KV Cache

The Memory Bottleneck That Defines Inference Optimization

KV caching stores key and value tensors from previous token positions so they don't need to be recomputed during autoregressive generation. Without caching, generating n tokens requires O(n^2) projection FLOPs. With KV cache, the projection cost drops to O(n) -- each step projects only the new token, appends K and V to the cache, and attends over the full cached history.

This demo covers:

1. Output equivalence: cached = uncached (bit-identical tokens)
2. Theoretical FLOP comparison at 7B model scale
3. Memory analysis: per-token cost, growth, 7B projections
4. Wall-clock timing benchmark
5. Prefill vs decode phase analysis
6. KV cache as THE memory bottleneck at production scale

Model config: V=256, d=64, layers=2, h=4, h_kv=2, d_ff=172
Random seed: 42
Number of visualizations: 6
Examples: 6

Mathematical Foundation

Without KV Cache (Naive)

At step t, recompute all projections for $[0, \ldots, t]$:

$$Q = XW_Q \in \mathbb{R}^{(t+1)\times d_k}, \quad K = XW_K, \quad V = XW_V$$

$$\text{scores} = \frac{QK^\top}{\sqrt{d_k}} \in \mathbb{R}^{(t+1)\times(t+1)}$$

$$\text{Projection FLOPs: } \sum_{i=1}^{n} i \cdot 3 \cdot d \cdot d_k = O(n^2 \cdot d)$$

With KV Cache

At step t, project only the new token x_t:

$$q_t = x_t W_Q \in \mathbb{R}^{1\times d_k}, \quad k_t = x_t W_K, \quad v_t = x_t W_V$$

$$K_c = \text{concat}(K_c,\ k_t) \in \mathbb{R}^{(t+1)\times d_k}$$

$$\text{scores} = \frac{q_t K_c^\top}{\sqrt{d_k}} \in \mathbb{R}^{1\times(t+1)}$$

$$\text{Projection FLOPs: } n \cdot 3 \cdot d \cdot d_k = O(n \cdot d)$$

Memory Analysis

$$\text{cache/layer} = 2 \cdot \text{seqlen} \cdot d_k \cdot \text{bytes}$$

$$\text{total} = n_{\text{layers}} \cdot n_{\text{heads}} \cdot 2 \cdot \text{seqlen} \cdot d_k \cdot \text{bytes}$$

7B model (FP16): $32 \times 32 \times 2 \times S \times 128 \times 2 = 524,288 \cdot S$ bytes

≈ 0.5 MB/token $\Rightarrow$ 4K ctx: 2 GB, 32K ctx: 16 GB, 128K ctx: 64 GB

Speedup

$$\text{Projection speedup } = \frac{\sum_{i=1}^{n}(P+i)}{P+n} \approx \frac{n}{2} \text{ for } n \gg P$$

Summary of Findings

1. Output Equivalence: Cached and uncached generation produce BIT-IDENTICAL token sequences across all tested configurations (single token, short/long prompts, batched). K and V for position i depend only on token i and the fixed weights -- caching avoids redundant recomputation without error.

2. FLOP Comparison: At 7B scale (32 layers, d=4096), projection FLOPs drop from O(n^2*d^2) to O(n*d^2). For 4096 generated tokens with P=512 prompt, this yields >1000x speedup in projection cost alone.

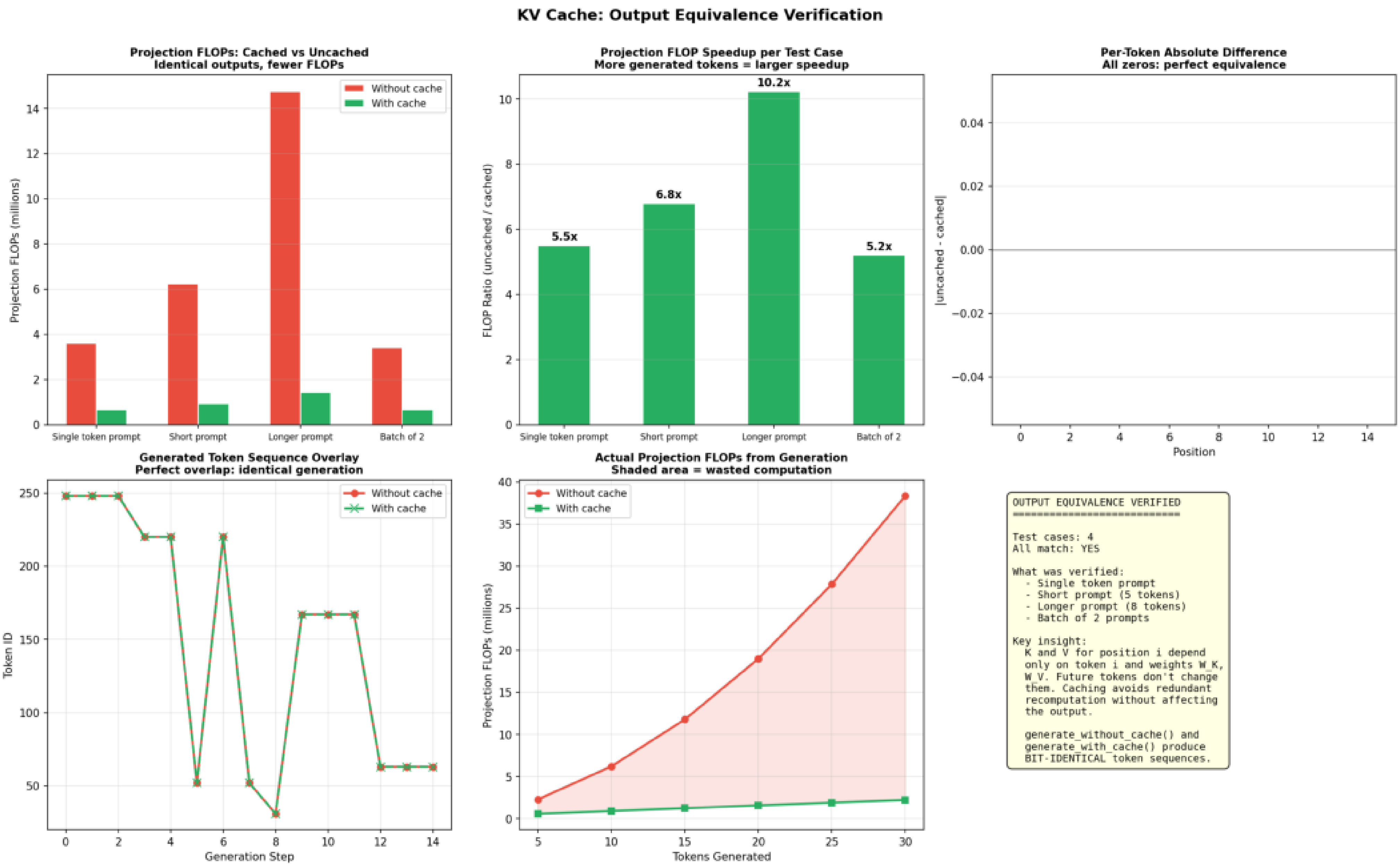
3. Memory Analysis: KV cache costs ~0.5 MB/token for a 7B model (FP16). At 4K context: 2 GB. At 32K context: 16 GB. At 128K context: 64 GB. With batch_size=32 and 4K context: 64 GB -- exceeding model weights (14 GB).

4. Timing Benchmark: Wall-clock measurements on our small model confirm speedup that grows with generation length. Per-token latency is nearly constant with cache; grows linearly without cache.

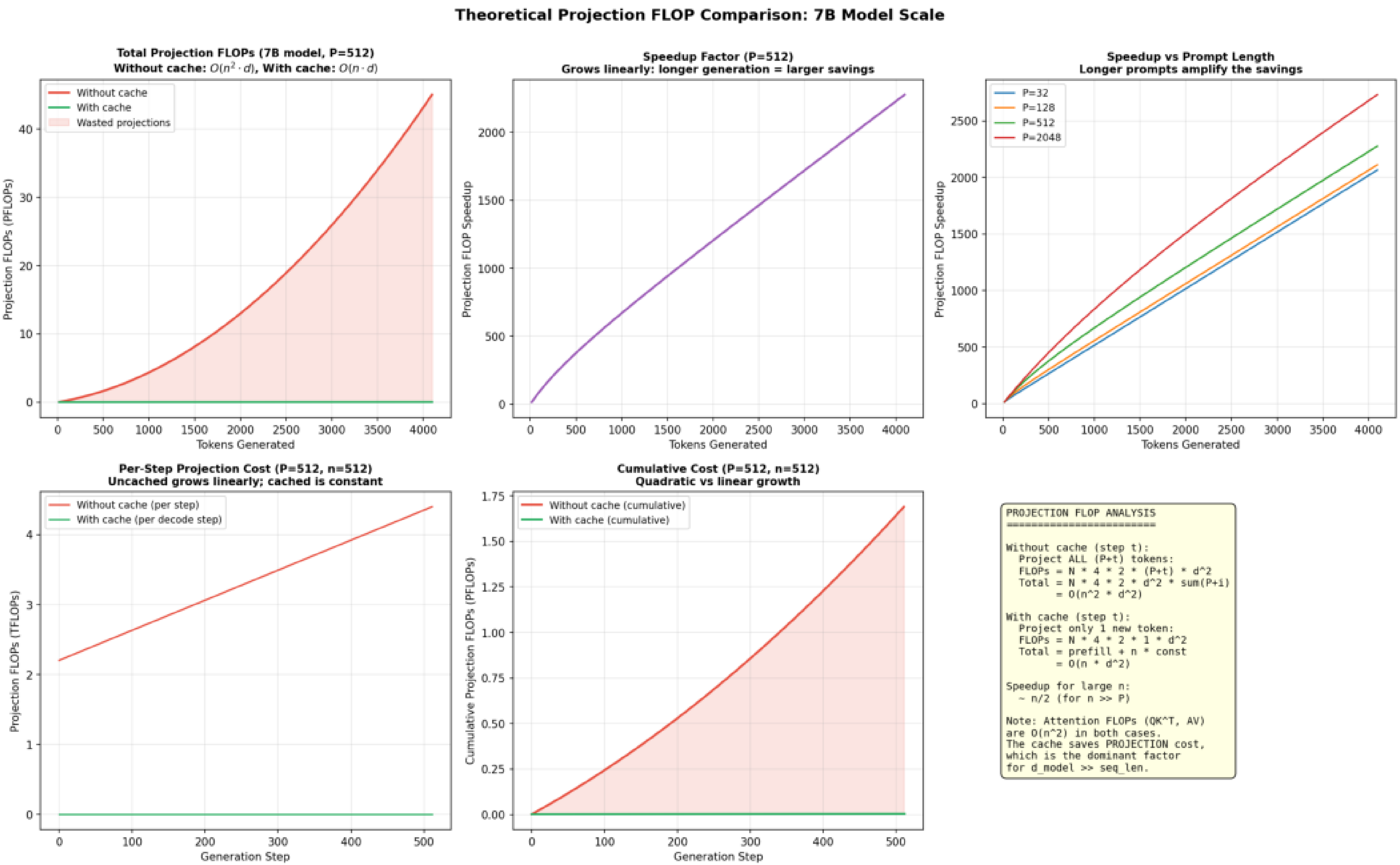
5. Prefill vs Decode: Prefill is compute-bound (high arithmetic intensity -- batch matmul over all prompt tokens). Decode is memory-bound (reads entire cache for each single-token attention step). This two-regime nature drives architecture decisions in inference systems (separate prefill/decode GPUs).

6. Memory Bottleneck: For long contexts and large batches, KV cache EXCEEDS model weight memory. A 7B model in FP16 is ~14 GB, but KV cache for batch_size=32 at 8K context is 128 GB. This motivates PagedAttention, GQA/MQA, KV cache quantization, and sliding window attention.

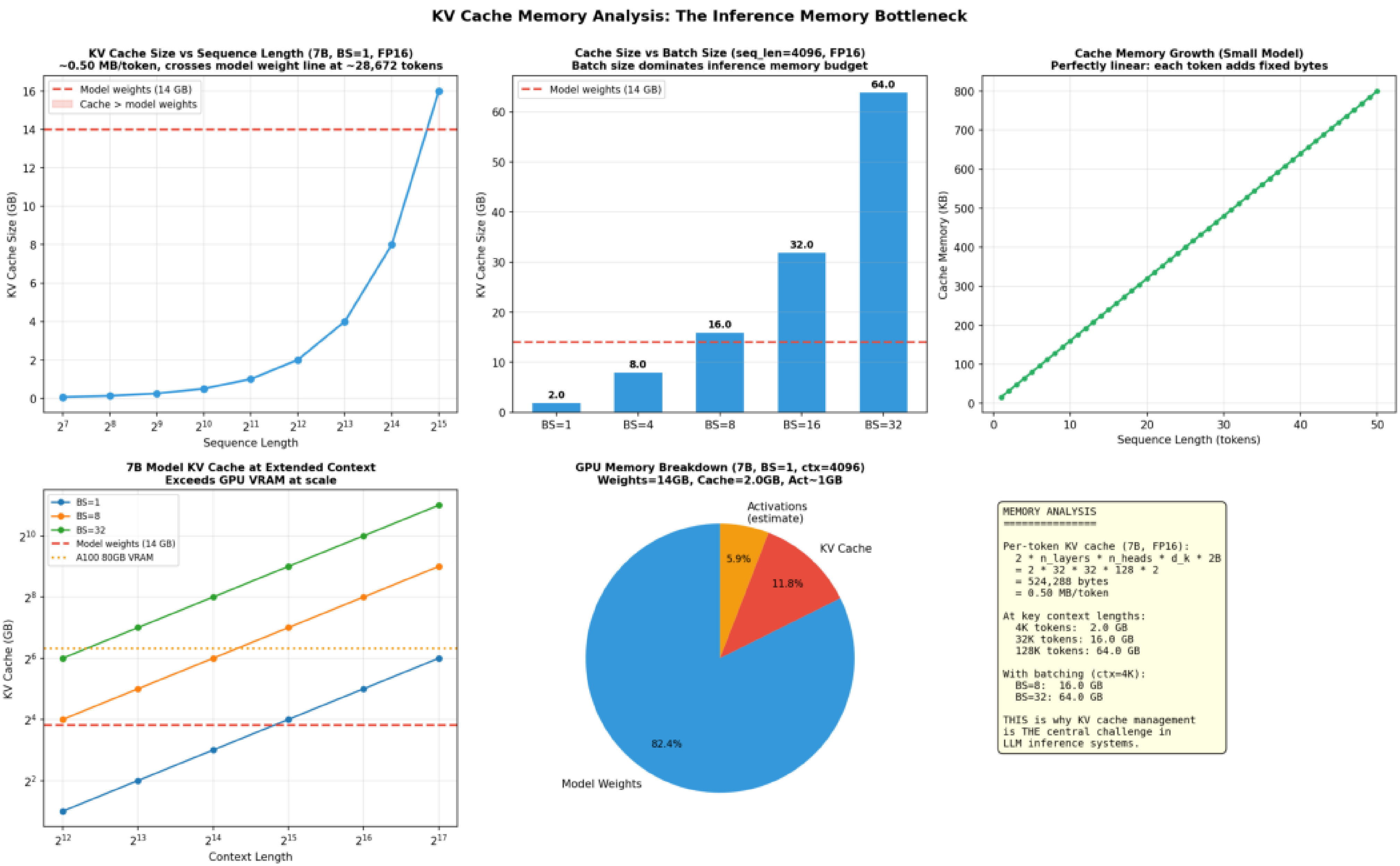
Example 1: Output Equivalence Verification



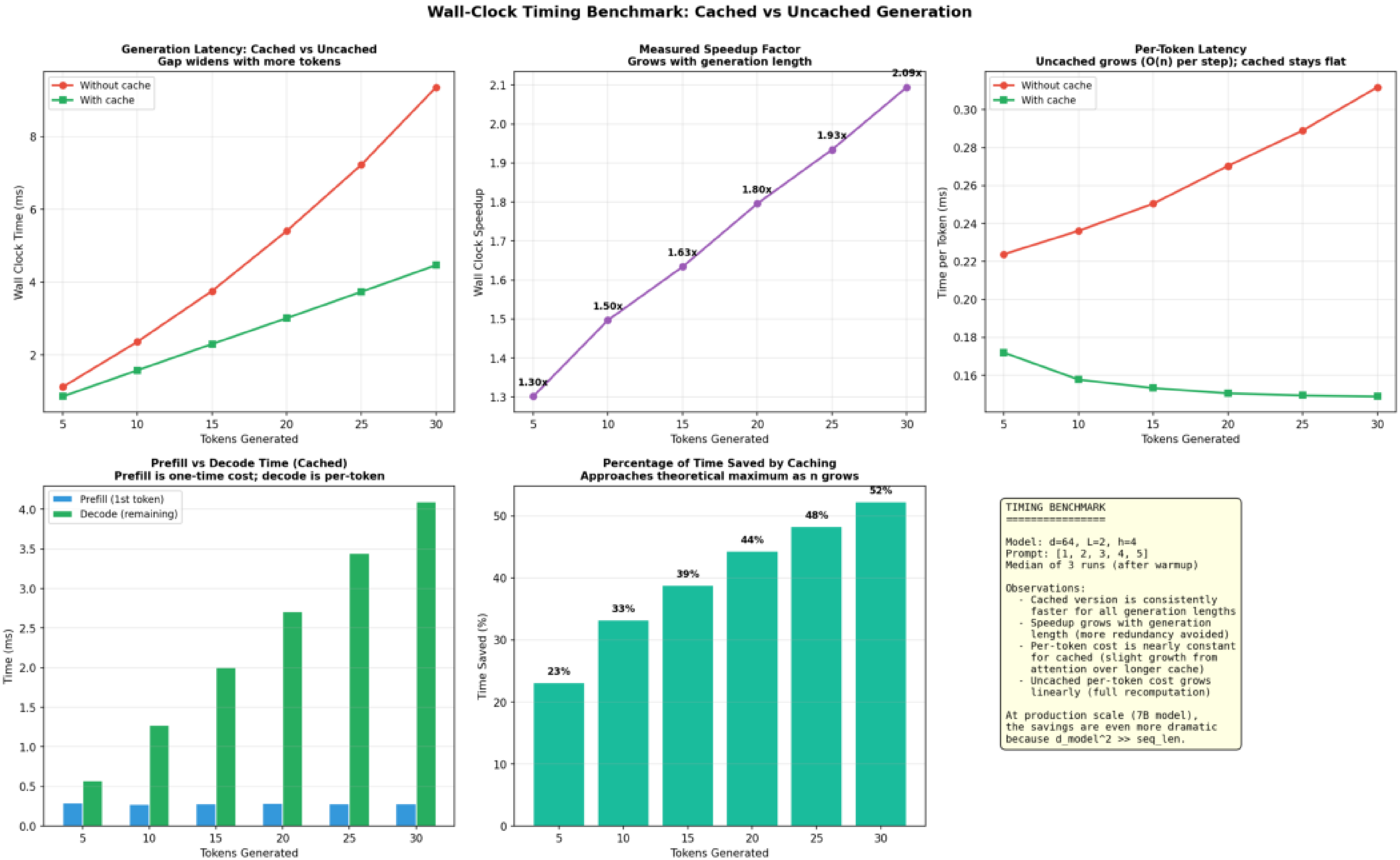
Example 2: Theoretical FLOP Comparison



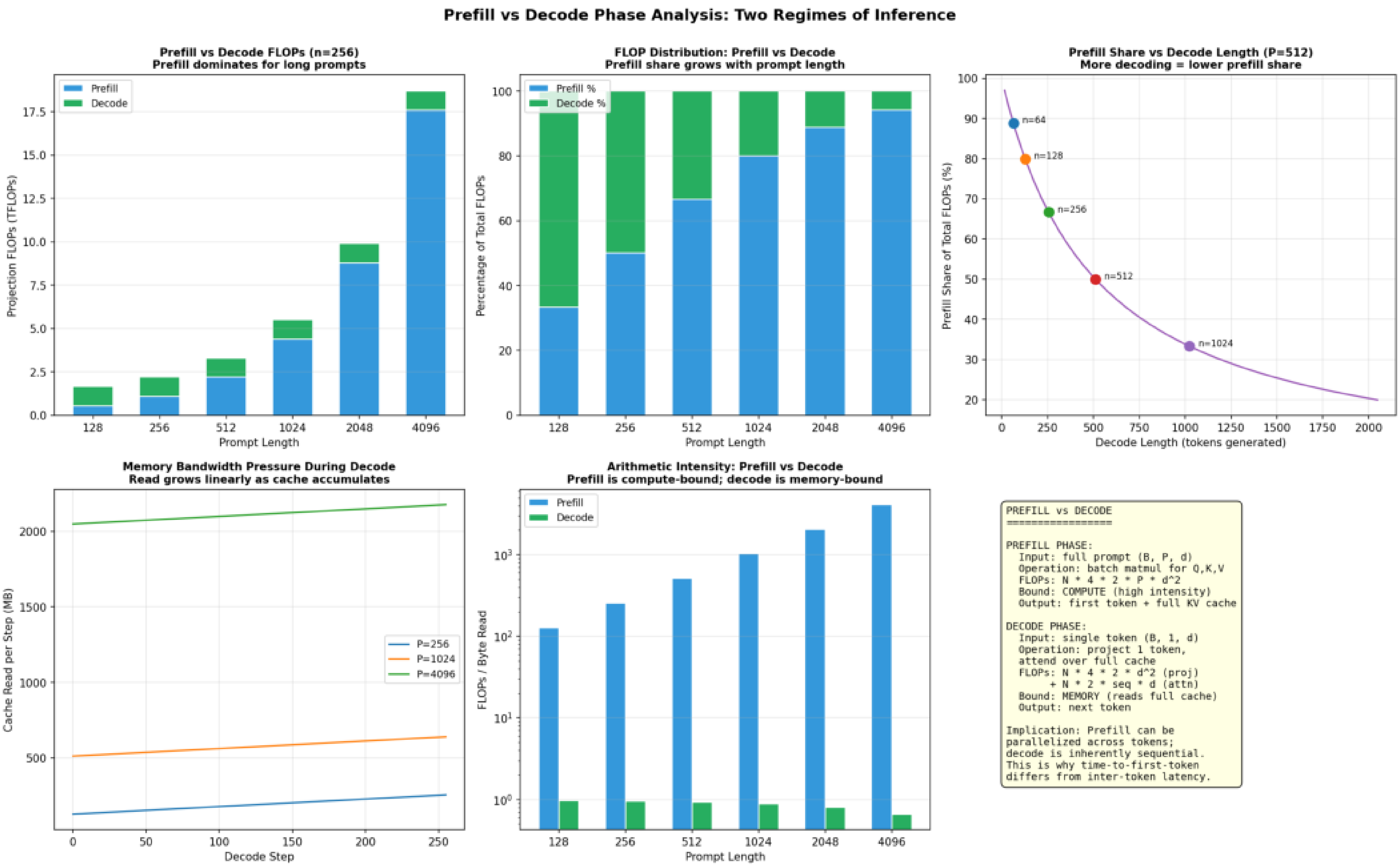
Example 3: Memory Analysis



Example 4: Timing Benchmark



Example 5: Prefill vs Decode Phase Analysis



Example 6: KV Cache as THE Memory Bottleneck