On the statistical mechanics of the $2D$ stochastic Euler equation.

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Abstract. The dynamics of vortices and large scale structures is qualitatively very different in two dimensional flows compared to its three dimensional counterparts, due to the presence of multiple integrals of motion. These are believed to be responsible for a variety of phenomena observed in Euler flow such as the formation of large scale coherent structures, the existence of meta-stable states and random abrupt changes in the topology of the flow. In this paper we study stochastic dynamics of the finite dimensional approximation of the $2D$ Euler flow based on Lie algebra $su(N)$ which preserves all integrals of motion. In particular, we exploit rich algebraic structure responsible for the existence of Euler's conservation laws to calculate the invariant measures and explore their properties and also study the approach to equilibrium. Unexpectedly, we find deep connections between equilibrium measures of finite dimensional $su(N)$ truncations of the stochastic Euler equations and random matrix models.

Our work can be regarded as a preparation for addressing the questions of large scale structures, meta-stability and the dynamics of random transitions between different flow topologies in stochastic $2D$ Euler flows.

1. Introduction

The dynamics of vortices and large scale structures is qualitatively very different in two dimensional flows compared to their three dimensional counterparts. Due to the existence of an infinite number of invariants of motion, for instance enstrophy or all higher moments of vorticity, the downscale energy cascade is prevented and energy flows towards the largest scale. In the regime of small forcing and dissipation, this inverse cascade reaches the box size and self-organized vortices and jets are formed. The long time dynamics of such large scale structure is particularly interesting due for instance to the switching phenomena between jets and vortices which have been observed Bouchet & Simonnet (2009); Bouchet & Venaille (2010). These phenomena are analogous to bi-stability observed for many experimental and geophysical turbulent flows (e. g. magnetic field reversal in MHD experiments Berhanu *et al.* (2007), oscillation between multiple equilibria in $2D$ turbulence experiments Sommeria (1986); Weeks *et al.* (1997), magnetic field reversal due to Earth core dynamics, in three dimensional turbulent Couette or Van-Karman flows Ravelet *et al.* (2004), oscillation between multiple equilibria in atmospheric flows Weeks *et al.* (1997), and in the paths of the Kuroshio current Pierini (2006); Schmeits & Dijkstra (2001)).

This phenomenon of the flow switching between multiple attractor is not yet understood for any turbulent flows, as no suitable non-equilibrium theoretical framework has been devised. The

relative simplicity of two-dimensional turbulence makes it a very interesting playground for the development of new theoretical ideas aimed at explaining the switching phenomenon.

The basic property of the $2D$ Euler equation responsible for the richness of natural phenomena which it successfully describes is the existence of infinitely many invariant measures. Therefore, the first step towards understanding the phenomenon of meta-stability in $2D$ Euler is to study these measures in stochastic setting. The present work does just that in the context of finite dimensional approximations of the dynamics, which preserve the underlying geometric structure behind the conservation laws of the $2D$ Euler equations.

2. Low dimensional approximations of attractors through Hamiltonian structure

The presence of multiple attractors is related to the multiplicity of Casimir invariants for the inertial dynamics of the $2D$ flow. Those are due to the degeneracy of the Hamiltonian structure of the $2D$ Euler equations Morrison (1998). Basing on finite dimensional truncations of the $2D$ Euler equations which preserve the Hamiltonian structure Zeitlin (1991), we propose finite dimensional approximations of the stpchastic flow dynamics possessing the same large scale attractor structure as the original $2D$ Euler and Navier-Stokes equations. Those approximations are related to the geometry of $su(N)$ matrix algebras.

Such finite dimensional approximations are amenable to a systematic statistical mechanics treatment. In our work we lay grounds for further research into stochastic Euler flows by addressing the following basic questions:

(i) Under which circumstances do degenerate Hamiltonian dynamics lead to a Liouville theorem?

(ii) If the dissipation and stochastic forcing are taken into account, which are the hypotheses leading to generalized Gibbs distributions for stochastic $su(N)$ Euler flows?

(iii) Can correlations functions of the $su(N)$ approximations of the $2D$ Euler equations be studied using the methods of random matrix theory?

3. Hamiltonian flows on Poisson manifolds equipped with Riemannian structure.

Let (M, J, g) be a smooth n-dimensional manifold equipped with Riemannian metric g and Poisson structure J. The following mathematical question turns out to be relevant in applications to fluid dynamics: how to construct a measure on M invariant with respect to any Hamiltonian vector field on M?

If the Poisson bi-vector field J is non-degenerate, the question admits a very simple answer: let $\omega = J^{-1}$ be the symplectic form on M. Then the symplectic measure $\omega^{n/2}$ is invariant in virtue of the Liouville theorem.

For degenerate Poisson structures the above construction does not apply. Instead we pose the following question: under which conditions the Riemannian measure $\mu_g(dx) = \sqrt{|g|}d^n x$ is invariant under all Hamiltonian flows on M:

$$div_{\sqrt{|g|}} J(H, \cdot) = 0 \tag{1}$$

Explicitly,

$$\left(\left(\frac{1}{\sqrt{|g|}}\partial_i \sqrt{|g|}\right) J^{ij} + \partial_i J^{ij}\right)\partial_j H = 0,$$

The above identity is satisfied for all functions H if the expression in brackets is equal to zero, which is equivalent to

$$\nabla_i J^{ij} = 0, \tag{2}$$

where ∇_i is the covariant derivative corresponding to metric tensor g.

Condition (2) is certainly satisfied if the Poisson structure is compatible with the metric structure, i. e. if

$$\nabla_k J^{ij} = 0. \tag{3}$$

While we are not able to analyze this compatibility condition in general, we can present an important class of examples of Poisson manifolds for which (3) is satisfied. As it turns out these are the most relevant examples in the context of fluid dynamics.

Namely, consider a semi-simple compact Lie algebra $\mathfrak{G}$ and the corresponding dual space $\mathfrak{G}^*$. Poisson structure on $\mathfrak{G}^*$ is given by Lie-Poisson bracket. Using explicit co-ordinates,

$$\{f, g\}(\omega) = C^i_{jk} \omega_i \frac{\partial f}{\partial \omega_j} \frac{\partial g}{\partial \omega_k}, \tag{4}$$

where C^i_{jk} are the structure constants of $\mathfrak{G}$. Let $g_{ij} = C^p_{iq} C^q_{pj}$ be the Killing form on $\mathfrak{G}^*$. It is positive definite. The corresponding scalar product on $\mathfrak{G}$ is $\mathfrak{G}$-invariant:

$$\langle [a, b], c \rangle + \langle b, [a, c] \rangle = 0, \text{ for any } a, b, c, \in \mathfrak{G}. \tag{5}$$

In terms of C and g tensors the invariance condition reads:

$$C_{i,jk} + C_{k,ji} = 0, \tag{6}$$

where $C_{i,jk} = g_{i,m} C^m_{jk}$. In other words there exists a basis of a semi-simple Lie algebra $\mathfrak{G}$ for which structure constants are completely anti-symmetric. Now it is very easy to prove that condition (3) is satisfied for Lie-Poisson bracket and Killing metric on $\mathfrak{G}^*$: the metric tensor g_{ij} is constant, therefore the corresponding Christoffel symbols are zero. Hence

$$\nabla^i J_{ij} = \partial^i J_{ij} = C^i_{ij} = \langle e^i, [e_i, e_j] \rangle = g^{im} \langle e_m, [e_i, e_j] \rangle = -g^{im} \langle [e_i, e_m], e_j \rangle = 0.$$

Here $\{e_i\}$ is a basis of $\mathfrak{G}$, is the corresponding dual basis of $\mathfrak{G}^*$.

Starting with the invariant Riemannian measure on $\mathfrak{G}^*$ it is easy to construct a class of probability measures with respect to a Hamiltonian flow with Hamiltonian H: for any integrable function $W = F(H, C_1, C_2, \ldots)$ on $\mathfrak{G}$ which depends only on the Hamiltonian H the Casimirs of the Lie-Poisson structure $C_1, C_2, \ldots$ the probability measure

$$\mu_W(dx) = \frac{1}{Z} W \sqrt{|g|} dx \tag{7}$$

is invariant with respect to $J(H, \cdot)$. Here Z is a normalization constant.

4. Gibbs measures for $su(N)$ approximations of stochastic $2D$ Euler equation.

The finite dimensional approximation to the $2D$ Euler equation developed in Zeitlin (1991) can be described as follows: Let $\mathfrak{G} = su(N)$, the algebra of traceless anti-Hermitean $N \times N$ matrices. We assume that N is an odd integer. We identify $su(N)$ with its dual space using the non-degenerate Killing form defined in terms of the trace:

$$< A, B >= -\frac{4}{N^3} Tr(AB)$$

There exists the following basis on $su(N)$:

$$C_N = \left\{ \vec{n} \in \mathbf{Z}^2 \mid \vec{n} \neq 0; -\frac{(N-1)}{2} \leq n_{1,2} \leq \frac{(N-1)}{2} \right\}. \tag{8}$$

Then any element $\Omega \in su(N)$ can be written in the form

$$\Omega = \sum_{\vec{n} \in C_N} \omega_{\vec{n}} J_{\vec{n}}, \tag{9}$$

where $\bar{\omega}_{\vec{n}} = \omega_{-\vec{n}}$ and the set of basic matrices $\{J_{\vec{n}}\}_{\vec{n} \in C(N)}$ has the following properties:

$$[J_{\vec{n}}, J_{\vec{m}}] = N \sin\left(\frac{2\pi}{N}\vec{n} \wedge \vec{m}\right) J_{\vec{n}+\vec{m}}, \tag{10}$$

$$J_{\vec{n}}^{\dagger} = -J_{-\vec{n}}, \tag{11}$$

$$Tr(J_{\vec{n}}) = 0, \quad <J_{\vec{n}}, J_{\vec{m}}> = \delta_{\vec{n}+\vec{m}}. \tag{12}$$

Consider the following system of ODE's on $su(N)$:

$$\dot{\Omega}(t) = [\Psi(t), \Omega(t)],$$

$$\Omega(0) = \Omega_0^{(N)}, \tag{13}$$

where

$$\Omega = \sum_{\vec{n} \in \mathbf{C}^N} \omega_{\vec{n}} J_{\vec{n}}, \Psi = -\sum_{\vec{n} \in \mathbf{C}^N} \frac{\omega_{\vec{n}}}{\vec{n}^2} J_{\vec{n}}, \tag{14}$$

As shown formally in Zeitlin (1991) and proved rigorously in Gallagher (2002) equations of motion for components $\omega_{\vec{n}}$ resulting from (13) converge to the $2D$ incompressible Euler equation on the torus $[0, 2\pi]^2$ written in terms of Fourier components $\{\omega_{\vec{n}}\}_{\vec{n} \in \mathbf{Z}^2}$. From now on we will also refer to the components of finite-dimensional matrix Ω in the basis of $J_{\vec{n}}$'s as Fourier components. Clearly, matrix Ω is related to the vorticity field, matrix Ψ - to the stream function.

It is easy to show that the nonlinear term $[\Psi, \Omega]$ in the right hand side of (13) is in fact Hamiltonian with respect to Lie-Poisson structure on $su(N)$ with the Hamiltonian

$$H = -<\Psi, \Omega>.$$

All Casimir functions of the Lie-Poisson bracket on $su(N)$ can presented in the form

$$C_n = -\frac{4}{N^3} Tr(\Omega^n), n = 2, 3, 4, \ldots \tag{15}$$

Therefore, the $su(N)$ approximation of Euler dynamics has the same set of integrals of motion as the full $N = \infty$ flow (but only finitely many of these integrals are independent).

Using the result of the previous section we conclude that any probability measure of the form

$$d\mu_W(d\Omega) = \frac{1}{Z} W(H(\Omega), C_2(\Omega), C_3(\Omega), \ldots) d\Omega \tag{16}$$

is invariant with respect to (13) for any integral of motion W. Here $d\Omega$ is the Haar measure on $su(N)$.

In order to fix the shape of function W we need to account for noise and dissipation. The simplest stochastic generalization of (13) is:

$$d\Omega = [\Psi, \Omega] dt - \nabla G + \sqrt{2\sigma^2} dB_t, \tag{17}$$

where G is an integral of motion and B_t is the Brownian motion on $su(N)$, which means that for any two matrices $P, Q \in su(N)$

$$\mathbb{E}\left(Tr(PB_t)Tr(QB_s)\right) = \langle P, Q \rangle s \wedge t. \tag{18}$$

It is straightforward to verify the following statement:

$$\mu_G(d\Omega) = \frac{1}{Z} e^{-\frac{G}{2\sigma^2}} d\Omega \tag{19}$$

is an invariant probability measure of the stochastic ODE (17) provided it is normalizable.

It is only possible to obtain such a result due to a special algebraic structure of the dissipative term and the stochastic forcing: function G is an integral of motion, the covariance matrix of noise is the Killing form which is an invariant of the adjoint action of $su(N)$. The large N-limit of this noise is space-time white, which can be too restrictive for applications and presents mathematical difficulties in extending our results to $N = \infty$ case.

Setting these important questions aside for now, we conclude that the invariant measure of the stochastic $su(N)$ Euler dynamics is given by the Hermitean matrix model (19) with the potential $G(H, C_2, \ldots)$.

5. Correlation functions for $su(N)$ approximations of stochastic $2D$ Euler equation.

We will now consider several special choices for the dissipation potential G and the discuss the properties of the associated invariant measures.

5.1. Energy-enstrophy ensemble.
Let G be a linear function of energy and enstrophy,

$$G = -\frac{1}{2}\nu_E < \Omega, \Psi > +\frac{1}{2}\nu_C < \Omega, \Omega >, \tag{20}$$

where $\nu_E > 0$ determines the strength of the 'energy' dissipative term ∇H and $\nu_C > 0$ - the strength of the 'enstrophy' dissipative term ∇C_2. The invariant measure is Gaussian,

$$\mu_G(d\Omega) = d\Omega \exp\left\{\frac{1}{2\sigma^2}\left(\nu_E < \Omega, \Psi > -\nu_C < \Omega, \Omega >)\right)\right\}, \tag{21}$$

All vorticity correlation functions can be computed by differentiating the generating functional:

$$\mathbb{E} \exp\left\{\sum_{\vec{n} \in C_N} \omega^{\vec{n}} j_{\vec{n}}\right\} = exp\left\{\frac{1}{2}\sum_{\vec{n} \in C_N} \frac{\sigma^2 \vec{n}^2}{\nu_E + \vec{n}^2 \nu_C} j_{\vec{n}} j_{-\vec{n}}\right\}. \tag{22}$$

In particular, the vorticity-vorticity correlation function in real space is

$$\mathbb{E}\left(\omega(\vec{r})\omega(\vec{0})\right) = \frac{\sigma^2}{\nu_C}\left(\delta_N(\vec{r}) - \sum_{\vec{n} \in C_N} \frac{e^{i\vec{n}\cdot\vec{r}}}{1 + \vec{n}^2 R}\right), \tag{23}$$

where

$$R = \frac{\nu_C}{\nu_E} \tag{24}$$

and $\delta_N(\vec{x} - \vec{y})$ is a δ-function acting on the space of mean-zero Fourier polynomials $f(\vec{y}) = \sum_{\vec{n} \in C_N} f_{\vec{n}} e^{\vec{n} \cdot \vec{y}}$. In the limit $N \to \infty$, the distribution δ_N converges to δ-function acting on the space of mean zero smooth functions on the torus.

In the limit $N \to \infty$ and $R << 1$ finite size effects can be neglected and the expression for the correlation function simplifies considerably:

$$\mathbb{E}(\omega(\vec{r})\omega(\vec{0})) = \frac{\sigma^2}{\nu_C}\left(\delta(\vec{r}) - \frac{1}{R}\int_0^\infty dx e^{-\cosh(x)\frac{r}{\sqrt{2\pi R}}}\right) = \frac{\sigma^2}{\nu_c}\left(\delta(\vec{r}) - \frac{1}{R}K_0\left(\frac{r}{\sqrt{2\pi R}}\right)\right), \quad (25)$$

where K_0 is the modified Bessel function of zero order. The small- and large-separation asymptotics immediately follow:

$$\mathbb{E}(\omega(\vec{r})\omega(\vec{0})) = \log\left(\frac{r}{2r_0}\right) - \gamma + O\left(\left(\frac{r}{r_0}\right)^2\right), 0 < r << r_0, \quad (26)$$

$$\mathbb{E}(\omega(\vec{r})\omega(\vec{0})) = -\sqrt{\frac{\pi r_0}{2r}}e^{-\frac{r}{r_0}}, r >> r_0, \quad (27)$$

where γ is the Euler's constant and

$$r_0 = \sqrt{2\pi\frac{\nu_C}{\nu_E}} \quad (28)$$

is the length scale created by the balance between two dissipative terms ∇H and ∇C_2. The slowly decaying vorticity correlations for $r < r_0$ is a sign of coherent structures present in the the steady state. The nature of these structures is a subject of further investigation.

5.2. Gaussian Unitary Ensembles.
If $G = \nu_C < \Omega, \Omega >$, the vorticity field in equilibrium is a two dimensional Gaussian white field with covariance

$$E(\omega(\vec{x})\omega(\vec{y})) = \frac{\sigma^2}{\nu_C}\delta(\vec{x} - \vec{y}). \quad (29)$$

The vorticity field has to be understood in the distributional sense: for any continuous function f on the torus define

$$\omega(f) = \int d^2x \omega(\vec{x})f(\vec{x}).$$

The defining property of ω is that $\omega(f)$ is a mean zero Gaussian random variable such that

$$\mathbb{E}(\omega(f)\omega(g)) = \frac{\sigma^2}{\nu_C}\int d^2x f(\vec{x})g(\vec{x})$$

for any $f, g \in C(\mathbf{T})$.

In particular, let

$$\Gamma(D) = \int_D d^2x \omega(\vec{x}) \quad (30)$$

be a circulation around ∂D, where D is a simply connected domain of the torus. We have the following results for the covariance of circulations:

$$\mathbb{E}\left[\Gamma(D)\Gamma(D')\right] = \frac{\sigma^2}{\nu_C}Area\left(D\bigcap D'\right). \quad (31)$$

The full equilibrium measure in the case $\nu_E = 0$ coincides with Gaussian Unitary Ensemble (GUE) of the random matrix theory, Mehta (1991):

$$d\mu(\Omega) = d\Omega \exp\left\{ \frac{2}{N^3\sigma^2} \left(\nu_C Tr(\Omega^2) \right) \right\}, \tag{32}$$

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ of matrix $i\Omega$ are invariants of deterministic Euler dynamics. In the presence of stochastic forcing, the equilibrium distribution of the eigenvalues can be obtained by integrating out unitary degrees of freedom in (32) which leads to the classical formula of random matrix theory:

$$d\mu(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \Delta(\lambda)^2 e^{-\frac{2}{N^3\sigma^2} \sum_{k=1}^{N} \lambda_k^2}. \tag{33}$$

Here $\Delta(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j)$ is the Van-der-Monde determinant. We see that eigenvalues behave like a system of particles in the confining harmonic potential which experience logarithmic repulsion.

5.2.1. GUE: Convergence to equilibrium. If the dissipation potential G is given by the enstrophy it is possible to calculate not only the stationary measure but also the one-dimensional measure $\mu_t(d\Omega)$ for all moments of time $t > 0$ for the initial condition on vorticity $\Omega = 0$. The idea of the calculation is to look for the solution as a function of integrals of motion thus reducing the problem to the linear one analyzed in Uhlenbeck & Ornstein (1930). The answer is

$$\mu_t(\Omega) = \frac{1}{Z(t)} e^{-\beta(t)<\Omega,\Omega>}, \tag{34}$$

where

$$\beta_t = -\frac{1}{\sigma^2} \frac{1}{1 - e^{-2\nu t}}, \tag{35}$$

$$Z_t = Z_0 \beta(t)^{-\frac{N^2-1}{2}}. \tag{36}$$

Here Z_0 is the normalization factor for the initial distribution concentrated at $\Omega = 0$. The above result illustrates the point reiterated repeatedly in Bouchet & Venaille (2010) that under many circumstances the non-equilibrium measures for the Euler dynamics remain close to the equilibrium ones.

Finally let us remark that the process which describes the evolution of eigenvalues of Ω for the case at hand corresponds to Dyson Brownian motions on $su(N)$:

$$d\lambda_t(i) = \sum_{j \neq i} \frac{2\sigma^2 dt}{\lambda_t(i) - \lambda_t(j)} - \nu_C \lambda_t(i) + \sqrt{2\sigma^2}[dB_t(i) - \frac{1}{N}\sum_{j=1}^{N} dB_t(j)], \tag{37}$$

where $\{B_t(j)\}_{j=1}^{N}$ are independent Brownian motions on the real line, see Biane (2009) for more details.

6. Conclusions.

We have developed a formalism for calculating the invariant measures for $su(N)$ truncations of the $2D$ Euler flow in a periodic box perturbed by dissipative and stochastic forces. We illustrated the formalism by considering the dissipation potential linear in energy and enstrophy

and calculating all correlation functions of vorticity in this case in the limit of large N. In the special case when the dissipation potential depends on the enstrophy only the Euler dynamics possess an extended $U(N)$-invariance which allows to calculate not just the invariant measure but also to study the approach to the equilibrium.

The stochastic forcing we used corresponds to space-time white noise in the limit $N \to \infty$. The corresponding vorticity field exists only in the distributional sense. An interesting mathematical problem is to prove rigorously that the measures we constructed are indeed the invariant measures of the $2D$ Euler equation driven by space-time white noise. A closely related problem of $2D$ Navier-Stokes equation driven by a rougher noise (space-time white noise in the velocity space) has been analyzed in DaPrato & Debussche (2002).

Let us conclude with two important remarks. Firstly the developed formalism allows one to construct non-Gaussian multi-modal measures and therefore model and study the phenomenon of random switching in $2D$ Euler flow using the formalism of large deviations, see Freidlin (1996). Secondly, an important further step would be to generalize those results to non-equilibrium stochastic forces (without detailed balance). We plan to address these two last points and related problems in the near future.

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