Waves and Vortices in Bose-Einstein Condensates

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Abstract

In this project the Nonlinear Schrödinger (NLS) equation and later the Gross-Pitaevskii (GP) equation will be considered as a model for Bose-Einstein Condensates. Bogoliubov's dispersion relation will be discussed in the context of the NLS equation and quantised vortices will be considered from the GP equation with the density profile of its vortex core analysed. Following this a brief discussion of the Magnus and Iordankskii forces on a vortex in a two-fluid model, and then moving on to the Aharonov-Bohm effect for the scattering of a phonon on a vortex. Finishing this section a mention of Berry Phase and its link to the Magnus force and then finally the section on the interactions of phonons on a vortex dipole, showing via the conservation of momentum how phonons change the distance between the vortices.

Contents

1 Introduction

1.1 History and motivation

The main objective of this project is to show the Aharonov-Bohm effect in Bose-Einstein Condensates via examples of wave scattering on a vortex. To achieve this I will mainly look at the non-dimensionalised version of the Nonlinear Schrödinger (NLS) Equation (1.1) or Gross-Pitaevskii (GP) equation as it is generally known as in reference to BEC, then use appropriate wave scattering approximations to show phonons scattering on a vortex.

$$i\dot{\psi} + \Delta\psi \pm |\psi|^2\psi = 0 \tag{1.1}$$

The Non-linear Schrödinger equation

1.2 What is Bose-Einstein condensation?

Bose-Einstein Condensation (BEC) was an idea first introduced in 1925 when Albert Einstein studied a paper written by the Indian physicist S.N. Bose (1924), who was devoted to the research in the statistical description of the quanta of light, who then predicted the occurrence of a phase transition in a gas of non-interacting atoms. This phase transition corresponds to the condensation of atoms in the state of their lowest energy level and is the consequence of quantum statistical effects.

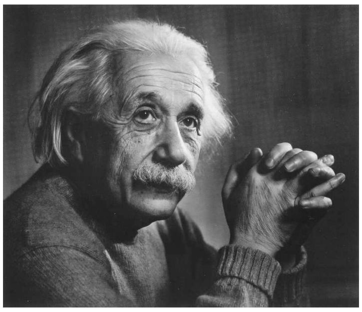

Figure 1: Albert Einstein (1979 - 1955)

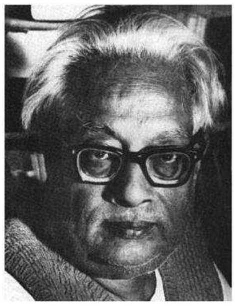

Figure 2: Satyendra Nath Bose (1894 - 1974)

F. London (1938) had the intuition that superfluidity could be a manifestation of BEC which then lead to the important development of the prediction of quantised vortices by L. Onsager (1949) and R. Feynman (1955). These vortices are quantised as they all have the same magnitude of circulation determined by mass of the boson used to form the condensate and Plank's constant $\hbar$.

1.3 The creation of BEC

BEC was first created in a laboratory in 1995 by Eric A. Cornell and Carl Wiemann at the University of Colorado. This was achieved by applying laser cooling and trapping followed by evaporative cooling used to bring clouds of rubidium atoms down to temperatures close to absolute zero (within microkelvin) needed to produce the macroscopic quantum effects that were predicted.

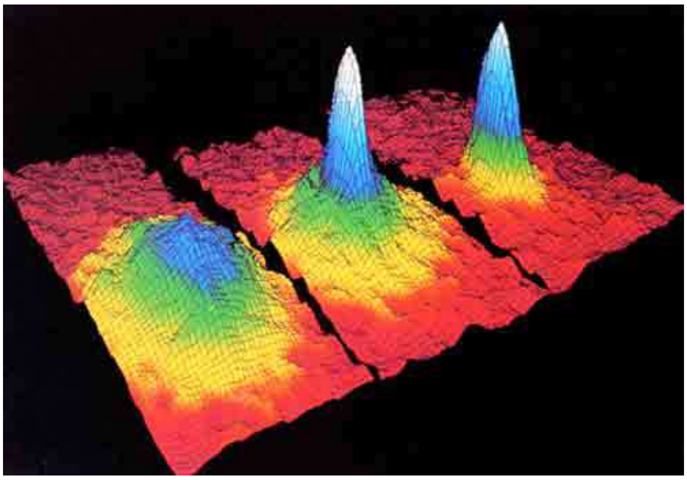

Figure 3: Successive occurrence of Bose-Einstein condensation in rubidium. From left to right is shown the atomic distribution in the cloud just prior to condensation, at the start of condensation and after full condensation. High peaks correspond to a large number of atoms. Silhouettes of the expanding atom cloud were recorded 6 ms after switching off the confining forces of the atom trap.

1.4 The Theory

The name of the NLS equation originates from the original Schrodinger Equation that is used in quantum mechanics. As the name suggests, the NLS equation has the non-linear term $\pm|\psi|^2\psi$ whereas the Schrödinger Equation is one of a linear form. When considering the NLS equation in the context of waves, the second order linear operator $\pm|\psi|^2\psi$ describes the dispersion and diffraction of the wave-packet.

The sign of this non-linear term determines if we want to discuss attractive (positive) or repulsive (negative) non-linearity of the wave. The NLS equation is also involve in Bose Condensation, a context where it is often called the Gross-Pitaevskii equation where it has solutions that describes vortex like structures that define states that can be observed in superfluid Helium.

This is a brief summary of the history about the areas that will be covered in this project and some background knowledge to give some motivation into why this area of applied mathematics is so important when it comes to vortex theory and quantum fluids.

1.5 Notation

For the purpose of this project it is desirable to define terms that will be used throughout this project that may or may not be known to the reader. The following symbols are used in the discussion of wave.

$\vec{r} = (x_1, x_2, x_3)$ this represents position in 3D Euclidean space $\mathbb{E}^3$
$\vec{k} = (k_1, k_2, k_3)$ is known as the Wavevector
ω is the frequency of the wave
λ is known as the Wavelength
$\vec{v} = (v_1, v_2, v_3)$ is velocity

The general Plane Wave solution is of the form $Ae^{i\vec{k}\cdot\vec{r}-i\omega t}$ where $A \in \mathbb{C}, t \in \mathbb{R}$ Another important quantity is Vorticity, this is defined to be the Curl of the velocity field.

$$Curl(\vec{v}) = \vec{\nabla} \times \vec{v}$$

where we define the gradient vector as:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

We define the Laplacian operator Δ as:

$$\Delta = \vec{\nabla} \cdot \vec{\nabla} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Throughout this project I will use simple notation to define the magnitude of a vector $\vec{k}$ as k where $k = |\vec{k}|$.

The time derivative of a function will denoted as $\frac{\partial \phi}{\partial t} = \dot{\phi}$.

2 The Nonlinear Schrödinger equation

In this chapter I will be mainly discussing the NLS equation (1.1). I will look at dispersion relations when considering a solution to this equation of the form of a plane wave and then finding the normal modes by considering an eigenvalue problem to this system. The NLS equation is relevant to this project as it is the basic equation associated to Bose-Einstein condensates. We will see later in the project that a more complex equation is generally used to describe BECs (the Gross-Pitaevskii equation), but this is a good foundation to start on.

2.1 Solution with no condensate

Now to make a clear comparison between a solution with no condensate and a solution that takes this into account, we will derive a simple dispersion relation by just considering a plane wave solution to equation (1.1) with no condensate that only takes small oscillations around zero, i.e. $|\psi| \ll 1$. A general plane wave solution takes the form $\psi = \psi(\vec{r}, t) = Ae^{i\vec{k}\cdot\vec{r} - i\omega t}$ where $A \in \mathbb{C}$.

As we are only considering ψ to small oscillations we can linearise (1.1) and so we can neglect the $|\psi|^2\psi$ term. This gives us

$$i\dot{\psi} + \Delta\psi = 0. \tag{2.1}$$

Now substituting in the general solution into (2.1) we get

$$i(-i\omega\psi) - k^2\psi = \omega\psi - k^2\psi = 0$$

by cancelling the ψ terms in this equation we get

$$\Rightarrow \omega = k^2. \tag{2.2}$$

This is the simple dispersion relation for small perturbations from a constant solution without condensate.

2.2 Solution with condensate

Now we go back to the NLS equation (1.1), this time we will consider condensate. To do this we must take a solution of the form $\psi = \psi_0(1 + \varphi)$ where we have $\psi_0 = \sqrt{\varrho}e^{\pm i\varrho t}$ with $\sqrt{\varrho} \in \mathbb{R}_+$ an exact solution to equation (1.1), where $\varphi = \varphi(\vec{r}, t)$ and let $|\varphi| \ll 1$. ϱ corresponds to the density of the condensate now being considered.

In order to work out a new equation in terms of φ we need to substitute our formula for ψ into the original NLS equation (1.1) and linearise with respect to φ.

By the product rule we get

$$\begin{aligned} \dot{\psi} &= \pm i\varrho\psi_0(1 + \varphi) + \psi_0\dot{\varphi} && (2.3) \\ \Delta\psi &= \psi_0\Delta\varphi && (2.4) \\ |\psi|^2 &= |\psi_0(1 + \varphi)|^2 = |\psi_0|^2|(1 + \varphi)|^2 && (2.5) \end{aligned}$$

As $|\psi_0| = |\sqrt{\varrho}e^{\pm i\varrho t}| = |\sqrt{\varrho}|$ this gives us

$$|\psi|^2 = \varrho|(1+\varphi)|^2. \tag{2.6}$$

Using (2.3), (2.4) and (2.6), we can substitute these into the NLS equation (1.1) and this will result in an equation in terms of φ.

So we get

$$\mp\varrho\psi_0(1+\varphi) + i\psi_0\dot{\varphi} + \psi_0\Delta\varphi \pm \varrho|(1+\varphi)|^2\psi_0(1+\varphi) = 0$$

we can now cancel out the ψ_0 terms from each expression

$$\Rightarrow \mp\varrho(1+\varphi) + i\dot{\varphi} + \Delta\varphi \pm \varrho|(1+\varphi)|^2(1+\varphi) = 0$$

expanding this equation out gives

$$\Rightarrow \mp\varrho(1+\varphi) + i\dot{\varphi} + \Delta\varphi \pm \varrho(1+\varphi+\bar{\varphi}+|\varphi|^2)(1+\varphi) = 0$$

$$\Rightarrow i\dot{\varphi} + \Delta\varphi \pm \varrho(\bar{\varphi} + 2|\varphi|^2 + \varphi|\varphi|^2 + \varphi + \varphi^2) = 0. \tag{2.7}$$

As we are only considering $|\varphi| \ll 1$ we can linearise the above equation w.r.t. φ to give us the equation we want

$$i\dot{\varphi} + \Delta\varphi \pm \varrho(\bar{\varphi} + \varphi) = 0. \tag{2.8}$$

So now we want to derive a dispersion relation for this problem, this is known as the Bogoliubov dispersion relation. We need to consider a solution for φ in the form of $\varphi = Be^{i\vec{k}\cdot\vec{r}-i\omega t} + Ce^{-i\vec{k}\cdot\vec{r}+i\omega t}$ where we have $C, B \in \mathbb{C}$ because you want to look for a wave like solution plus its conjugate in a continuously changing index of refraction where the local wave length does not change appreciably over a distance of one wavelength.

Then by substituting in this solution into Equation (2.8) we have

$$\omega Be^{i\vec{k}\cdot\vec{r}-i\omega t} - \omega Ce^{-i\vec{k}\cdot\vec{r}+i\omega t} - k^2Be^{i\vec{k}\cdot\vec{r}-i\omega t} - k^2Ce^{-i\vec{k}\cdot\vec{r}+i\omega t}$$

$$\pm B\varrho e^{i\vec{k}\cdot\vec{r}-i\omega t} \pm C\varrho e^{-i\vec{k}\cdot\vec{r}+i\omega t} \pm \bar{B}\varrho e^{i\vec{k}\cdot\vec{r}-i\omega t} \pm \bar{C}\varrho e^{-i\vec{k}\cdot\vec{r}+i\omega t} = 0$$

Now as this equation is true for all time t and position $\vec{r}$ we can equate the $e^{i\vec{k}\cdot\vec{r}-i\omega t}$ and $e^{-i\vec{k}\cdot\vec{r}+i\omega t}$ terms

$$(\omega B - k^2B \pm B\varrho \pm \bar{C}\varrho)e^{i\vec{k}\cdot\vec{r}-i\omega t} = 0 \tag{2.9}$$

and

$$(-\omega C - k^2C \pm C\varrho \pm \bar{B}\varrho)e^{-i\vec{k}\cdot\vec{r}+i\omega t} = 0. \tag{2.10}$$

We can now divide through by the exponentials as these are always non-zero

$$\Rightarrow B(\omega - k^2 \pm \varrho) = \mp\bar{C}\varrho \tag{2.11}$$

and

$$C(-\omega - k^2 \pm \varrho) = \mp \bar{B}\varrho. \tag{2.12}$$

We can now complex conjugate one of these equations to enable us to eliminate some of the variables, i.e. Equation (2.11) now becomes

$$\bar{B}(\omega - k^2 \pm \varrho) = \mp C\varrho \tag{2.13}$$

as ω, k and ϱ are all real valued.

Then by multiplying equations (2.13) and (2.12) together we get

$$C\bar{B}(\omega - k^2 \pm \varrho)(-\omega - k^2 \pm \varrho) = \varrho^2 C\bar{B}$$

We can then multiply this out and divide through by $C\bar{B}$

$$\Rightarrow -\omega^2 + k^4 + \varrho^2 \mp 2k^2\varrho = \varrho^2$$

$$\Rightarrow \omega^2 = k^4 \mp 2k^2\varrho \tag{2.14}$$

This is the Bogoliubov dispersion relation.

2.3 The Bogoliubov dispersion relation

Now that we have derived Bogoliubov's dispersion relation we will consider the physical meaning of this relation when we vary the values of the variables. When the sign in the relation is "+" this represents defocusing nonlinearity of the wave. This can be seen when considering BEC in repulsive Rubidium atoms.

If we consider $k^2 \gg 2\varrho$ then the relation behaves approximately like $\omega = k^2$ which is the dispersion relation for the non-condensate solution. If we know look at the other case, if $k^2 \ll 2\varrho$ then we can neglect the k^4 term and so the dispersion relation reduced to $\omega = (\sqrt{2\varrho})|k|$. This represents sound waves, or more precisely acoustic waves. The $\sqrt{2\varrho}$ is then equal to c_s which is the speed of sound.

Lets consider when we have the " $-$ " in (2.14). This corresponds to wave focusing, this can be seen as an attractive potential in BEC, e.g. in Lithium gas. Now if we have $k^2 < 2\varrho$ this implies that ω^2 is negative.

$$\omega^2 = -(2\varrho - k^2)k^2$$

$$\Rightarrow \omega = i|k|\sqrt{2\varrho - k^2}$$

This is called Modulational Instability as our wave solution ψ becomes

$$\psi(\vec{r}, t) = e^{i\vec{k}\cdot\vec{r} - i\omega t} = e^{|k|\sqrt{2\varrho - k^2}} e^{i\vec{k}\cdot\vec{r}}.$$

This solution has exponential growth as seen in the first exponential and so will show signs of instability when this exponent tends to infinity.

2.4 Normal Modes

Normal modes are very important in an oscillatory system where all parts of the system are oscillating with the same frequency, these frequencies are called normal frequencies or allowed frequencies.

The normal modes are in fact the eigenvectors of the eigenvalue problem of the system, the corresponding eigenvalues are called normal frequencies.

Hence in this section we will consider the eigenvalue problem and then find the eigenvalues and eigenvectors of this.

To start we need to consider equation (2.8):

$$i\dot{\varphi} + \Delta\varphi \pm \varrho(\bar{\varphi} + \varphi) = 0.$$

As $\varphi(\vec{r}, t) : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ we can consider the Real and Imaginary parts of φ. Let the real part of φ equal $\Re(\varphi) = R$, and the imaginary part to be $\Im(\varphi) = I$. Then from (2.8) we get:
Real part of (2.8)

$$-\dot{I} + \Delta R \pm 2\varrho R = 0 \tag{2.15}$$

Imaginary part of (2.8)

$$\dot{R} + \Delta I = 0. \tag{2.16}$$

Now lets define the Fourier Transform of a function f to be $\hat{f}$ such that

$$\hat{f}(k) := \int_{\mathbb{R}^n} e^{-i\vec{k}\cdot\vec{r}} f(\vec{r}) \mathrm{d}\vec{r}$$

where $\vec{k} \in \mathbb{R}^n$ and $\vec{k} \cdot \vec{x} = \sum_{j=1}^{n} k_j x_j$

So now lets take the Fourier Transform with respect to $\vec{r}$ of (2.15) and (2.16)

$$-\dot{\hat{I}} - k^2\hat{R} \pm 2\varrho\hat{R} = 0$$

$$\Rightarrow \dot{\hat{I}} = -k^2\hat{R} \pm 2\varrho\hat{R} \tag{2.17}$$

and

$$\dot{\hat{R}} - k^2\hat{I} = 0$$

$$\Rightarrow \dot{\hat{R}} = k^2\hat{I}. \tag{2.18}$$

Now consider a vector $\vec{v} = \binom{\hat{R}}{\hat{I}}$, rewriting equations (2.18) and (2.17) in the form of $\dot{\vec{v}} = M\vec{v}$ where $M \in \mathbb{R}^{2\times 2}$.

So

$$\dot{\vec{v}} = \begin{pmatrix} \dot{\hat{R}} \\ \dot{\hat{I}} \end{pmatrix} = \begin{pmatrix} 0 & k^2 \\ -k^2 \pm 2\varrho & 0 \end{pmatrix} \begin{pmatrix} \hat{R} \\ \hat{I} \end{pmatrix} = M\vec{v}$$

Now that we have found the matrix M, we need to now find the eigenvalues and thus the eigenvectors for this system. To find the eigenvalues λ we need to consider $\det(M - \lambda I_2) = 0$.

$$\det(M - \lambda I_2) = \det\left(\begin{pmatrix} -\lambda & k^2 \\ -k^2 \pm 2\varrho & -\lambda \end{pmatrix}\right) = \lambda^2 + k^4 \mp 2\varrho k^2 = 0$$

So we have

$$\lambda^2 = -k^4 \pm 2\varrho k^2$$
$$\Rightarrow \lambda_\pm = \pm|k|\sqrt{-k^2 \pm 2\varrho} = \pm i|k|\sqrt{k^2 \mp 2\varrho} = \pm i\omega_B.$$

Here I denote $\lambda_\pm$ as the two values of λ corresponding to the $\pm$ created from taking the square root and where ω_B is Bogoliubov's dispersion relation

$$\omega_B = \omega = |k|\sqrt{k^2 \mp 2\varrho}.$$

So now we need to find the corresponding eigenvectors $\vec{v}_+$ and $\vec{v}_-$.

Consider the eigenvalue equations $\lambda_+\vec{v}_+ = M\vec{v}_+$ and $\lambda_-\vec{v}_- = M\vec{v}_-$ where $\lambda_\pm$ are the eigenvalues and $\vec{v}_\pm$ are the corresponding eigenvectors to be found.

Let

$$\vec{v}_\pm = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

So

$$\lambda_\pm \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & k^2 \\ -k^2 \pm 2\varrho & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{aligned} \Rightarrow \quad \lambda_\pm v_1 &= k^2 v_2 \\ \lambda_\pm v_2 &= (-k^2 \pm 2\varrho) v_1 \end{aligned}$$

Solving these two simultaneous equations gives

$$\vec{v}_\pm = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \hat{R} \\ \hat{I} \end{pmatrix} = \begin{pmatrix} k^2 \\ \lambda_\pm \end{pmatrix}$$

so therefore our eigenvalues $\lambda_\pm$ and eigenvectors $v_\pm$ for this problem are:

$$\lambda_+ = i|k|\sqrt{k^2 \mp 2\varrho} \qquad v_+ = \begin{pmatrix} k^2 \\ i|k|\sqrt{k^2 \mp 2\varrho} \end{pmatrix}$$
$$\lambda_- = -i|k|\sqrt{k^2 \mp 2\varrho} \qquad v_- = \begin{pmatrix} k^2 \\ -i|k|\sqrt{k^2 \mp 2\varrho} \end{pmatrix}.$$

Now let's define a quantity z_- such that its a linear combination of the terms in our eigenvectors.

$$z_- = \hat{R} - i\frac{k^2}{\omega_B}\hat{I}. \tag{2.19}$$

We get the coefficients for the $\hat{R}$ and $\hat{I}$ terms by taking the values in the eigenvector.

Then

$$\dot{z}_- = \dot{\hat{R}} - i\frac{k^2}{\omega_B}\dot{\hat{I}}$$

Now using (2.17) and (2.18)
We have

$$\dot{z}_- = k^2\hat{I} - i\frac{(-k^2 \pm 2\varrho)k^2}{\omega_B}\hat{R} = -i\omega_B\hat{R} + k^2\hat{I} = -i\omega_B z_- = \lambda_- z_-.$$

Also if we define a quantity z_+ to be

$$z_+ = \hat{R} + i\frac{k^2}{\omega_B}\hat{I} \tag{2.20}$$

then similarly taking the time derivative of this gives

$$\dot{z}_+ = \dot{\hat{R}} + i\frac{k^2}{\omega_B}\dot{\hat{I}} = k^2\hat{I} + i\frac{(-k^2 \pm 2\varrho)k^2}{\omega_B}\hat{R} = i\omega_B\hat{R} - k^2\hat{I} = i\omega_B z_+ = \lambda_+ z_+$$

Therefore we have found the two vectors $z_\pm$ corresponding to the eigenvalues $\lambda_\pm$, or by rearranging $z_\pm$ we get

$$\hat{R} = \frac{1}{2}(z_+ + z_-), \qquad \hat{I} = \frac{i\omega_B}{2k^2}(z_+ - z_-).$$

3 Fluidic properties of the NLS Equation

In this section I will look at another representation of the NLS equation (1.1), formally close to the equations of hydrodynamics.

$$\frac{\partial\theta}{\partial t}+\frac{p}{\rho}+\frac{1}{2}(\vec{\nabla}\theta)^2+\lambda=f(t) \tag{3.1}$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \tag{3.2}$$

These are the equations of Hydrodynamics for a compressible inviscid fluid, (3.1) is the time dependent Bernoulli equation for irrotational flow, this can be derived from the Euler equation by considering a potential, which in this case is θ, and (3.2) is the Continuity equation for incompressible fluid, and where $\vec{v}$ is the velocity.

First we need to start with the NLS equation (1.1) below, and then transform it by using the Madelung's transformation.

$$i\dot{\psi}+\Delta\psi\pm|\psi|^2\psi=0$$

Madelung's transformation:

$$\psi=\sqrt{\rho}e^{i\theta},\qquad \rho=|\psi|^2,\qquad \vec{v}=\vec{\nabla}\theta$$

This transformation was originally introduced in the context of the linear schrödinger equation for quantum mechanics.

So we will start by applying the transformation to each term in the NLS equation:

$$\begin{aligned}
\vec{\nabla}\psi &= \vec{\nabla}(\sqrt{\rho})e^{i\theta}+i(\vec{\nabla}\theta)\psi\\
\Delta\psi &= \vec{\nabla}\cdot(\vec{\nabla}\psi)=\Delta(\sqrt{\rho})e^{i\theta}+2i(\vec{\nabla}\theta)\cdot(\vec{\nabla}\sqrt{\rho})e^{i\theta}-(\vec{\nabla}\theta)^2\sqrt{\rho}e^{i\theta}+i\Delta\theta\psi\\
\dot{\psi} &= (\dot{\sqrt{\rho}})e^{i\theta}+\sqrt{\rho}i\dot{\theta}e^{i\theta}=\tfrac{1}{2\sqrt{\rho}}\dot{\rho}e^{i\theta}+\sqrt{\rho}i\dot{\theta}e^{i\theta}.
\end{aligned}$$

So substituting these equations into (1.1) we get:

$$\begin{aligned}
&\tfrac{i}{2\sqrt{\rho}}\dot{\rho}e^{i\theta}-\sqrt{\rho}\dot{\theta}e^{i\theta}+\Delta(\sqrt{\rho})e^{i\theta}+2i(\vec{\nabla}\theta)\cdot(\vec{\nabla}\sqrt{\rho})e^{i\theta}\\
&-(\vec{\nabla}\theta)^2\sqrt{\rho}e^{i\theta}+i(\Delta\theta)\sqrt{\rho}e^{i\theta}\pm\rho^{\frac{3}{2}}e^{i\theta}=0
\end{aligned} \tag{3.3}$$

Now we can cancel out the $e^{i\theta}$ from each term in the equation above, and then equate real and imaginary parts of this equation.

Equating real parts of (3.3)

$$-\sqrt{\rho}\dot{\theta}+\Delta(\sqrt{\rho})-(\vec{\nabla}\theta)^2\sqrt{\rho}\pm\rho^{\frac{3}{2}}=0$$

Now using the fact that $\Delta(\sqrt{\rho})=\frac{1}{2\sqrt{\rho}}\Delta\rho$ and then dividing through by $\sqrt{\rho}$ gives

$$-\dot{\theta}+\frac{1}{2\rho}\Delta\rho-(\vec{\nabla}\theta)^2\pm\rho=0, \tag{3.4}$$

so by rescaling time so that $t \mapsto \frac{t}{2}$ gives

$$-\dot{\theta} + \frac{1}{4\rho}\Delta\rho - \frac{1}{2}(\vec{\nabla}\theta)^2 \pm \frac{\rho}{2} = 0.$$

Now by rewriting in the following form, this gives us an equation that resembles the time dependent Bernoulli equation for irrotational flow (3.1) with a "quantum pressure" term on the right.

$$\dot{\theta} + \frac{1}{2}(\vec{\nabla}\theta)^2 \pm \frac{\rho}{2} = \frac{1}{4\rho}\Delta\rho. \tag{3.5}$$

Equating imaginary parts of (3.3)

$$\frac{1}{2\sqrt{\rho}}\dot{\rho} + 2(\vec{\nabla}\theta)\cdot(\vec{\nabla}\sqrt{\rho}) + (\Delta\theta)\sqrt{\rho} = 0$$

$$\Rightarrow \dot{\rho} = -2(\vec{\nabla}\theta)\cdot(\vec{\nabla}\rho) - 2\rho\Delta\theta = -2\vec{\nabla}\cdot(\rho\vec{\nabla}\theta).$$

Similarly by rescaling time by a factor of 2 we get an equation that is exactly (3.2)

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{\nabla}\theta) = 0. \tag{3.6}$$

This model is relevant for the description of superfluid Helium II at absolute zero, viewed as an imperfect Bose condensate with local interactions. It is useful to note that Madelung's transformation is singular when ψ vanishes. Since the field is complex, these 'topological defects' are located on points in 2-dimensions and on lines in 3-dimensions. When talking about superfluidity, these defects are know as quantum vortices. The circulation of the velocity $\vec{v} = \vec{\nabla}\theta$ around each of these vortices is $\pm 2\pi$.

The quantum-mechanical pressure induces a dispersive effect when equations (3.5) and (3.6) are linearised about the solution $\rho = 1$, $\vec{\nabla}\theta = 0$. This creates a long-wavelength reductive perturbation expansion that leads to the Korteweg-de Vries equation in 1-dimension and to the Kadomtsev-Petviashvili equation in higher dimensions.

4 Gross-Pitaevskii Theory

4.1 The Gross-Pitaevskii equation

When specifically considering a Bose-Einstein Condensate we tend to consider the GP Equation. This is very similar to the NLS equation that we have already studied, but with an extra term proportional to ψ. This term corresponds to the use of a potential when studying BEC experimentally. The GP equation was derived independently by Gross and Pitaevskii in 1961, and it is still the main theoretical tool when investigating nonuniform dilute Bose gases at very low temperatures. The full GP Equation is:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + V(\vec{r},t)\psi(\vec{r},t) + g|\psi(\vec{r},t)|^2\psi(\vec{r},t). \tag{4.1}$$

The external potential $V(\vec{r},t)$ allows us to model the action of the external environment on the fluid. In particular the effect of a trap. Generally the bosons are confined in a small region of space. This is created by using a trapping potential with a minimum in the central region. This forces the bosons to collect in this well.

In this section we will study the nondimensionalised version of the GP equation as this will simplify the analysis done. This is given by

$$i\frac{\partial\psi(\vec{r},t)}{\partial t} = -\Delta\psi(\vec{r},t) + V(\vec{r},t)\psi(\vec{r},t) + |\psi(\vec{r},t)|^2\psi(\vec{r},t) \tag{4.2}$$

where the density of the condensate is $\varrho(\vec{r}) = |\psi(\vec{r})|^2$.

4.2 Hamiltonian form of the GP equation

The GP equation can be expressed as a Hamiltonian system, this can be very useful in some circumstances as Hamiltonian systems are simpler and thus easier to work with.

The Lagrangian can easily be worked out from the GP equation itself, this is

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\dot{\psi} - \psi\dot{\bar{\psi}}\right) - \nabla\psi\nabla\bar{\psi} - \frac{|\psi|^4}{2} - V|\psi|^2 \tag{4.3}$$

The Hamiltonian density is the integrand of the *Energy functional* or *Hamiltonian* of the system $\mathcal{H}$. This is given by the formula

$$\text{Hamiltonian Density} = \frac{i}{2}\left(\bar{\psi}\dot{\psi} - \psi\dot{\bar{\psi}}\right) - \mathcal{L}.$$

So integrating this density over the space we get the Hamiltonian of the system $\mathcal{H}$ as

$$\mathcal{H} = \int\left(|\nabla\psi|^2 + V|\psi|^2 + \frac{|\psi|^4}{2}\right)\mathrm{d}\vec{r}. \tag{4.4}$$

So now we can rewrite the Gross-Pitaevskii equation (4.2) in the form below.

$$i\frac{\partial\psi(\vec{r},t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta\bar{\psi}(\vec{r},t)}.$$

Now if we multiply (4.2) by $\bar{\psi}$ we get

$$i\bar{\psi}\dot{\psi} = -\bar{\psi}\Delta\psi + V|\psi|^2 + |\psi|^4$$

then by subtracting the complex conjugate of this equation from itself gives

$$i\bar{\psi}\dot{\psi} + i\psi\dot{\bar{\psi}} = -\bar{\psi}\Delta\psi + \psi\Delta\bar{\psi} \tag{4.5}$$

which happens to be the continuity equation

$$\frac{\partial\varrho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{4.6}$$

where $\varrho = |\psi|^2$ and $\vec{j} = i\left(\psi\vec{\nabla}\bar{\psi} - \bar{\psi}\vec{\nabla}\psi\right)$. Check:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{j} &= \vec{\nabla}\cdot\left(i\psi\vec{\nabla}\bar{\psi} - i\bar{\psi}\vec{\nabla}\psi\right) \\
&= i\vec{\nabla}\psi\cdot\vec{\nabla}\bar{\psi} + i\psi\Delta\bar{\psi} - i\vec{\nabla}\bar{\psi}\cdot\vec{\nabla}\psi - i\bar{\psi}\Delta\psi \\
&= i\left(\psi\Delta\bar{\psi} - \bar{\psi}\Delta\psi\right)
\end{aligned}$$

using (4.5)

$$\Rightarrow i\left(\psi\Delta\bar{\psi} - \bar{\psi}\Delta\psi\right) = -\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi = -\frac{\partial\varrho}{\partial t}$$

which yields (4.6).

From equation (3.2) we can compare this with (4.6). Then using Madelung's transformation we get the relationship

$$\vec{j}(\vec{r},t) = \varrho\vec{\nabla}\theta(\vec{r},t) \tag{4.7}$$

where we have from the definition of the Madelung's transformation

$$\psi(\vec{r},t) := \sqrt{\varrho}e^{i\theta(\vec{r},t)}.$$

By comparing the terms in the classical continuity equation (3.2) it immediately follows that the velocity of the condensate flow is

$$v_s(\vec{r},t) = \vec{\nabla}\theta(\vec{r},t) \tag{4.8}$$

and this is clearly irrotational as

$$\mathrm{Curl}(v_s) = \vec{\nabla}\times(\vec{\nabla}\theta(\vec{r},t)) \equiv 0. \tag{4.9}$$

4.3 Conservation of Energy

Also the energy of the system $\mathcal{H}$ is conserved as

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial t} &= \frac{\partial}{\partial t}\int \left(|\vec{\nabla}\psi|^2 + V(\vec{r})|\psi|^2 + \tfrac{1}{2}|\psi|^4\right) \mathrm{d}\vec{r} \\
&= \int \frac{\partial}{\partial t}\left(|\vec{\nabla}\psi|^2 + V(\vec{r})|\psi|^2 + \tfrac{1}{2}|\psi|^4\right) \mathrm{d}\vec{r} \\
&= \int \left(\vec{\nabla}\dot{\psi}\cdot\vec{\nabla}\bar{\psi} + \vec{\nabla}\dot{\bar{\psi}}\cdot\vec{\nabla}\psi + V\dot{\psi}\bar{\psi} + V\dot{\bar{\psi}}\psi + \dot{\psi}\psi\bar{\psi}^2 + \dot{\bar{\psi}}\bar{\psi}\psi^2\right) \mathrm{d}\vec{r}
\end{aligned}$$

using Integration By Parts

$$\begin{aligned}
&= \int \left(-\dot{\psi}\Delta\bar{\psi} - \dot{\bar{\psi}}\Delta\psi + V\dot{\psi}\bar{\psi} + V\dot{\bar{\psi}}\psi + \dot{\psi}\psi\bar{\psi}^2 + \dot{\bar{\psi}}\bar{\psi}\psi\right) \mathrm{d}\vec{r} \\
&= \int \left(\dot{\psi}(-\Delta\bar{\psi} + V\bar{\psi} + |\psi|^2\bar{\psi}) + \dot{\bar{\psi}}(-\Delta\psi + V\psi + |\psi|^2\psi)\right) \mathrm{d}\vec{r}
\end{aligned}$$

now using the Gross-Pitaevskii Equation (4.2)

$$= \int \left(\dot{\psi}(-i\dot{\bar{\psi}}) + \dot{\bar{\psi}}(i\dot{\psi})\right) \mathrm{d}\vec{r} = 0.$$

(Note: the energy is only conserved when V is time independent)

4.4 Quantised vortices in BEC

The vortical solution of the GP equation exhibits nontrivial features due to the occurrence of a core region where the density tends to zero. The size of this region turns out to be of the order of the healing length. The healing length is the length over which the wave function can vary while still minimising the free energy. This means that normal classical hydrodynamics cannot be used and quantum pressure terms become important in the calculations.

A superfluid cannot rotate as a normal fluid. In usual systems the velocity field corresponding to a rotation is given by the rigid body form $\vec{v} = \vec{\Omega}\times\vec{r}$, and is characterised by a diffused vorticity; Curl $(\vec{v}) = 2\vec{\Omega} \neq 0$. This velocity field contradicts the irrotationality condition of the superfluid (4.9).

Take a gas confined in a macroscopic cylinder vessel of radius R and length L and lets look for a solution of the GP equation corresponding to a rotation around the axis of the cylinder. Such a solution can be found to be of the form similar to the representation of ψ in Madelung's Transformation:

$$\psi(r) = |\psi(r)|e^{is\theta},$$

where we are in cylindrical coordinates (r, θ, z) and $|\psi| = \sqrt{\varrho}$. The parameter s is an integer in order to ensure the wave function ψ is single valued. Also as seen in Section 1 we had to rescale time to get the time dependent Euler equation and so this factor s is just another rescaling.

This wave function represents a fluid rotating around the z-axis with the following tangential velocity using (4.8)

$$\vec{v}_s = \vec{\nabla}(s\theta) = \frac{s}{r}$$

where

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \phi}{\partial z}\hat{e}_z$$

The circulation of the velocity field over the closed contour around the z-axis is given by

$$\kappa = \oint \vec{v}_s \cdot \mathrm{d}\vec{l} = 2\pi s$$

This is actually quantised in units independent of the radius of the contour. This is in fact a consequence of the vorticity being concentrated along the z-axis according to the law

$$\mathrm{Curl}(\vec{v}_s) = 2\pi s \delta_2(\vec{r})\hat{e}_z$$

where $\hat{e}_z$ is the unit vector in the z-direction. The last two results show that the irrotationality criterion associated with BEC is satisfied everywhere except on the vortex line.

One can define the number of particles in the Bose-Einstein condensate by simply integrating the density w.r.t. the volume, hence

$$N = \int |\psi_0|^2 \mathrm{d}\vec{r}$$

Now like we did with the NLSE, let's plug $\psi(\vec{r},t) = \sqrt{\varrho}e^{is\theta(\vec{r},t)}$ into our GP equation (nondimensionalised version) (4.2). We will get (3.4) but with an extra term that corresponds to V:

$$\frac{\partial \theta}{\partial t} + v_s^2 + V + \varrho - \frac{1}{\sqrt{\varrho}}\Delta\sqrt{\varrho} = 0. \tag{4.10}$$

Neglecting the quantum pressure term in (4.10), this is known as the Thomas-Fermi approximation. We can neglect this because the quantum pressure term is of the order $\sim R^{-2}$ for R much larger than the characteristic length. Then take the gradient of (4.10)

$$\frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla}\left(v_s^2 + V + \varrho\right) = 0 \tag{4.11}$$

Now if we take $v_s = 0$ in (4.11) we obtain

$$\vec{\nabla}\left(V + \varrho\right) = 0$$

$$\Rightarrow V(\vec{r}) + \varrho(\vec{r}) = \mu \tag{4.12}$$

where μ is actually the ground state chemical potential. (4.12) represents the condition of local equilibrium for the system.

4.5 Radial solution around a vortex

The solution of the GP equation takes a special form when considering stationary solutions where the condensate wave function evolves in time according to

$$\psi(\vec{r},t) = \psi(\vec{r})e^{-i\mu t} \tag{4.13}$$

substituting this into equation (4.2) gives

$$\left(-\Delta + V - \mu + |\psi(\vec{r})|^2\right)\psi(\vec{r}) = 0. \tag{4.14}$$

Now consider a radial solution, such that ψ depends on r rather than $\vec{r}$, so substituting $\psi(\vec{r}) = |\psi(r)|e^{is\theta}$ into (4.14) and using cylindrical coordinates (r,θ,z), where

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

gives (after dividing through by $e^{is\theta}$)

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}|\psi|}{\mathrm{d}r}\right) + \frac{s^2}{r^2}|\psi| + |\psi|^3 + (V-\mu)|\psi| = 0 \tag{4.15}$$

(Note: we now have ordinary derivatives and not partial)

At large distances from the vortex line, the density of the gas must approach its unperturbed uniform value ϱ, and hence $|\psi| \to \sqrt{\varrho}$.

Now lets introduce a dimensionless function f such that

$$|\psi| = \sqrt{\varrho}f(\eta) \tag{4.16}$$

where we define $\eta = \frac{r}{\xi}$, $\xi = \frac{1}{\sqrt{\varrho}}$ = healing length. The Healing length is the length over which the wave function can vary while still minimising the free energy of the system (nondimensionalised GP equation).

Now substitute (4.16) into (4.15) to get an equation in terms of f and its derivatives.

First let's consider the first term of (4.15).

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}|\psi|}{\mathrm{d}r}\right) = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\left(\sqrt{\varrho}f(\eta)\right)}{\mathrm{d}r}\right) = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sqrt{\varrho}f'(\eta)\frac{\mathrm{d}\eta}{\mathrm{d}r}\right)$$

where $f'(\eta) = \frac{\mathrm{d}f}{\mathrm{d}\eta}$ and as $\eta = r\sqrt{\varrho} \Rightarrow \frac{\mathrm{d}\eta}{\mathrm{d}r} = \sqrt{\varrho}$.

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sqrt{\varrho}f'(\eta)\frac{\mathrm{d}\eta}{\mathrm{d}r}\right) = -\frac{\sqrt{\varrho}}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\eta f'(\eta)\right).$$

Using the chain rule $\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}\eta}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}\eta}$, and the fact that $\eta = r\sqrt{\varrho}$ gives

$$-\frac{\sqrt{\varrho}}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\eta f'(\eta)\right) = -\frac{\varrho}{r}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f'(\eta)\right) = -\frac{\varrho\sqrt{\varrho}}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f'(\eta)\right).$$

Applying the same principle to the other terms gives the equation in this new form

$$-\frac{\varrho^{\frac{3}{2}}}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)+\frac{\varrho^{\frac{3}{2}}s^2}{\eta^2}f+\varrho^{\frac{3}{2}}f^3+(V-\mu)\sqrt{\varrho}f=0$$

Now from (4.12) we get that $V-\mu=-\varrho$. So now we can divide the above equation by $\varrho^{\frac{3}{2}}$ as it now appears in all terms to get the equation

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)+\left(1-\frac{s^2}{\eta^2}\right)f-f^3=0 \tag{4.17}$$

Now f must satisfy a few conditions. We know at large distances away from the vortex line $|\psi|\to\sqrt{\varrho}$, so this corresponds to $f(\infty)=1$. As $\eta\to 0$ the physical solution of (4.17) tends to zero as $f\sim\eta^{|s|}$, so the density $\varrho(r)=|\psi(r)|^2$ of the gas tends to zero on the axis of the vortex.

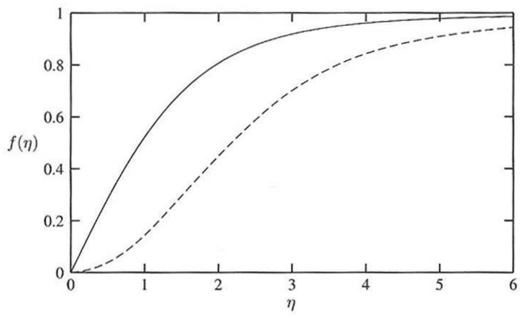

Figure 4: Vortical solution (s=1 solid line, s=2 dashed line) of the Gross-Pitaevskii equation as a function of the radial coordinate $\eta=\frac{r}{\xi}$. The density of the Bose-Einstein condensate is given by $\varrho(\vec{r})=\varrho f^2$.

5 Forces on a Vortex in a Superfluid

Before we go on to discuss the Aharonov-Bohm effect, which is going to be the main objective of this project, lets first consider the Magnus Force. It is worth quickly considering the magnus force in classical hydrodynamics. It has been know for a fairly long time that if a vortex moves with respect to a liquid there is a force on the vortex normal to the vortex velocity, this is what is known as the Magnus Force. This is the special case of a force on a body immersed in a fluid with a flow circulation around it. This is shown in the Kutta-Joukowski Lift Theorem.

Theorem 1 (Kutta-Joukowski Theorem)**.** *Consider a steady flow past a 2D body, with a cross-section of which is some simple closed curve C. Let the flow be uniform at infinity, with speed U in the x-direction, and let the circulation around the body be* Γ*, then*

$$F_x = 0, F_y = -\rho U \Gamma$$

where F_x *is the force in the x-direction and* F_y *is the force in the y-direction.*

The key role of the magnus force in vortex dynamics became clear from the very beginning of studies in superfluid hydrodynamics. In an article by Hall and Vinen, they defined the superfluid magnus force as a force between a vortex and a superfluid, therefore it was proportional to the superfluid density ρ_s. However in two-fluid hydrodynamics, the superfluid magnus force is not the only force on the vortex transverse to its velocity; there was also a transverse force between the vortex and quasiparticles moving with respect to the vortex.

The transverse force from rotons was found by Lifshitz and Pitaevskii and a transverse force from phonons discovered by Iordanskii. Ao and Thouless pointed out a link between the magnus force and the Berry phase which is the phase variation of the quantum-mechanical wave function resulting from the transport of the vortex round a closed loop.

Ao and Thouless concluded that the effective magnus force is proportional to the superfluid density, and that there is no transverse force on the vortex from quasiparticles and impurities, which disagrees with previous ideas within the field.

5.1 Magnus Force in Classical Hydrodynamics

A simple example of the magnus force seen in everyday life, is when a ball travelling in one direction curves through the air, e.g. Football or tennis ball. This is because the ball is rotating while it is travelling is a straight line, this creates a force on the ball perpendicular to its motion that overall moves the ball in a nonlinear way. As seen in Figure 5 the ball is moving to the right with velocity $\vec{v}$ and thus has a corresponding drag force F_d in the opposite direction. However the ball also has an angular velocity Ω, this causes a force F_m; the Magnus force on the ball perpendicular to the motion of the ball that makes the ball appear to curve through the air.

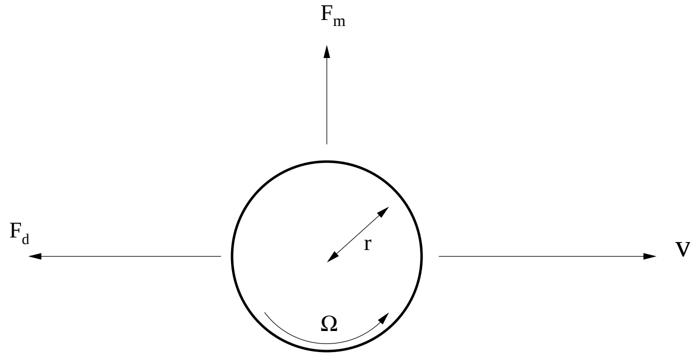

Figure 5: A ball travelling to the right with spin creates a magnus force pointing upwards.

As seen in [3], let us consider an isolated straight vortex line in an incompressible inviscid liquid. The line along the axis z induces the velocity field $\vec{v}_v$. This can been seen using the example of a cylinder radius r_0 immersed in an incompressible inviscid fluid. There is a potential circular flow around the cylinder with velocity $\vec{v}_v$ such that

$$\vec{v}_v(\vec{r}) = \frac{\vec{\kappa} \times \vec{r}}{2\pi r^2} \tag{5.1}$$

where $\vec{r}$ is the position vector in the xy-plane and the z-axis is the axis of the cylinder and κ is the circulation vector around z where

$$\kappa = \oint \vec{v}_v \cdot \mathrm{d}\vec{l}.$$

There is also a uniform flow of fluid past the cylinder with transport velocity $\vec{v}_{tr}$ and so the net velocity is

$$\vec{v}(\vec{r}) = \vec{v}_v(\vec{r}) + \vec{v}_{tr} \tag{5.2}$$

this expression is valid for distances r much larger than the cylinder radius r_0.

Now for an incompressible inviscid fluid let us consider the Euler equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P \tag{5.3}$$

Now lets assume that the cylinder is moving with constant velocity $\vec{v}_L$, we get by replacing $\vec{r}$ with $\vec{r} - t\vec{v}_L$:

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v}_L \cdot \vec{\nabla})\vec{v} \tag{5.4}$$

Then using Bernoulli's Theorem:

Theorem 2 (Bernoulli's Theorem)**.** *For a steady flow:*

$$\vec{\omega} \times \vec{v} = -\vec{\nabla}H$$

where

$$H = \frac{P}{\rho} + \frac{\vec{v}^2}{2} + \lambda \tag{5.5}$$

With λ constant. Then on taking the dot product with v we obtain

$$(\vec{v} \cdot \vec{\nabla})H = 0$$

and so if an ideal fluid is in steady flow, the H is constant along a streamline.

So using this theorem, (5.2) and (5.5) we get

$$\begin{aligned} P \quad = P_0 - \tfrac{1}{2}\rho(\vec{v}(\vec{r}) - \vec{v}_L)^2 \quad &= -\frac{1}{2}\rho((\vec{v}_v(\vec{r}))^2 - 2\vec{v}_v(\vec{r}) \cdot \vec{v}_{tr} + 2\vec{v}_{tr} \cdot \vec{v}_L + 2\vec{v}_v(\vec{r}) \cdot \vec{v}_L \\ &\quad +(\vec{v}_{tr})^2 + (\vec{v}_L)^2) \\ &= P_0' - \frac{1}{2}\rho(\vec{v}_v(\vec{r}))^2 - \rho\vec{v}_v(\vec{r}) \cdot (\vec{v}_{tr} - \vec{v}_L) \end{aligned} \tag{5.6}$$

where P_0 is a constant and $P_0' = P_0 - \frac{1}{2}\rho(\vec{v}_{tr} - \vec{v}_L)^2$ is also a constant. According to Bernoulli's theorem, the pressure is higher in the region where velocity is lower. As a result of this, the fluid produces a force on the cylinder perpendicular to the relative velocity of the fluid, this is the Magnus Force.

In order to find the whole force, we must consider the momentum balance for the cylinder region of a radius r_0 around the cylinder. The momentum conservation law requires the external force $\vec{F}$ on the cylinder to equal the momentum flux through the entire cylindrical boundary in the reference frame moving with the vortex velocity $\vec{v}_L$.

The momentum-flux tensor is defined as:

$$\Pi_{ij} = P\delta_{ij} + \rho v_i(\vec{r})v_j(\vec{r})$$

Therefore for this example (in reference frame with vortex velocity $\vec{v}_L$) we get the following momentum flux tensor

$$\Pi'_{ij} = P\delta_{ij} + \rho(v_i(\vec{r}) - (v_L)_i)(v_j(\vec{r}) - (v_L)_j) \tag{5.7}$$

The momentum flux through the surface of the cylinder at radius r_0 is

$$\int \Pi'_{ij} \mathrm{d}S_j$$

Where $\mathrm{d}S_j$ are the components of the vector $\mathrm{d}\vec{S}$ directed along the outer normal to the boundary of the cylindrical region and equal to the elementary area of the boundary magnitude. Then using (5.1), (5.6) and (5.7), the momentum balance law yields the following relation:

$$\rho[(\vec{v}_L - \vec{v}_{tr}) \times \vec{\kappa}] = \vec{F} \tag{5.8}$$

In the absence of the external force the cylinder moves with the transport velocity $\vec{v}_L = \vec{v}_{tr}$, (this is an example of Helmholtz Theorem).

5.2 The Magnus Force in a Superfluid

In the superfluid state liquid motion is described by two-fluid hydrodynamics, this means the liquid consists of the superfluid and the normal component with the superfluid, and normal densities ρ_s and ρ_n and normal velocities $\vec{v}_s$ and $\vec{v}_n$ respectively.

Hall and Vinen suggested that the magnus force is connected with the superfluid density ρ_s and that the superfluid velocity $\vec{v}_s(\vec{r}) = \vec{v}_v(\vec{r}) + \vec{v}_{s(tr)}$, so we get

$$\rho_s[(\vec{v}_L - \vec{v}_{s(tr)}) \times \vec{\kappa}] = \vec{F}. \tag{5.9}$$

The 'external' force $\vec{F}$ is in fact not external for the whole liquid, but for only its superfluid part. The force appears due to interactions with quasiparticles which constitute the normal part of the liquid and therefore is proportional to the relative velocity $\vec{v}_L - \vec{v}_n$.

The Euler equation for superfluid component after adding the external δ-component force $\vec{F}$ applied at the vortex line is

$$\frac{\partial \vec{v}_s}{\delta t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\vec{\nabla}\mu + \frac{\vec{F}}{\rho_s}\delta_2(\vec{r}) \tag{5.10}$$

where μ is the chemical potential.

Let us consider a vortex line in a neutral (non-charged) superfluid with velocity field (5.1).

Now the velocity $\vec{v}_{tr}$ is the superfluid velocity $\vec{v}_s$ far from the vortex line and the Bernoulli law is used for variation of the chemical potential, but the momentum of the superfluid component is not conserved because of interactions with quasiparticles in the vicinity of the vortex line.

One may assume that all of these interactions are incorporated by the external force $\vec{F}$ localised at the vortex line. Then we get equation (5.9), where we drop the (tr) from $\vec{v}_{s(tr)}$.

The force $\vec{F}$, which enters the theory as a δ-function force is distributed over a small vicinity of the vortex line in reality. Replacing the external force by the magnus force the Euler equation becomes

$$\frac{\partial \vec{v}_s}{\delta t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\vec{\nabla}\mu + [(\vec{v}_L - \vec{v}_s) \times \vec{\kappa}]\delta_2(\vec{r}) \tag{5.11}$$

Now using the vector identity

$$(\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = \vec{\nabla}\frac{\vec{v}_s^2}{2} - \vec{v}_s \times [\vec{\nabla} \times \vec{v}_s]$$

For neutral superfluids, vorticity is concentrated on the vortex line, so

$$(\vec{\nabla} \times \vec{v}_s) = \vec{\kappa}\delta_2(\vec{r})$$

Then the Euler equation becomes

$$\frac{\partial \vec{v}_s}{\delta t} = -\vec{\nabla}(\mu + \frac{\vec{v}_s^2}{2}) + [\vec{v}_L \times \vec{\kappa}]\delta_2(\vec{r}) \tag{5.12}$$

This analysis demonstrates that the total external force on the superfluid in the vicinity of the vortex line is exactly balanced by the superfluid magnus force $\rho_s[(\vec{v}_L - \vec{v}_s) \times \vec{\kappa}]$.

However the Euler equation is not sufficient to get a full description of superfluid motion as an additional equation for the vortex velocity $\vec{v}_L$ is needed. In order to obtain this additional equation one should specify the force $\vec{F}$ by considering the momentum balance of the whole fluid, not only just its superfluid part.

5.3 The Iordanskii Force

When considering a Galilean invariant fluid in the situation of the two-fluid model, we can write the transverse force in it's most general form as

$$\vec{F} = A\,[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)] + B\,[\vec{\kappa} \times (\vec{v}_L - \vec{v}_n)] \tag{5.13}$$

Where the motion of the vortex is given by $\vec{v}_L$, $\vec{v}_s$ and $\vec{v}_n$ denote the velocity of the superfluid and normal components of the fluid and $\vec{\kappa}$ is the circulation vector along the z-axis or vortex line.

So in the absence of any normal component of the fluid we get that

$$\vec{F} = \rho_{\text{Tot}}\,[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)] \tag{5.14}$$

This is known as the Magnus Force. However if we consider a normal component in the fluid we should consider a total transverse force in the form of (5.13).

The values of A and B have been discussed in detail over the past few years, it is generally accepted that $A = \rho_s$.

Wexter gave a thermodynamic argument with the help of Thouless that the coefficient of $\vec{v}_L$ in (5.13) is also ρ_s. This coefficient is $A + B$ so this would imply that $B = 0$. Thus ruling out the second term in (5.13) which is what is know as the Iordanskii Force.

However Edouard Sonin whose work I have based my project on does not agree with this result. He gives the coefficient of $-\vec{v}_n$ as ρ_n, thus having $B = \rho_n$.

Therefore (5.13) now becomes

$$\vec{F} = \rho_s\,[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)] + \rho_n\,[\vec{\kappa} \times (\vec{v}_L - \vec{v}_n)] \tag{5.15}$$

This is the most commonly accepted expression for the transverse force, but we can rewrite this in a way to get a term for momentum transfer to the vortex due to the condensate motion and a term for the force on the vortex due to phonon scattering.

To do this, note that $\rho_{\text{Tot}} = \rho_s + \rho_n$. Therefore we can rewrite (5.15) as

$$\begin{aligned}
\vec{F} &= \rho_s \left[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)\right] + \rho_n \left[\vec{\kappa} \times (\vec{v}_L - \vec{v}_n)\right] \\
&= \rho_{\text{Tot}} \left[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)\right] + \rho_n \left[\vec{\kappa} \times (\vec{v}_L - \vec{v}_n)\right] - \rho_n \left[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)\right] \\
&= \underbrace{\rho_{\text{Tot}} \left[\vec{\kappa} \times (\vec{v}_L - \vec{v}_s)\right]}_{\{1\}} + \underbrace{\rho_n \left[\vec{\kappa} \times (\vec{v}_s - \vec{v}_n)\right]}_{\{2\}}
\end{aligned}$$

Now, Part {1} of the above equation is the momentum transfer to the vortex due to the condensate motion of the fluid. Part {2} is the force on the vortex due to the scattering of phonons. This force does not depend not depend on $\vec{v}_L$, the motion of the vortex line relative to either component of the fluid.

5.4 The Born approximation

Before the next section it is worth just discussing some of the techniques used later on. When considering the scattering of a Phonon on a vortex, I will use the Born approximation. In general the Born approximation is a technique used in scattering theory in particular in quantum mechanics.

The Born approximation is a method used when considering a weak scattering of waves, it can only be used for this purpose because it is basically a linearisation of the scattering effect.

The Born approximation is the most commonly used approach to the application of vortex-sound interactions. There are however various ways to use this, the main way is by considering the Mach number to be small, Ma$\ll 1$. This was first introduced by Lighthill to calculate the noise generated by turbulence. A small Mach number implies the there is a weak scattering effect of sound by the vortex, so that the Born approximation can be used to give a good description. The Mach number represents the ratio of the speed of an object to the speed of sound in the surrounding medium.

6 Scattering of Phonons by a Vortex (Aharonov-Bohm effect)

6.1 The Aharonov-Bohm effect

The Aharonov-Bohm effect is commonly associated with wave scattering by a vortex and electron scattering by the magnetic-flux tube. However it was first studied in classical hydrodynamics for water surface waves. Later the scattering of light by a vortex was also studied and a similar result was seen. These three scattering effects are in general the same Aharonov-Bohm effect but seen in different situations.

The Aharonov-Bohm (AB) effect was first predicted in 1949 by Werner Ehrenberg and R.E. Sidey and then similar effects were later re-discovered by Aharonov and Bohm in 1959.

The most common description of the AB effect is actually the AB Solenoid effect, this is when the wave function of a charged particle passing around a long solenoid experiences a phase shift as a direct result of the enclosed magnetic field.

In this project I will look at the effect of a phonon scattering as it passes a vortex line, this was initially studied by Pitaevskii and Fetter. A phonon is quasiparticle which is a quantised sound waves, these appear in compressible fluids.

We will consider a sound wave propagating in the xy-plane with the vortex line in the z-plane. We consider a fluid that is moving at a uniform speed of $\vec{v}_0$, but around the vortex line we now consider a velocity of $\vec{v}_v(\vec{r})$ due to the effects from this vortex.

Now we consider a set of linearised hydrodynamic equations

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = -\vec{v}_v \cdot \vec{\nabla} \rho_1 \tag{6.1}$$

$$\frac{\partial \vec{v}_1}{\partial t} + \frac{c_s^2}{\rho_0} \vec{\nabla} \rho_1 = -[(\vec{v}_v \cdot \vec{\nabla}) \vec{v}_1 + (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_v] \tag{6.2}$$

where these are derived by taking the normal set of hydrodynamical equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{6.3}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} \mu. \tag{6.4}$$

Then consider a plane sound wave propagating through this liquid generating the phase variation $\phi(\vec{r}, t) = \phi_0 e^{i\vec{k}\cdot\vec{r} - i\omega t}$ where we get that $\omega = c_s k + \vec{k} \cdot \vec{v}_0$. Then consider the perturbed fluid density and velocity as

$$\rho(\vec{r}, t) = \rho_0 + \rho_1(\vec{r}, t), \qquad \vec{v}(\vec{r}, t) = \vec{v}_0 + \vec{v}_1(\vec{r}, t) \tag{6.5}$$

So substituting (6.5) into equations (6.3) and (6.4) gives

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot [(\rho_0 + \rho_1)(\vec{v}_0 + \vec{v}_1)] = 0$$

expanding out

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 + \vec{v}_0 \cdot \vec{\nabla} \rho_1 + \vec{\nabla} \cdot (\rho_1 \vec{v}_1) = 0$$

The last term of the above equation is nonlinear, so we can neglect this when linearising, and so we derive (6.1) after replacing $\vec{v}_0$ by $\vec{v}_v$. For the Euler equation (6.4)

$$\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_0 + \vec{v}_1) \cdot \vec{\nabla}(\vec{v}_0 + \vec{v}_1) = -\vec{\nabla}\mu$$

Now expanding and using the fact that $\mu = \frac{c_s^2}{\rho_0}\rho_1$

$$\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_0 \cdot \vec{\nabla})\vec{v}_1 + (\vec{v}_1 \cdot \vec{\nabla})\vec{v}_0 + (\vec{v}_1 \cdot \vec{\nabla})\vec{v}_1 + (\vec{v}_0 \cdot \vec{\nabla})\vec{v}_0 = -\frac{c_s^2}{\rho_0}\vec{\nabla}\rho_1$$

So by neglecting the nonlinear terms and now replacing $\vec{v}_0$ with $\vec{v}_v$ we get (6.2).

Now lets consider (6.1) and (6.2). Using the following vector identity on $\vec{v} = \vec{v}_v + \vec{v}_1$,

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla}\frac{v^2}{2} - \vec{v} \times [\vec{\nabla} \times \vec{v}] \tag{6.6}$$

and then linearising we get that

$$(\vec{v}_v \cdot \vec{\nabla})\vec{v}_1 + (\vec{v}_1 \cdot \vec{\nabla})\vec{v}_v = \vec{\nabla}(\vec{v}_v \cdot \vec{v}_1) - [\vec{v}_1 \times \vec{\kappa}]\delta_2(\vec{r}) \tag{6.7}$$

This is because the velocity field $\vec{v}_1$ is irrotational and so it's curl is zero and that $\vec{\nabla} \times \vec{v}_v = \vec{\kappa}\delta_2(\vec{r})$ because the vorticity is concentrated on the vortex line. So now (6.2) becomes

$$\frac{\partial \vec{v}_1}{\partial t} + \frac{c_s^2}{\rho_0}\vec{\nabla}\rho_1 = -\vec{\nabla}(\vec{v}_v \cdot \vec{v}_1) + [\vec{v}_1 \times \vec{\kappa}]\delta_2(\vec{r}) \tag{6.8}$$

We get the delta function because the vortex line is not at rest when the sound wave propagates past the vortex (the subscript on the delta is used to clarify the fact that we are considering it in two-dimensions). We can however reduce the singularity by making the vortex velocity time-dependent as a zero-order approximation to the velocity field. Then we must replace $\vec{r}$ with $\vec{r} - \vec{v}_L t$, where $\vec{v}_L$ is the velocity of the liquid. Then we get by using the vector identity above (6.7)

$$\frac{\partial \vec{v}_v}{\partial t} = -(\vec{v}_L \cdot \vec{\nabla})\vec{v}_v = -\vec{\nabla}(\vec{v}_L \cdot \vec{v}_v) + [\vec{v}_L \times \vec{\kappa}]\delta_2(\vec{r}). \tag{6.9}$$

Since we aren't considering any external force on the liquid, the vortex will move with the same velocity as the sound wave, so $\vec{v}_L = \vec{v}_1(0, t)$. However this means that the overall acceleration is now

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{v}_v}{\partial t} + \frac{\partial \vec{v}_1}{\partial t}$$

So now we need to add this extra acceleration term into (6.8)

$$\frac{\partial \vec{v}_1}{\partial t} + \frac{\partial \vec{v}_v}{\partial t} + \frac{c_s^2}{\rho_0}\vec{\nabla}\rho_1 = -\vec{\nabla}(\vec{v}_v \cdot \vec{v}_1) + [\vec{v}_1 \times \vec{\kappa}]\delta_2(\vec{r}).$$

Using (6.9) this now becomes

$$\frac{\partial \vec{v}_1}{\partial t} + \frac{c_s^2}{\rho_0}\vec{\nabla}\rho_1 = -\vec{\nabla}(\vec{v}_v \cdot \vec{v}_1) + \vec{\nabla}(\vec{v}_v \cdot \vec{v}_L). \tag{6.10}$$

Note that $[\vec{v}_1 \times \vec{\kappa}]\delta_2(\vec{r})$ is identical to $[\vec{v}_L \times \vec{\kappa}]\delta_2(\vec{r}) = [\vec{v}_1(0) \times \vec{\kappa}]\delta_2(\vec{r})$ because of the δ-function and so these cancel in the above calculation. Now using the fact that $\vec{v}_1(\vec{r},t) = \frac{\kappa}{2\pi}\vec{\nabla}\phi$ (when considering a BEC $\kappa = \pm\pi$), so now we can now find a solution for ρ_1 from equation (6.10)

$$\rho_1 = -\frac{\rho_0}{c_s^2}\frac{\kappa}{2\pi}\left[\frac{\partial \phi}{\partial t} + \vec{v}_v \cdot (\vec{\nabla}\phi(\vec{r}) - \vec{\nabla}\phi(0))\right]. \tag{6.11}$$

Now substituting (6.11) into equation (6.1) which is our continuity equation yields

$$\begin{aligned} & -\tfrac{\rho_0}{c_s^2}\tfrac{\kappa}{2\pi}\left[\tfrac{\partial^2 \phi}{\partial t^2} + \vec{v}_v \cdot \vec{\nabla}\tfrac{\partial}{\partial t}\left(\phi(\vec{r}) - \phi(0)\right)\right] + \rho_0\tfrac{\kappa}{2\pi}\Delta\phi \\ = \; & -\vec{v}_v \cdot \left(-\tfrac{\rho_0}{c_s^2}\tfrac{\kappa}{2\pi}\left[\tfrac{\partial}{\partial t}\vec{\nabla}\phi(\vec{r}) + \vec{\nabla}(\vec{v}_v(\vec{r}) \cdot [\vec{\nabla}\phi(\vec{r}) - \vec{\nabla}\phi(0)])\right]\right). \end{aligned}$$

Which after cancellation and linearisation gives the following linear inhomogeneous wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c_s^2\Delta\phi = -2\vec{v}_v(\vec{r}) \cdot \vec{\nabla}\frac{\partial}{\partial t}\left[\phi(\vec{r}) - \frac{1}{2}\phi(0)\right]. \tag{6.12}$$

Substituting in the plane wave solution ϕ into the right-hand side of (6.12) gives an inhomogeneous wave equation that can be solved for ϕ using Green's and Hankel functions. The solution is stated in [5] and is

$$\phi = \phi_0 e^{-i\omega t}\left\{e^{i\vec{k}\cdot\vec{r}} - \frac{ik}{4c_s}\int H_0^{(1)}(k|\vec{r}-\vec{r}_1|)\vec{k}\cdot\vec{v}_v(\vec{r}_1)[2e^{i\vec{k}\cdot\vec{r}_1} - 1]\mathrm{d}\vec{r}_1\right\} \tag{6.13}$$

Where $H_0^{(1)}(z)$ is the zero-order Hankel function of the first kind, and where we find $\frac{i}{4}H_0^{(1)}(k|\vec{r}-\vec{r}_1|)$ is the Green's function for the 2D wave equation.

The Green's function $G(\vec{r})$for a 2D wave equation satisfies

$$-(k^2 + \Delta)G(\vec{r}) = \delta_2(\vec{r} - \vec{r}_1)$$

In standard scattering theory they use an asymptotic expression for the Hankel function at large values of z:

$$\lim_{z\to\infty} H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i(z-\frac{\pi}{4})}$$

If the perturbation is confined to a region close to the vortex line, then $r_1 \ll r$ (Noting that $r = |\vec{r}|$) and

$$|\vec{r} - \vec{r}_1| \approx r - \frac{(\vec{r}\cdot\vec{r}_1)}{r}$$

After integrating the integral in equation (6.13) the wave at $kr \gg 1$ becomes a superposition of the incident plane wave proportional to $e^{i\vec{k}\cdot\vec{r}}$ and the scattered wave proportional to e^{ikr}:

$$\phi = \phi_0 e^{-i\omega t}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{ia(\varphi)}{\sqrt{r}}e^{ikr}\right] \tag{6.14}$$

where here $a(\varphi)$ is the scattering amplitude which is a function of φ which is the angle between the initial wave vector $\vec{k}$ and the wave vector $\vec{k}' = \frac{k\vec{r}}{r}$ after scattering.

Now if we use the Born approximation we can get an explicit expression for $a(\varphi)$. The working can be seen in [3] with the result:

$$\begin{aligned} a(\varphi) &= \sqrt{\frac{k}{2\pi}}\frac{1}{c_s}e^{i\frac{\pi}{4}}[\vec{\kappa}\times\vec{k}']\cdot\vec{k}\frac{1}{q^2}\left(1-\frac{q^2}{2k^2}\right) \\ &= -\frac{1}{2}\sqrt{\frac{k}{2\pi}}\frac{\kappa}{c_s}e^{i\frac{\pi}{4}}\frac{\sin\varphi\cos\varphi}{1-\cos\varphi} \end{aligned} \tag{6.15}$$

where $\vec{q} = \vec{k} - \vec{k}'$ is the momentum transferred by the scattered phonon to the vortex and $q^2 = 2k^2(1-\cos\varphi)$.

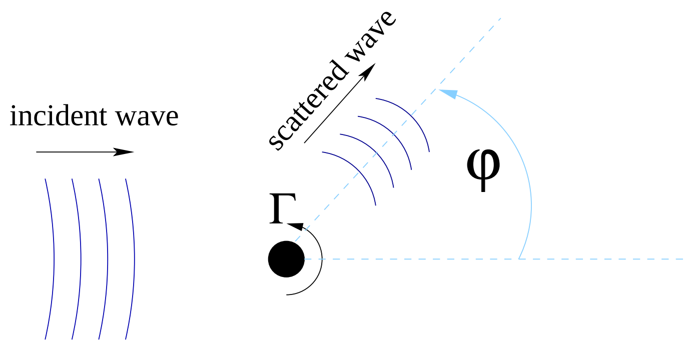

Figure 6: The Aharonov-Bohm Effect

Therefore we get a scattered wave as the incident wave passes the vortex. This scattering produces a force upon the vortex and if the perturbation by the vortex is confined to a finite vicinity of the vortex line we get the force to be

$$\vec{F}^{ph} = \sigma_{\shortparallel} c_s \vec{j}^{ph} - \sigma_{\perp} c_s[\hat{z}\times\vec{j}^{ph}]. \tag{6.16}$$

Which is determined by two effective cross-sections; the transport cross-section for the dissipative force $\sigma_{\shortparallel}$ and the transverse cross-section for the transverse force $\sigma_{\perp}$ where we define these as

$$\sigma_{\shortparallel} = \int \sigma(\varphi)(1-\cos\varphi)\mathrm{d}\varphi \tag{6.17}$$

$$\sigma_{\perp} = \int \sigma(\varphi) \sin\varphi \mathrm{d}\varphi \tag{6.18}$$

where we define $\sigma(\varphi) = |a(\varphi)|^2$ and

$$\vec{j}^{ph} = \rho_0 \phi_0^2 \frac{\kappa^2 k}{8\pi^2 c_s} \vec{k} \tag{6.19}$$

is the mass current of phonons in the fluid which is the phonon momentum density in the reference frame moving with the average liquid velocity $\vec{v}_0$. Using the Born approximation it is quite common to get the transverse cross section $\sigma_{\perp}$ to vanish due to the fact that $\sigma(\varphi)$ is quadratic in the circulation κ and φ.

However because of a very slow decrease of the velocity $\vec{v}_v \backsim \frac{1}{r}$ far from the vortex, the scattering amplitude is divergent at very small values of φ, as from (6.15) we get that

$$\frac{1}{2}\frac{\sin\varphi\cos\varphi}{1-\cos\varphi} = \frac{\sin 2\varphi}{4-4\cos\varphi} = \frac{2\varphi - 8\frac{\varphi^3}{3!} + \ldots}{4-4+\frac{4\varphi^2}{2!} - \ldots}$$

so now we get

$$\lim_{\varphi\to 0} a(\varphi) = -\sqrt{\frac{k}{2\pi}}\frac{\kappa}{c_s} e^{i\frac{\pi}{4}} \frac{1}{\varphi} \tag{6.20}$$

This divergence is integrable for $\sigma_{\shortparallel}$ (6.17). However for $\sigma\perp$ (6.18) the integral has a pole at $\varphi = 0$ which requires further analysis, that won't be considered here.

6.2 Small angle scattering

At small scattering angles $\varphi \ll \frac{1}{\sqrt{kr}}$ the asymptotic expansion (6.14) does not hold. An accurate calculation of (6.14) can been seen in [10] for small scattering angles. This results states that for $\varphi \ll 1$

$$\phi = \phi_0 e^{-i\omega t + i\vec{k}\cdot\vec{r}} \left[1 + \frac{i\kappa k}{2c_s} \Phi\left(\varphi\sqrt{\frac{kr}{2i}} \right) \right]. \tag{6.21}$$

Where the error integral $\Phi(z)$ has an asymptotic expression

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \mathrm{d}t \to \frac{z}{|z|} - \frac{1}{\sqrt{\pi} z} e^{-z^2} \tag{6.22}$$

as $|z| \to \infty$, and for angles $\frac{1}{\sqrt{kr}} \ll \varphi \ll 1$

$$\phi = \phi_0 e^{-i\omega t} \left[e^{i\vec{k}\cdot\vec{r}} \left(1 + \frac{i\kappa k}{2c_s}\frac{\varphi}{|\varphi|} \right) + \frac{i\kappa}{c_s}\sqrt{\frac{k}{2\pi r}} \frac{1}{\varphi} e^{ikr + i\frac{\pi}{4}} \right]. \tag{6.23}$$

The second term in the square brackets coincides with scattering at small angles $\varphi \ll 1$ when the scattering amplitude is given as (6.20). However from the above equation we see that the incident plane wave changes sign when the scattering angle φ goes through zero. The coefficient in front of the term responsible of this, $\frac{\kappa k}{2c}\frac{\varphi}{|\varphi|}$, is exactly the phase shift of the sound wave along the trajectories past the vortex on the right and left sides. This is a consequence of the Aharonov-Bohm effect, the sound wave after its interaction with the vortex velocity field has different phases on the left and on the right of the vortex line. This difference in phase results in interference.

6.3 Berry Phase

Berry phase, named after Michael Berry in 1984 is a concept that is used in many quantum mechanical circumstances. It modifies the motion of vortices in Superfluids and Superconductors as a result of wave-particle duality; where particles have wave like properties.

6.4 Magnus Force and Berry Phase

There is a link between the transverse force found in the previous subsection and that with the Berry phase. To see this relation we must first consider the Madelung transformation applied to the Lagrangian of the Nonlinear Schrödinger equation.

To quickly recall the Madelung transformation is

$$\psi = \sqrt{\rho}e^{i\theta}, \qquad \rho = |\psi|^2$$

and the Lagrangian for the Nonlinear Schrödinger equation is

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\dot{\psi} - \psi\dot{\bar{\psi}}\right) - |\vec{\nabla}\psi|^2 - \frac{|\psi|^4}{2}.$$

So we get the Lagrangian as

$$\mathcal{L} = -\rho\frac{\partial\theta}{\partial t} - (\vec{\nabla}\theta)^2 - \frac{\rho^2}{2}. \tag{6.24}$$

The first term of this expression is called the Wess-Zumino term and is responsible for the Berry phase $\Theta = \frac{\Delta S_B}{\hbar}$, which is the variation of the phase of the quantum mechanical wave function for an adiabatic motion of the vortex around a closed loop.

$$\Delta S_B = \int \rho\frac{\partial\theta}{\partial t}\mathrm{d}\vec{r}\mathrm{d}t = -\int(\vec{v}_L\cdot\vec{\nabla}_L)\theta\mathrm{d}\vec{r}\mathrm{d}t$$

is the classical action variation around a loop where $\vec{\nabla}_L\theta$ is the gradient of the phase $\theta(\vec{r}-\vec{r}_L(t))$ w.r.t. the vortex position vector $\vec{r}_L(t)$. Also it is clear to see that $\vec{\nabla}_L\theta = -\vec{\nabla}\theta$, where $\vec{\nabla}\theta$ is the gradient w.r.t. $\vec{r}$.

Then considering an integral around a close curve yields the total current: $\vec{j} = \rho\vec{\nabla}\theta$ for points inside the loop, but vanishes for points outside. As a result the Berry phase action is given by

$$\Delta S_B = V\oint \vec{j}\cdot\mathrm{d}\vec{l}$$

where V is the volume inside the loop.

Since the Berry phase is proportional to the current circulation which determines the transverse force, there is a direct link between the Berry phase and the amplitude of the transverse force on a vortex. However the Berry phase does not yield the total transverse force, since the Iordanskii force is presented by the viscous momentum flux, which cannot be obtained when considering a Lagrangian.

7 Phonon-Vortex interactions

7.1 Phonons on a vortex

In this section I will consider a point vortex in a 2D fluid with positive circulation Γ and the effect upon this by a wall of phonons moving along the x-axis. I will use the conservation of momentum to show that the phonons will have a change in their y-component of momentum, and thus change this momentum in the vortex, resulting in the movement of the vortex in the y-direction.

I will use the ray acoustic approach to do this, and so the equations for acoustic rays in a moving weakly inhomogeneous medium are

$$\dot{\vec{r}} = \vec{\nabla}_{\vec{k}} H \tag{7.1}$$

$$\dot{\vec{k}} = -\vec{\nabla}_{\vec{r}} H \tag{7.2}$$

where H is the Hamiltonian function or frequency such that

$$H = \omega \equiv c_s k + \vec{v} \cdot \vec{k}. \tag{7.3}$$

Here $\vec{r} = (x, y)$ and $\vec{k} = (k_x, k_y)$ are the coordinate and wave vector of the phonon, and $\vec{v} = \vec{v}(\vec{r}, t)$ and c_s are the velocity field produced by the vortex and the local speed of sound respectively.

As we are considering a point vortex centred at $(0, 0)$, it has the velocity field

$$\vec{v}(\vec{r}, t) = (v_x, v_y) = \left(-\frac{\Gamma y}{2\pi(x^2 + y^2)}, \frac{\Gamma x}{2\pi(x^2 + y^2)} \right) \tag{7.4}$$

Now we want to consider the movement of the phonons using polar coordinates (r, ϕ). So using polar coordinates we find that

$$\dot{r} = \cos\phi \dot{x} + \sin\phi \dot{y} \tag{7.5}$$

$$\dot{\phi} = \frac{\cos\phi \dot{y} - \sin\phi \dot{x}}{r}. \tag{7.6}$$

From (7.1) we get that

$$\dot{\vec{r}} = (\dot{x}, \dot{y}) = \left(c_s \frac{k_x}{k} + v_x, c_s \frac{k_y}{k} + v_y \right). \tag{7.7}$$

So using this with equations (7.5) and (7.6) we get that

$$\begin{aligned} \dot{r} &= \frac{\cos\phi c_s k_x}{k} + \cos\phi v_x + \frac{\sin\phi c_s k_y}{k} + \sin\phi v_y \\ &= \frac{c_s}{k}(k_x \cos\phi + k_y \sin\phi) - \frac{\Gamma \cos\phi \sin\phi}{2\pi r} + \frac{\Gamma \cos\phi \sin\phi}{2\pi r} \\ &= \frac{c_s k_r}{k} \end{aligned} \tag{7.8}$$

$$
\begin{aligned}
\dot{\phi} &= \frac{c_s k_y \cos\phi}{kr} + \frac{\cos\phi v_y}{r} - \frac{c_s k_x \sin\phi}{kr} - \frac{\sin\phi v_x}{r} \\
&= \frac{c_s}{kr}(\cos\phi k_y - \sin\phi k_x) + \frac{\Gamma\cos^2\phi}{2\pi r^2} + \frac{\Gamma\sin^2\phi}{2\pi r^2} \\
&= \frac{c_s k_\phi}{kr} + \frac{\Gamma}{2\pi r^2}. \qquad (7.9)
\end{aligned}
$$

We must find out what k_r and k_ϕ are, to do this simply substitute the vortex velocity field in polar coordinates and the fact that the z-component of angular momentum is conserved in (7.3). The conservation of the z-component of angular momentum M_z is given by

$$M_z \equiv (\vec{k} \times \vec{r})_z = -k_\phi r = Const. \qquad (7.10)$$

So we get

$$H = c_s\sqrt{k_x^2 + k_y^2} + v_x k_x + v_y k_y.$$

$$\Rightarrow c_s^2(k_x^2 + k_y^2) = H^2 - 2Hv_x k_x - 2Hv_y k_y + v_x^2 k_x^2 + 2v_x v_y k_x k_y + v_y^2 k_y^2.$$

Using the fact that $k^2 = k_x^2 + k_y^2 = k_r^2 + k_\phi^2$ and the values for v_x and v_y gives us

$$c_s^2(k_r^2 + k_\phi^2) = H^2 + \frac{2H\Gamma y k_x}{2\pi r^2} - \frac{2H\Gamma x k_y}{2\pi r^2} + \frac{\Gamma^2 y^2 k_x^2}{4\pi^2 r^4} - \frac{2\Gamma^2 xy k_x k_y}{4\pi^2 r^4} + \frac{\Gamma^2 x^2 k_y^2}{4\pi^2 r^4}.$$

Now $M_z = (\vec{k} \times \vec{r})_z = k_x y - k_y x$, so now

$$c_s^2(k_r^2 + k_\phi^2) = H^2 + \frac{2H\Gamma M_z}{2\pi r^2} + \frac{\Gamma^2 M_z^2}{4\pi^2 r^4}. \qquad (7.11)$$

Now $k_\phi = -\frac{M_z}{r}$ so

$$k_r^2 = \frac{H^2}{c_s^2} + \frac{H\Gamma M_z}{\pi c_s^2 r^2} + \frac{\Gamma^2 M_z^2}{4c_s^2 \pi^2 r^4} - \frac{M_z^2}{r^2}. \qquad (7.12)$$

A quick note is that $|\vec{k}| = k$ is

$$k = \left(\frac{H}{c_s} + \frac{\Gamma M_z}{2\pi r^2}\right). \qquad (7.13)$$

So now let's go back to the equation for $\dot{\phi}$ (7.9) by substituting in the values for k_ϕ and k

$$\dot{\phi} = -\frac{c_s M_z}{\gamma r^2 + \alpha} + \frac{\Gamma}{2\pi r^2} \qquad (7.14)$$

where we have

$$\alpha = \frac{\Gamma M_z}{2\pi c_s}, \qquad \gamma = \frac{H}{c_s}.$$

Now consider a position vector of the phonon as $r(t) = \sqrt{x(t)^2 + y^2}$ where $x(t) = c_s t$, this now give $\dot{\phi}$ as

$$\dot{\phi} = -\frac{c_s M_z}{\gamma c_s^2 t^2 + y^2\gamma + \alpha} + \frac{\Gamma}{2\pi c_s^2 t^2 + 2\pi y^2} \tag{7.15}$$

Integrating this from $-\infty$ to ∞ gives us the overall change of ϕ in the phonon with the addition of $+\pi$ as this corresponds to the loss of ϕ as the phonons move in a straight line, I will denote this as $\Delta\phi$.

$$\begin{aligned}
\Delta\phi &= \int_{-\infty}^{\infty} -\frac{c_s M_z}{\gamma c_s^2 t^2 + y^2\gamma + \alpha} + \frac{\Gamma}{2\pi c_s^2 t^2 + 2\pi y^2}\mathrm{d}t - \pi \\
&= \left[-\frac{M_z}{\gamma c_s}\sqrt{\frac{\gamma c_s^2}{\gamma y^2 + \alpha}}\arctan\left(\frac{\gamma c_s^2 t}{\gamma y^2 + \alpha}\right) + \frac{\Gamma}{2\pi c_s y}\arctan\left(\frac{c_s t}{y}\right)\right]_{-\infty}^{\infty} - \pi \\
&= -\frac{M_z \pi}{\gamma\sqrt{y^2 + \frac{\alpha}{\gamma}}} + \frac{\Gamma}{2c_s y} - \pi. \qquad (7.16)
\end{aligned}$$

Now that we have calculated $\Delta\phi$, we need to integrate over all phonons to find out the change in the y-component of momentum.

The change in the y-component of momentum is proportional to Δk_y, where this is the change in k_y; the change in the y-component of the wavevector $\vec{k}$. So we get that

$$\Delta k_y = k_0 \Delta\phi$$

where k_0 is the initial values of k_y. However when we consider this integral, it will clearly diverge due to the fact that $\Delta\phi \sim \frac{1}{y}$. So I will make a few assumptions. Seen in [7], S. Nazarenko showed that for some value, say y_0 we get the phonons spiralling into the vortex and thus getting absorbed by the vortex due to the effects of viscosity within the vortex core. Hence I will neglect any phonon within this region as they do not contribute any y-momentum in our integral. Also another assumption I will use is that the effects of the vortex upon the phonon can be neglected after a finite length, this in general is not true, however I will assume that other vortices can be considered in this fluid, and so as distances from the vortex being considered increase, the effects from other vortices upon the phonons are greater. I will assume that the effects of this vortex can be neglected at a distance of y_∞ along the y-axis from $y = 0$. I shall now consider a more detailed problem involving a vortex dipole.

7.2 Phonons on a vortex dipole

Now that we have worked out an explicit formula for Δk_y, let's consider a new problem. Let there be a vortex dipole moving with velocity v_d in the positive x-direction. Now consider the effect of phonons initially travelling parallel to the x-axis arriving from both directions with speed c_s. So using the calculations in the previous subsection, we need to alter the values of c_s in the formula for $\Delta\phi$.

So by considering a moving frame of reference, we have that from the left we have phonons travelling with velocity $c_s - v_d$ in the x-direction, and from the right phonons travelling with velocity $-(c_s + v_d)$. By considering the top vortex, we can neglect the effects from this vortex when y reaches a point say $y = -y_1$, as the effects from the other vortex in the dipole will be dominant. Therefore considering the phonons that are approaching from the left of the dipole we get

$$\int \Delta^L k_y \mathrm{d}y = \int_{y_0}^{y_\infty} k_0 \Delta^L \phi \mathrm{d}y + \int_{-y_1}^{-\epsilon} k_0 \Delta^L \phi \mathrm{d}y \tag{7.17}$$

where now

$$\Delta^L \phi = -\frac{M_z \pi (c_s - v_d)}{H\sqrt{y^2 + \delta}} + \frac{\Gamma}{2(c_s - v_d)y} - \pi \tag{7.18}$$

where $\delta = \frac{\Gamma M_z}{2\pi H}$, so now

$$\begin{aligned}
\int \Delta^L k_y \mathrm{d}y \quad &= \quad k_0 \Big\{ \Big[-\tfrac{M_z\pi(c_s - v_d)}{H} \log\left(y + \sqrt{y^2+\delta}\right) + \tfrac{\Gamma}{2(c_s - v_d)} \log(y) - \pi y \Big]_{y_0}^{y_\infty} \\
&\quad + \Big[-\tfrac{M_z\pi(c_s - v_d)}{H} \log\left(y + \sqrt{y^2+\delta}\right) + \tfrac{\Gamma}{2(c_s - v_d)} \log(y) - \pi y \Big]_{-y_1}^{-\epsilon} \Big\} \\
&= \quad k_0 \Big\{ -\tfrac{M_z\pi(c_s - v_d)}{H} \log\left(y_\infty + \sqrt{y_\infty^2+\delta}\right) + \tfrac{\Gamma}{2(c_s - v_d)} \log(y_\infty) - \pi y_\infty \\
&\quad + \tfrac{M_z\pi(c_s - v_d)}{H} \log\left(y_0 + \sqrt{y_0^2+\delta}\right) - \tfrac{\Gamma}{2(c_s - v_d)} \log(y_0) + \pi y_0 \\
&\quad - \tfrac{M_z\pi(c_s - v_d)}{H} \log\left(-\epsilon + \sqrt{\epsilon^2+\delta}\right) + \tfrac{\Gamma}{2(c_s - v_d)} \log(\epsilon) + \pi \epsilon \\
&\quad + \tfrac{M_z\pi(c_s - v_d)}{H} \log\left(-y_1 + \sqrt{y_1^2+\delta}\right) - \tfrac{\Gamma}{2(c_s - v_d)} \log(y_1) - \pi y_1 \Big\}
\end{aligned}$$

The change in the y-component of momentum P of the phonons approaching from the left can now be calculated, we use the formula

$$\dot{P}_y^L = (c_s - v_d) N \int \Delta^L k_y \mathrm{d}y \tag{7.19}$$

where $\dot{P}_y^L$ denotes the change w.r.t time of the y-component of momentum from the phonons approaching from the left and N is the density of phonons per unit area.

This is now

$$\begin{aligned}
\dot{P}_y^L \quad &= \quad -\frac{M_z \pi k_0 N (c_s - v_d)^2}{H} \log\left(y_\infty + \sqrt{y_\infty^2 + \delta}\right) + \frac{\Gamma N k_0}{2} \log(y_\infty) \\
&\quad + \frac{M_z k_0 N \pi (c_s - v_d)^2}{H} \log\left(y_0 + \sqrt{y_0^2 + \delta}\right) - \frac{\Gamma N k_0}{2} \log(y_0) \\
&\quad - \frac{M_z k_0 N \pi (c_s - v_d)^2}{H} \log\left(-\epsilon + \sqrt{\epsilon^2 + \delta}\right) + \frac{\Gamma k_0 N}{2} \log(\epsilon) \\
&\quad + \frac{M_z k_0 N \pi (c_s - v_d)^2}{H} \log\left(-y_1 + \sqrt{y_1^2 + \delta}\right) - \frac{\Gamma k_0 N}{2} \log(y_1) \\
&\quad - \pi k_0 N (c_s - v_d)(y_1 + y_\infty - y_0 - \epsilon)
\end{aligned} \tag{7.20}$$

Now we must do the same with the phonons approaching from the right, again we will be able to neglect the effects of this vortex for $y < -y_1$, also we will still have this collapse region for $0 > y > -y_0$. This is where trajectories of phonons will spiral into the vortex filament.

So now

$$\int \Delta^R k_y \mathrm{d}y = \int_{\epsilon}^{y_\infty} k_0 \Delta^R \phi \mathrm{d}y + \int_{-y_1}^{-y_0} k_0 \Delta^R \phi \mathrm{d}y \tag{7.21}$$

where

$$\Delta^R \phi = \frac{M_z \pi (c_s + v_d)}{H\sqrt{y^2+\delta}} - \frac{\Gamma}{2(c_s+v_d)y} + \pi. \tag{7.22}$$

Note that we get $+\pi$ now because for a straight trajectory approaching from the right and travelling left, we get loss of π that needs to be compensated for.

So

$$\begin{aligned}
\int \Delta^R k_y \mathrm{d}y &= k_0 \Bigg\{ \left[\tfrac{M_z\pi(c_s+v_d)}{H} \log\left(y+\sqrt{y^2+\delta}\right) - \tfrac{\Gamma}{2(c_s+v_d)} \log(y) + \pi y \right]_{\epsilon}^{y_\infty} \\
&\quad + \left[\tfrac{M_z\pi(c_s+v_d)}{H} \log\left(y+\sqrt{y^2+\delta}\right) - \tfrac{\Gamma}{2(c_s+v_d)} \log(y) + \pi y \right]_{-y_1}^{-y_0} \Bigg\} \\
&= k_0 \Bigg\{ \tfrac{M_z\pi(c_s+v_d)}{H} \log\left(y_\infty+\sqrt{y_\infty^2+\delta}\right) - \tfrac{\Gamma}{2(c_s+v_d)} \log(y_\infty) + \pi y_\infty \\
&\quad - \tfrac{M_z\pi(c_s+v_d)}{H} \log\left(\epsilon+\sqrt{\epsilon^2+\delta}\right) + \tfrac{\Gamma}{2(c_s+v_d)} \log(\epsilon) - \pi\epsilon \\
&\quad \tfrac{M_z\pi(c_s+v_d)}{H} \log\left(-y_0+\sqrt{y_0^2+\delta}\right) - \tfrac{\Gamma}{2(c_s+v_d)} \log(y_0) - \pi y_0 \\
&\quad - \tfrac{M_z\pi(c_s+v_d)}{H} \log\left(-y_1+\sqrt{y_1^2+\delta}\right) + \tfrac{\Gamma}{2(c_s+v_d)} \log(y_1) + \pi y_1 \Bigg\}.
\end{aligned}$$

Now we can calculate the change in the y-component of momentum for phonons approaching from the right, this will be

$$\dot{P}_y^R = -(c_s + v_d) N \int \Delta^R k_y \mathrm{d}y, \tag{7.23}$$

and is

$$\begin{aligned}
\dot{P}_y^R &= -\frac{M_z k_0 N \pi (c_s+v_d)^2}{H} \log\left(y_\infty+\sqrt{y_\infty^2+\delta}\right) + \frac{\Gamma k_0 N}{2}\log(y_\infty) \\
&\quad + \frac{M_z k_0 N \pi (c_s+v_d)^2}{H} \log\left(\epsilon+\sqrt{\epsilon^2+\delta}\right) - \frac{\Gamma k_0 N}{2}\log(\epsilon) \\
&\quad - \frac{M_z k_0 N \pi (c_s+v_d)^2}{H} \log\left(-y_0+\sqrt{y_0^2+\delta}\right) + \frac{\Gamma k_0 N}{2}\log(y_0) \\
&\quad + \frac{M_z k_0 N \pi (c_s+v_d)^2}{H} \log\left(-y_1+\sqrt{y_1^2+\delta}\right) - \frac{\Gamma k_0 N}{2}\log(y_1) \\
&\quad - (c_s+v_d) k_0 N \pi (y_\infty - \epsilon - y_0 + y_1).
\end{aligned} \tag{7.24}$$

The overall y-component momentum change is $\dot{P}_y^L + \dot{P}_y^R$, so adding (7.20) and (7.24) together and letting $\epsilon \to 0$ we get that

$$
\begin{aligned}
\dot{P}_y^L + \dot{P}_y^R \;\; = \;\; & -\frac{2M_z k_0 N\pi(c_s^2+v_d^2)}{H}\log\left(y_\infty + \sqrt{y_\infty^2+\delta}\right) + \Gamma k_0 N \log\left(\frac{y_\infty}{y_1}\right) \\
& -\frac{M_z k_0 N\pi(c_s+v_d)^2}{H}\log\left(-y_0 + \sqrt{y_0^2+\delta}\right) \\
& +\frac{M_z k_0 N\pi(c_s-v_d)^2}{H}\log\left(y_0 + \sqrt{y_0^2+\delta}\right) \\
& +\frac{2M_z k_0 N\pi(c_s^2+v_d^2)}{H}\log\left(-y_1 + \sqrt{y_1^2+\delta}\right) \\
& -2c_s k_0 N\pi(y_\infty + y_1 - y_0) \qquad (7.25)
\end{aligned}
$$

We can make a few assumptions about each term of (7.25). First note that y_∞ can be as large as possible, as this corresponds to the values of y above the vortex. Second, note that y_0 and y_1 are finite values, this is because y_0 is the cut off value for phonons that collapse into the vortex, and so we can assume this to be close to the vortex, and y_1 is the point were the second vortex in the dipole has a more dominant effect upon the phonons than the vortex being considered.

As we can see, the first two terms in the above equation have logarithmic growth to $-\infty$ and ∞ respectively, the following three terms are all finite by the reasoning in the previous paragraph, and then finally the last term has polynomial divergences to $-\infty$ as $y_\infty \to \infty$, so by taking y_∞ large enough we get that $\dot{P}_y^L + \dot{P}_y^R < 0$.

Now by the assumption of conservation of momentum in the system, we get that the vortex dipole will gain momentum in the y-direction, and so the two vortices will move apart from each other. However this contradicts what is suggested by qualitative reasoning seen in Figure 7.

7.3 Conclusion

So now we have an interesting problem, was our original suggestion that the vortices in the dipole move together incorrect or is our analysis of the problem flawed?

Well clearly the analytical argument is just an approximation and so does not fully represent everything that is going on in this dipole system. So in theory we should be able to refine the calculations done in this section and hopefully determine whether or not the dipole converge to each other or not.

However Figure 7 suggests that the dipole does converge, but does the fact that as phonon 2 is travelling faster than phonon 1 imply that more momentum is transferred? or does it just mean the effect of the vortex upon it is far less than of phonon 1? Well I believe that calculations done are inaccurate, and should show that the phonons gain momentum in the y-direction. Further analysis of this problem will be done at a later date and further conclusions can be made.

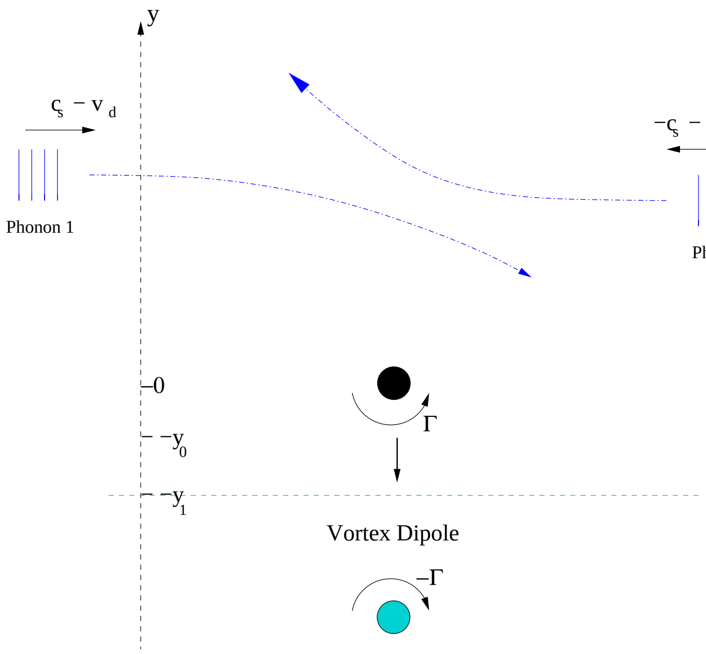

Figure 7: As phonon 2 has greater speed than phonon 1, the figure suggests that due to the effect from the black vortex, phonon 2 loses more y momentum to the vortex than phonon 1, so the vortex should gain net momentum in the opposite direction to phonon 2 and thus move closer to the blue vortex.

8 Summary

So in conclusion what I have looked at in this project has mainly been the relationship between the effects of waves upon vortices in fluids. I have looked specifically at BEC when needed, by using the Gross-Pitaevskii equation (GP equation) (4.2) as a basic model for this, then moved onto the effect that phonons have on vortices in general fluids.

So to summarise what this project is about I will quickly give a brief idea of what each chapter was about and it's relevance in the area of research in BEC.

At the beginning I decided to study the Nonlinear Schrödinger equation (NLS equation) (1.1). This equation is often referred to as the GP equation when considering BEC, but is still a slightly simpler version of the full GP equation that contains an extra term corresponding to the use of a trapping potential. In chapter 2, I looked at the dispersion relations obtained by considering a plane-wave solution to the equation when linearised and then when left nonlinear. A simple relation $\omega = k^2$ was obtained when I only considered the linearised version, where a more complicated relation called the Bogoliubov dispersion relation $\omega^2 = k^4 \pm 2\varrho k^2$ was obtained with the nonlinear equation. I also mentioned the importance of this relation in optics and BEC. Finally I went on to discuss the normal modes of the NLS equation. This was done by taking the Fourier Transform of the real and imaginary parts of the wave function and then considered an eigenvalue problem.

Chapter 3 is a very important section as it shows the link to hydrodynamics. This is achieved simply via using Madelung's transformation to view the NLS equation as the time dependent Bernoulli equation with quantum pressure and the Continuity equation for fluids. This concept forms the foundation for using the GP equation as a model for BEC.

In Chapter 4 the Hamiltonian form of the GP equation was considered, and then I looked at why we have quantised vortices in BEC. Then by considering a radial solution to the fluid moving around a vortex using the GP equation I was able to show the density profile of the fluid surrounding the vortex core.

Chapter 5 saw a brief overview of the main forces involved on a vortex in an incompressible fluid. I started by quickly looking at the Magnus force in classical hydrodynamics, and then moving on to consider the two-fluid model of a superfluid. In this situation we get the usual Magnus force but now another force called the Iordanskii force that is a result of phonon scattering on the vortex. I then gave a brief description of the Born approximation as it was used in the next section.

Chapter 6 forms the basis of my project, the discussion of the Aharonov-Bohm effect. This effect (in the situation of phonons in fluid) is when we get a scattering of waves as they pass a vortex line. This was considered by taking a phonon and showing that it changed angle of trajectory as it past the vortex line. Using the Born approximation to analysis the situation, results were found about how this angle related to the original phonon considered. I ended this chapter with a brief subsection on Berry Phase and its link with the Magnus force.

The final section focuses on phonon-vortex interactions, I showed the effects of phonons travelling along the x-axis with uniform density and constant velocities approaching a vortex dipole from both directions can change the distance between the vortices by using the conservation of momentum of the system. However analysis of the problem showed to contradict what was originally predicted qualitatively, showing that the vortices actually move apart from each other.

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