicator for algorithms defined on these structures. The polynomial Hirsch conjecture postulates the existence of a polynomial bound on the diameter of polytopes. The diameter is a significant indicator of complexity for polytopes and is also an indication of how fast linear programming can be done over that polytope. Polytopes are usually thought of as round objects, there are several ways to quantify roundness leading to different theories. It is particularly interesting to investigate how these geometric notions of roundness affect the combinatorial invariants of polytopes. For example, is it possible to construct an inscribed polytope for every possible f-vector?

Research Progress. Manneville, Santos and I showed that the techniques developed by Adiprasito and Benedetti [AB14] to prove the Hirsch bound for flag manifolds cannot be extended to give a polynomial bound on the diameter of simplicial polytopes by constructing an exotic family of polytopes [LMS17]. With my colleagues Doolittle, Lange, Sinn, Spreer, and Ziegler, we found the first f-vector that cannot be the f-vector of a polytope with vertices on a sphere [Doo+20]. This vector was obtained via a combinatorial obstruction present in the face lattice of polytopes and constitutes the first combinatorial obstructions to inscribability in dimension larger than 3. Previously, obstructions made use of the graph of the polytope only and gave rather weak conditions for inscribability in higher dimensions.

Since February 2017, the polyhedral geometry capacities of `Sagemath` has improved significantly. Currently, `Sagemath` offers the leading algorithms of `e-antic`, `polymake`, `Normaliz`, and `latte` and its own cutting-edge implementation of combinatorial invariants of polyhedral objects. These developments were paramount in the study of obstructions to inscribability of polytopes using algebraic coordinates [Doo+20].

Research Plans. These results call for a better understanding of flag spheres and spheres with small missing faces: their asymptotical enumeration, possible number of faces, and diameter. My colleague Goodarzi and I aim at finding many exotic examples of flag spheres that exhibit extremal combinatorial properties. Along with my colleague Samper, I aim to study variations of algebraic shifting and deepen the theory by explaining its internal mechanisms. Continuing the study of roundness, with my collaborator Moci, we aim to uncover how the fiber polytope construction of associahedra delivers inscribed realizations.

With my colleagues Bruns, Delecroix, and Köppe, we plan to finish our novel implementation framework for fast exact computations on algebraic polytopes. Further, the team of developers that I formed in my current workgroup will continue to expand computational capacities of `Sagemath`.

B) Subword Complexes of Coxeter Groups and Applications

Summary. Coxeter groups are abstractions of groups generated by reflections in a vector space. Through their simple definition based on the concept of symmetry, they provide a flexible combinatorial framework to study properties of geometric or topological objects. Among these objects is the family of simplicial complexes called *subword complexes*, which are homeomorphic to balls or spheres. Subword complexes were introduced by Knutson and Miller in the context of Gröbner geometry of Schubert varieties [KM04]. In [CLS14], Ceballos, Stump and I exposed a strong relationship between subword complexes and cluster complexes in the theory of cluster algebras which led to a fruitful research avenue. In particular, subword complexes have shown further connections to cluster algebras [CP15], Hopf algebras [BC17], and many geometric structures [Jon05, SW09, PP12, Esc16, BCL17, EM18]. At the intersection of algebraic combinatorics and discrete geometry, the following open question resisted for the last 16 years [KM04, Question 6.4]:

Are the spherical subword complexes realizable as the boundary of simplicial polytopes?

Checking the polytopal realizability of a spherical simplicial complex is an NP-hard problem. An answer to the above question carries striking consequences: The existence of such polytopes would provide a distinguished family of polytopes with exceptional combinatorial properties and connections to many areas of mathematics, thus opening the door to the use of new discrete geometric tools and the study of their associated toric varieties, for example. On the other hand, if subword complexes are not all polytopal, they would constitute a large and combinatorially simple family of vertex-decomposable simplicial spheres arising *naturally* that *are not* polytopal. This would be of extraordinary interest, as presently such obstructions are usually obtained using brute force enumeration or *ad hoc* constructions.

Answering this question would also settle three conjectures simultaneously [KM04, Jon05, SW09]. Two of these conjectures have been open since the beginning of the 2000's and yet little progress has been made. This research framework's goal is to overturn this situation.

Research Progress. In the article [BCL15], Bergeon, Ceballos and I lay down necessary conditions for the polytopality of subword complexes. The first step consisted in showing the existence of a certain sign function, which is then used to formulate sign conditions on minors of matrices to obtain *signature matrices.* This sign function is a natural generalization of the notion of odd and even permutations in the symmetric group and relate to work on scattering amplitudes in quantum physics [Ark+16] and permutation patterns in algebraic combinatorics [Ten17]. Then, a combinatorial construction is given that provides signature matrices and it was possible to prove that they lead to complete simplicial fans for subword complexes of type A_3 and for certain cases in type A_4. In spite of these positive results, the reason *why* the construction works is still mysterious. More recently, Manneville gave a complete fan construction of certain subword complexes and conjectured that the construction works in general [Man18].

To improve the general knowledge on subword complexes I explored certain combinatorial aspects of reduced words. I implemented a large-scale database storing sign functions and subwords, that made it possible to analyze new large examples. Using tools from multilinear algebra, Schur functions and combinatorics of Coxeter groups, I layed down the foundation of the convex geometry of subword complexes and present a family of oriented matroids that *realizes them all*, i.e., whose realizability is equivalent to the realizability of "essentially all" subword complexes [Lab20].

Research Plans. This project's goal is to establish an explicit pathway to determine whether subword complexes are boundaries of polytopes. Further, it aims to clarify how Coxeter groups generalize cyclic polytopes through subword complexes.

In relation with Research Area A, some intermediate goals include the determination of their face vectors (f-vector, h-vector and γ-vector) and the study of their diameters. Studying the f-vector and the diameter directly calls for a deep understanding of reduced words of Coxeter groups. In turn, the structure of reduced words of Coxeter groups has a dramatic effect on the geometrical realizations of subword complexes as polytopes and therefore on potential properties of the associated Schubert varieties. New words statistics of reduced words should be determined, for example describe the possible number of occurrences of generators in them.

In [LM06], Landsberg and Manivel describe an exceptional Lie algebra called $E_{7\frac{1}{2}}$ with similar properties as the Lie algebras E_7 and E_8. They also study some other subexceptional Lie algebras whose dimension, in a certain case, correspond to the number of facets of multi-cluster complexes of type H_3. I will study the internal symmetries of the related Lie algebras and determine how they relate to the symmetries of the relevant subword complexes to uncover the relation between them.

As for the polytopality conjecture, the recent developments layed out in the article [Lab20] suggest to describe explicitly the realization space of rank-3 subword complexes using Gale duality and Schur functions.

C) Combinatorics, Dynamics, and Geometry of Coxeter Groups

Summary. Coxeter groups enjoy a rich combinatorial and geometric structure. On the one hand, the weak order poset encodes the combinatorial structure of Coxeter groups, and is useful to study the Bruhat order, Hecke algebras, and Kazhdan–Lusztig polynomials. On the other hand, root systems are a prime tool to represent Coxeter groups as reflection groups and many combinatorial aspects of the Coxeter groups translate into properties of polytopes associated to root systems. The interplay between root systems and the weak order lead to fascinating relations, for example, between combinatorial objects (for example, automata and reduced words), algebraic objects (for example semi-simple Lie algebras, and hyperbolic groups), and geometric objects (for example, billiards trajectories, and limit sets).

In contrast with the finite case, the infinite weak order is delicate: it is only a meet-semilattice. However, in my doctoral thesis I showed that it is possible to embed it into a bigger lattice for Coxeter groups of rank $\leqslant 3$ [Lab13, Theorem 2.35] and this allows to describe the join operation in the weak order geometrically [HL16]. Nevertheless, a better geometric understanding of the weak order is desirable.

When studying and visualizing root systems of infinite Coxeter groups, we obtained intriguing fractal-like pictures using `Sagemath`. These pictures show that roots tend to the isotropic cone of the vector space. The investigations of these pictures led to many developments surrounding infinite Coxeter groups of which the following problems are still open.

(1) How to understand the weak order of Coxeter groups geometrically in general?
(2) How to unite the combinatorial and geometric weak order?
(3) How to characterize Lorentzian Coxeter groups by their graphs?

Research Progress. *Limit roots* were introduced to obtain a geometric understanding of the asymptotic behavior of roots shown in the pictures above [HLR14], and computational and visualization tools were implemented in `Sagemath` [Lab17]. It turns out that the join operation in the weak order can be described geometrically using the corresponding inversion set and their relative position with respect to the imaginary cone [Lab13, HL16]. Thus far, limit roots provide an innovative approach to the study of the weak order

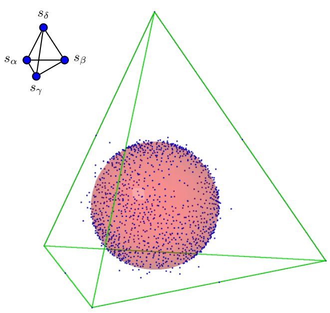

FIGURE 1. The normalized isotropic cone (sphere) and the first thousand normalized roots (dots) for the root system with diagram the complete graph with labels 3.

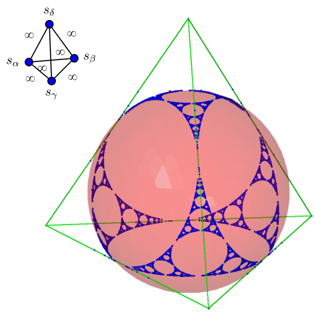

FIGURE 2. The normalized isotropic cone (sphere) and the first thousand normalized roots (dots) for the based root system with diagram the complete graph with labels ∞.

and have shown connections with sphere packings [CL15], the imaginary cone [DHR16], and Garside theory [DH16]. This approach is both geometric through its definition and combinatorial in its close connection to infinite reduced words. *Infinite reduced words* are right-infinite words with reduced finite prefixes. They were introduced by Lam and Pylyavskyy [LP13, LT15].

Given a random vector in space, what is the limit set of its orbit under the action of a Coxeter group? In the case of Lorentz spaces, Chen and I have characterized these limit sets using eigenspaces of infinite order element, and showed that to an infinite reduced word corresponds a unique limit root [CL17]. Although very natural, the general relationship between infinite reduced words and limit roots is not yet fully understood. This approach exhibits peculiar properties of the associate infinite arrangements of hyperplanes and yet another deep relationship between the geometry of the limit set with the combinatorics of the weak order that deserves to be explored in more detail.

Research Plans. The long term objective is to expand the theory of geometric representations of Coxeter groups. First, we seek an effective geometric characterizations of limit roots, limit weights and infinite order elements of Coxeter groups. These characterizations will help to reveal further the structure of Tits cones. Another step is to overcome the difficulties when passing from Lorentz space to the general case. Indeed, many of the current results rely deeply on the geometry of Lorentz space, see e.g. [CL17]. Therefore calling for a better understanding of the dynamics of infinite Coxeter groups in general. The theory of primitive matrices seems to fit the problem well [LL15], and further research in this direction should lead to an explicit description of their spectra.

When a Coxeter group acts on a Lorentz space, it is called *Lorentzian*. Which Coxeter groups are Lorentzian? Vinberg discussed this long standing problem in geometry at the ICM in 1983 in Warsaw [Vin84]. Although a lot of progress has been made since then [FT14], the problem still eludes our comprehension. It is possible to encode Coxeter graphs as certain set of points in a vector space. I propose to classify Lorentzian Coxeter groups using strategies from lattice polytope theory and Robertson-Seymour's theorem about forbidden minors [RS04]. Indeed, the property of being Lorentzian can be phrased in the language of forbidden minors. Thus there exists a finite set of obstruction which describes the non-Lorentzian Coxeter groups. Obtaining a list of these obstructions seems to be within reach.

The peculiar incidences in infinite hyperplane arrangements observed in [CL17] suggest a deeper study of their geometric and combinatorial properties. There is currently very little knowledge on infinite hyperplane arrangements in non-Euclidean spaces. What can be said about infinite and not locally finite hyperplane arrangements in Lorentz space? This suggests the introduction of an infinite poset of flats of finite rank that would describe infinite Coxeter arrangements as oriented matroids do in the finite case.

Project-related publications of the author

[Doo+20] Joseph Doolittle, **Jean-Philippe Labbé**, Carsten Lange, Rainer Sinn, Jonathan Spreer, and Günter M. Ziegler, *Combinatorial inscribability obstructions for higher-dimensional polytopes*, Mathematika **66** (2020) no. 4, 927–953.

[Lab20] **Jean-Philippe Labbé**, *Combinatorial foundations for geometric realizations of subword complexes of Coxeter groups*, arXiv:2003.02753 (2020) 34 pp.

[BCL17] Sarah B. Brodsky, Cesar Ceballos, and **Jean-Philippe Labbé**, *Cluster algebras of type D_4, tropical planes, and the positive tropical Grassmannian*, Beitr. Algebra Geom. **58** (2017) no. 1, 25–46.

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[Lab17] **Jean-Philippe Labbé**, *Brocoli: Sagemath package dealing with LImit ROots of COxeter groups*, https://github.com/jplab/brocoli (2017) version 1.0.0 3500 lines.

[LMS17] **Jean-Philippe Labbé**, Thibault Manneville, and Francisco Santos, *Hirsch polytopes with exponentially long combinatorial segments*, Math. Program. **165** (2017) no. 2, Ser. A, 663–688.

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